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Advanced Analytic Number Theory: L-Functions

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Dedicated to my teachers
Leon Ehrenpreis
and
Harold N. Shapiro

Contents

Preface

The purpose of this book is to give an exposition of the analytic theory of L-functions following the ideas of harmonic analysis inaugurated by Tate and Weil. The central theme is the exploitation of the Local Langlands' Correspondence for GL_n to obtain results that apply to both Artin-Hecke L-functions associated to representations of the Weil group and to automorphic L-functions of principal type on GL_n. In addition to establishing functional equations, explicit formulas and non-vanishing theorems, essential ingredients in any discussion of generalized prime number theorems, we also derive lower bound estimates for discriminants and conductors.

The author's intention has been to make available to a broad mathematical audience those aspects of the theory of L-functions that are closely related to the modern interconnections between the analytic theory of numbers and the theory of group representations.

A noteworthy characteristic property of number fields that distinguishes them from function fields is the existence of archimedean primes. These primes not only make their appearance as gamma factors, but also play a crucial role in controlling the analytic growth of L-functions and in the distribution of zeros and poles. In this spirit, we have placed a great deal of emphasis in the study of archimedean L-factors.

A detailed description of the contents of each chapter can be obtained from the introductory section. A survey of the local theory of root numbers is also included as an appendix.

This book is based on lectures given by the author over a period of several years first at the University of Illinois and more recently at the Graduate School and University Center of the City University of New York. The author acknowledges the help he has received from many of his colleagues and students. The appendix, written in collaboration with Aaron Wan, is the result of many fruitful discussions about root numbers, and for these the author is particularly thankful.

Carlos Julio Moreno

North Salem, December 14, 2004

Advice to the Reader

It is necessary for the reader of this survey to be familiar with the following topics:

(a) He must have a good elementary knowledge of the theory of analytic functions of one complex variable, as contained for instance in L. Ahlfors, *Complex Analysis* [**1**].

(b) He must know the basic definitions in the theory of functions of a real variable at the level of H. Royden, *Real Analysis* [**150**] (see also Rudin [**151**]).

(c) He must have taken a graduate level course in number theory at a level comparable to that in E. Hecke, *Lectures in the Theory of Algebraic Numbers* [**76**] (see also [**104**],[**105**],[**207**]). In particular, he must know the three fundamental results of algebraic number theory: the unique factorization of ideals, the finiteness of the class group, and the Dirichlet unit theorem.

If he wishes to profit from the reading of the proofs in the individual chapters, each written in an increasing order of sophistication, he must also have an acquaintance with certain concepts and results which unfortunately appear scattered throughout the mathematical literature. In the following we suggest a road map that can facilitate his reading of the relevant literature and prepare him for further study and research of the relevant topics.

Chapter I. The rudimentary knowledge of abstract harmonic analysis needed can be acquired by selectively reading those chapters in L. Loomis, *An Introduction to Abstract Harmonic Analysis* [**115**] or in the short and elegant monograph by G. Bachman, *Elements of Abstract Harmonic Analysis* [**8**], which deal specifically with topological groups, Haar measure, character and dual groups, and Fourier analysis on locally compact abelian groups. An exposition of these topics that is still worth reading from a historical point of view can be found in A. Weil, *L'integration dans les groupes topologiques et ses applications* [**209**] and in L. Pontrjagin, *Topological Groups* [**144**].

An excellent introduction to the basic theory of distributions can be gleaned from the first two chapters in L. Hormander, *The Analysis of Linear Partial Differential Operators I* [**80**].

The Appendix in Chapter I, §3 on principal L-functions on $GL(n)$ is meant to serve only as an outline of how the Hecke-Tate theory on $GL_1(k)$ generalizes to $GL_n(k)$, and requires some concepts from representation theory not covered in the earlier sections, but which are essential ingredients in the understanding of the theory of automorphic L-functions on $GL_n(k)$. A more detailed presentation can be found in A. Weil, *Dirichlet Series and Automorphic Forms* [**210**], and in H. Jacquet and R.P. Langlands, *Automorphic Forms on GL(2)* [**86**] for the theory on $GL(2)$; a treatment of the higher dimensional case, $GL(r), r \geq 3$, can be found in H. Jacquet, *Principal L-functions of the Linear Group* [**84**] and in R. Godement and H. Jacquet, *Zeta Functions of Simple Algebras* [**69**].

Chapter II. The reader is expected to be familiar with the basic theory of linear representations of finite groups up to and including knowledge of Brauer's Theorem as contained for instance in J.-P. Serre, *Linear Representations of Finite Groups* [**164**], §§1-10.

For a deeper discussion of Weil and Weil-Deligne groups the reader can supplement his study by consulting J. Tate, *Number Theoretic Background*, [**187**].

The ramification theory needed to understand the properties of conductors from the point of view of the Herbrand distribution is given in C.J. Moreno, *Advanced Analytic Number Theory* [**127**]. The definitions and elementary properties of the absolute Weil group of a number field given in Chapter II, §2.3 are taken from the report in A. Weil, *Sur la theorie de corps de classes* [**211**] and from the detailed presentation in [**212**]. A modern exposition is also given in J. Tate's article referred to above [**187**].

The descriptive survey of the local Langlands correspondence for $GL(n)$ given in Chapter II, §15 uses standard terminology about group representations; for these the reader can consult A. Kirillov, *Elements of the Theory of Representations* [**101**] or the excellent *Encyclopedic Dictionary of Mathematics*, second edition, published by the Mathematical Society of Japan. For a more detailed and rigorous presentation the reader can consult the excellent treatment in A. Knapp, *Representation Theory of Semisimple Groups*, [**100**].

Chapter III. The essential requirements for this chapter have been kept to a minimum. An acquaintance with the classical Hadamard theory of entire functions of order 1 and their associated Weierstrass products would be sufficient. This material is found in L. Ahlfors, *Complex Analysis* [**1**], Chapter IV, §3, and Chapter V, §§1, 2 on harmonic and subharmonic functions. The principal results of the chapter deal directly with the analytic properties of archimedean L-factors, also known as gamma factors; for these the reader cannot do better than consult the classical treatment in E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* [**215**], particularly Chapters XII and XIII.

Chapter IV. Some acquaintance with the classical Mellin transform as well as knowledge of the conditions under which its inverse exists is needed. The basic theory can be deduced from that of the Fourier transform on the real line. The latter is developed in Y. Katznelson, *An Introduction to Harmonic Analysis* [**98**], Chapter VI (see also [**115**]). A more classical treatment of the Mellin transform is in E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* [**194**].

The integral formulas for the Herbrand distribution used in this chapter are discussed in great detail in the author's monograph [**127**] already cited above, particularly Chapter IX.

Chapter V. Those properties of the gamma and related digamma functions, which are not proved in this chapter, can be found in the treatise of Whittaker and Watson [**215**].

Chapter VI. For an understanding of the geometric example in §1.1, the reader should be acquainted with the elementary theory of zeta and L-functions of algebraic curves over finite fields, particularly as it applies to elliptic curves. The theory of these functions is explained in great detail in the author's monograph [**127**], Chapters III and IV. Some background on the origin and significance of the arithmetic example in §1.1 can be acquired in J.-P. Serre, (i) *A Course in Arithmetic*, [**169**] Chapter VII, and (ii) *Abelian ℓ-adic Representations*, [**165**] Chapter I, (see also G. H. Hardy, *Twelve Lectures on the Work of Ramanujan* [**71**], Chapter XII). In (ii) the reader will also find a useful but brief discussion of the relations between equipartition of conjugacy classes, L-functions, and Chebotarev's density theorem.

The theory of irreducible complex linear representations of a locally compact group G needed in Chapter VI, §1.2, is an extension of the classical theory for compact groups. The reader can find an elementary introduction to the theory of linear complex representations of compact Lie groups, including the unitary group $SU(n)$ and the Peter-Weyl theorem, in W. Fulton and J. Harris, *Representation Theory: A First Course* [**55**], Chapter XX. The reader will also find an elementary treatment of the Bohr compactification in Y. Katznelson, *Harmonic analysis* [**98**], p. 192.

