

Mathematical
Surveys
and
Monographs
Volume 117

Self-Similar Groups

Volodymyr Nekrashevych



American Mathematical Society

Self-Similar Groups

**Mathematical
Surveys
and
Monographs**
Volume 117

Self-Similar Groups

Volodymyr Nekrashevych



American Mathematical Society

EDITORIAL COMMITTEE

Jerry L. Bona
Michael G. Eastwood
Peter S. Landweber
Michael P. Loss
J. T. Stafford, Chair

2000 *Mathematics Subject Classification*. Primary 20F65, 37B10;
Secondary 37F20, 37F15, 20E08, 22A22.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-117

Library of Congress Cataloging-in-Publication Data

Nekrashevych, Volodymyr, 1975–
Self-similar groups / Volodymyr Nekrashevych.
p. cm. — (Mathematical surveys and monographs, ISSN 0076-5376 ; v. 117)
Includes bibliographical references and index.
ISBN 0-8218-3831-8 (alk. paper)
1. Geometric group theory. 2. Symbolic dynamics. 3. Self-similar processes. I. Title. II. Mathematical surveys and monographs ; no. 117.

QA183 .N45 2005
512'.2—dc22

2005048021

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2005 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Printed in the United States of America.

⊗ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 10 09 08 07 06 05

Contents

Preface	vii
Chapter 1. Basic Definitions and Examples	1
1.1. Rooted tree X^* and its boundary X^ω	1
1.2. Groups acting on rooted trees	2
1.3. Automata	3
1.4. Wreath products	9
1.5. Self-similar actions	10
1.6. The Grigorchuk group	12
1.7. The adding machine and self-similar actions of \mathbb{Z}^n	16
1.8. Branch groups	17
1.9. Other examples	21
1.10. Bi-reversible automata and free groups	23
Chapter 2. Algebraic Theory	31
2.1. Permutational bimodules	31
2.2. Bases of a covering bimodule and wreath recursions	32
2.3. Tensor products and self-similar actions	33
2.4. The left G -space $\mathfrak{M}^{\otimes\omega}$	36
2.5. Virtual endomorphisms	37
2.6. The linear recursion	40
2.7. Invariant subgroups and the kernel of a self-similar action	41
2.8. Recurrent actions	44
2.9. Example: free abelian groups	46
2.10. Rigidity	50
2.11. Contracting actions	57
2.12. Finite-state actions of \mathbb{Z}^n	62
2.13. Defining relations and word problem	64
Chapter 3. Limit Spaces	71
3.1. Introduction	71
3.2. The limit G -space \mathcal{X}_G	73
3.3. Digit tiles	78
3.4. Axiomatic description of \mathcal{X}_G	82
3.5. Connectedness of \mathcal{X}_G	91
3.6. The limit space \mathcal{J}_G	92
3.7. Limit spaces of self-similar subgroups	96
3.8. The limit space \mathcal{J}_G as a hyperbolic boundary	97
3.9. Groups of bounded automata	102
3.10. One-dimensional subdivision rules	110

3.11. Uniqueness of the limit space	113
Chapter 4. Orbispaces	117
4.1. Pseudogroups and étale groupoids	117
4.2. Orbispaces	119
4.3. Open sub-orbispaces and coverings	122
4.4. Coverings and skew-products	124
4.5. Partial self-coverings	127
4.6. The limit orbispace \mathcal{J}_G	128
4.7. Paths in an orbispace	131
Chapter 5. Iterated Monodromy Groups	137
5.1. Definition of iterated monodromy groups	137
5.2. Standard self-similar actions of $\text{IMG}(p)$ on X^*	142
5.3. Iterated monodromy groups of limit dynamical systems	146
5.4. Length structures and expanding maps	148
5.5. Limit spaces of iterated monodromy groups	150
5.6. Iterated monodromy group of a pull-back	154
5.7. The limit solenoid and inverse limits of self-coverings	156
Chapter 6. Examples and Applications	161
6.1. Expanding self-coverings of orbifolds	161
6.2. Limit spaces of free Abelian groups	165
6.3. Examples of self-coverings of orbifolds	169
6.4. Rational functions	174
6.5. Combinatorial equivalence and Thurston's Theorem	176
6.6. "Twisted rabbit" question of J. Hubbard	179
6.7. Abstract kneading automata	185
6.8. Topological polynomials and critical portraits	190
6.9. Iterated monodromy groups of complex polynomials	193
6.10. Polynomials from kneading automata	196
6.11. Quadratic polynomials	203
6.12. Examples of iterated monodromy groups of polynomials	208
6.13. Matings	215
Bibliography	223
Index	229

Preface

Self-similar groups (groups generated by automata) appeared in the early eighties as interesting examples. It was discovered that very simple automata generate groups with complicated structure and exotic properties which are hard to find among groups defined by more “classical” methods.

For example, the Grigorchuk group [Gri80] can be defined as a group generated by an automaton with five states over an alphabet of two letters. This group is a particularly simple example of an infinite finitely generated torsion group and is the first example of a group whose growth is intermediate between polynomial and exponential. Another interesting example is a group discovered in [GŻ02a], which is generated by a three-state automaton over the alphabet of two letters. This group can be defined as the *iterated monodromy group* of the polynomial $z^2 - 1$ (see Chapter 5 of this book). It is the first example of an amenable group (see [BV]), which cannot be constructed from groups of sub-exponential growth by the group-theoretical operations preserving amenability.

Many other interesting groups were constructed using self-similar actions and automata. This includes, for instance, groups of finite width, groups of non-uniform exponential growth, new just-infinite groups, etc.

The definition of a self-similar group action is as follows. Let X be a finite alphabet and let X^* denote the set of all finite words over X . A faithful action of a group G on X^* is said to be *self-similar* if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w)$$

for all words $w \in X^*$. Thus, self-similar actions agree with the self-similarity of the set X^* given by the shift map $xw \mapsto w$.

One of the aims of these notes is to show that self-similar groups are not just isolated examples, but that they have close connections with dynamics and fractal geometry.

We will show, for instance, that self-similar groups appear naturally as *iterated monodromy groups* of self-coverings of topological spaces (or *orbispaces*) and encode combinatorial information about the dynamics of such self-coverings. Especially interesting is the case of a post-critically finite rational function $f(z)$. We will see that iterated monodromy groups give a convenient algebraic way of characterizing combinatorial (Thurston) equivalence of rational functions and that the Julia set of f can be reconstructed from its iterated monodromy group.

In the other direction, we will associate a *limit dynamical system* to every *contracting* self-similar action. The limit dynamical system consists of the *limit (orbi)space* \mathcal{J}_G and of a continuous finite-to-one surjective map $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$, which becomes a partial self-covering if we endow \mathcal{J}_G with a natural orbispace structure.

Since the main topics of these notes are geometry and dynamics of self-similar groups and algebraic interpretation of self-similarity, we do not go deep into the rich and various algebraic aspects of groups generated by automata such as just-infiniteness, branch groups, computation of spectra, Lie methods, etc. A reader interested in these topics may read the surveys [BGŠ03, Gri00, BGN03].

The first chapter, “Basic definitions and examples”, serves as an introduction. We define the basic terminology used in the study of self-similar groups: automorphisms of rooted trees, automata and wreath products. We define the notion of a self-similar action by giving several equivalent definitions and conclude with a sequence of examples illustrating different aspects of the subject.

