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To our wives Regina, Liza, Veronica, and Olga

Contents

Preface

This book summarizes recent developments in the theory of classical Painlevé equations based on the so-called Riemann-Hilbert, or Isomonodromy, method. The emphasis is on the use of this method for the global asymptotic analysis of the Painlevé transcendents. Specifically, we will study the *connection problem* for the Painlevé transcendents, i.e., the problem of finding explicit formulae relating the relevant asymptotic parameters at different critical points. In more detail we will present:

(i) A complete description of all types of possible asymptotic behavior of the solutions of the important particular cases of Painlevé II and III equations on the real line.
(ii) The derivation of the associated connection formulae.
(iii) A complete description of the asymptotic behavior in the complex plane; quasi-linear and nonlinear Stokes phenomena.
(iv) The distributions of zeros and poles of the solutions.

In addition, we will show how the isomonodromy approach can be used for the study of algebraic aspects of the Painlevé theory. These include the study of the symmetry properties of Painlevé equations (Bäcklund transformations) and the derivation of their elementary solutions and the solutions expressed in terms of classical special functions. In other words, although the asymptotic analysis is the primary theme of this book, several other aspects of modern Painlevé theory will be presented as well. One of our methodological goals is to stress the unique role of the Riemann-Hilbert approach as a universal tool which provides the most effective treatment of many aspects of the theory. Another methodological message is that the Riemann-Hilbert method is, essentially, the only tool available for the study of the problem which is analytically the most challenging, namely the problem of global asymptotics.

It should be emphasized that before the emergence of the Riemann-Hilbert approach, it was thought that the explicit global analysis of the solutions was possible only for the linear differential equations associated with the classical special functions of the hypergeometric type. We will show that the Riemann-Hilbert formalism brings both the *nonlinear special functions*, i.e., the Painlevé functions, and the *linear special functions*, such as Bessel and Airy functions, under the same analytic umbrella.

The Riemann-Hilbert method is based on the remarkable observation, which goes back to the classical works of Garnier and Fuchs, that the Painlevé equations describe the isomonodromy deformations of certain systems of linear differential equations with rational coefficients. This implies that solving a Painlevé equation is equivalent to solving the inverse monodromy problem, i.e., *the Riemann-Hilbert* *problem* for that particular linear system which is associated to the given Painlevé equation and whose monodromy data are the first integrals of the latter. What was apparently missed in the classical works is that this monodromy relation provides a powerful tool for analyzing the Painlevé transcendents. Indeed, as will be demonstrated in this book, these *Riemann-Hilbert representations* of the Painlevé functions play the same efficient role that the contour integral representations play in the theory of linear special functions.

This book is the second attempt in the literature to present systematically the theory of Painlevé equations from the Riemann-Hilbert point of view. The first attempt was made by A. Its and V. Novokshenov in 1986 in the monograph *Isomonodromic Method in the Theory of Painlevé Transcendents.* Since then the field has developed considerably. The three most important and interrelated advances are: (1) the asymptotic analysis of the Painlevé equations has been extended to the entire complex plane, (2) the Riemann-Hilbert methodology has been made rigorous, (3) the method has been enhanced by the powerful *nonlinear steepest descent* asymptotic techniques introduced in the beginning of the 1990s by P. Deift and X. Zhou. Also, many new concrete asymptotic results have been obtained and many new exciting applications, notably in random matrix theory, have appeared. In the present book we will attempt to present some of these new developments as fully as possible.

We have tried to make the book self-contained so that no prior knowledge of the Painlevé equations or of the Riemann-Hilbert problems is needed. In fact, we begin the book with a detailed introduction to the monodromy theory of systems of linear ordinary differential equations with rational coefficients and with a discussion of the associated Riemann-Hilbert problems. In this context, the Painlevé equations appear as the isomonodromy deformations of the first nontrivial examples of linear systems. This point of view will allow us to put the theory of Painlevé transcendents from the beginning into the proper analytic context. Indeed, in this approach the Painlevé equations will appear simultaneously with the Riemann-Hilbert representations for their solutions. It is these representations, i.e., the representations in terms of solutions of certain matrix Riemann-Hilbert problems, that will allow us to obtain all the results mentioned earlier.

We also present some of the recent applications of Painlevé transcendents in physics and mathematics. We hope that this, together with the lists of the asymptotic formulae obtained and discussed in the book, will make the monograph valuable, not only for the readers interested in theoretical aspects of the Painlevé equations and of the Riemann-Hilbert method, but also for the broad scientific community as a source of reference material on Painlevé transcendents. A small part of this material is included in the introduction so that the reader can quickly appreciate the type of results contained in this book.

APPENDIX A

Proof of Theorem 3.4

This Appendix is based on the paper [**BolIK**].

Let us define the function $\Psi(\lambda, x)$ by the equations,

$$\Psi(\lambda, x)\big|_{\overline{\Omega_1^0}} = Y_1(\lambda, x)e^{-\theta(\lambda,x)\sigma_3}, \tag{A.1}$$

$$\Psi(\lambda, x)\big|_{\overline{\Omega_k^0}} = Y_k(\lambda, x)e^{-\theta(\lambda,x)\sigma_3}\mathrm{S}_{k-1}^{-1}\dots\mathrm{S}_1^{-1}, \quad k = 1, \dots, 6, \tag{A.2}$$

where the branch of the $\ln\lambda$ is fixed by the condition,

$$\Big|\arg\lambda - \frac{k-1}{3}\pi\Big| < \frac{\pi}{6} \quad \text{when} \quad \lambda \in \Omega_k^0.$$

Slightly abusing the terminology, we will call the function $\Psi(\lambda, x)$, along with the function $Y(\lambda, x)$, *the solution of the Riemann-Hilbert problem (*3.3.98*)-(*3.3.99*).* In terms of the function$\Psi(\lambda, x)$, Theorem 3.4 can be reformulated as follows.

THEOREM A.0 *Let $s \equiv (\mathrm{S}_1, \mathrm{S}_2, \dots, \mathrm{S}_6; \nu)$ be the given monodromy data, i.e., the complex number ν and the set of six 2×2 matrices satisfying equations (*3.3.117*) and (*3.3.118*). Then there exists the countable set $X_s = \{x_j \equiv x_j(s)\} \subset \mathbb{C}$, $\quad x_j \to \infty$ and the unique matrix function $\Psi(\lambda, x)$ with the following properties.*

- **(a)** *$\Psi(\lambda, x)$ is holomorphic in $\mathbb{C} \times (\mathbb{C} \setminus X_s)$ and meromorphic along $\mathbb{C} \times X_s$ (i.e., it has poles at $x_j \in X_s$ and the coefficients of the corresponding Laurent series are entire functions of λ).*
- **(b)** *For every $x \in \mathbb{C} \setminus X_s$, the following asymptotic relation takes place,*

$$\Psi(\lambda, x)\mathrm{S}_1\dots\mathrm{S}_{k-1} \sim \Big(I + \sum_{j=1}^{\infty}\frac{\Psi_j(x)}{\lambda^j}\Big)e^{-\theta(\lambda,x)\sigma_3}, \tag{A.3}$$

$$\lambda \to \infty, \quad \lambda \in \overline{\Omega_k^0}, \quad \Big|\arg\lambda - \frac{k-1}{3}\pi\Big| < \frac{\pi}{6},$$

where

$$\theta(\lambda, x) = \frac{4i}{3}\lambda^3 + ix\lambda + \nu\ln\lambda,$$

and the coefficient functions $\Psi_j(x)$ are meromorphic in x and have the set X_s as the set of their poles.