The theory of Eisenstein series for maximal parabolic subgroups of $GL(n)$ used in the proof of the Jacquet-Shalika non-vanishing theorem is a generalization of the Hecke-Riemann method for proving functional equations and uses the properties of theta series developed in Chapter I. The reader who is unfamiliar with these topics or who wishes to acquire a working knowledge of the theory may want to consult the following treatises on the general theory of Eisenstein series: (i) T. Kubota, *Elementary Theory of Eisenstein Series* [**102**], (ii) A. Borel, *Automorphic Forms on* $SL_2(\mathbf{R})$ [**16**], (iii) C. Moeglin and J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series* [**125**], (iv) H.-Chandra, *Automorphic Forms on Semisimple Lie Groups* [**74**]. A third alternative approach to the theory of Eisenstein series is due to A. Selberg (for the group $SL(3)$) and uses the Fredholm theory of operator equations. This has been developed independently by J. Bernstein (unpublished) and in Shek-Tung Wong [**216**].

The reader who wishes to understand the technical aspects behind the powerful Langlands-Shahidi method can consult the original exposition in F. Shahidi, *Functional Equation Satisfied by Certain L-Functions* [**178**]. The approach there is quite general and applies with minor modifications to Chevalley groups. The reader can acquire the necessary knowledge of Whittaker models and Whittaker functions in J. Shalika, *The Multiplicity One Theorem for* GL_n [**170**] and in H. Jacquet, *Fonctions de Whittaker associees aux groupes de Chevalley* [**85**].

Finally we must describe the method followed for cross-references. Theorems have been numbered continuously throughout each chapter; the same is true for lemmas, for definitions, and for the numbered formulas. The few corollaries and propositions appearing in each chapter have not been numbered; all concepts, except those that are assumed to be known, are listed in the index at the end of the book, with reference to the page(s) where they appear. Formulas are numbered for the sole purpose of reference. A reference given as "Chapter IV, §3, Theorem 3" refers to Theorem 3 in §3 of Chapter IV of this book; a reference such as "[16], Chapter IV, §5" refers to Chapter IV, §5 in item [16] of the bibliography. A number of parenthetical remarks appear throughout the text. For the most part, these are meant to amplify some points in the foregoing discussion and can be skipped on a first reading.

Introduction

Chapter I. The pioneering work of Euler on the zeta function and Dirichlet's subsequent introduction of L-functions provided a major conceptual advance in our understanding of the set of all prime numbers. The identity

$$\prod_p \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which exhibits clearly the multiplicative and the additive properties of the ordinary integers, is a cornerstone of the modern analytic theory of numbers, a theory which evolved in the course of the last century into a vast and powerful enterprize. Within this new framework, formal infinite products indexed by the primes, often called "*product formulas*", are intended to be the carriers of both algebraic and geometric as well as analytic and spectral information. The genesis of these ideas can be traced back to Hilbert's formulation of his **reciprocity law** for the residue symbol, as well as to the work done by Herbrand, Chevalley and Weil who forged the old arithmetic concepts of unique factorization with the topological concepts present in Tychonoff's Theorem on infinite products of locally compact spaces to produce the basic tools of **harmonic analysis** on the groups of ideles and adeles.

On the analytic side, the study of zeta and L-functions dates back to one of Riemann's three proofs of the functional equation for $\zeta(s)$ - the one based on the transformation formula for the elliptic theta function:

(**Theta Transformation**) $$\sum_{n\in\mathbf{Z}} e^{-(n+\alpha)^2\pi/x} = \sqrt{x}\sum_{n\in\mathbf{Z}} e^{-n^2\pi x+2\pi i n\alpha}.$$

To gain a better understanding of the origin of these ideas, we recall that a proof of the theta transformation formula is based on the solution to the following[1].

"**Initial value problem for heat conduction**":

On a closed linearly extended heat conductor (a wire for instance) of length 1, find a solution $u(x,t)$ to the heat equation

$$u_{xx} - u_t = 0,$$

with continuous derivatives up to the second order for all values of the variable x and for all $t > 0$, having a prescribed set of

[1]See Courant-Hilbert [**37**]: *Methods of Mathematical Physics* , vol. II, p. 197

values

$$u(x, 0) = \psi(x) \qquad at \qquad t = 0.$$

The function $\psi(x)$ is assumed to be everywhere continuous and bounded. It being assumed that both u and ψ as functions of x are periodic of period 1.

From physical considerations based on the superposition principle and the idea of separation of variables one is lead to the two solutions

$$u(x,t) = \int_0^1 \psi(\xi)\Big\{1 + 2\sum_{\nu=1}^{\infty} e^{-4\pi^2\nu^2 t}\cos 2\pi\nu(x-\xi)\Big\}d\xi,$$

and

$$u(x,t) = \int_0^1 \psi(\xi)\Big\{\frac{1}{\sqrt{\pi t}}\sum_{\nu=-\infty}^{\infty} e^{-(x-\xi-\nu)^2/4t}\Big\}d\xi.$$

Using an elementary lemma from the calculus of variations, the uniqueness of the solution u, itself a consequence of the energy estimate $\frac{d}{dt}\int_0^1 u^2 dx \leq 0$, leads immediately to an equivalent form of the theta formula. The functional equation of the Riemann zeta function in its usual form is then obtained by applying a suitable Mellin transform to both sides of the theta identity.

New ideas in functional analysis and the rise of the theory of distributions made it possible to develop the above elementary argument into a powerful technique capable of new applications to two closely related branches of mathematics: to zeta and L-functions and to infinite dimensional group representations.

Weil was the first to formulate in the local to global language of **Tate's Thesis** the relation between functional equations and the uniqueness of certain zeta distributions. At the heart of his new interpretation is an old idea of Weil concerning the determination of *relative invariant measures*, which had already appeared in his classic work on integration over topological groups. The subsequent full development of these ideas by Weil himself, and the light they shed on Siegel's insights into the Poisson Summation Formula, established beyond doubt the fruitfulness of this new point of view. **Chapter I** is an elementary introduction to this circle of ideas. It includes a detailed presentation of Tate's Thesis together with Weil's ideas about distributions and zeta functions. In this framework, the local calculations at the infinite prime are simply an elaboration of the well known results of Hadamard and M. Riesz on homogeneous distributions. Our elementary presentation can serve as an introduction to the theories of Sato concerning hyperfunctions defined by complex powers of polynomials and zeta functions on prehomogeneous spaces and to the theory of the Bernstein polynomial. We have also given a brief outline of the Jacquet-Godement theory of principal L-functions which is the natural generalization to GL(n) of Hecke's L-functions viewed as automorphic objects on GL(1). The principal results of Chapter I are those of Tate and Weil concerning the uniqueness of local and global zeta distributions (Lemma 9, Theorem 22) and the functional equation (Theorem 23).

Chapter II. The essential nature of Artin's reciprocity law, in Chevalley's formulation, is the identification of two seemingly distinct objects of interest in the arithmetic of number fields:

(i) *The Pontrjagin dual of the Galois group of the maximal abelian extension of a number field k,*

(ii) *The characters of finite order of the idele class group* $\mathcal{C}\ell(k) = k_{\mathbf{A}}^{\times}/k^{\times}$,

together with the attendant functoriality properties describing the behavior of the objects in (i) and (ii) under finite extensions (restriction) and the norm map (induction).

As is well known, Artin constructed L-functions that generalize those of Dirichlet in the non-abelian case. The main gluing ingredient being the Artin-Brauer theorem, which describes how the characters of a finite group can be expressed as linear combinations of those arising by induction from abelian groups. This, and the fact that Hecke had constructed L-functions associated to arbitrary characters of the idele class group, lead to the possibility of amalgamating these two types of L-functions (Artin's and Hecke's) into a single analytic package. This challenge was taken up by Weil, who based his construction on the existence and uniqueness of the group extension

$$1 \longrightarrow \mathcal{C}\ell(K) \longrightarrow W(K/k) \longrightarrow Gal(K/k) \longrightarrow 1$$

associated to the fundamental class of class field theory. Weil's construction of the Artin-Hecke L-functions associated to finite dimensional representations of $W(K/k)$ requires a generalization of the Artin-Brauer theorem, which works for infinite non-abelian locally compact groups like $W(K/k)$. This theory was complemented by work of Tamagawa on conductors and archimedean L-factors. The main results of this theory are explained in **Chapter II** which includes the functional equation for the L-functions of representations of $W(K/k)$ (Lemma 3 and the Main Theorem in §13). We also include a brief description of the Dwork-Langlands' theorem on the decomposition of the root number into local factors, a result of fundamental significance for the modern theory of automorphic L-functions on GL(n), and the new non-abelian reciprocity laws of Langlands. A brief survey of the basic Langlands functoriality principle as it applies to $GL(n)$ is included. This principle implies among other things that the Galois class of Artin-Hecke L-functions studied in Chapter II coincides with a subset of the class of principal L-functions studied in Chapter I.