The second chapter, “Algebraic theory”, studies self-similarity of groups from the algebraic point of view. We show that self-similarity can be interpreted as a *permutational bimodule*, i.e., a set with two commuting (left and right) actions of the group. The bimodule associated to a self-similar action is defined as the set \mathfrak{M} of transformations $v \mapsto xg(v)$ of the set of words X^* , where $x \in X$ is a letter and $g \in G$ is an element of the self-similar group. It follows from the definition of a self-similar action that for every $m \in \mathfrak{M}$ and $h \in G$ the compositions $m \cdot h$ and $h \cdot m$ are again elements of \mathfrak{M} . We get in this way two commuting (left and right) actions of the self-similar group G on \mathfrak{M} . The bimodule \mathfrak{M} is called the *self-similarity bimodule*. The self-similarity bimodules can be abstractly described as bimodules for which the right action is free and has a finite number of orbits. A self-similarity bimodule together with a choice of a *basis* (an orbit transversal) of the right action uniquely determines the self-similar action. Change of a basis of the bimodule changes the action to a conjugate one.

Virtual endomorphisms are another convenient tool used to construct permutational bimodules and hence self-similar actions. A virtual endomorphism ϕ of a group G is a homomorphism from a subgroup of finite index $\text{Dom } \phi \leq G$ to G . We show that the set of formal expressions of the form $\phi(g)h$ (with natural identifications) is a permutational bimodule and that one gets a self-similar action in this way. If we start from a self-similar action, then the *associated virtual endomorphism* ϕ is defined on the stabilizer G_x of a letter $x \in X$ in G by the condition that

$$g(xw) = x\phi(g)(w)$$

for every $w \in X^*$ and $g \in \text{Dom } \phi = G_x$.

For example, the *adding machine* action, i.e., the natural action of \mathbb{Z} on the ring of diadic integers $\mathbb{Z}_2 \geq \mathbb{Z}$, where \mathbb{Z}_2 is encoded in the usual way by infinite binary sequences, is the self-similar action defined by the virtual endomorphism $\phi : n \mapsto n/2$. In this sense self-similar actions may be viewed as generalizations of *numeration systems*. In Section 2.9 of Chapter 2, we apply the developed technique to describe self-similar actions of the free abelian groups \mathbb{Z}^n , making the relation between self-similar actions and numeration systems more explicit.

Section 2.11 introduces the main class of self-similar actions for these notes. It is the class of the so-called *contracting actions*. An action is called contracting if the associated virtual endomorphism ϕ asymptotically shortens the length of the elements of the group. Contraction of a self-similar action corresponds to the condition of expansion of a dynamical system. We show in the next chapters that if a self-covering of a Riemannian manifold (or orbifold) is expanding, then its iterated monodromy group is contracting with respect to a *standard* self-similar action.

The *limit spaces* and the *limit dynamical systems* of contracting self-similar actions are constructed and studied in Chapter 3. If \mathfrak{M} is the permutational bimodule associated to a self-similar action of a group G , then its tensor power $\mathfrak{M}^{\otimes n}$ is defined in a natural way. It describes the action of G on the set of words of length n and is interpreted as the n th iteration of the self-similarity of the group. Passing to the (appropriately defined) limits as n goes to infinity, we get the left G -module (G -space) $\mathfrak{M}^{\otimes\omega} = \mathfrak{M} \otimes \mathfrak{M} \otimes \dots$ and the right G -module $\mathfrak{M}^{\otimes-\omega} = \dots \otimes \mathfrak{M} \otimes \mathfrak{M}$. The left G -space $\mathfrak{M}^{\otimes\omega}$ is naturally interpreted as the action of G on the space of infinite words $X^\omega = \{x_1x_2\dots : x_i \in X\}$.

The right G -space $\mathcal{X}_G = \mathfrak{M}^{\otimes-\omega}$ (if the action is contracting) is a finite-dimensional metrizable locally compact topological space with a proper co-compact right action of G on it. The limit space \mathcal{X}_G can also be described axiomatically as the unique proper co-compact G -space with a *contracting self-similarity* (Theorem 3.4.13). A right G -space \mathcal{X} is called *self-similar* if the actions (\mathcal{X}, G) and $(\mathcal{X} \otimes_G \mathfrak{M}, G)$ are topologically conjugate. For the notion of a contracting self-similarity see Definition 3.4.11.

Another construction is the quotient (orbispace) \mathcal{J}_G of \mathcal{X}_G by the action of G (Section 3.6). The limit space \mathcal{J}_G can be alternatively defined as the quotient of the space of the left-infinite sequences $X^{-\omega} = \{\dots x_2x_1 : x_i \in X\}$ by the equivalence relation, which identifies two sequences $\dots x_2x_1$ and $\dots y_2y_1$ if there exists a bounded sequence $g_k \in G$ such that $g_k(x_k\dots x_1) = y_k\dots y_1$ for all k . Here a sequence is called bounded if it takes a finite set of values. One can prove that this equivalence is described by a finite graph labeled by pairs of letters and that equivalence classes are finite. This gives us a nice symbolic presentation of the space \mathcal{J}_G .

The limit space \mathcal{J}_G comes together with a natural *shift map* $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$ and with a Markov partition of the dynamical system (\mathcal{J}_G, s) . The shift is induced by the usual shift $\dots x_2x_1 \mapsto \dots x_3x_2$, and the elements of the Markov partition are the images of the cylindrical sets of the described symbolic presentation of \mathcal{J}_G . The elements of the Markov partition are called (*digit*) *tiles*. Digit tiles can also be defined for the limit G -space \mathcal{X}_G , and they are convenient tools for the study of the topology of \mathcal{X}_G .

The most well-studied contracting groups are the self-similar groups generated by *bounded automata*. They can be defined as the groups whose digit tiles have finite boundary. We show that this condition is equivalent to a condition studied by S. Sidki in [Sid00] and show an iterative algorithm which constructs approximations of the limit spaces \mathcal{J}_G of such groups. Groups generated by bounded automata are defined and studied in Section 3.9; their limit spaces are considered in Section 3.10 and Section 3.11, where we prove that in some cases the limit spaces depend only on the algebraic structure of the group and thus can be used to distinguish the groups up to isomorphisms.

Chapter 4, “Orbispaces”, is a technical chapter in which we collect the basic definitions related to the theory of orbispaces. Orbispaces are structures represented locally as quotients of topological spaces by finite homeomorphism groups. They are generalizations of a more classical notion of an *orbifold* introduced by W. Thurston (see [Thu90] and [Sco83]). A similar notion of a V-manifold was introduced earlier by I. Satake [Sat56]. We use in our approach pseudogroups and étale groupoids, following [BH99]. Most constructions in this chapter are well known, though we

present some new (and we hope natural) definitions, like the definition of an open map between orbispaces and the notion of an open sub-orbispaces. We also define the *limit orbispace* \mathcal{J}_G of a contracting self-similar action and show that the shift map $\mathfrak{s} : \mathcal{J}_G \rightarrow \mathcal{J}_G$ is a covering of the limit orbispace by an open sub-orbispaces (is a *partial self-covering*).

The orbispace structure on \mathcal{J}_G comes from the fact that the limit space \mathcal{J}_G is the quotient of the limit space $\mathcal{X}_G = \mathfrak{M}^{\otimes -\omega}$ by the action of the group G . Introduction of this additional structure on \mathcal{J}_G makes it possible to reconstruct the group G itself from the partial self-covering \mathfrak{s} of \mathcal{J}_G as the iterated monodromy group $\text{IMG}(\mathfrak{s})$ (see Theorem 5.3.1). Hence, if we want to be able to go back and forth between self-similar groups and dynamical systems, then we need to define iterated monodromy groups in the general setting of orbispace mappings.

One cannot avoid using orbispaces even in more classical situations like iterations of rational functions. W. Thurston associated with every post-critically finite rational function its *canonical orbispace*, playing an important role in the study of dynamics (see [DH93, Mil99]).