- **(c)** *The asymptotics (*A.3*) is uniform, i.e., for any positive number N and for any compact subset $\mathbb{K} \in \mathbb{C} \setminus X_s$, there exists a constant $C > 0$ such that*

$$\Big\|\Psi(\lambda, x)\mathrm{S}_1\dots\mathrm{S}_{k-1}e^{-\theta(\lambda,x)\sigma_3} - I - \sum_{j=1}^{N}\frac{\Psi_j(x)}{\lambda^j}\Big\| < \frac{C}{\lambda^{N+1}}, \tag{A.4}$$

for all $|\lambda| > 1$, $\lambda \in \overline{\Omega_k^0}$ and $x \in \mathbb{K}$.

(d) *The asymptotics* (A.3) *is differentiable with respect to* λ *and* x.

(e) $\det \Psi(\lambda, x) \equiv 1$.

REMARK A.1. Because of the "correct" triangular structure of the Stokes matrices S_k, the asymptotic condition (A.3) actually holds in the open sectors Ω_k defined in (2.1.90). Indeed, we have that for any positive number N, for any compact subset $\mathbb{K} \in \mathbb{C} \setminus X_s$, and for any closed subsector $\Omega'_k \subset \Omega_k$, there exists a constant $C > 0$, such that

$$\left\| \Psi(\lambda, x) S_1 \dots S_{k-1} e^{-\theta(\lambda,x)\sigma_3} - I - \sum_{j=1}^{N} \frac{\Psi_j^0}{\lambda^j} \right\| \leq \frac{C}{\lambda^{N+1}},$$

$$\forall (\lambda, x) \in \Omega'_k \times \mathbb{K}, \quad |\lambda| > 1.$$

In what follows we will prove Theorem A.0. Theorem 3.4 will then follow by using equations (A.1) and (A.2) to define the function $Y(\lambda, x)$ via the already known function $\Psi(\lambda, x)$.

We split the proof into three steps. In the first step, we establish the existence of the local, in both λ and x, solution of the Riemann-Hilbert problem. This means we prove that, given a neighborhood Ω of $\lambda = \infty$ and a disk D on the x-plane, there exists a matrix function, which we denote by $M(\lambda, x)$, which is holomorphically invertible in $\Omega \times D$, and which satisfies the asymptotic condition (A.3) with all $\Psi_j(x)$ holomorphic in D (see Theorem A.1 below). We note, that for the standard case that the dependence of RH on a parameter is not addressed, this result follows from the Sibuya Theorem [**Si2**] concerning the local solvability of the general Riemann-Hilbert-Birkhoff problem for a linear system near its irregular singularity. Therefore, we call this step *the Sibuya Theorem with a parameter.* The principal construction used in this step, is the introduction of the explicit model functions $g_k(\lambda, x)$, such that each of them factorizes the corresponding Stokes matrix S_k. It is essentially this technique, together with the application of the Birkhoff-Grothendieck theorem (the second step), that allows us to avoid using the L^2-Fredholm theory of singular integral operators, which is the principal tool used in the works [**BS**] and [**FoZ**].

In the second step, we apply the classical Birkhoff-Grothendieck theorem, appropriately generalized to the case that the RH depends on a parameter, to the function $M(\lambda, x)$. This yields a global in λ, but still local in x, solution of the original RH problem (see Theorem A.3 below). This is the most important step, therefore for the convenience of the reader we will present a detailed proof of the Birkhoff-Grothendieck theorem with a parameter, following [**Bol4**], in Appendix B.

Finally, in the third step, we cover the x-plane by a countable number of small disks, and glue up the corresponding local solutions into a single, global in both λ and x, solution of the inverse monodromy problem, which completes the proof of Theorem A.0 and hence Theorem 3.4.

Before we proceed, we make one technical assumption concerning the formal monodromy exponent ν. We assume that

$$\nu \notin \mathbb{Z} \setminus \{0\}. \tag{A.5}$$

The following lemma shows that we can make this assumption, without loss of generality.

LEMMA A.1. *Let $\Psi(\lambda, x) \equiv \Psi_1(\lambda, x)$ be the solution of the RH problem with the monodromy data $s = (\mathrm{S}_1, \dots, \mathrm{S}_6; \nu)$, and let $u(x)$ and $v(x)$ be the corresponding functional parameters of the manifold $\mathcal{A}$ of the singular data. Then, the functions*

$$\hat{\Psi}(\lambda, x) := \begin{pmatrix} \lambda + u_x/2iu & -u/2 \\ 2/u & 0 \end{pmatrix} \Psi(\lambda, x),$$

and

$$\tilde{\Psi}(\lambda, x) := \begin{pmatrix} 0 & 2/v \\ -v/2 & \lambda - v_x/2iv \end{pmatrix} \Psi(\lambda, x),$$

solve RH problems with the monodromy data

$$\hat{s} = (\mathrm{S}_1, \dots, \mathrm{S}_6; \nu - 1),$$

and

$$\tilde{s} = (\mathrm{S}_1, \dots, \mathrm{S}_6; \nu + 1),$$

respectively.

Note, that only the asymptotic property (3.3.125) needs to be proven. The proof can be obtained by a direct calculation using equations (3.3.132) and (3.3.133), relating the matrix coefficients Ψ_j of the asymptotic series for Ψ and the functions u, v, $w = u_x$, and $z = v_x$[1].

0.1. The Sibuya theorem with a parameter. Throughout this section we will use the following notations. For any set $\mathbb{M} \subset \mathbb{C}^N$, with nonempty interior, we denote $H(\mathbb{M})$ the Banach space of the 2×2 matrix functions holomorphic in the interior of $\mathbb{M}$ and continuous in $\overline{\mathbb{M}}$. We will denote by $H^0(\mathbb{M})$ the subset of $H(\mathbb{M})$ consisting of holomorphically invertible matrix functions. We shall call such functions holomorphic in $\mathbb{M}$ and holomorphically invertible in $\mathbb{M}$ respectively.

