Fourier Analysis in Convex Geometry

Alexander Koldobsky
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10 9 8 7 6 5 4 3 2 1 10 09 08 07 06 05
To my teachers, Evgeniy Alekseevich Gorin and Aleksandr Isaakovich Plotkin


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The study of the geometry of convex bodies based on information about sections and projections of these bodies has important applications in many areas of mathematics and science. In this book, a new Fourier analysis approach is discussed. The idea is to express certain geometric properties of bodies in terms of Fourier analysis and to use harmonic analysis methods to solve geometric problems.

One of the results discussed in the book is Ball's theorem, establishing the exact upper bound for the \((n - 1)\)-dimensional volume of hyperplane sections of the \(n\)-dimensional unit cube (it is \(\sqrt{2}\) for each \(n \geq 2\)). Another is the Busemann-Petty problem: if \(K\) and \(L\) are two convex origin-symmetric \(n\)-dimensional bodies and the \((n - 1)\)-dimensional volume of each central hyperplane section of \(K\) is less than the \((n - 1)\)-dimensional volume of the corresponding section of \(L\), is it true that the \(n\)-dimensional volume of \(K\) is less than the volume of \(L\)? (The answer is positive for \(n \leq 4\) and negative for \(n > 4\).)

The book is suitable for all mathematicians interested in geometry, harmonic and functional analysis, and probability. Prerequisites for reading this book include basic real, complex, and functional analysis.