Chapter 5 defines and studies iterated monodromy groups. If $p : \mathcal{M}_1 \rightarrow \mathcal{M}$ is a covering of a topological space (or an orbispace) \mathcal{M} by an open subset (an open sub-orbispaces) \mathcal{M}_1 , then the fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally by the monodromy action on the set of preimages $p^{-n}(t)$ of the basepoint under the n th iteration of p . Let us denote by K_n the kernel of the action. Then the *iterated monodromy group* of p (denoted $\text{IMG}(p)$) is the quotient $\pi_1(\mathcal{M}, t) / \bigcap_{n \geq 0} K_n$.

The disjoint union $T = \bigsqcup_{n \geq 0} p^{-n}(t)$ of the sets of preimages has a natural structure of a rooted tree. It is the tree with the root t , where a vertex $z \in p^{-n}(t)$ is connected by an edge with the vertex $p(z) \in p^{-(n-1)}(t)$. The iterated monodromy group acts faithfully on this tree in a natural way.

We define a special class of isomorphisms of the tree of preimages T with the tree of words X^* using preimages of paths in \mathcal{M} . After conjugation of the natural action of $\text{IMG}(p)$ on T by such an isomorphism, we get a *standard* faithful self-similar action of $\text{IMG}(p)$ on X^* . The standard action depends on a choice of paths connecting the basepoint to its preimages, but a different choice of paths corresponds to a different choice of a basis of the associated self-similarity bimodule. In particular, two different standard actions of $\text{IMG}(p)$ are conjugate, and if the actions are contracting, then the limit spaces $\mathcal{X}_{\text{IMG}(p)}$ and $\mathcal{J}_{\text{IMG}(p)}$ (and the limit dynamical system) depend only on the partial self-covering p .

The main result of the chapter is Theorem 5.5.3, which shows that the limit space $\mathcal{J}_{\text{IMG}(p)}$ of the iterated monodromy group of an expanding partial self-covering $p : \mathcal{M}_1 \rightarrow \mathcal{M}$ is homeomorphic to the Julia set of p (to the attractor of the backward orbits) and, moreover, that the limit dynamical system $\mathfrak{s} : \mathcal{J}_{\text{IMG}(p)} \rightarrow \mathcal{J}_{\text{IMG}(p)}$ is topologically conjugate to the restriction of p onto the Julia set. The respective orbispace structures of the Julia set and the limit space also agree.

The last chapter shows different examples of iterated monodromy groups and their applications. We start with the case when a self-covering $p : \mathcal{M} \rightarrow \mathcal{M}$ is defined on the whole (orbi)space \mathcal{M} . The case when \mathcal{M} is a Riemannian manifold and p is expanding was studied by M. Shub, J. Franks and M. Gromov. They showed that \mathcal{M} is in this case an *infra-nil* manifold and that p is induced by an expanding automorphism of a nilpotent Lie group (the universal cover of \mathcal{M}). We show how results of M. Shub and J. Franks follow from Theorem 5.5.3, also proving

them in a slightly more general setting. A particular case, when \mathcal{M} is a torus $\mathbb{R}^n/\mathbb{Z}^n$, corresponds to numeration systems on \mathbb{R}^n and is related to self-affine *digit tilings* of the Euclidean space, which were studied by many mathematicians.

Another interesting class of examples are the iterated monodromy groups of post-critically finite rational functions. A rational function $f(z) \in \mathbb{C}(z)$ is called *post-critically finite* if the orbit of every critical point under the iterations of f is finite. If P is the union of the orbits of the critical points, then f is a partial self-covering of the punctured sphere $\widehat{\mathbb{C}} \setminus P$. Then the iterated monodromy group of f is, by definition, the iterated monodromy group of this partial self-covering.

The closure of the iterated monodromy group of a rational function f in the automorphism group of the rooted tree is isomorphic to the Galois group of an extension of the field of functions $\mathbb{C}(t)$. This is the extension obtained by adjoining the solutions of the equation $f^{\circ n}(x) = t$ to $\mathbb{C}(t)$ for all n . These Galois groups were considered by Richard Pink, who was the first to define the profinite iterated monodromy groups.

Every post-critically finite rational function is an expanding self-covering of the associated *Thurston orbifold* by an open sub-orbifold, so Theorem 5.5.3 can be applied, and we get a symbolic presentation of the action of the rational function on the Julia set.

Iterated monodromy groups are rather exotic from the point of view of group theory. The only known finitely presented examples are the iterated monodromy groups of functions with “smooth” Julia sets: z^d , Chebyshev polynomials and Lattè examples. Some iterated monodromy groups of rational functions are groups of intermediate growth (for instance $\text{IMG}(z^2 + i)$), while some are essentially new examples of amenable groups (like $\text{IMG}(z^2 - 1)$).

Chapter 6 concludes with a complete description of automata generating iterated monodromy groups of polynomials and with an example showing how iterated monodromy groups can be used to construct and to understand plane-filling curves originating from matings of polynomials.

Acknowledgments. I was introduced to the fascinating subject of groups generated by automata by Rostislav Grigorchuk and Vitalij Sushchansky, to whom I am very grateful for their tremendous support of my research.

I also want to use this opportunity to thank Laurent Bartholdi, Yevgen Bondarenko, Anna Erschler, Yaroslav Lavreniuk, Kevin Pilgrim, Richard Pink, Dierk Schleicher, Said Sidki and the referees for helpful suggestions and collaboration.

A great part of this work was done during visits to Geneva University, sponsored by the Swiss National Science Foundation, during the stay in Heinrich Heine University of Düsseldorf as a fellow of the Alexander von Humboldt Foundation and during the stay at International University Bremen. I gratefully acknowledge the support of these foundations and institutions and want especially to thank Pierre de la Harpe for invitations and hospitality during my visits to Geneva and Fritz Grunewald for hosting my visit to Düsseldorf.