Let

$$\Omega_\rho = \{\lambda \in \mathbb{C}: \quad |\lambda| > \rho\}$$

be a neighborhood of $\lambda = \infty$. We define the following subsets of Ω_ρ:

$$\Gamma_k^\rho = \Big\{\lambda \in \mathbb{C}: \quad |\lambda| \geq 2\rho, \quad \arg\lambda = \frac{2k-1}{6}\pi\Big\}, \quad k = 1, \dots, 6,$$

$$\Omega_k^\rho = \Big\{\lambda \in \mathbb{C}: \quad |\lambda| > \rho, \quad \Big|\arg\lambda - \frac{2k-1}{6}\pi\Big| < \varepsilon\Big\},$$

$$\mathcal{L}_k = \Big\{\lambda \in \mathbb{C}: \quad |\lambda| \geq 3\rho, \quad \Big|\arg\lambda - \frac{k-1}{3}\pi\Big| \leq \frac{\pi}{6}\Big\}, \quad k = 1, \dots, 6.$$

---

[1] The transformations $\Psi \mapsto \hat{\Psi}$ and $\Psi \mapsto \tilde{\Psi}$ are examples (found by A. Kitaev) of the *Schlesinger transformations* (see e.g. **[JMU]**; see also Chapter 6). They induce the following *Bäcklund transformations*,

$$(u, v) \mapsto \Big(-\frac{u}{4}(uv + x - u_x^2/u^2), \ \frac{4}{u}\Big),$$

and

$$(u, v) \mapsto \Big(\frac{4}{v}, \ -\frac{v}{4}(uv + x - v_x^2/v^2)\Big),$$

on the solution space of the system (2.1.86).

We assume that the contours Γ_k^ρ are oriented towards the ∞, and that ε is a sufficiently small positive number such that $\Omega_k^\rho \cap \Omega_{k+1}^\rho = \emptyset$. At the moment, we do not impose any restrictions on the positive number ρ. The contours Γ_k^ρ and the sets Ω_k^ρ are depicted in Figure A.1. The sectors $\mathcal{L}_k$ are shown in Figure A.2.

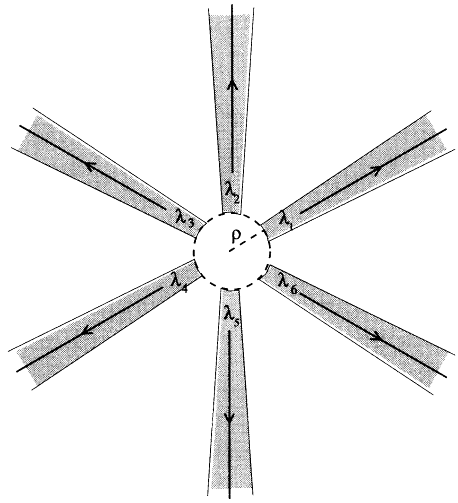

FIGURE A.1. The contours Γ_k^ρ and the sets Ω_k^ρ

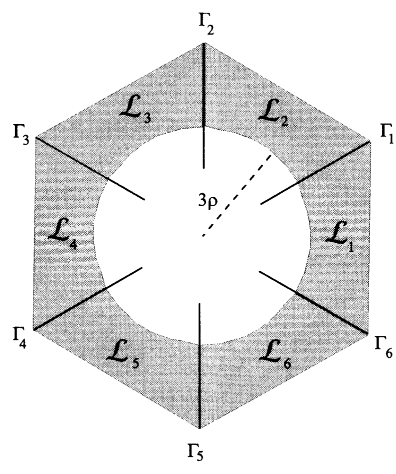

FIGURE A.2. The sectors $\mathcal{L}_k$

The goal of this section is to prove the following theorem.

THEOREM A.1 (The Sibuya theorem with a parameter). *Let $s = (S_1, \ldots, S_6; \nu) \in \mathcal{M}$ be the given monodromy data, i.e., the complex number ν and the set of six 2×2 matrices satisfying equations (3.3.117) and (3.3.118). Let $\mathcal{K}$ be a closed disk on the complex x-plane. Then, there exists a number $\rho_0 \equiv \rho_0(\mathcal{K})$, and a matrix function $M(\lambda, x)$, such that for all $\rho \geq \rho_0$:*

1. $M(\lambda, x) \in H^0(\Omega_{3\rho} \times \mathcal{K})$.
2. *For every* $x \in \mathcal{K}$,

$$M(\lambda, x)\mathrm{S}_1 \ldots \mathrm{S}_{k-1} \sim \Big(I + \sum_{j=1}^{\infty} \frac{\Psi_j^0(x)}{\lambda^j}\Big) e^{\left\{-\frac{4}{3}i\lambda^3\sigma_3 - ix\lambda\sigma_3 - \nu \ln \lambda\sigma_3\right\}}, \tag{A.6}$$

$$\lambda \to \infty, \quad \lambda \in \mathcal{L}_k, \quad \left|\arg \lambda - \frac{k-1}{3}\pi\right| \le \frac{\pi}{6},$$

where the coefficients $\Psi_j^0(x)$ *are holomorphic in* $\mathcal{K}$.
3. *The asymptotics* (A.6) *is uniform, i.e., for any positive number* l, *there exists a constant* $C > 0$, *such that*

$$\left\| M(\lambda, x)\mathrm{S}_1 \ldots \mathrm{S}_{k-1} \exp\left\{\frac{4}{3}i\lambda^3\sigma_3 + ix\lambda\sigma_3 + \nu \ln \lambda\, \sigma_3\right\} - I - \sum_{j=1}^{l} \frac{\Psi_j^0}{\lambda^j} \right\| \le \frac{C}{\lambda^{l+1}},$$

$$\forall (\lambda, x) \in \mathcal{L}_k \times \mathcal{K}.$$

4. *The asymptotics* (A.6) *is differentiable with respect to both* x *and* λ.

In an equivalent way, Theorem A.1 can be reformulated as follows.