Chapter III. The finite primes in a number field, that is to say, those associated to non-archimedean valuations, are to some extent determined by the well known theorem of Wedderburn concerning the commutativity of finite division rings. In contrast, the infinite primes in a number field, those corresponding to archimedean valuations, are controlled by the fundamental theorem of Gelfand, which identifies the field of complex numbers, up to topological isomorphism, as the only complex commutative Banach division ring with identity. In this context, the main characteristic property which distinguishes number fields from function fields of positive characteristic is the existence of archimedean primes. An old saying in analytic number theory declares that the central questions will remain open until we have gained a better understanding of the nature of all the primes including the archimedean ones. It is in this spirit that the verification of the integrality of L-functions remains a difficult problem intimately related to the presence of gamma factors. A similar instance is the location and distribution of the zeros of L-functions, a problem which rests ultimately on bounds for the local archimedeam L-factors. An important result in this connection is Riemann's product formula:

(**Product Formula**) $$\pi^{-s/2}\Gamma(\frac{s}{2})\prod_p \frac{1}{1-\frac{1}{p^s}} = \frac{1}{s(s-1)}\prod_\rho{}'(1-\frac{s}{\rho}),$$

where the product on the right-hand side is taken over all the zeros of $\zeta(s)$ in the strip $0 < Re(s) < 1$, it being understood that ρ and $1-\rho$ go together.

The study in depth of the archimedean primes and their local L-factors starts with the well known characterization theorem of Bohr-Mollerup:

The Euler gamma function $\Gamma(s)$ is the only function defined for real $s > 0$, which is positive, is 1 at s=1, satisfies the functional equation $s\Gamma(s)=\Gamma(s+1)$, and is logarithmically convex, that is, $\log \Gamma(s)$ is a convex function on the real line.

The uniqueness of the multiplication formula for the gamma function, viewed as an identity between local L-factors, is the arithmetic manifestation of the basic fact that the real archimedean prime admits a unique non-trivial extension: the complex one. The analytic properties of the gamma function stem from the realization that $\log|\Gamma(s)| = Re\, \log\Gamma(s)$ is a harmonic function, a fact already used by Rademacher (and to some extent also by Siegel) to obtain strong Phragmen-Lindelof estimates for $\Gamma(s)$. This circle of ideas have applications to growth estimates for L-functions with at most a finite number of poles.

The above remarks set the tone for the emphasis given to number fields in this book. **Chapter III** is an exposition of these ideas, classically known as convexity estimates. It includes estimates for the automorphic L-functions of principal type on GL(n), as well as for the Artin-Hecke L-functions. The main results in this chapter are Riemann's Formula in Theorem 3 and the estimates in Theorems 14A, 14B, and 14C.

Chapter IV. The linearization of Riemann's product formula, obtained by differentiating the logarithms of both sides, leads to explicit relations between the primes (finite and archimedean) and the non-trivial zeros of $\zeta(s)$. These relations, when integrated against particular functions $\Phi(s)$ over suitable domains in the s-plane provide new identities that form the core of many applications of zeta functions to questions about number fields. The internal symmetries possessed by these formulas seemed to have been first observed by Riemann himself, but were not developed to any great extent until the middle of the last century when Guinand, then Delsarte, and finally Weil found generalizations which exhibited some kind of duality between zeros and primes. In its modern formulation, Weil's explicit formula is an equality between two distributions, one associated to the zeros of an L-function, and another - its "Fourier Dual"- associated to the primes. It has been a goal of many number theorists, a goal which remains unfulfilled to this day, to formulate a "natural operator problem" whose solution has a well defined trace given by both sides of the explicit formulas. An answer to this problem will certainly be accompanied with new insights into the dual relation between the primes and zeros of $\zeta(s)$.

There is no doubt that the explicit formulas have proven their worth in the study of the distribution of prime numbers. They have also been very suggestive in the study of group representations, as already noted by Selberg [2] in reference to the trace formula for $SL(2)$:

> *"... it has a rather striking analogy to certain formulas that arise in analytic number theory from the zeta and L-functions of algebraic number fields."*

An observation that lead him to the introduction of the Selberg zeta function.

Given the new work of Connes, which adds significance to the general notion of *trace*, we expect that the explicit formulas will continue to exercise an ever increasing role in the study of L-functions.

We remark that in the meantime it is possible, using an observation of Delsarte, to calculate certain invariants of operators, e.g. the regularized expression

$$\exp\,(\sum_{\rho} \log\ \rho),$$

an expression that attaches a sense to the notion of determinant associated to zeta and L-functions. Another important application of the explicit formulas, not considered in this book, is Montgomery's work on the *Pair Correlation Hypothesis*, a deep insight on how the zeros of $\zeta(s)$ are jointly distributed. The work of Sarnak and Rudnick has demonstrated that this is a very general phenomenon that applies to the larger class of automorphic L-functions on $GL(n)$. The explicit formulas of Weil are studied in **Chapter IV**. The main results are Theorem 4A and Theorem 4B.

Chapter V. The diophantine properties of number fields are intimately connected with the presence of ramification as was early realized by Kronecker. The well known theorem of Minkowski:

$$|disc(k)| > 1, \qquad k \neq \mathbf{Q},$$

has a representation theoretic interpretation, as follows from a result of Hamburger, to the effect that the trivial representation is the only Galois representation

$$r : Gal(\mathbf{Q}^{sep}/\mathbf{Q}) \longrightarrow GL_n(\mathbf{C}),$$

which is unramified everywhere, including at the archimedean primes. Artin noted that the zeta functions of simple algebras over number fields could be calculated explicitly, and that their precise form, including the exact location of their poles, could be used advantageously to prove the central theorem of class field theory:

[2]See A. Selberg, *"Harmonic Analysis and Discontinuous Groups ..."*, Indian Jour., p. 75.

A simple algebra A over a number field k is trivial if and only if it is everywhere unramified, i.e. if and only if $A_v = A \otimes_k k_v$ *is trivial over the completions* k_v *of k at all the places.*

In the opposite direction, as soon as ramification is allowed, as in the case of quaternion algebras, the internal structure of the zeta function is more complicated and the explicit form of the global root number, the holder of ramification information, becomes relevant. Siegel, and in a more explicit way Stark, discovered analytic formulas (relatives of the Poisson summation formula in the additive case and variants of the explicit formulas in the multiplicative case) capable of exhibiting the non-triviality of the discriminant of a simple algebra over a number field. The explicit form of these analytic approaches to the study of ramification, which uses increasingly more information about zeros and poles of zeta functions, was developed by Stark, Odlyzko, Serre and others. A noteworthy refinement of these ideas involves the use of the explicit formulas; for example, Mestre has obtained lower bounds for the conductor of certain automorphic L-functions whose archimedean components are discrete series. We give in **Chapter V** a development of these ideas. The results in this chapter are best viewed as contributions to arithmetic-algebraic geometry in the sense of Arakelov, Faltings, and Szpiro. The principal results are The Main Formula in §4, Lemmas 2 and 3 and Theorem 6.

Chapter VI. Norbert Wiener's main contribution to analytic number theory was his tauberian theorem which establishes the equivalence of prime number theorems and the non-vanishing theorems for zeta functions. Whether one is interested in the proof of a classical prime number theorem, or in one of its modern versions, e.g. Chebotarev's density theorem, Sato-Tate distributions for the eigenvalues of Hecke operators, analytic proofs of strong multiplicity one, the Deligne or Katz-Sarnak monodromy distribution theorems etc., the key problem in the analytic approach is the proof of the non-vanishing of L-functions on the boundary of the region of absolute convergence.

The most significant progress in this direction has been the generalization by Deligne of the **method of Hadamard and de la Vallée Poussin**, which establishes non-vanishing for L-functions of a wide class that includes all the classical ones (Riemann zeta, Dirichlet L-functions, Artin-Hecke, etc.) and has the potential for further applicability to arbitrary automorphic L-functions, once a weak version of Langlands' functoriality principle is available - *analytic continuation of the L-functions of Langlands* $L(s, r, \pi)$ *for* $Re(s) \geq 1$ *for all finite dimensional representations r of the L-group, except possibly for a simple pole at* $s = 1$ *when* $r = 1$.