Bibliography

- [Ale83] S. V. Aleshin, *A free group of finite automata*, Moscow University Mathematics Bulletin **38** (1983), 10–13.
- [Ban91] Christoph Bandt, *Self-similar sets. V: Integer matrices and fractal tilings of \mathbb{R}^n* , Proc. Am. Math. Soc. **112** (1991), no. 2, 549–562.
- [Bar98] Laurent Bartholdi, *The growth of Grigorchuk’s torsion group*, Internat. Math. Res. Notices **20** (1998), 1049–1054.
- [Bar01] ———, *Lower bounds on the growth of Grigorchuk’s torsion group*, Internat. J. Algebra Comput. **11** (2001), no. 1, 73–88.
- [Bar03a] ———, *Endomorphic presentations of branch groups*, Journal of Algebra **268** (2003), no. 2, 419–443.
- [Bar03b] ———, *A Wilson group of non-uniformly exponential growth*, C. R. Acad. Sci. Paris. Sér. I Math. **336** (2003), no. 7, 549–554.
- [Bau93] Gilbert Baumslag, *Topics in combinatorial group theory*, Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag, Basel, 1993.
- [Bea91] Alan F. Beardon, *Iteration of rational functions. Complex analytic dynamical systems*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991.
- [Bel03] Igor Belegradek, *On co-Hopfian nilpotent groups*, Bull. London Math. Soc. **35** (2003), 805–811.
- [BFH92] Ben Bielefeld, Yuval Fisher, and John H. Hubbard, *The classification of critically preperiodic polynomials as dynamical systems*, Journal of the A.M.S. **5** (1992), no. 4, 721–762.
- [BG00a] Laurent Bartholdi and Rostislav I. Grigorchuk, *Lie methods in growth of groups and groups of finite width*, Computational and Geometric Aspects of Modern Algebra (Michael Atkinson et al., eds.), London Math. Soc. Lect. Note Ser., vol. 275, Cambridge Univ. Press, Cambridge, 2000, pp. 1–27.
- [BG00b] ———, *On the spectrum of Hecke type operators related to some fractal groups*, Proceedings of the Steklov Institute of Mathematics **231** (2000), 5–45.
- [BG02] ———, *On parabolic subgroups and Hecke algebras of some fractal groups*, Serdica Math. J. **28** (2002), 47–90.
- [BGN03] Laurent Bartholdi, Rostislav Grigorchuk, and Volodymyr Nekrashevych, *From fractal groups to fractal sets*, Fractals in Graz 2001. Analysis – Dynamics – Geometry – Stochastics (Peter Grabner and Wolfgang Woess, eds.), Birkhäuser Verlag, Basel, Boston, Berlin, 2003, pp. 25–118.
- [BGŠ03] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunić, *Branch groups*, Handbook of Algebra, Vol. 3, North-Holland, Amsterdam, 2003, pp. 989–1112.
- [BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer, Berlin, 1999.
- [Bha95] Meenaxi Bhattacharjee, *The ubiquity of free subgroups in certain inverse limits of groups*, J. Algebra **172** (1995), 134–146.
- [BJ99] Ola Bratelli and Palle E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, vol. 139, Memoirs of the American Mathematical Society, no. 663, A. M. S., Providence, Rhode Island, 1999.
- [BL01] Hyman Bass and Alexander Lubotzky, *Tree lattices*, Progress in Mathematics, vol. 176, Birkhäuser Boston Inc., Boston, MA, 2001, With appendices by Bass, L. Carbone, Lubotzky, G. Rosenberg and J. Tits.
- [BN03] Evgen Bondarenko and Volodymyr Nekrashevych, *Post-critically finite self-similar groups*, Algebra and Discrete Mathematics **2** (2003), no. 4, 21–32.

- [BN05a] Laurent Bartholdi and Volodymyr V. Nekrashevych, *Iterated monodromy groups of quadratic polynomials* (preprint), 2005.
- [BN05b] Laurent Bartholdi and Volodymyr V. Nekrashevych, *Thurston equivalence of topological polynomials* (preprint), 2005.
- [BORT96] Hyman Bass, Maria Victoria Otero-Espinar, Daniel Rockmore, and Charles Tresser, *Cyclic renormalization and automorphism groups of rooted trees*, Lecture Notes in Mathematics, vol. 1621, Springer-Verlag, Berlin, 1996.
- [Bou71] Nicolas Bourbaki, *Éléments de mathématique. Topologie générale. Chapitres 1 à 4*, Hermann, Paris, 1971.
- [BP04] Kai-Uwe Bux and Rodrigo Pérez, *On the growth of iterated monodromy groups* (preprint), 2004.
- [BS97] Andrew M. Brunner and Said N. Sidki, *On the automorphism group of the one-rooted binary tree*, J. Algebra **195** (1997), 465–486.
- [BS98] ———, *The generation of $GL(n, Z)$ by finite state automata*, Internat. J. Algebra Comput. **8** (1998), no. 1, 127–139.
- [BS02a] Henk Bruin and Dierk Schleicher, *Symbolic dynamics of quadratic polynomials*, Institut Mittag-Leffler, Report No. 7, 2001/2002.
- [BS02b] Andrew M. Brunner and Said N. Sidki, *Wreath operations in the group of automorphisms of the binary tree*, J. Algebra **257** (2002), 51–64.
- [BV] Laurent Bartholdi and Bálint Virág, *Amenability via random walks*, to appear in Duke Math Journal.
- [CDP90] Michel Coornaert, Thomas Delzant, and Athanase Papadopoulos, *Geometrie et theorie des groupes: Les groupes hyperboliques de Gromov*, Lectures Notes in Mathematics, vol. 1441, Springer Verlag, 1990.
- [CJ85] Alain Connes and Vaughan Jones, *Property T for von Neumann algebras*, Bull. London Math. Soc. **17** (1985), 57–62.
- [Con94] Alain Connes, *Noncommutative geometry*, San Diego, CA: Academic Press, 1994.
- [CSGH99] Tullio Ceccherini-Silberstein, Rostislav I. Grigorchuk, and Pierre de la Harpe, *Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces*, Trudy Mat. Inst. Steklov. **224** (1999), no. Algebra. Topol. Differ. Uravn. i ikh Prilozh., 68–111, Dedicated to Academician Lev Semenovich Pontryagin on the occasion of his 90th birthday (Russian).
- [Day57] Mahlon M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
- [DH84] Adrien Douady and John H. Hubbard, *Étude dynamique des polynômes complex. (Première partie)*, Publications Mathématiques d’Orsay, vol. 02, Université de Paris-Sud, 1984.
- [DH85a] ———, *Étude dynamique des polynômes complex. (Deuxième partie)*, Publications Mathématiques d’Orsay, vol. 04, Université de Paris-Sud, 1985.
- [DH85b] Adrien Douady and John H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. Éc. Norm. Supér. IV. Sér. **18** (1985), 287–343.
- [DH93] ———, *A proof of Thurston’s topological characterization of rational functions*, Acta Math. **171** (1993), no. 2, 263–297.
- [Dra00] Alexander N. Dranishnikov, *Asymptotic topology*, Uspekhi Mat. Nauk **55** (2000), no. 6(336), 71–116.
- [Eil74] Samuel Eilenberg, *Automata, languages and machines*, vol. A, Academic Press, New York, London, 1974.
- [Eng68] Ryszard Engelking, *Outline of general topology*, Amsterdam: North-Holland Publishing Company, 1968.
- [Eng77] ———, *General topology*, Monografie Matematyczne, vol. 60, Państwowe Wydawnictwo Naukowe, Warszawa, 1977.
- [Ers04] Anna Erschler, *Boundary behaviour for groups of subexponential growth*, Annals of Mathematics **160** (2004), 1183–1210.
- [FG91] Jacek Fabrykowski and Narain D. Gupta, *On groups with sub-exponential growth functions. II*, J. Indian Math. Soc. (N.S.) **56** (1991), no. 1-4, 217–228.
- [For81] Otto Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, New York – Heidelberg – Berlin: Springer-Verlag, 1981.
- [Fra70] John M. Franks, *Anosov diffeomorphisms*, Global Analysis, Berkeley, 1968, Proc. Symp. Pure Math., vol. 14, Amer. Math. Soc., 1970, pp. 61–93.