THEOREM A.1′. *Let* $s = (\mathrm{S}_1, \ldots, \mathrm{S}_6; \nu) \in \mathcal{S}$ *be the given monodromy data, i.e., the complex number* ν *and the set of six* 2×2 *matrices satisfying equations* (3.3.117) *and* (3.3.118). *Let* $\mathcal{K}$ *be a closed disk on the complex* x*-plane. Then, there exists a number* $\rho_0 \equiv \rho_0(\mathcal{K})$, *and a collection of six matrix functions* $Y_k(\lambda, x)$, $k = 1, \ldots, 6$, *such that for all* $\rho \ge \rho_0$:

1. $Y_k(\lambda, x) \in H^0(\mathcal{L}_k \times \mathcal{K})$.
2. $Y_{k+1}(\lambda, x) = Y_k(\lambda, x)G_k(\lambda, x)$, $\quad \lambda \in \Gamma_k^{\rho} \cap \{|\lambda| \ge 3\rho\}$ $\quad k = 1, \ldots 6$, *where*

$$G_k(\lambda, x) = e^{-\theta(\lambda,x)\sigma_3}\mathrm{S}_k e^{\theta(\lambda,x)\sigma_3}, \quad \theta(\lambda, x) = i\frac{4}{3}\lambda^3 + ix\lambda + \nu \ln \lambda, \tag{A.7}$$

$$\arg \lambda = \frac{2k-1}{6}\pi \quad \textit{if} \quad \lambda \in \Gamma_k^{\rho},$$

and $Y_7(\lambda, x) \equiv Y_1(\lambda, x)^2$.
3. *For every* $x \in \mathcal{K}$, *and* $k = 1, \ldots, 6$,

$$Y_k(\lambda, x) \sim I + \sum_{j=1}^{\infty} \frac{\Psi_j^0(x)}{\lambda^j}, \tag{A.8}$$

$$\lambda \to \infty, \quad \lambda \in \mathcal{L}_k,$$

where the coefficients $\Psi_j^0(x)$ *are holomorphic in* $\mathcal{K}$.
4. *The asymptotics* (A.8) *is uniform, i.e., for any positive number* l, *there exists a constant* $C > 0$, *such that*

$$\left\| Y_k(\lambda, x) - I - \sum_{j=1}^{l} \frac{\Psi_j^0}{\lambda^j} \right\| \le \frac{C}{\lambda^{l+1}},$$

$$\forall (\lambda, x) \in \mathcal{L}_k \times \mathcal{K}, \quad \textit{and for all } k.$$

5. *The asymptotics* (A.8) *is differentiable with respect to both* x *and* λ.

---

[2]We emphasize that the sectors $\mathcal{L}_k$ belong to $\mathbb{C}$ but *not* to the universal covering of $\mathbb{C} \setminus \{0\}$.

To see that Theorem A.1$'$ implies Theorem A.1, let us define (cf. (A.1) and (A.2)) the function $M(\lambda, x)$ by the equations,

$$M(\lambda,x)\big|_{\mathcal{L}_1} = Y_1(\lambda,x)e^{-\theta(\lambda,x)\sigma_3},$$

$$M(\lambda,x)\big|_{\mathcal{L}_k} = Y_k(\lambda,x)e^{-\theta(\lambda,x)\sigma_3}\mathrm{S}_{k-1}^{-1}\dots\mathrm{S}_1^{-1}, \quad k>1,$$

where the branch of the $\ln\lambda$ is fixed by the condition,

$$\left|\arg\lambda - \frac{k-1}{3}\pi\right| \le \frac{\pi}{6} \quad \text{when} \quad \lambda\in\mathcal{L}_k.$$

The function $M(\lambda, x)$ defined above satisfies all the statements of the Theorem A.1. Indeed, because of the property 2 of $Y_k(\lambda, x)$, and of the cyclic constraint (3.3.118) on S_k, the function $M(\lambda,x)$ has no jumps across any of Γ_k^ρ and hence it is holomorphic in the whole $\Omega_{3\rho}$. The remaining of the properties stated in Theorem A.1 follows directly from the analogous properties of the functions $Y_k(\lambda, x)$.

In what follows, we prove Theorem A.4$'$. Our approach is based on replacing the jump conditions posed on the rays Γ_k^ρ, with some equivalent jump conditions posed on the horseshoe shape domains around the rays. To this end we introduce the following covering of the neighborhood Ω_ρ:

$$\Omega_\rho \subset \Omega_+ \cup \Omega_-,$$

where the sets Ω_+ and Ω_- are depicted in Figure A.3 and Figure A.4 respectively. These sets have the following properties:

- The set Ω_+ is obtained by deleting from the complex plane $\mathbb{C}$ a collection of thin neighborhoods, Ω_+^k, of the contours Γ_k^ρ. The closure of the neighborhood Ω_+^k is a subset of Ω_k^ρ. In other words,

$$\Omega_+ = \mathbb{C}\setminus\cup_{j=1}^6\Omega_+^k, \quad \Gamma_k^\rho\subset\Omega_+^k, \quad \overline{\Omega_+^k}\subset\Omega_k^\rho.$$

- The set Ω_- is the union of the (slightly bigger) neighborhoods, Ω_-^k, of the contours Γ_k^ρ, whose closures are also the subsets of the respective Ω_k^ρ. In other words,

$$\Omega_- = \cup_{k=1}^6\Omega_-^k, \quad \Gamma_k^\rho\subset\Omega_+^k, \quad \overline{\Omega_+^k}\subset\Omega_-^k, \quad \overline{\Omega_-^k}\subset\Omega_k^\rho.$$

- Let $\gamma^+ := \partial\Omega_+$ be the boundary of the set Ω_+. Then

$$\gamma^+ = \cup_k\gamma_k^+, \quad \gamma_k^+ = -\partial\Omega_+^k,$$

  and we assume that each γ_k^+ is a simple analytic curve which encircles the ray Γ_k^ρ, and which is asymptotic to the rays $\Gamma_{(k;\,\pm\varepsilon/3)}$, where

$$\Gamma_{(k;\,\delta)} := \left\{\lambda\in\mathbb{C}: \quad \arg\lambda - \frac{2k-1}{6}\pi = \delta\right\}.$$

- Let $\gamma^- := \partial\Omega_-$ be the boundary of the set Ω_-. Then

$$\gamma^- = \cup_k\gamma_k^-, \quad \gamma_k^- = \partial\Omega_-^k,$$

  and we assume that each γ_k^- is a simple analytic curve which encircles the ray Γ_k^ρ, and which is asymptotic to the rays $\Gamma_{(k;\,\pm\varepsilon/2)}$.

In addition to the sets Ω_+ and Ω_- we introduce their intersection,

$$\Omega_0 := \Omega_+\cap\Omega_-.$$

The set Ω_0 is depicted in Figure A.5. Observe that the above listed properties of the sets Ω_+ and Ω_- imply the following properties of the set Ω_0.