There is a second method for proving non-vanishing which rests on the analytic properties of Eisenstein series. In one version it is based on the fact that the **Eisenstein series** $E(s, g)$ of a maximal parabolic subgroup of a connected reductive algebraic group G, as a function of the complex variable s, has analytic continuation to the unitary axis $i\mathbf{R}$, and on the theory of the **Whittaker functional**, which generalizes the theory of Fourier coefficients, and which for a non-trivial additive character ψ of the unipotent group $N_{\mathbf{A}}/N_k$, gives a Fourier coefficient

$$\int_{N_{\mathbf{A}}/N_k} \psi(n)E(s,gn)dn = \varphi(s,g,\psi)\frac{L(s,\pi)}{L(1+s,\pi)},$$

where $\varphi(s,g,\psi) \neq 0$ for $s \in i\mathbf{R}$. This in turn implies that

$$L(1+it,\pi) \neq 0,$$

for all real values of t. An exposition of these ideas, including a brief introduction to the powerful Langlands-Shahidi method is given in **Chapter VI**. In addition, we have included a discussion of a little known result on the non-vanishing of $\zeta(s)$ based on elementary Hilbert space techniques. The reader will also find here a brief discussion to the **generalized Ramanujan conjecture**, a very significant problem, full of implications for the future development of analytic number theory. Some exciting new results have been obtained in this area by Iwaniec and Sarnak and their collaborators ([**155**]) by using methods akin to those presented in **Chapter IV**.

There is another application of the theory of Eisenstein series on metaplectic groups to non-vanishing theorems for L-functions, which is not presented in this book, but because of its non-classical nature deserves to be mentioned in this introductory paragraph. In its simplest manifestation, it arises from the Fourier expansion of Eisenstein series of half integral weight on the Hecke group $\Gamma_0(4N)$. The Fourier coefficients turn out to be essentially Dirichlet L-functions associated to quadratic extensions of the field of rational numbers. By a sieving procedure, average information about these coefficients can be extracted from knowledge of the singularities of the relevant Eisenstein series. In the special case of automorphic L-functions $L(s,\pi)$ on $GL(2)$, a key fundamental idea is Waldspurger's characterization of the values of $L(\frac{1}{2},\pi\otimes\chi)$ in terms of lifts $\tilde{\pi}$ to the metaplectic cover of $GL(2)$. In one of the most striking applications of these ideas to number theory, it has been possible to show the non-vanishing of the twisted Hasse-Weil zeta function $L(s,E\otimes\chi)$ of elliptic curves at the point $s=1$, (the central point of the critical line) for infinitely many quadratic characters χ, as well as for the derivatives. In contrast to the classical situation, this type of non-vanishing has direct applications to problems of diophantine analysis, a situation that is controlled by the well known Birch and Swinnerton-Dyer conjecture. For an excellent exposition of the ideas surrounding this type of non-vanishing, the reader should consult the survey article by Bump, Friedberg, and Hoffstein in [**24**].

Appendix. The functional equation of an Artin L-function, which relates its values at s and at $1-s$, contains an arithmetic factor, known as the root number, whose behavior resembles that of a quadratic Gauss sum

$$\sum_{n\in\mathbf{Z}/N\mathbf{Z}} e^{-2\pi i n^2/N} = \frac{1+i^N}{1+i}\cdot\sqrt{N}.$$

Hasse, who knew well the factorization of the Gaussian sum as a product of similar sums with the integer N replaced by its prime power divisors, suggested that the root number appearing in the functional equation of Artin L-functions should have analogous factorizations into locally defined root numbers. Tate's harmonic interpretation of Hecke's functional equation using ideles established Hasse's conjecture for abelian L-functions. For non-abelian L-functions, the factorization of the root number into a product of local root numbers was achieved by Dwork up to sign, and in full generality by Langlands. Greatly simplified analytic proofs were subsequently produced by Deligne and by Tate.

Langlands' original approach, itself a lengthy and intricate induction which builds on earlier work of Dwork as well as on his own new Gauss sum identities is noteworthy because in it the existence of local root numbers is established by purely local techniques, suggestive of the existence of a non-abelian local class field theory, a theme which is at the heart of Langlands' Functoriality Principle.

The **Appendix** surveys in broad outline the two main components of the local theory of root numbers: (a) a description of the form of the three main local root number identities, (b) the determination of the corresponding three generators for the kernel of Brauer induction for solvable groups. The main goal of this **Appendix** is to introduce the reader to one of the most interesting and profound tools for the study of L-functions and automorphic forms.

APPENDIX A

THE LOCAL THEORY OF ROOT NUMBERS: A Survey

In A.1 and A.2 we review briefly in a simplified form some definitions and results that have been discussed in Chapter I and Chapter II.

1. Artin L-Functions and their Functional Equations

Let K/k be a Galois extension of number fields of degree n. Let

$$\rho : Gal(K/k) \longrightarrow GL(V_{\mathbf{C}}),\ d = dim\,\rho,\ \chi = Tr\,\rho$$

be a finite dimensional Galois representation of degree d and character χ. Let v be a prime of k and w a prime of K above v. Let K_w be the completion of K with respect to the valuation associated to w, and k_v the completion of k with respect to v. Let $\mathbf{F}_w$ be the residue class field of K_w and $\mathbf{F}_v$ that of k_v. K_w/k_v is a Galois extension of local fields and its Galois group, also known as the decomposition group of w, and denoted by $D(w)$, fits into an exact sequence

$$1 \longrightarrow I(w) \longrightarrow D(w) \longrightarrow Gal(\mathbf{F}_w/\mathbf{F}_v) \longrightarrow 1.$$

The group $Gal(\mathbf{F}_w/\mathbf{F}_v)$ is cyclic and is generated by the automorphism $x \mapsto x^{Nv}$, $Nv = \text{Card}\ \mathbf{F}_v$. A choice of pre-image in $D(w)$ will be called the Frobenius element and denoted by Fr_w in the following. For each integer $i \geq -1$, we define a subgroup of $D(w)$ by

$$G_i := \{\sigma \in D(w) :\ \text{ord}_w(\frac{\pi_w^\sigma}{\pi_w} - 1) \geq i\},$$

(π_w a local uniformizer for K_w). In particular $G_{-1} = D(w)$ and $G_0 = I(w)$, the inertia group. The Artin conductor of the representation ρ is defined by

$$f(\rho) = \sum_{i \geq 0} \frac{|G_i|}{|G_0|}(\chi(1) - \chi(G_i)),$$

where $\chi(G_i) = \frac{1}{|G_i|}\sum_{g \in G_i} \chi(g)$. If P_w denotes the maximal ideal of K_w, we define the local conductor ideal of ρ to be $\mathcal{F}_{\rho,w} = P_w^{f(\rho)}$ and note that the global conductor of ρ is given by the ideal in k

$$\mathcal{F}_\rho = \prod_v N_{K_w/k_v}(\mathcal{F}_{\rho,w}).$$

In order to define the Artin L-function associated to the representation ρ we must introduce local factors at each of the primes v of k. Let $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbf{C}}(s) = (2\pi)^{-s}\Gamma(s)$ be the well known gamma functions of the complex variable s. If v is an archimedean prime, we define a local L-factor by the prescription:

(I): If $K_w = k_v = \mathbf{R}$, then $L_v(s,\rho) = \Gamma_{\mathbf{R}}(s)^{dim\,\rho}$.
(II): If $K_w = k_v = \mathbf{C}$, then $L_v(s,\rho) = \Gamma_{\mathbf{C}}(s)^{dim\,\rho}$.
(III): If $K_w/k_v = \mathbf{C}/\mathbf{R}$, then complex conjugation breaks the vector space V into an invariant subspace V^+ and a subspace V^-, where the action is multiplication by -1. In this case we put $L_v(s,\rho) = \Gamma_{\mathbf{R}}(s)^{d^+}\Gamma_{\mathbf{C}}(s)^{d^-}$, where $d^+ = dim\,V^+$ and $d^- = dim\,V^-$.

For v a non-archimedean prime we put

$$L_v(s,\rho) = \frac{1}{det(1_d - N(v)^{-s}\rho(Fr_w)|V^{I(w)})},$$

where $V^{I(w)}$ is the subspace of invariants in $V_{\mathbf{C}}$ under the action of the inertia group $I(w)$. The Artin L-function is then defined by

$$L(s,\rho) = \prod_v L_v(s,\rho),$$

where the product is taken over all primes of k. Standard estimates from analytic number theory and Brauer's fundamental induction theorem show that, as a function of the complex variable s, $L(s,\rho)$, originally defined only in the half plane $Re(s) > 1$, has a meromorphic continuation to the whole s-plane and satisfies a relation of the type

$$L(s,\rho) = W(\rho)(d_k^{dim\,\rho}N(\mathcal{F}_\rho))^{(\frac{1}{2}-s)}L(1-s,\hat{\rho}),$$

where $W(\rho)$, **the global Artin root number**, is a complex number of absolute value 1, and $\hat{\rho}$ is the contragradient representation of ρ, i.e., the representation acting on the dual space of $V_{\mathbf{C}}$; d_k is the absolute discriminant of k. The three classic problems associated with the definition of $L(s,\rho)$ are[1]

(1) The conductor exponent $f(\rho)$ is a positive integer.
(2) The root number $W(\rho)$ is an algebraic integer.
(3) $L(s,\rho)$ is an entire function, unless ρ is the trivial representation.