- [Gan59] F. R. Gantmacher, *The theory of matrices. Vols. 1, 2*, translated by K. A. Hirsch, Chelsea Publishing Co., New York, 1959.
- [Gel95] Götz Gelbrich, *Self-similar tilings and expanding homomorphisms of groups*, Arch. Math. **65** (1995), no. 6, 481–491.
- [GH90] Étienne Ghys and Pierre de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser Boston Inc., Boston, MA, 1990, Papers from the Swiss Seminar on Hyperbolic Group held in Bern, 1988.
- [GLSŽ00] Rostislav I. Grigorchuk, Peter Linnell, Thomas Schick, and Andrzej Żuk, *On a question of Atiyah*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 9, 663–668.
- [GM03] Yair Glasner and Shahar Mozes, *Automata and square complexes*, to appear in Geom. Dedicata, 2005.
- [GNS00] Rostislav I. Grigorchuk, Volodymyr V. Nekrashevich, and Vitalii I. Sushchanskii, *Automata, dynamical systems and groups*, Proceedings of the Steklov Institute of Mathematics **231** (2000), 128–203.
- [GNS01] Piotr W. Gawron, Volodymyr V. Nekrashevych, and Vitaly I. Sushchansky, *Conjugation in tree automorphism groups*, Int. J. of Algebra and Computation **11** (2001), no. 5, 529–547.
- [Gri80] Rostislav I. Grigorchuk, *On Burnside's problem on periodic groups*, Functional Anal. Appl. **14** (1980), no. 1, 41–43.
- [Gri83] ———, *On the Milnor problem of group growth*, Dokl. Akad. Nauk SSSR **271** (1983), no. 1, 30–33.
- [Gri85] ———, *Degrees of growth of finitely generated groups and the theory of invariant means*, Math. USSR Izv. **25** (1985), no. 2, 259–300.
- [Gri88] ———, *Semigroups with cancellations of polynomial growth*, Mat. Zametki **43** (1988), no. 3, 305–319, 428.
- [Gri90] ———, *On the Hilbert-Poincaré series of graded algebras that are associated with groups*, Math. USSR-Sb. **66** (1990), no. 1, 211–229.
- [Gri98] ———, *An example of a finitely presented amenable group that does not belong to the class EG*, Mat. Sb. **189** (1998), no. 1, 79–100.
- [Gri99] ———, *On the system of defining relations and the Schur multiplier of periodic groups generated by finite automata*, Groups St. Andrews 1997 in Bath, I, Cambridge Univ. Press, Cambridge, 1999, pp. 290–317.
- [Gri00] ———, *Just infinite branch groups*, New Horizons in pro- p Groups (Aner Shalev, Marcus P. F. du Sautoy, and Dan Segal, eds.), Progress in Mathematics, vol. 184, Birkhäuser Verlag, Basel, 2000, pp. 121–179.
- [Gro81] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. I. H. E. S. **53** (1981), 53–73.
- [Gro87] ———, *Hyperbolic groups*, Essays in Group Theory (S. M. Gersten, ed.), M.S.R.I. Pub., no. 8, Springer, 1987, pp. 75–263.
- [Gro93] ———, *Asymptotic invariants of infinite groups*, Geometric Group Theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.
- [GS83] Narain D. Gupta and Said N. Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), 385–388.
- [GW03] Rostislav I. Grigorchuk and John S. Wilson, *The uniqueness of the actions of certain branch groups on rooted trees*, Geom. Dedicata **100** (2003), 103–116.
- [GŽ01] Rostislav I. Grigorchuk and Andrzej Żuk, *The lamplighter group as a group generated by a 2-state automaton and its spectrum*, Geom. Dedicata **87** (2001), no. 1–3, 209–244.
- [GŽ02a] ———, *On a torsion-free weakly branch group defined by a three state automaton*, Internat. J. Algebra Comput. **12** (2002), no. 1, 223–246.
- [GŽ02b] Rostislav I. Grigorchuk and Andrzej Żuk, *Spectral properties of a torsion-free weakly branch group defined by a three state automaton*, Computational and Statistical Group Theory (Las Vegas, NV/Hoboken, NJ, 2001), Contemp. Math., vol. 298, Amer. Math. Soc., Providence, RI, 2002, pp. 57–82.
- [Hae01] André Haefliger, *Groupoids and foliations*, Groupoids in Analysis, Geometry, and Physics. AMS-IMS-SIAM joint summer research conference, University of Colorado, Boulder, CO, USA, June 20–24, 1999 (Arlan Ramsay et al., eds.), Contemp. Math, vol. 282, Providence, RI: A.M.S., 2001, pp. 83–100.

- [Har00] Pierre de la Harpe, *Topics in geometric group theory*, University of Chicago Press, 2000.
- [Hir70] Morris W. Hirsch, *Expanding maps and transformation groups*, Global Analysis, Proc. Sympos. Pure Math., vol. 14, American Math. Soc., Providence, Rhode Island, 1970, pp. 125–131.
- [HR32] H. Hopf and W. Rinow, *Über den Begriff der vollständigen differentialgeometrischen Fläche*, Comment. Math. Helv **3** (1932), 209–225.
- [HS94] John H. Hubbard and Dierk Schleicher, *The spider algorithm*, Complex Dynamical Systems. The Mathematics Behind the Mandelbrot and Julia Sets (Robert L. Devaney, ed.), Proceedings of Symposia in Applied Mathematics, vol. 49, 1994, pp. 155–180.
- [JM04] John J. Milnor, *Pasting together Julia sets: a worked out example of mating*, Experiment. Math. **13** (2004), no. 1, 55–92.
- [JS97] Vaughan Jones and V.S. Sunder, *Introduction to subfactors*, London Mathematical Society Lecture Note Series, vol. 234, Cambridge University Press, 1997.
- [Kai03] Vadim A. Kaimanovich, *Random walks on Sierpiński graphs: hyperbolicity and stochastic homogenization*, Fractals in Gratz 2001 (W. Woess, ed.), Trends Math., Birkhäuser, Basel, 2003, pp. 145–183.
- [Kam01] Atsushi Kameyama, *The Thurston equivalence for postcritically finite branched coverings*, Osaka J. Math. **38** (2001), no. 3, 565–610.
- [Kel00] Karsten Keller, *Invariant factors, Julia equivalences and the (abstract) Mandelbrot set*, Lecture Notes in Mathematics, vol. 1732, Springer, 2000.
- [Ken92] Richard Kenyon, *Self-replicating tilings*, Symbolic Dynamics and Its Applications (P. Walters, ed.), Contemp. Math., vol. 135, Amer. Math. Soc., Providence, RI, 1992, pp. 239–264.
- [Kig92] Jun Kigami, *Laplacians on self-similar sets — analysis on fractals*, Transl., Ser. 2, Amer. Math. Soc. 161, 75-93 (1994); translation from Sugaku 44, No.1, 13-28 (1992) (1992).
- [Kig01] ———, *Analysis on fractals*, Cambridge Tracts in Mathematics, vol. 143, Cambridge University Press, 2001.
- [KL05] Vadim A. Kaimanovich and Mikhail Lyubich, *Conformal and harmonic measures on laminations associated with rational maps*, vol. 173, Memoirs of the A.M.S., no. 820, A.M.S., Providence, Rhode Island, 2005.
- [KM79] M. I. Kargapolov and Ju. I. Merzljakov, *Fundamentals of the theory of groups*, Graduate Texts in Mathematics, vol. 62, Springer-Verlag, New York, Heidelberg, Berlin, 1979.
- [Knu69] Donald E. Knuth, *The art of computer programming, Vol. 2, Seminumerical algorithms*, Addison-Wesley Publishing Company, 1969.
- [Kur61] Kazimierz Kuratowski, *Topologie*, vol. II, Warszawa, 1961.
- [Lan87] Serge Lang, *Elliptic functions. Second edition*, Graduate Texts in Mathematics, vol. 112, Springer-Verlag, New York, 1987.
- [Lat18] S. Lattès, *Sur l'itération des substitutions rationnelles et les fonctions de Poincaré*. C. R. Acad. Sci. Paris **166** (1918), 26–28.
- [Lav99] Yaroslav Lavreniuk, *Automorphisms of wreath branch groups*, Visnyk Kyivskogo Universytetu (1999), no. 1, 50–57 (in Ukrainian).
- [Leo00] Yuriĭ G. Leonov, *On a lower bound for the growth function of the Grigorchuk group*, Mat. Zametki **67** (2000), no. 3, 475–477.
- [Lin90] Tom Lindstrøm, *Brownian motion on nested fractals*, Mem. Am. Math. Soc. **420** (1990), 128 pp.
- [LM89] Alexander Lubotzky and Avinoam Mann, *Residually finite groups of finite rank*, Math. Proc. Cambridge Philos. Soc. **106** (1989), no. 3, 385–388.
- [LM97] Mikhail Lyubich and Yair Minsky, *Laminations in holomorphic dynamics*, J. Differ. Geom. **47** (1997), no. 1, 17–94.
- [LMZ94] Alexander Lubotzky, Shahar Mozes, and Robert J. Zimmer, *Superrigidity for the commensurability group of tree lattices*, Comment. Math. Helvetici **69** (1994), 523–548.
- [LN02] Yaroslav V. Lavreniuk and Volodymyr V. Nekrashevych, *Rigidity of branch groups acting on rooted trees*, Geom. Dedicata **89** (2002), no. 1, 155–175.
- [LPS88] Alexander Lubotzky, Ralph Philips, and Peter Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), no. 3, 261–277.