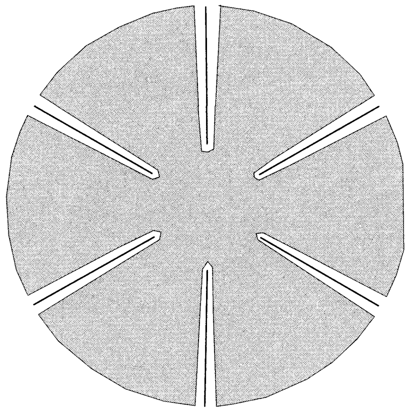

FIGURE A.3. The set $\Omega_+ = \mathbb{C} \setminus \cup_{j=1}^6 \Omega_+^k$

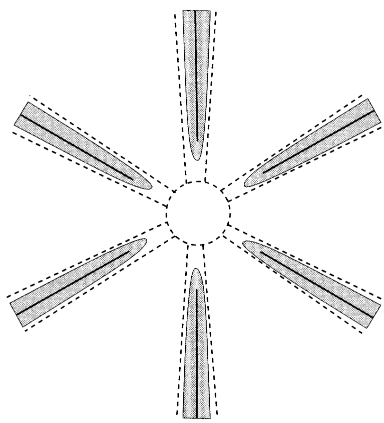

FIGURE A.4. the set $\Omega_- = \cup_{k=1}^6 \Omega_-^k$

- $\Omega_0 = \cup_{k=1}^6 \omega_k$, $\overline{\omega_k} \subset \Omega_k^\rho \setminus \Gamma_k^\rho$, and each subset ω_k is an open and connected set.
- $\partial\omega_k = \gamma_k^+ + \gamma_k^-$.
- $\mathrm{dist}\big(\gamma_k^+, \gamma_k^-\big) \geq c_\varepsilon$.

We are now ready to introduce the main ingredient of our proof, namely the model functions $g_k(\lambda, x)$. Assuming that $\lambda \in \Omega_k^\rho \setminus \Gamma_k^\rho$, we define g_k by

$$g_k(\lambda, x) = \exp\Big\{-\frac{1}{2\pi i}\int_{\Gamma_k^\rho} \frac{\ln G_k(\mu, x)}{\mu - \lambda}\, d\mu\Big\}$$

$$= \begin{cases} \begin{pmatrix} 1 & -\frac{s_k}{2\pi i}\int_{\Gamma_k^\rho} \frac{e^{-2\theta(\mu,x)}}{\mu-\lambda}\, d\mu \\ 0 & 1 \end{pmatrix}, & k = 2l, \\ \begin{pmatrix} 1 & 0 \\ -\frac{s_k}{2\pi i}\int_{\Gamma_k^\rho} \frac{e^{2\theta(\mu,x)}}{\mu-\lambda}\, d\mu & 1 \end{pmatrix}, & k = 2l-1, \end{cases} \tag{A.9}$$

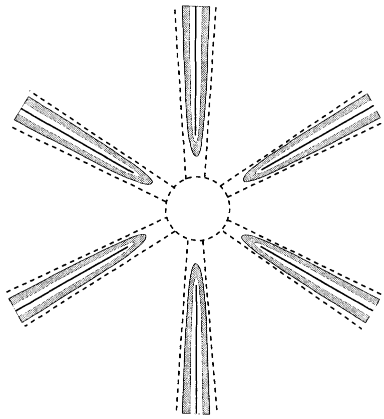

FIGURE A.5. The set $\Omega_0 = \Omega_- \cap \Omega_+ = \cup_{k=1}^6 \omega_k$

where the jump matrices $G_k(\lambda, x)$ and the phase function $\theta(\lambda, x)$ are given in (A.7). The characteristic property of $g_k(\lambda, x)$ is the jump relation,

$$g_{k+}(\lambda, x) = g_{k-}(\lambda, x) G_k^{-1}(\lambda, x), \quad \lambda \in \Gamma_k^\rho \cap \{|\lambda| \geq 2\rho + \delta\}, \tag{A.10}$$

where $g_{k+}(\lambda, x)$ and $g_{k-}(\lambda, x)$ denote the boundary values of $g_k(\lambda, x)$ as λ approaches Γ_k^ρ from the left and from the right, respectively.

On the set $\Omega_0 \times \mathcal{K}$, the function $g(\lambda, x)$ can be defined by setting

$$g(\lambda, x)\big|_{\omega_k} = g_k(\lambda, x). \tag{A.11}$$

The following are the direct consequences of the *explicit* formulae (A.9) and (A.11), as well as of the geometric properties of the set Ω_0 indicated above.

(i) $g(\lambda, x)$ is holomorphic in both λ and x. Indeed,

$$g(\lambda, x) \in H^0(\Omega_0 \times \mathcal{K}).$$

(ii) The following uniform estimate is valid,

$$\sup_{\substack{\lambda \in \Omega_o \\ x \in \mathcal{K}}} \|\lambda(g(\lambda, x) - I)\| \leq c_{\rho, \mathcal{K}}. \tag{A.12}$$

Moreover,

$$c_{\rho, \mathcal{K}} \to 0 \quad \text{as} \quad \rho \to \infty. \tag{A.13}$$

The next lemma, is one of the local versions of the Birkhoff-Grothendieck theorem with a parameter. This is the main technical step in our proof of the Sibuya Theorem A.1$'$. The basic idea is the uniform boundness of the relevant Cauchy operators (see Proposition A.1 below), which is the main reason for moving from the rays Γ_k^ρ to the horseshoe domains ω_k. This idea was first suggested in [**AnoBol**] (see also [**Bol4**] and Appendix B).

LEMMA A.2 (The basic factorization lemma). *For sufficiently large $\rho > 0$, there exist functions $\Psi_{\pm}(\lambda, x)$, such that:*

1. $\Psi_{\pm}(\lambda, x) \in H^0(\Omega_{\pm} \times \mathcal{K})$.
2. $\sup_{(\lambda,x)\in\Omega_{\pm}\times\mathcal{K}} \|\lambda(I - \Psi_{\pm}(\lambda, x))\| \le C.$
3. $\Psi_+(\lambda, x)g(\lambda, x) = \Psi_-(\lambda, x) \quad \forall\lambda \in \Omega_0, \quad \forall x \in \mathcal{K}.$

PROOF. Let us introduce three more Banach spaces.

$$H = \Big\{f \in H(\Omega_0 \times \mathcal{K}), \quad \sup_{(\lambda,x)\in\Omega_0\times\mathcal{K}} \big|\lambda f(\lambda, x)\big| < \infty\Big\},$$

$$H_{\pm} = \Big\{f \in H(\Omega_{\pm} \times \mathcal{K}), \quad \sup_{(\lambda,x)\in\Omega_{\pm}\times\mathcal{K}} \big|(1 + |\lambda|)f(\lambda, x)\big| < \infty\Big\} \subset H,$$

$$\|f\|_H \equiv \sup_{(\lambda,x)\in\Omega_0\times\mathcal{K}} |\lambda f(\lambda, x)|,$$

$$\|f\|_{H_{\pm}} \equiv \sup_{(\lambda,x)\in\Omega_{\pm}\times\mathcal{K}} |(1 + |\lambda|)f(\lambda, x)|.$$

Given

$$f \in H,$$

we can define the Cauchy operators:

$$\big(P_{\pm}f\big)(\lambda, x) = \frac{1}{2\pi i}\int_{\gamma^{\pm}} \frac{f(\mu, x)}{\mu - \lambda}\, d\mu, \quad \lambda \in \Omega_{\pm}.$$

We recall that

$$\gamma^{\pm} = \partial\Omega_{\pm},$$

and that

$$\partial\Omega_0 = \gamma^+ + \gamma^- = \sum_{k=1}^{6} \partial\omega_k = \sum_{k=1}^{6}(\gamma_k^+ + \gamma_k^-).$$

Note that the Cauchy formula,

$$f(\lambda, x) = \big(P_+f\big)(\lambda, x) + \big(P_-f\big)(\lambda, x), \quad \lambda \in \Omega_0, \tag{A.14}$$

is valied for the functions from H.