Artin's proof of the assertion (1) is the crowning achievement of his theory of conductors. The proof of (2) was a veritable tour de force first accomplished by Dwork in the middle 50's. Its solution is intimately related to the existence of local root numbers $W_v(\rho)$ and to the decomposition

$$W(\rho) = \prod_v W_v(\rho),$$

a fact first established by Langlands. Of course problem (3) remains an open question, one whose full significance can only be appreciated from the point of view of Langlands' basic functoriality principle, which in this case ascertains that in fact $L(s,\rho)$ is expected to be an automorphic L-function, one associated to an automorphic representation of the adele group $GL_d(k_{\mathbf{A}})$.

The rest of this appendix will attempt to explicate Langlands' local theory of root numbers, a significant and crucial component of the functoriality principle. Its original development by Langlands, dating back to 1967, has remained for a

[1]These were apparently first formulated by Hasse in his Zahl-Bericht of 1927, but were of course also known to Artin.

long time an almost impenetrable manuscript, dependent on unpublished work of Dwork, some of which is possibly no longer extant.[2]

Our aim is to convey to the reader the beautiful and elegant architecture which underlies Langlands' original local approach to the theory of root numbers. It is based on Deligne's version of Langlands' determination of the relations in the kernel of Brauer induction for solvable groups[3]. In a monograph presently under preparation, Langlands' local theory of root numbers will be fully developed; this will include details of the necessary Dwork relations. It is hoped that by doing so interest will be generated in pursuing the study of Langlands' insight into the possible intertwined and parallel local approach to the development of the local functoriality principle and the local theory of root numbers.

2. The Theory of Local Root Numbers

Let k be a local field, let $\psi_k : k^+ \longrightarrow \mathbf{C}^\times$ be the standard additive character and let dx be the self-dual Haar measure on k^+ with respect to ψ_k, so that

$$d(\alpha x) = \omega_1(\alpha) dx$$

with $\omega_1 : k^\times \longrightarrow \mathbf{C}^\times$ the normalized absolute value on k. We put

$$\hat{f}(y) = \int_k f(x)\psi_k(xy)dx$$

whenever $f \in L^1(k)$. For k a non-archimedean local field (of characteristic 0 and residue class field $\mathbf{F}_q$) we define the local L-function as follows: If $\chi : k^\times \longrightarrow \mathbf{C}^\times$ is a ramified character (e.g. not trivial on the unit group U_k of k), $L(\chi) = 1$. Otherwise, we put

$$L(\chi) = \frac{1}{1 - \chi(\pi_k)},$$

(π_k a local uniformizing parameter for k).

Tate's Local Functional Equation. Take dx to be the self-dual Haar measure with respect to the standard additive character ψ_k on k^+ and put $d^\times x = \frac{dx}{x}$. Let $\chi : k^\times \longrightarrow \mathbf{C}^\times$ be a quasi-character of $k^\times$, and let $\hat{\chi} = \omega_1/\chi$. We then have

$$\frac{\int_{k^\times} \hat{\chi}(x)\hat{f}(x)d^\times x}{L(\hat{\chi})} = \varepsilon(\chi, \psi_k, dx)\frac{\int_{k^\times} \chi(x)f(x)d^\times x}{L(\chi)},$$

where

$$\varepsilon(\chi, \psi_k, dx) = (N\gamma)^{-\frac{1}{2}} W(\chi, \psi_k),$$

$\gamma \in k^\times$ with $ord_k\, \gamma = ord_k \mathcal{D}_k + ord_k \mathcal{F}_\chi$, and

$$W(\chi, \psi) = \frac{1}{\sqrt{N\mathcal{F}_\chi}} \sum_{u \in U_k/(1+\mathcal{F}_\chi)} \hat{\chi}(u/\gamma)\psi_k(u/\gamma).$$

($\mathcal{D}_k$ = different of k, $\mathcal{F}_\chi$ = conductor of χ).

REMARK. Note that $W(\chi, \psi_k)$ is not necessarily of absolute value 1.

[2]Deligne, who seems to have been the first to understand the intricate logic of the group theoretic component of Langlands' argument, was lead to a greatly simplified proof based on analytic techniques. Deligne's proof was further simplified by Tate who used the full extent of the analytic methods first inaugurated by Dwork in his original work on root numbers.

[3]The details of this approach form part of Aaron Wan's Ph.D. CUNY thesis (2004).

Let K/k be a finite Galois extension of local fields and let $W(K/k)$ be the relative Weil group of K/k, e.g. the group extension associated to the canonical class of local class field theory:

$$1 \longrightarrow K^\times \longrightarrow W(K/k) \longrightarrow Gal(K/k) \longrightarrow 1.$$

The absolute Weil group $W_k := W(k_s/k)$, k_s a separable Galois closure of k, is defined as a quasi-compact group (isomorphic to the direct product of $\mathbf{R}$ and a compact group) satisfying the usual "Galois-type" properties: If W_k^c is the closure of the commutator subgroup, then W_k/W_k^c is isomorphic to $k^\times$; if K is a Galois extension of degree d of k, W_k has a subgroup of index d isomorphic to W_K and W_k/W_K is canonically isomorphic to $Gal(K/k)$.

These and all the facts we use about Weil groups are found in Chapters I and II. See also A. Weil, "*Dirichlet Series and Automorphic Forms*", [**210**], as well as the references given there in Chapter XI (see also P. Deligne [**45**]).

In formulating the main result of the local theory we shall use the following lemma due to Weil and proved in Chapter II, §1. (See also Lemma 4, in A. Weil, [**205**]).

LEMMA. (Weil) *Let k be a local field and W_k its absolute Weil group. Let*

$$\rho : W_k \longrightarrow GL(V_{\mathbf{C}})$$

be an irreducible representation in a complex vector space $V_{\mathbf{C}}$. Then there exists a complex number $s \in \mathbf{C}$ such that the twisted representation $\rho \otimes \omega_{-s}$ has kernel, which is of finite index in W_k and therefore has a realization as a representation of $Gal(K/k)$ for some finite extension K of k.(Here we have put $\omega_s = \omega_1^s$).

The above lemma reduces the study of local root numbers on Weil groups to root numbers on Galois groups. ($\omega_s = \omega_1^s$ is an unramified quasi-character.)

The Main Result.

THEOREM. (Existence of Local Constants) *There exists a unique function $W(V, dx)$, satisfying the conditions (1)-(4) below, which attaches a number $W(V, dx)$ in $\mathbf{C}^\times$ to each isomorphism class of quintuples (K, K_s, dx, V, ρ) consisting of a local field K, a separable closure K_s of K, a Haar measure dx on K^+, a complex vector space V of finite dimension and a representation*

$$\rho : Gal(K_s/K) \longrightarrow GL(V).$$

(1) *For every exact sequence of representations*

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

one has $W(V, dx) = W(V', dx)W(V'', dx)$. This condition guarantees that $W(V, dx)$ is well defined on the class of V in the Grothendieck group of representations of $Gal(K_s/K)$ and gives a sense to $W(V, dx)$ for any V which is a virtual representation.

(2) $W(V, adx) = a^{\dim V} W(V, dx)$.

In particular, if the standard character ψ_k is replaced by a non-trivial twist, the behavior of the dual Haar measure dx is also determined. Furthermore, for V virtual of dimension 0, $W(V, dx)$ is independent of the choice of dx. We denote it by $W(V)$.

(3) *If L/K is a finite Galois extension and V_K is the virtual representation of $Gal(L/K)$ induced by a virtual representation V_L, one has*

$$W(V_K) = W(V_L).$$

(4) *If $\dim v = 1$, ρ being defined by a quasi-character χ of $K^\times$, one has*

$$W(V, dx) = W(\chi, \psi_K, dx), \text{ Tate's root number.}$$

The proof of the main result is long and complex and is based upon the following theories:

(i) Weil's generalization of the Brauer induction theorem to W_K (Deligne's version suffices).
(ii) Langlands' determination of the relations that generate the kernel of Brauer induction for finite solvable groups (Deligne's version, which applies to virtual representations, avoids the difficult theory of λ-constants)
(iii) The three fundamental identities for root numbers: Dwork's First and Second Identities and Langlands' Identity.