- [Lys85] Igor G. Lysionok, *A system of defining relations for the Grigorchuk group*, Mat. Zametki **38** (1985), 503–511.
- [Mal49] A. I. Malcev, *On a class of homogeneous spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 9–32.
- [Mer83] Yuriĭ I. Merzlyakov, *Infinite finitely generated periodic groups*, Dokl. Akad. Nauk SSSR **268** (1983), no. 4, 803–805.
- [Mil99] John W. Milnor, *Dynamics in one complex variable. Introductory lectures*, Wiesbaden: Vieweg, 1999.
- [MNS00] Olga Macedońska, Volodymyr V. Nekrashevych, and Vitalii I. Sushchansky, *Commensurators of groups and reversible automata*, Dopov. Nats. Akad. Nauk Ukr., Mat. Pryr. Tekh. Nauky (2000), no. 12, 36–39.
- [Nek99] Volodymyr V. Nekrashevych, *Uniformly bounded spaces*, Voprosy Algebra **14** (1999), 47–97.
- [Nek00] ———, *Stabilizers of transitive actions on locally finite graphs*, Int. J. of Algebra and Computation **10** (2000), no. 5, 591–602.
- [Nek02] ———, *Virtual endomorphisms of groups*, Algebra and Discrete Mathematics **1** (2002), no. 1, 96–136.
- [Nek04] ———, *Cuntz-Pimsner algebras of group actions*, Journal of Operator Theory **52** (2004), no. 2, 223–249.
- [Neu86] Peter M. Neumann, *Some questions of Edjvet and Pride about infinite groups*, Illinois J. Math. **30** (1986), no. 2, 301–316.
- [NS04] Volodymyr Nekrashevych and Said Sidki, *Automorphisms of the binary tree: state-closed subgroups and dynamics of 1/2-endomorphisms*, Groups: Topological, Combinatorial and Arithmetic Aspects (T. W. Müller, ed.), LMS Lecture Notes Series, vol. 311, 2004, pp. 375–404.
- [Oli98] Andriy S. Oliĭnyk, *Free groups of automatic permutations*, Dop. NAS Ukraine (1998), no. 7, 40–44 (in Ukrainian).
- [Oli99] ———, *Free products of C_2 as groups of finitely automatic permutations*, Voprosy Algebra (Gomel) **14** (1999), 158–165.
- [Pil00] Kevin M. Pilgrim, *Dessins d'enfants and Hubbard trees*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 5, 671–693.
- [Pil03a] ———, *An algebraic formulation of Thurston's combinatorial equivalence*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3527–3534.
- [Pil03b] ———, *Combinations of complex dynamical systems*, Lecture Notes in Mathematics, vol. 1827, Springer, 2003.
- [Pil04] ———, *A Hurwitz-like classification of Thurston combinatorial classes*, Osaka J. Math. **41** (2004), 131–143.
- [Poi93] Alfredo Poirier, *On Post Critically Finite Polynomials. Part One: Critical Portraits*, arXiv:math.DS/9305207 v1, 1993.
- [Pri80] Stephen J. Pride, *The concept of "largeness" in group theory*, Word Problems II (S. I. Adian, W. W. Boone, and G. Higman, eds.), Studies in Logic and Foundations of Math., 95, North-Holland Publishing Company, 1980, pp. 299–335.
- [Roe03] John Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, Rhode Island, 2003.
- [Röv02] Claas E. Röver, *Commensurators of groups acting on rooted trees*, Geom. Dedicata **94** (2002), 45–61.
- [Roz96] A. V. Rozhkov, *Finiteness conditions in automorphism groups of trees*, Cheliabinsk, 1996, Habilitation thesis.
- [Rub89] Matatyahu Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, Trans. Amer. Math. Soc. **312** (1989), no. 2, 487–538.
- [Sab97] Christophe Sabot, *Existence and uniqueness of diffusions of finitely ramified self-similar fractals*, Ann. Sci. Éc. Norm. Supér., IV. Sér. **30** (1997), no. 5, 605–673.
- [Sat56] Ichiro Satake, *On a generalization of the notion of a manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363.
- [Sco83] Peter Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.
- [Ser80] Jean-Pierre Serre, *Trees*, New York: Springer-Verlag, 1980.

- [Shu69] Michael Shub, *Endomorphisms of compact differentiable manifolds*, Am. J. Math. **91** (1969), 175–199.
- [Shu70] ———, *Expanding maps*, Global Analysis, Proc. Sympos. Pure Math., vol. 14, American Math. Soc., Providence, Rhode Island, 1970, pp. 273–276.
- [Sid87a] Said N. Sidki, *On a 2-generated infinite 3-group: subgroups and automorphisms*, J. Algebra **110** (1987), no. 1, 24–55.
- [Sid87b] ———, *On a 2-generated infinite 3-group: the presentation problem*, J. Algebra **110** (1987), no. 1, 13–23.
- [Sid97] ———, *A primitive ring associated to a Burnside 3-group*, J. London Math. Soc. (2) **55** (1997), 55–64.
- [Sid98] ———, *Regular trees and their automorphisms*, Monografias de Matematica, vol. 56, IMPA, Rio de Janeiro, 1998.
- [Sid00] ———, *Automorphisms of one-rooted trees: growth, circuit structure and acyclicity*, J. of Mathematical Sciences (New York) **100** (2000), no. 1, 1925–1943.
- [Sid04a] ———, *Finite automata of polynomial growth do not generate a free group*, Geom. Dedicata **108** (2004), 193–204.
- [Sid04b] ———, *Tree-wreathing applied to generation of groups by finite automata* (preprint), 2004.
- [SS] Pedro V. Silva and Benjamin Steinberg, *On a class of automata groups generalizing lamplighter groups*, to appear in Internat. J. Algebra Comput.
- [Sus79] Vitalii I. Sushchansky, *Periodic permutation p -groups and the unrestricted Burnside problem*, DAN SSSR. **247** (1979), no. 3, 557–562 (in Russian).
- [Sus98] ———, *Groups of automatic permutations*, Dop. NAN Ukrainy (1998), no. 6, 47–51 (in Ukrainian).
- [Sus99] ———, *Groups of finitely automatic permutations*, Dop. NAN Ukrainy (1999), no. 2, 29–32 (in Ukrainian).
- [SW03] Said N. Sidki and John S. Wilson, *Free subgroups of branch groups*, Arch. Math. **80** (2003), 458–463.
- [Tan92] Lei Tan, *Matings of quadratic polynomials*, Ergodic Theory Dynam. Systems **12** (1992), no. 3, 589–620.
- [Thu89] William P. Thurston, *Groups, tilings and finite state automata* (AMS Colloquium Lecture Notes), 1989.
- [Thu90] ———, *Three-dimensional geometry and topology*, Univ. of Minnesota Geometry Center preprint, 1990.
- [Vin95] Andrew Vince, *Rep-tiling Euclidean space*, Aequationes Mathematicae **50** (1995), 191–213.
- [Vin00] ———, *Digit tiling of Euclidean space*, Directions in Mathematical Quasicrystals, Amer. Math. Soc., Providence, RI, 2000, pp. 329–370.
- [Wil71] John S. Wilson, *Groups with every proper quotient finite*, Math. Proc. Cambridge Philos. Soc. **69** (1971), 373–391.
- [Wil00] ———, *On just infinite abstract and profinite groups*, New Horizons in pro- p Groups (Aner Shalev, Marcus P. F. du Sautoy, and Dan Segal, eds.), Progress in Mathematics, vol. 184, Birkhäuser Verlag, Basel, 2000, pp. 181–203.
- [Wil04a] ———, *Further groups that do not have uniformly exponential growth*, Journal of Algebra **279** (2004), 292–301.
- [Wil04b] ———, *On exponential growth and uniform exponential growth for groups*, Inventiones Mathematicae **155** (2004), 287–303.
- [Yac73] M. V. Yacobson, *On the question of topological classification of rational mappings of the Riemann sphere*, Uspekhi Mat. Nauk **28** (1973), no. 2, 247–248.
- [Yac80] ———, *Markov partitions for rational endomorphisms of the Riemann sphere*, Multicomponent Random Systems, Dekker, New York, 1980, pp. 381–396.