Put

$$g(\lambda, x) = I + \overset{o}{g}(\lambda, x), \quad \overset{o}{g}(\lambda, x) \in H,$$

and introduce the operators $P_{\pm}^g$ by the equation,

$$P_{\pm}^g(f) := P_{\pm}(f\, g^o).$$

PROPOSITION A.1. *$P_{\pm}^g$ are bounded operators from H to $H_{\pm}$.*

PROOF OF THE PROPOSITION (an adaptation of the technique of [**AnoBol**]). Each ω_k can be represented as the union of the domains ω_k^+ and ω_k^- according to the Figure 6. Assume that $\lambda \in \Omega_+ \setminus \cup_k \omega_k^+$ and hence

$$\frac{|\lambda|}{\operatorname{dist}\{\lambda, \gamma_+\}} \le C_{\epsilon},$$

for some positive constant C_{ϵ}. Then

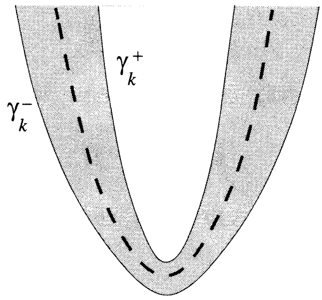

FIGURE A.6. $\omega_k = \omega_k^+ \cup \omega_k^-$

$$\lambda\big(P_+^g f\big)(\lambda, x) = \frac{\lambda}{2\pi i}\int_{\gamma^+} \frac{f(\mu,x)g^o(\mu,x)}{\mu-\lambda}\,d\mu,$$

$$\begin{aligned}\big|\lambda\big(P_+^g f\big)(\lambda,x)\big| &\le \frac{|\lambda|}{2\pi}\|f\|_H\|g^o\|_H\int_{\gamma^+}\frac{|d\mu|}{|\mu|^2|\mu-\lambda|}\\ &\le \frac{|\lambda|}{2\pi\,\mathrm{dist}\,\{\lambda,\gamma_+\}}\|f\|_H\|g^o\|_H\int_{\gamma^+}\frac{|d\mu|}{|\mu|^2}\\ &\le \frac{C_\epsilon}{2\pi}\|f\|_H\|g^o\|_H\int_{\gamma^+}\frac{|d\mu|}{|\mu|^2} \equiv C_+\cdot\|f\|_H\,.\end{aligned}$$

Similarly, if $\lambda \in \Omega_-\setminus \cup_k\,\omega_k^-$, then

$$\big|\lambda\big(P_-^g f\big)(\lambda,x)\big| \le C_-\cdot\|f\|_H\,. \tag{A.15}$$

Suppose now that $\lambda \in \omega_k^+$. Then, we can use identity (A.14) and rewrite $\lambda\big(P_+^g f\big)(\lambda,x)$ as

$$\lambda\big(P_+^g f\big)(\lambda,x) = \lambda f(\lambda,x)g^o(\lambda,x) - \lambda\big(P_-^g f\big)(\lambda,x). \tag{A.16}$$

On the other hand, $\lambda \in \omega_k^+ \Rightarrow \lambda \in \Omega_-\setminus\cup_k\,\omega_k^-$, and hence we can use (A.15) in the right-hand side of (A.16). Therefore (cf. [**AnoBol**] and the proof of of Lemma B.1 in Appendix B), we obtain,

$$\big|\lambda\big(P_+^g f\big)(\lambda,x)\big| \le C'_+\cdot\|f\|_H,$$

if $\lambda \in \omega_k^+$. In other words, we have the inequality

$$\big|\lambda\big(P_+^g f\big)(\lambda,x)\big| \le C''\cdot\|f\|_H,\qquad C'' := \max\{C_+, C'_+\} \tag{A.17}$$

for all $(\lambda, x) \in \overline{\Omega_+}\times\mathcal{K}$.

Assuming that $|\lambda| \le \rho/2$ we obtain at once that

$$\begin{aligned}\big|\big(P_+^g f\big)(\lambda,x)\big| &\le \frac{1}{2\pi}\|f\|_H\|g^o\|_H\int_{\gamma^+}\frac{|d\mu|}{|\mu|^2|\mu-\lambda|}\\ &\le \frac{1}{2\pi\,\mathrm{dist}\,\{\lambda,\gamma_+\}}\|f\|_H\|g^o\|_H\int_{\gamma^+}\frac{|d\mu|}{|\mu|^2}\equiv \tilde{C}_+\cdot\|f\|_H\,.\end{aligned}$$

This estimate together with (A.17) imply

$$\big\|P_+^g f\big\|_{H_+} \le C\cdot\|f\|_H,\qquad C := \max\{C'', \tilde{C}_+\}$$

Similarly,

$$\left\|P_-^g f\right\|_{H_-} \leq C' \cdot \|f\|_H$$

and the proposition is proved. □

Two additional facts should be noted. First, all the C-constants above contain the norm of g^o, i.e., $\left\|g^o\right\|_H$, as a factor. Hence, in virtue of estimates (A.12) and (A.13), the norm of the operators $P_\pm^g$ can be made as small as we wish by choosing ρ large enough. Secondly, from the integral formula (A.9) we derive the existence of the asymptotic expansion of the function $g^0(\lambda, x)$ over λ^{-1} as $\lambda \to \infty$, $\lambda \in \Omega_0$. This, in particular, implies that there exist the matrix function $c(x)$ and the constant C_0 such that

$$g^o(\lambda, x) = \frac{c(x)}{\lambda} + \tilde{g}(\lambda, x), \quad \left|\tilde{g}(\lambda, x)\right| \leq \frac{C_0}{|\lambda|^2}, \tag{A.18}$$

for all $(\lambda, x) \in \overline{\Omega_0} \times \mathcal{K}$. This estimate allows us to claim that

$$P_+ g^o \in H_+. \tag{A.19}$$

To see (A.19), we note that

$$\left(P_+ g^o\right)(\lambda, x) = \frac{c(x)}{2\pi i} \int_{\gamma^+} \frac{d\mu}{\mu(\mu - \lambda)} + \frac{1}{2\pi i} \int_{\gamma^+} \frac{\tilde{g}(\mu, x)}{(\mu - \lambda)}\, d\mu. \tag{A.20}$$

Denote the second integral as $g_r(\lambda, x)$. In virture of the estimate (A.18) and repeating the same line of arguments as in the proof of Proposition A.1, we conclude that

$$g_r \in H_+, \quad \text{and moreover} \quad \left\|g_r\right\|_{H_+} \to 0, \quad \rho \to \infty.$$

Since the first integral in (A.20) is obviously zero for all $\lambda \in \Omega_+$, we arrive at (A.19). Moreover, we have that

$$\left\|P_+ g^0\right\|_{H_+} \to 0, \quad \rho \to \infty. \tag{A.21}$$

We are ready now to complete the proof of Lemma A.2.