In the remaining sections of this report we shall give in broad outline the essential structure of the proof.

2.1. Definition of Root Numbers. Recall that if L/K is an extension of local fields, and ψ_K is the standard additive character on K, then $\psi_L = \psi_K \circ Tr_{L/K}$ is the standard character on L. If dx_L is a self-dual Haar measure on L with respect to the standard additive character ψ_L, then the restriction of dx_L to K coincides with the self-dual Haar measure $dx = dx_K$ on K relative to the standard additive character ψ_K. Now let $V = \sum_{i,j} n_{i,j}(H_i, \chi_{ij})$ be a Brauer decomposition of a virtual representation in $R_+^0(Gal(K_s/K)$ (see Deligne [**45**], for the meaning of this and related notations). We suppose the H_i are not pairwise conjugate. Let L_i be the extension defined by H_i and denote by (the same symbol) $\chi_{i,j}$ also the quasi-character of $L_i^\times$ corresponding by local class field theory to $\chi_{i,j}$. With the choice of standard additive characters and Haar measures as above, we put

$$W(V) = \prod_{i,j} W(\chi_{i,j}, dx_{L_i})^{n_{ij}},$$

where $W(\chi_{i,j}, dx_{L_i})$ is the Tate root number.

The main problem is to show uniqueness, that is to say, the definition of $W(V)$ is independent of the Brauer decomposition used in the definition. This is equivalent to showing that any relation in the kernel of Brauer induction (which must necessarily correspond to a virtual representation of degree 0) is assigned the trivial value 1. By Langlands' results on the kernel of Brauer induction this is equivalent to showing that the root numbers assigned to a relation of type I, type II, or type III, as described below, corresponds to the root number identities of Dwork and Langlands'. We will explicate this next. For this purpose we shall utilize Deligne's version of Langlands' determination of the kernel of Brauer induction.

2.2. The First Dwork Identity. Let E/F be an abelian extension, let $\chi_F : F^\times \longrightarrow \mathbf{C}^\times$ be a quasi-character and put $\chi_{E/F} = \chi_F \circ N_{E/F}$. Let $S(E/F)$ be the set of local class field quasi-characters of $F^\times$ associated with E/F, i.e. $\mu : F^\times \longrightarrow \mathbf{C}^\times$ and μ is trivial on $N_{E/F}(E^\times)$. If we use the convention that $W(1_{E/F}, \psi_{E/F}) = 1$, that is, the Tate root number associated with the trivial character is 1, we can express the first identity in the form

(Identity I)
$$\frac{W(\chi_{E/F}, \psi_{E/F})}{W(1_{E/F}, \psi_{E/F})} = \prod_{\mu \in S(E/F)} \frac{W(\mu\chi_F, \psi_F)}{W(\mu, \psi_F)}.$$

This we can express in the equivalent form, working in the context of the Grothendieck group,

$$W(\chi_{E/F} - 1_{E/F}, \psi_{E/F}) = \prod_{\mu \in S(E/F)} W((\chi_F - 1_F) \cdot \mu, \psi_F).$$

This then corresponds to the relation of **Type I** in the kernel of Brauer induction:

(Type I)
$$\chi_{E/F} - 1_{E/F} = \sum_{\mu \in S(E/F)} (\chi_F - 1_F) \cdot \mu.$$

Note that this is equivalent, using the usual notation from the theory of induced representations, to

$$\mathrm{Ind}(\chi_F - 1_F)_{E/F} = \sum_{\mu \in S(E/F)} (\chi_F - 1_F) \cdot \mu.$$

REMARK. This is the content of Dwork's Ph.D. Thesis at Columbia University.

2.3. The Second Dwork Identity. The situation here deals specifically with a "*diamond*" of extensions of degree ℓ:

$$\begin{tikzcd} & K \arrow[dl, dash, "\ell"'] \arrow[dr, dash, "\ell"] & \\ L_1 \arrow[dr, dash, "\ell"'] & & L_2 \arrow[dl, dash, "\ell"] \\ & F & \end{tikzcd}$$

with quasi-characters χ_{L_1} of $L_1^\times$ and χ_{L_2} of $L_2^\times$ satisfying $\chi_{L_1} \circ N_{K/L_1} = \chi_{L_2} \circ N_{K/L_2}$.

We then have

(Identity II)
$$W(\chi_{L_1}, \psi_{L_1}) \prod_{\mu \in S(L_1/F)} W(\mu, \psi_F) = W(\chi_{L_2}, \psi_{L_2}) \prod_{\mu' \in S(L_2/F)} W(\mu', \psi_F).$$

A particular case of this identity when $\ell = 2$ is given by

$$\frac{W(\chi_{L_1}, \psi_{L_1/F})}{W(\tau_{2L_1/F}, \psi_{L_1/F})} \cdot \frac{W(\tau_1\tau_2, \psi_F)}{W(\tau_1, \psi_F)} = \frac{W(\chi_{L_2}, \psi_{L_2/F})}{W(\tau_{1L_2/F}, \psi_{L_2/F})} \cdot \frac{W(\tau_1\tau_2, \psi_F)}{W(\tau_2, \psi_F)},$$

where τ_1 is the nontrivial character in $S(L_1/F)$ and τ_2 is the nontrivial character in $S(L_2/F)$. Working in the context of the Grothendieck group we can rewrite this in the more suggestive form:

$$W(\chi_{L_1} - \tau_{2L_1/F}, \psi_{L_1/F})W(\tau_1(\tau_2 - 1), \psi_F)$$
$$= W(\chi_{L_2} - \tau_{1L_2/F}, \psi_{L_2/F})W(\tau_2(\tau_1 - 1), \psi_F).$$

The above corresponds to the relation of Type II in the kernel of Brauer induction:

(Type II) $$(\chi_{L_1} - \tau_{2L_1/F}) + \tau_1(\tau_2 - 1) = (\chi_{L_2} - \tau_{1L_2/F}) + \tau_2(\tau_1 - 1),$$

which is clearly equivalent to the standard relation of type II that occurs in the statement of Langlands' theorem; namely

$$S : (H_1, \chi_1) - (H_2, \chi_2).$$

2.4. Langlands' Third Identity. Let G be the semidirect product $H \cdot C$, χ a quasi-character of $F^\times$, the fixed field of G, $\chi_{E/F}$ the quasi-character of $E^\times$ obtained by composing a quasi-character χ_F of $F^\times$ with the norm map $N_{E/F}$. Suppose the normal subgroup C is minimal among the normal abelian subgroups of G. For a character $\mu \in S(K/L)$ viewed as a character of $Gal(K/L) = C$, let G_μ be the isotropy subgroup and let F_μ be the fixed field of G_μ. Let $\mu' \cdot \chi_{F_\mu/F}$ be the quasi-character of $F_\mu^\times$ obtained by restricting μ to μ' on $F_\mu^\times$ and composing χ_F with $N_{F_\mu/F}$. With the proviso that the root number associated to the trivial character is 1, Langlands' third identity is

(Identity III) $$\frac{W(\chi_{E/F}, \psi_{E/F})}{W(1_{E/F}, \psi_{E/F})} = \frac{W(\chi_F, \psi_F)}{W(1_F, \psi_F)} \cdot \prod_{\mu \in T} \frac{W(\mu'\chi_{F_\mu/F}, \psi_{F_\mu/F})}{W(\mu', \psi_{F_\mu/F})},$$

or equivalently in the Grothendieck group

$$W(\chi_{E/F} - 1_{E/F}, \psi_{E/F}) = W(\chi_F - 1_F, \psi_F) \prod_{\mu \in T} W(\mu'(\chi_{F_\mu/F} - 1_{F_\mu/F}), \psi_{F_\mu/F}).$$

(Here T denotes a complete set of orbit representatives for the action of G on the non-trivial characters of C.) This identity corresponds to the relation of type III in the kernel of Brauer induction:

(Type III) $$(\chi_{E/F} - 1_{E/F}) = (\chi_F - 1_F) + \sum_{\mu \in T} \mu'(\chi_{F_\mu/F} - 1_{F_\mu/F}).$$

REMARK. Note that in Deligne's notation we have $\{\chi, \mu\} = \mu' \cdot \chi$.