Index

- action
 - contracting, 57
 - defined by a bimodule and a basis, 35
 - finite-state, 11
 - fractal, 45
 - level-transitive, 2
 - monodromy, 136
 - recurrent, 45
 - self-similar, 10
 - standard, 142, 143
- adding machine, 16, 196
- airplane, 116
- asymptotic equivalence, 73
- atlas, 121
- automaton, 4
 - bi-reversible, 23
 - complete, 11
 - dual, 6
 - invertible, 8
 - kneading, 187
 - planar, 196
 - reduced, 8
- bad isotropy groups, 201
- basilica, 209
- basis of a bimodule, 32
- Belyi polynomial, 210
- Bernoulli measure, 50
- bimodule
 - $\phi(G)G$, 39
 - associated to a self-covering, 140
 - associated to an action, 32
 - covering, 31
 - d -fold, 31
 - hyperbolic, 59
 - irreducible, 38
 - over algebras, 40
 - permutational, 31
- boundary
 - of a hyperbolic space, 101
 - of a rooted tree, 1, 50
- bounded automatic transformation, 106
- branched covering, 174
- Chebyshev polynomials, 211
- cocycle, 125
- covering defined by a cocycle, 126
- covering of orbispaces, 124
- critical portrait, 190
- cycle diagram, 185
- cycle graph, 186
- DBP, 210
- depth of a finitary automorphism, 104
- digit system, 45
- digit tile, 78
- Douady rabbit, 116
- dragon curve, 172
- embedding of orbispaces, 123
- equivalence of groupoids, 120
- Euclidean orbifold, 170
- Euler characteristic, 170, 178
- expanding self-covering, 149
- extended ray, 194
- external ray, 194
- Fabrykowski-Gupta group, 213
- Fatou component, 193
- Fatou set, 193
- finitary automorphism, 104
- \mathcal{G} -path, 131
- \mathcal{G} -set, 118
- graded covering, 125
- Grigorchuk group, 13
- Grigorchuk groups G_w , 55
- Gromov-hyperbolic space, 99
- group
 - of A -adic vectors, 49
 - basilica, 209
 - of bounded automata, 104
 - branch, 3
 - contracting, 57
 - Fabrykowski-Gupta, 213
 - of finitary automorphisms, 104
 - of finite automata, 8
 - finite-state, 11

- of functionally recursive automorphisms, 12
- generated by an automaton, 12
- Grigorchuk, 13
- Gupta-Sidki, 18
- G_w , 55
- Heisenberg, 173
- of intermediate growth, 213
- isotropy, 118
- iterated monodromy, 137
- just-infinite, 17
- lamplighter, 22
- level-transitive, 2
- profinite iterated monodromy, 137
- recurrent, 45
- regular branch, 66
- self-similar, 10
- weakly branch, 3
- groupoid, 117
 - of action, 118
 - of changes of charts, 120, 121
 - étale, 117
 - free, 120
 - of germs, 118
 - proper, 119
- Gupta-Sidki group, 18
- Heighway dragon, 172
- Heisenberg group, 173
- hyperbolic space, 99
- index of a virtual homomorphism, 37
- internal ray, 194
- isotropy group, 118
- iterated monodromy action, 138, 140
- iterated monodromy group, 137, 138
- Julia set, 149
- kneading automaton, 187, 191
- kneading sequence, 203, 207
- L -presentation, 66
- Lattès examples, 171
- length structure, 148
- level-transitive
 - automorphism, 25
 - group, 2
 - tree, 50
- limit dynamical system, 93
- limit G -space, 73
- limit solenoid, 156
- limit space
 - \mathcal{J}_G , 92
 - \mathcal{X}_G , 73
- linear recursion, 41
- localization of a groupoid, 121
- mating, 215
- monodromy action, 136
- Moore diagram. 5
 - dual, 7
- nucleus, 57
- odometer, 16
- open map of orbispaces, 122
- open set condition, 80
- orbifold, 122
- orbispace, 121
- orbit of a groupoid, 118
- output function, 4
- parameter ray, 205
- partial self-covering, 127
- path in an orbispace, 131
- path-connected
 - groupoid, 132
 - orbispace, 132
- portrait
 - of an automorphism, 4
 - critical, 190
- post-critical point, 174
- post-critically finite, 174, 175
- pseudogroup, 117
 - proper, 119
- pseudogroup of changes of charts, 121
- pull back of a partial self-covering, 127
- pull-back, 126
- quasi-isometry, 97
- rabbit, 116
- ray
 - extended, 194
 - external, 194
 - internal, 194
 - supporting, 194
- restriction, 4, 35
 - of a groupoid, 120, 121
 - of a partial self-covering, 127
- rigid orbispace, 122
- rigid stabilizer, 2
- rooted automorphism, 10
- saturated isomorphism, 54
- Schreier graph, 94
- sectors (of a critical portrait), 190
- self-covering, 127
- self-similarity graph, 97
- shift, 93
- Sierpinski gasket, 112
- skew product, 125
- spider, 190
- stabilizer
 - of a level, 2
 - rigid, 2
 - of a vertex, 2
- standard action, 142, 143
- sub-hyperbolic rational function, 176

- subgroup
 - ϕ -invariant, 41
 - ϕ -semi-invariant, 41
 - self-similar, 42, 96
- supporting ray, 194

- tame twin dragon, 167
- tensor power of an action, 35
- tensor product of bimodules, 33
- Thurston map, 175
- Thurston orbifold, 175
- tile, 78
- tile diagram, 110
- topological polynomial, 190
- transition function, 4
- tree-like set of permutations, 185
- twin dragon, 167

- underlying space, 121
- uniformizing map, 121
- union of atlases, 122
- universal covering, 133

- virtual endomorphism, 37
 - associated to an action, 38
 - associated to a bimodule, 38
 - associated to a self-covering, 141
- virtual homomorphism, 37