We are looking for

$$\Psi_\pm(\lambda, x) = I + \overset{o}{\Psi}_\pm(\lambda, x), \quad \overset{o}{\Psi}_\pm(\lambda, x) \in H_\pm,$$

such that

$$I + \overset{o}{\Psi}_- = \left(I + \overset{o}{\Psi}_+\right)\left(I + \overset{o}{g}\right),$$

or

$$\overset{o}{\Psi}_- = \overset{o}{\Psi}_+ + \overset{o}{\Psi}_+ \overset{o}{g} + \overset{o}{g}.$$

Applying P_+, we have

$$0 = \overset{o}{\Psi}_+ + P_+\left(\overset{o}{\Psi}_+ \overset{o}{g}\right) + P_+(\overset{o}{g}).$$

Hence, we need just to solve the following integral equation in the space H_+:

$$\overset{o}{\Psi}_+ + P_+^g\left(\overset{o}{\Psi}_+\right) = -P_+ \overset{o}{g}.$$

As we have already indicated, because of estimates (A.12) and (A.13), we conclude that the operator,

$$I + P_+^g$$

is invertible, for sufficiently large ρ, in the Banach space H_+, and, due to (A.21), the norm of $\Psi^o{}_+$ can be made small enough to ensure the invertability of matrix $\Psi_+(\lambda, x)$ for all $(\lambda, x) \in \Omega_+ \times \mathcal{K}$. This completes the proof of the Lemma. □

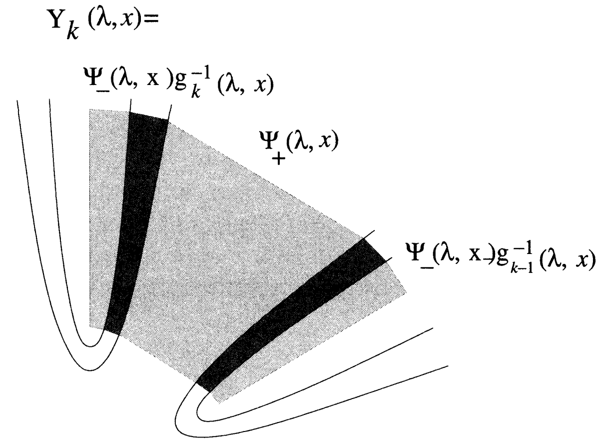

FIGURE A.7. Functions $Y_k(\lambda, x)$

PROOF OF THEOREM A.1′. Define the functions $Y_k(\lambda, x)$ as shown in Figure A.7. We claim that these functions satisfy all the properties stated in the theorem. The only properties which do not follow directly from the definition of the functions $Y_k(\lambda, x)$, and hence need a proof, are the asymptotic statements 3–5. The proof is based on the use of the appropriate integral representations for the functions $Y_k(\lambda, x)$; these representations can be obtained as follows (cf. [**Pa**]).

We first note that property 2 of the functions $\Psi_\pm(\lambda, x)$ implies the estimate

$$\sup_{(\lambda,x)\in\mathcal{L}_k\times\mathcal{K}} \|\lambda\big(I - Y_k(\lambda, x)\big)\| \le C, \tag{A.22}$$

which allows us to use the Cauchy formula in the sectors $\mathcal{L}_k$, and to obtain the equations,

$$Y_k(\lambda, x) = I + \int_{\partial\mathcal{L}_k} \frac{Y_k(\mu, x) - I}{\mu - \lambda}\frac{d\mu}{2\pi i}, \quad \lambda \in \overset{o}{\mathcal{L}}_k, \tag{A.23}$$

and

$$0 = \int_{\partial\mathcal{L}_k} \frac{Y_k(\mu, x) - I}{\mu - \lambda}\frac{d\mu}{2\pi i}, \quad \lambda \notin \mathcal{L}_k, \tag{A.24}$$

where $\mathcal{L}^o{}_k$ denotes the interior of $\mathcal{L}_k$. Denote by

$$\Gamma_k := \Gamma_k^\rho \cap \{|\lambda| \ge 3\rho\},$$

and let C_k be the arc of the circle $|\lambda| = 3\rho$, between the rays Γ_{k-1} and Γ_k (see Figure A.2). Assume that C_k is oriented counterclockwise. Then,

$$\partial\mathcal{L}_k = -\Gamma_k - C_k + \Gamma_{k-1},$$

and we can rewrite (A.23) as,

$$Y_k(\lambda, x) = I - \int_{C_k} \frac{Y_k(\mu, x) - I}{\mu - \lambda}\frac{d\mu}{2\pi i}$$

$$-\int_{\Gamma_k} \frac{Y_k(\mu, x) - I}{\mu - \lambda}\frac{d\mu}{2\pi i} + \int_{\Gamma_{k-1}} \frac{Y_k(\mu, x) - I}{\mu - \lambda}\frac{d\mu}{2\pi i}, \tag{A.25}$$

where $\lambda \in \mathcal{L}^o{}_k$. Observe that from property 2 of Y_k, we have that

$$Y_k(\lambda, x) - I = -Y_k(\lambda, x)\big(G_k(\lambda, x) - I\big) + Y_{k+1}(\lambda, x) - I, \quad \lambda \in \Gamma_k,$$

and

$$Y_k(\lambda, x) - I = Y_{k-1}(\lambda, x)\big(G_{k-1}(\lambda, x) - I\big) + Y_{k-1}(\lambda, x) - I, \quad \lambda \in \Gamma_{k-1}.$$

Therefore, (A.25) can be transformed into

$$Y_k(\lambda, x) = I - \int_{C_k} \frac{Y_k(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i}$$

$$+ \int_{\Gamma_k} \frac{Y_k(\mu, x)\big(G_k(\mu, x) - I\big)}{\mu - \lambda} \frac{d\mu}{2\pi i} + \int_{\Gamma_{k-1}} \frac{Y_{k-1}(\mu, x)\big(G_{k-1}(\mu, x) - I\big)}{\mu - \lambda} \frac{d\mu}{2\pi i}$$

$$- \int_{\Gamma_k} \frac{Y_{k+1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i} + \int_{\Gamma_{k-1}} \frac{Y_{k-1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i}. \tag{A.26}$$