Alert: In the next section, in describing the **Type III** relations in the kernel of Brauer induction it will be found convenient to include in the definition of the set T the trivial character $\mu = 1$.

3. The Relations in the Kernel of Brauer Induction

The starting point of the group theoretic component in the development of the local approach is Brauer's theorem:

THEOREM. *Let ρ be a finite dimensional representation (over the complex numbers) of a finite group G. If $\chi_\rho = Trace(\rho)$, then*

$$\chi_\rho = \sum_i n_i \, Ind_{H_i}^G (\chi_i)$$

where the sum is finite, n_i is an integer, H_i is an elementary subgroup of G and $\chi_i \in H_i^* = Hom(H_i, \mathbf{C}^\times)$.

However, these n_i, H_i, χ_i are not uniquely determined by χ_ρ. In other words, there are **relations**:

$$\sum_i n_i \, Ind_{H_i}^G (\chi_i) = 0.$$

Question: Find a characterization of all such relations.

In the context of the local theory of root numbers, it is sufficient to investigate the above question for solvable G. Deligne (after Langlands) gives an answer to this solvable case for slightly more general relations in the sense that the subgroups H_i need not be nilpotent. Also, instead of characterizing every

$$R = \sum_i n_i \, Ind_{H_i}^G (\chi_i) = 0$$

Deligne works with the equivalence class of R defined by

$$[R] = \Big\{ \sum_i n_i \, Ind_{g_i H_i g_i^{-1}}^G (\chi_i^{g_i}) \;\Big|\; g_i \in G \Big\}.$$

Here it is understood $\chi_i^{g_i}(x) = \chi_i(g_i^{-1} x g_i)$. In order to facilitate the discussion of Deligne's version of Langlands' idea, we address three particular cases of the above question: 1) G is abelian, 2) G is nilpotent, 3) G is solvable.

3.1. The abelian case. The key relation when G is abelian is given by

$$(A) \qquad Ind_H^G (\chi) = \sum_{\substack{\varphi \in G^* \\ \varphi|_H}} = \chi\varphi.$$

In fact, every relation

$$R = \sum_i n_i \, Ind_{H_i}^G (\chi_i) = \sum_i n_i \Big[Ind_{H_i}^G (\chi_i) - \sum_j \varphi_{i,j} \Big].$$

So R is generated by relation (A). It is possible to iterate this argument and write R as

$$\sum_i n_i \Big[Ind_{H_i'}^G \Big(Ind_{H_i}^{H_i'}(\chi_i) - \sum_j \varphi_{i,j}' \Big) + \sum_j Ind_{H_i''}^G \Big(Ind_{H_i'}^{H_i''}(\varphi_{i,j}') - \sum_k \varphi_{i,j,k}'' \Big) + \dots \Big]$$

where H_i'/H_i, $H_i''/H_i' \dots$ are cyclic of prime order. If relation (A) with G/H cyclic is called **Type I**, then we just obtain

THEOREM. *When G is abelian, all the relations are generated by* Type I *and* Ind(Type I).

3.2. The nilpotent case. Besides Type I , there is another type of generator when G is nilpotent:

$$\textbf{(Type II)} \qquad Ind_H^G(\chi) = Ind_C^G(\mu) \qquad (H \neq C)$$

where
1) both H and C have prime index ℓ in G,
2) the center of G, denoted by Z, coincides with $H \cap C$,
3) G/Z is isomorphic to the direct sum of 2 cyclic groups of order ℓ, i.e.

$$1 \to Z \to G \to Z_\ell \oplus Z_\ell \to 1,$$

4) $\chi|_Z = \mu|_Z$,
5) $\chi \notin Res(G^*)$ if $G^* \xrightarrow{Res} H^*$ denotes restriction of character.[1]

The complete set of generators includes not only Ind(Type I) , Ind(Type II) but also pullbacks of these relations in the following sense:

$$(B) \qquad Ind_H^{HZ}(\chi) = \sum_{\substack{\varphi \in (HZ)^* \\ \varphi|_H = \chi}} \varphi$$

is a generalization of relation (A). Observe that $H/ker(\chi)$ is cyclic, while $(HZ)/ker(\chi)$ must be abelian. Consequently,

$$(R1) \qquad Ind_{H/ker(\chi)}^{(HZ)/ker(\chi)}(\chi) - \sum \varphi$$

can be expressed as a Z-linear combination of Type I and Ind(Type I) according to the previous theorem. Meanwhile (B) is the pullback of the above relation $(R1)$ under the canonical projection

$$HZ \xrightarrow{\text{pr}} (HZ)/ker(\chi).$$

Hence, we have

Claim. The pullbacks of Type I and Ind(Type I) generate (B).

In broader terms, if λ is a character of subgroup B that annihilates all of $A \triangleleft G$, then $Ind_{B/A}^{G/A}(\lambda)$ makes sense, and we call $Ind_B^G(\lambda)$ the pullback of $Ind_{B/A}^{G/A}(\lambda)$.

THEOREM. *When G is nilpotent, a relation R is generated by* Type I *and* Type II *together with all their pullbacks and Ind.*

[1]It may be helpful to point out that 5) holds if and only if $\chi|_Z \neq \lambda|_Z$ for all $\lambda \in G^*$.

A particularly interesting example arises from Dedekind's quarternion extension over Q:

$$\begin{tikzcd}
 & Q\left(\sqrt{(1+\sqrt{2})(\sqrt{2}+\sqrt{3})\sqrt{2}\sqrt{3}} \right) \arrow[d, dash] & \\
 & Q(\sqrt{2},\sqrt{3}) \arrow[dl, dash] \arrow[dr, dash] & \\
Q(\sqrt{2}) \arrow[dr, dash] & & Q(\sqrt{3}) \arrow[dl, dash] \\
 & Q &
\end{tikzcd}$$

By localizing at $p = 2$, we obtain a totally and wildly ramified Galois extension over Q_2, the field of 2-adic numbers, such that its Galois group is the quarternion group of order 8:

$$H_8 = \{ \pm 1, \ \pm i, \ \pm j, \ \pm k \}.$$

For notational simplicity, let

$$\begin{aligned}
B &= Q_2\left(\sqrt{(1+\sqrt{2})(\sqrt{2}+\sqrt{3})\sqrt{2}\sqrt{3}} \right), \\
K &= Q_2(\sqrt{2},\sqrt{3}), \\
L_1 &= Q_2(\sqrt{2}), \\
L_2 &= Q_2(\sqrt{3}), \\
G &= Gal(B/Q_2).
\end{aligned}$$

Notice that $Gal(B/L_i)$ must be cyclic of order 4 because all 3 subgroups of index 2 in H_8 are cyclic of order 4. Given a generator χ_{L_i} for the character group of $Gal(B/L_i)$, we conclude:

1) $Gal(B/L_i)$ is a normal subgroup of prime index in G.
2) $Gal(B/L_1) \cap Gal(B/L_2) = Gal(B/K)$ is the center of G.
3) $1 \to Gal(B/K) \to G \to Z_2 \oplus Z_2 \to 1$.
4) $\chi_{L_1}|_{Gal(B/K)} = \chi_{L_2}|_{Gal(B/K)}$.
5) None of the 3 non-trivial characters of $G \simeq H_8$ coincides with χ_{L_1} on $Gal(B/L_1)$.

It follows that

$$Ind^G_{Gal(B/L_1)}(\chi_{L_1}) = Ind^G_{Gal(B/L_2)}(\chi_{L_2})$$

is a type II relation.

If we abbreviate $Gal(B/L_1)$ to H and $Gal(B/L_2)$ to C, then the above relation together with (A) imply

$$Ind_H^G\,(\chi_{L_1}-1) \;+\; \sum_{\substack{\mu\,\in\,G^*\\ \mu|_H\,=\,1}} (\mu-1) \;=\; Ind_C^G\,(\chi_{L_2}-1) \;+\; \sum_{\substack{\mu'\,\in\,G^*\\ \mu'|_C\,=\,1}} (\mu'-1),$$

which is translated into the root number identity

$$W(\chi_{L_1}) \prod_{\mu\,\in\,S(L_1/Q_2)} W(\mu) \;=\; W(\chi_{L_2}) \prod_{\mu'\,\in\,S(L_2/Q_2)} W(\mu').$$

It turns out that the difference between the nilpotent case and the solvable case lies in the existence of an abelian normal subgroup C of a non-abelian nilpotent G such that

1) $C \supset Z \neq \{1\}$,
2) $[C:Z] = \ell$ is a prime,
3) $C/Z \subseteq Z(G/Z)$, the center of G/Z.