- wreath product, 9
- wreath recursion, 10, 33

Titles in This Series

- 117 **Volodymyr Nekrashevych**, Self-similar groups, 2005
- 116 **Alexander Koldobsky**, Fourier analysis in convex geometry, 2005
- 115 **Carlos Julio Moreno**, Advanced analytic number theory: L-functions, 2005
- 114 **Gregory F. Lawler**, Conformally invariant processes in the plane, 2005
- 113 **William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith**, Homotopy limit functors on model categories and homotopical categories, 2004
- 112 **Michael Aschbacher and Stephen D. Smith**, The classification of quasithin groups II. Main theorems: The classification of simple QTKE-groups, 2004
- 111 **Michael Aschbacher and Stephen D. Smith**, The classification of quasithin groups I. Structure of strongly quasithin K -groups, 2004
- 110 **Bennett Chow and Dan Knopf**, The Ricci flow: An introduction, 2004
- 109 **Goro Shimura**, Arithmetic and analytic theories of quadratic forms and Clifford groups, 2004
- 108 **Michael Farber**, Topology of closed one-forms, 2004
- 107 **Jens Carsten Jantzen**, Representations of algebraic groups, 2003
- 106 **Hiroyuki Yoshida**, Absolute CM-periods, 2003
- 105 **Charalambos D. Aliprantis and Owen Burkinshaw**, Locally solid Riesz spaces with applications to economics, second edition, 2003
- 104 **Graham Everest, Alf van der Poorten, Igor Shparlinski, and Thomas Ward**, Recurrence sequences, 2003
- 103 **Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré**, Lusternik-Schnirelmann category, 2003
- 102 **Linda Rass and John Radcliffe**, Spatial deterministic epidemics, 2003
- 101 **Eli Glasner**, Ergodic theory via joinings, 2003
- 100 **Peter Duren and Alexander Schuster**, Bergman spaces, 2004
- 99 **Philip S. Hirschhorn**, Model categories and their localizations, 2003
- 98 **Victor Guillemin, Viktor Ginzburg, and Yael Karshon**, Moment maps, cobordisms, and Hamiltonian group actions, 2002
- 97 **V. A. Vassiliev**, Applied Picard-Lefschetz theory, 2002
- 96 **Martin Markl, Steve Shnider, and Jim Stasheff**, Operads in algebra, topology and physics, 2002
- 95 **Seiichi Kamada**, Braid and knot theory in dimension four, 2002
- 94 **Mara D. Neusel and Larry Smith**, Invariant theory of finite groups, 2002
- 93 **Nikolai K. Nikolski**, Operators, functions, and systems: An easy reading. Volume 2: Model operators and systems, 2002
- 92 **Nikolai K. Nikolski**, Operators, functions, and systems: An easy reading. Volume 1: Hardy, Hankel, and Toeplitz, 2002
- 91 **Richard Montgomery**, A tour of subriemannian geometries, their geodesics and applications, 2002
- 90 **Christian Gérard and Izabella Łaba**, Multiparticle quantum scattering in constant magnetic fields, 2002
- 89 **Michel Ledoux**, The concentration of measure phenomenon, 2001
- 88 **Edward Frenkel and David Ben-Zvi**, Vertex algebras and algebraic curves, second edition, 2004
- 87 **Bruno Poizat**, Stable groups, 2001
- 86 **Stanley N. Burris**, Number theoretic density and logical limit laws, 2001
- 85 **V. A. Kozlov, V. G. Maz'ya, and J. Rossmann**, Spectral problems associated with corner singularities of solutions to elliptic equations, 2001

TITLES IN THIS SERIES

- 84 **László Fuchs and Luigi Salce**, Modules over non-Noetherian domains, 2001
- 83 **Sigurdur Helgason**, Groups and geometric analysis: Integral geometry, invariant differential operators, and spherical functions, 2000
- 82 **Goro Shimura**, Arithmeticity in the theory of automorphic forms, 2000
- 81 **Michael E. Taylor**, Tools for PDE: Pseudodifferential operators, paradifferential operators, and layer potentials, 2000
- 80 **Lindsay N. Childs**, Taming wild extensions: Hopf algebras and local Galois module theory, 2000
- 79 **Joseph A. Cima and William T. Ross**, The backward shift on the Hardy space, 2000
- 78 **Boris A. Kupershmidt**, KP or mKP: Noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems, 2000
- 77 **Fumio Hiai and Dénes Petz**, The semicircle law, free random variables and entropy, 2000
- 76 **Frederick P. Gardiner and Nikola Lakic**, Quasiconformal Teichmüller theory, 2000
- 75 **Greg Hjorth**, Classification and orbit equivalence relations, 2000
- 74 **Daniel W. Stroock**, An introduction to the analysis of paths on a Riemannian manifold, 2000
- 73 **John Locker**, Spectral theory of non-self-adjoint two-point differential operators, 2000
- 72 **Gerald Teschl**, Jacobi operators and completely integrable nonlinear lattices, 1999
- 71 **Lajos Pukánszky**, Characters of connected Lie groups, 1999
- 70 **Carmen Chicone and Yuri Latushkin**, Evolution semigroups in dynamical systems and differential equations, 1999
- 69 **C. T. C. Wall (A. A. Ranicki, Editor)**, Surgery on compact manifolds, second edition, 1999
- 68 **David A. Cox and Sheldon Katz**, Mirror symmetry and algebraic geometry, 1999
- 67 **A. Borel and N. Wallach**, Continuous cohomology, discrete subgroups, and representations of reductive groups, second edition, 2000
- 66 **Yu. Ilyashenko and Weigu Li**, Nonlocal bifurcations, 1999
- 65 **Carl Faith**, Rings and things and a fine array of twentieth century associative algebra, 1999
- 64 **Rene A. Carmona and Boris Rozovskii, Editors**, Stochastic partial differential equations: Six perspectives, 1999
- 63 **Mark Hovey**, Model categories, 1999
- 62 **Vladimir I. Bogachev**, Gaussian measures, 1998
- 61 **W. Norrie Everitt and Lawrence Markus**, Boundary value problems and symplectic algebra for ordinary differential and quasi-differential operators, 1999
- 60 **Iain Raeburn and Dana P. Williams**, Morita equivalence and continuous-trace C^* -algebras, 1998
- 59 **Paul Howard and Jean E. Rubin**, Consequences of the axiom of choice, 1998
- 58 **Pavel I. Etingof, Igor B. Frenkel, and Alexander A. Kirillov, Jr.**, Lectures on representation theory and Knizhnik-Zamolodchikov equations, 1998
- 57 **Marc Levine**, Mixed motives, 1998
- 56 **Leonid I. Korogodski and Yan S. Soibelman**, Algebras of functions on quantum groups: Part I, 1998
- 55 **J. Scott Carter and Masahico Saito**, Knotted surfaces and their diagrams, 1998

For a complete list of titles in this series, visit the
AMS Bookstore at www.ams.org/bookstore/.

Self-similar groups (groups generated by automata) appeared initially as examples of groups that are easy to define but that enjoy exotic properties like nontrivial torsion, intermediate growth, etc.

The book studies the self-similarity phenomenon in group theory and shows its intimate relation with dynamical systems and more classical self-similar structures, such as fractals, Julia sets, and self-affine tilings. The relation is established through the notions of the iterated monodromy group and the limit space, which are the central topics of the book.

A wide variety of examples and different applications of self-similar groups to dynamical systems and vice versa are discussed. It is shown in particular how Julia sets can be reconstructed from the respective iterated monodromy groups and that groups with exotic properties appear now not just as isolated examples but as naturally defined iterated monodromy groups of rational functions.

The book is intended to be accessible to a wide mathematical readership, including graduate students interested in group theory and dynamical systems.

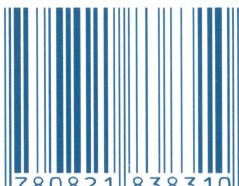


For additional information
and updates on this book, visit

www.ams.org/bookpages/surv-117

AMS on the Web
www.ams.org

ISBN 0-8218-3831-8



9 780821 838310

SURV/117