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Dedicated to Professor WENYUAN CHEN

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Preface

We present in this book a geometric theory for incompressible flows and its applications to fluid dynamics. This study was initiated by the authors in the mid-1990s with the original motivation to contribute to the understanding of oceanic dynamics in physical space. The development of the theory and its applications have gone well beyond the original motivation since then.

The main objective of the work presented in this book is to study the stability and transitions of the structure of incompressible flows and their applications to fluid dynamics and geophysical fluid dynamics. This book addresses both kinematic and dynamic theories for incompressible flows and their applications.

Part of this book was used for a one-semester graduate topic course in the Mathematics Department at Indiana University. This book contains six chapters. The first three chapters are devoted to the classification and stability of topological structures of divergence-free vector fields on two-dimensional compact manifolds. The last three chapters deal with the classification, stability and evolution of topological structures of the solutions of hydrodynamical equations such as the Euler equations and the Navier-Stokes equations. Examples with numerical studies are given in the last chapter.

We would like to acknowledge explicitly the singular influence of Professors Ciprian Foias and Roger Temam, in our studies of mathematical fluid mechanics. We have greatly benefitted from discussions with Jerry Bona, Ciprian Foias, Susan Friedlander, Michael Ghil, Robert Glassey, David Hoff, Darryl Holm, James McWilliams, Paul Newton, George Papanicoulaou, Jie Shen, Roger Temam, Cheng Wang, Mohammed Ziane, and Kevin Zumbrun. Our warm thanks to all of them. Also, we are grateful to Wen Masters and Reza Malek-Madani of the Office of Naval Research for their constant support and encouragement. We express our sincerest thanks to John Ewing and Edward Dunne of the American Mathematical Society for their great effort, support, encouragement and confidence in this book project.

Finally, nothing would have been possible without the understanding and patience of Li and Ping, and special thanks go to Jiao, Wayne and Melinda, for all the fun they have brought us.

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Tian Ma and Shouhong Wang
Bloomington, April 5, 2005

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