It is sufficient to consider two special cases of the theorem:

(i) all H_i in R contain Z.
(ii) all H_i in R contain C.

Let us use (B) to rewrite R as follows:

$$R \;=\; \sum_i n_i\, Ind_{H_iZ}^G \underbrace{\Big[Ind_{H_i}^{H_iZ}\,(\chi_i) \;-\; \sum_j \varphi_{i,j}\Big]}_{\text{see claim}} \;+\; \underbrace{\sum_{i,j} n_i\, Ind_{H_iZ}^G\,(\varphi_{i,j})}_{(i)}.$$

Now, (i) is proved by induction on $|G/Z|$, and the induction hypothesis will yield (ii) due to the following lemma.

LEMMA. *Suppose C^*/G consists of all orbits of the G-action on C^* by conjugation. Let T be a set of representatives of the orbit in C^*/G. If $\sum n_i\, Ind_{H_i}^G(\chi_i) = 0$ such that $H_i \supseteq C$ and $\chi_i|_C \in T$ for all i, then for any fixed $\mu \in T$, we have*

$$\underbrace{\sum_{\chi_i|_C\,=\,\mu} n_i\, Ind_{H_i}^{G_\mu}\,(\chi_i)}_{R_\mu} \;=\; 0$$

where G_μ is the stabilizer of μ under the G-action.

It is also important to note that any relation R with $H_i \supseteq C$ can be written

$$R \;=\; \sum_T Ind_{G_\mu}^G\,(R_\mu) \qquad \text{up to a equivalence.}$$

To apply the induction hypothesis to R_μ, simply observe that

$$C/ker(\mu) \;\subseteq\; Z(\,G_\mu/ker(\mu)\,),$$

which implies

$$|\,(\,G_\mu/ker(\mu)\,)\;/\;Z(\,G_\mu/ker(\mu)\,)\,| \;<\; |G/Z|.$$

Together with (ii), the following relation completes the proof of (i).

If $G = HC$, then[2]

$$(C) \qquad Ind_H^{HC}(\chi) = \sum_{\substack{\mu \in C^*/H \\ \mu|_{H\cap C} = \chi|_{H\cap C}}} Ind_{H_\mu C}^{HC}(\{\chi,\mu\}).$$

As before, H acts on C^* by conjugation. We denote the set of orbits of this H-action by C^*/H. Here H_μ is the stabilizer of μ under the H-action, and $\{\chi,\mu\} \in (H_\mu C)^*$ is the extension of μ and $\chi|_{H_\mu}$.

In fact, given a relation R with $H_i \supseteq Z$, it is a consequence of (C) that

$$R = \sum_i n_i\, Ind_{H_iC}^G \left[Ind_{H_i}^{H_iC}(\chi_i) - \sum_\mu Ind_{(H_i)_\mu C}^{H_iC}(\{\chi_i,\mu\})\right] + \underbrace{\sum_{i,\mu} n_i\, Ind_{(H_i)_\mu C}^G(\{\chi_i,\mu\})}_{(ii)}.$$

Because

$$A_i = \bigcap_{g\in H_iC} g^{-1}ker(\chi_i)g$$

is normal in H_iC and $A_i \subseteq (H_i)_\mu C$, by considering

$$(R2) \qquad Ind_{H_i}^{H_iC}(\chi_i) - \sum_\mu Ind_{(H_i)_\mu C}^{H_iC}(\{\chi_i,\mu\})$$

as the pullback of a similar relation for H_iC/A_i, the induction hypothesis[3] shows $(R2)$ is also a Z-linear combination of the prescribed generators in our theorem, unless all of the followings are true:

1) $G/A_i = (H_iC)/A_i$,
2) $A_i \subseteq Z$,
3) $Z(G/A_i) = (ZA_i)/A_i$

which implies $|C/Z| = \ell = |H/Z|$ and $(H_i)_\mu C \neq H_iC$. As a result, when the induction hypothesis is not applicable, $(R2)$ turns out to be the pullback of a Type II relation

$$Ind_{H_i/A_i}^{G/A_i}(\chi_i) = Ind_{C/A_i}^{G/A_i}(\mu).$$

3.3. The solvable case. Unlike nilpotent groups, solvable G may have a trivial center. In order to describe the set of generators for all the relations in this case, we will require that every generator tensored with a character of G must also be a generator. Recall

$$R \otimes \varphi = \sum_i n_i Ind_{H_i}^G(\chi_i \otimes \varphi|_{H_i}) \qquad \text{for } \varphi \in G^*.$$

This operation is called torsion by φ.

[2]Relation (C) remains valid as long as the subgroup C is an abelian normal in G.

[3]Specifically, we have

$$|H_iC/Z| \geq |(H_iC)/(ZA_i)| = |\,(H_iC/A_i)\,/\,(ZA_i/A_i)\,| \geq |\,(H_iC/A_i)\,/\,Z(H_iC/A_i)|.$$

THEOREM. *When* G *is solvable, the relations are generated by* Type I, Type II, *a version of* (C) *below, all their pullbacks, torsion by* $\varphi \in G^*$, *and* Ind.

Let C be a minimal abelian normal subgroup of $G = HC$ (this implies $H \cap C = \{1\}$). Then (C) becomes

(Type III) $$Ind_H^{HC}(\chi) = \sum_{\mu \in C^*/H} Ind_{H_\mu C}^{HC}(\{\chi, \mu\}).$$

This theorem is proved by induction on $|G|$. If the center Z is trivial, we choose C to be minimal and repeat the course of argument for the nilpotent case after the lemma[4]. If Z is non-trivial, we can assume G is not nilpotent. By Brauer,

$$1_{G/Z} = \sum_j m_j Ind_{K_j/Z}^{G/Z}(\tau_j)$$

where $K_j \neq G$ for all j. Then the relation

$$S = 1 - \sum_j m_j Ind_{K_j}^{G}(\tau_j)$$

is a Z-linear combination of the prescribed generators due to the induction hypothesis. Tensor S by any relation $R = \sum n_i Ind_{H_i}^G(\chi_i)$. We obtain

$$\sum_i n_i\, S \otimes Ind_{H_i}^G(\chi_i) = R - \sum_j m_j Ind_{K_j}^G(\tau_j) \otimes R,$$

or equivalently,

$$\sum_i n_i Ind_{H_i}^G(\underbrace{Res_{H_i}(S) \otimes \chi_i}_{:=(R3)}) = R - \sum_j m_j Ind_{K_j}^G(\underbrace{\tau_j \otimes Res_{K_j}(R)}_{:=(R4)}).$$

Notice that the induction hypothesis applies to relation $(R4)$ because $|K_j| < |G|$ and

$$Res_{K_j}(Ind_{H_i}^G(\chi_i)) = \sum_{s \in K_j \backslash G/H_i} Ind_{K_j \cap sH_is^{-1}}^{K_j}(\chi_i^s).$$

Relation $(R3)$ can be handled similarly when $H_i \neq G$. In the case $H_i = G$, we recall that the set of generators is closed under torsion. This concludes the group theoretic component.

To complete the proof of the existence of the local root numbers one need only observe that to each of the relations of Type I, II, and III, which generate the kernel of Brauer induction one applies the corresponding root number identity: Type I and Type II are consequences of Dwork's Identity I and Identity II. Type III is a consequence of Langlands' Identity. In the process of carrying out the argument, there arises a delicate point which we have not discussed in this brief summary, and which must be dealt in the complete proof. Namely, when invoking the induction hypothesis, one must be able to twist the relations in the kernel of Brauer induction with abelian characters and have the corresponding twisted root number identities

[4]Because $Z = \{1\}$, we have

$$|G_\mu / ker(\mu)| < |G|$$

which calls for the induction hypothesis.

ready for use in the inductive step. This technical point is handled by Langlands by working in the setting of the local Weil groups. The details will appear in [**197**].

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Since the pioneering work of Euler, Dirichlet, and Riemann, the analytic properties of L-functions have been used to study the distribution of prime numbers. With the advent of the Langlands Program, L-functions have assumed a greater role in the study of the interplay between Diophantine questions about primes and representation theoretic properties of Galois representations. The present book provides a complete introduction to the most significant class of L-functions: the Artin-Hecke L-functions associated to finite-dimensional representations of Weil groups and to automorphic L-functions of principal type on the general linear group. In addition to establishing functional equations, growth estimates, and non-vanishing theorems, a thorough presentation of the explicit formulas of Riemann type in the context of Artin-Hecke and automorphic L-functions is also given.

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