Using the second Cauchy relation (A.24), we can replace the last two integrals by

$$\int_{C_{k+1}} \frac{Y_{k+1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i} + \int_{\Gamma_{k+1}} \frac{Y_{k+1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i},$$

and

$$- \int_{C_{k-1}} \frac{Y_{k-1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i} + \int_{\Gamma_{k-2}} \frac{Y_{k-1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i},$$

respectively. Thus, equation (A.26) can be further transformed into the equation,

$$Y_k(\lambda, x) = I - \int_{C_{k+1}} \frac{Y_{k+1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i} - \int_{C_k} \frac{Y_k(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i}$$
$$- \int_{C_{k-1}} \frac{Y_{k-1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i}$$

$$+ \int_{\Gamma_k} \frac{Y_k(\mu, x)\big(G_k(\mu, x) - I\big)}{\mu - \lambda} \frac{d\mu}{2\pi i} + \int_{\Gamma_{k-1}} \frac{Y_{k-1}(\mu, x)\big(G_{k-1}(\mu, x) - I\big)}{\mu - \lambda} \frac{d\mu}{2\pi i}$$

$$- \int_{\Gamma_{k+1}} \frac{Y_{k+1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i} + \int_{\Gamma_{k-2}} \frac{Y_{k-1}(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i}. \tag{A.27}$$

Comparing the two integrals in the last row of (A.27) with the analogous integrals of (A.25), we see that the effect of the manipulations that led us from (A.25) to (A.27) is, the increase of the index k in the first integral and its decrease in the second one. Therefore, repeating several times the above procedure, we arrive at the following final integral representation for $Y_k(\lambda, x)$,

$$Y_k(\lambda, x) = I - \sum_{l=1}^{6} \int_{C_l} \frac{Y_l(\mu, x) - I}{\mu - \lambda} \frac{d\mu}{2\pi i}$$

$$+ \sum_{j=1}^{6} \int_{\Gamma_l} \frac{Y_l(\mu, x)\big(G_l(\mu, x) - I\big)}{\mu - \lambda} \frac{d\mu}{2\pi i}, \tag{A.28}$$

for all $\lambda \in \mathcal{L}^o{}_k$.

Each term of the first sum in the r.h.s. of (A.28) is holomorphic in the neighborhood of $\lambda = \infty$, while the integrand of each term in the second sum decays exponentially as $\lambda \to \infty$. In fact, there exist positive constants C and c such that

$$\|Y_l(\mu, x)\big(G_l(\mu, x) - I\big)\| < Ce^{-c|\lambda|^3}, \quad \mu \in \Gamma_l, \quad x \in \mathcal{K}.$$

This means that, we indeed have for $Y_k(\lambda, x)$, uniformly in $x \in \mathcal{K}$, the full asymptotic series (A.8) with coefficients given by the equation,

$$\Psi_j^0(x) = \sum_{l=1}^{6} \int_{C_l} \big(Y_l(\mu, x) - I\big)\mu^{j-1}\frac{d\mu}{2\pi i}$$

$$-\sum_{l=1}^{6} \int_{\Gamma_l} Y_l(\mu, x)\big(G_l(\mu, x) - I\big)\mu^{j-1}\frac{d\mu}{2\pi i}. \tag{A.29}$$

¿From this equation, in particular, it follows that all $\Psi_j^0(x)$ are holomorphic functions.

By standard arguments, the integral representation (A.28) implies the x and λ-differentiability of the asymptotics (A.8). In fact, the λ and x-differentiability of the asymptotics (A.8) is a direct consequence of the fact that it is uniformly valid (see, e.g. [**W1**]).

Finally, we note that the asymptotic expansion (A.8) is valid in the closed sector $\mathcal{L}_k$ (in fact, in some sector which includes $\mathcal{L}_k$). Indeed, because of property 2 each of the functions $Y_k(\lambda, x)$ can be analytically continued beyond the boundaries of $\mathcal{L}_k$, so that $Y_k(\lambda, x)$ becomes holomorphic and satisfies the estimate (A.22) in a sector which includes $\mathcal{L}_k$. Therefore, in the representation (A.28) we can slightly rotate the rays Γ_k and Γ_{k-1} so that we can make the representation, and hence the asymptotics (A.8) valid for all $\lambda \in \mathcal{L}_k$. □

0.2. The local solution of the RH problem. Denote by Ω_0, Ω_∞, and ω the following subsets of the Riemann sphere $\mathbb{C}P^1$:

$$\Omega_0 = \{\lambda \in \mathbb{C} : |\lambda| \le R\},$$
$$\Omega_\infty = \{\lambda \in \mathbb{C} \cup \infty : |\lambda| \ge 4\rho\},$$
$$\omega = \Omega_0 \cap \Omega_\infty,$$

where $R > 4\rho$. Also denote by $D_\delta(x_0)$ the closed disk in the x-plane of radius δ and centered at x_0, i.e.,

$$D_\delta(x_0) = \{x \in \mathbb{C} : |x - x_0| \le \delta\}.$$

Let $M(\lambda, x)$ be the Sibuya function of the previous section corresponding to the choice

$$\mathcal{K} = D_\delta(x_0).$$

Observe that

$$M(\lambda, x) \in H^0\Big(\big(\Omega_\infty \setminus \{\infty\}\big) \times D_\delta(x_0)\Big), \tag{A.30}$$

for all

$$\rho \ge \rho_0(D_\delta(x_0)) \equiv \rho_0(x_0, \delta).$$

In particular,

$$M(\lambda, x) \in H^0(\omega \times D_\delta(x_0)). \tag{A.31}$$

By a direct application to the function $M(\lambda, x)$ of the Birkhoff-Grothendieck theorem with a parameter, which is formulated and proven in Appendix B, we arrive at the following result.

THEOREM A.2. *Let $x_0 \in \mathbb{C}$. Then there exists a number $\delta_0 > 0$, the finite set $X^0 \equiv \{x_{0j}\} \subset D_{\delta_0}(x_0)$, and the matrix functions $T(\lambda, x)$ and $\Phi(\lambda, x)$ such that:*

- *$T(\lambda, x)$ and $T^{-1}(\lambda, x)$ are holomorphic in $\Omega_\infty \times \big(D_{\delta_0}(x_0) \setminus X^0 0\big)$ and meromorphic along $\Omega_\infty \times X^0$ (i.e., they have poles at $x_{0j} \in X^0$ and the coefficients of the corresponding Laurent series are holomorphic in D_∞).*
- *$\Phi(\lambda, x)$ and $\Phi^{-1}(\lambda, x)$ are holomorphic in $\Omega_0 \times \big(D_{\delta_0}(x_0) \setminus X^0\big)$ and meromorphic along $\Omega_0 \times X^0$.*
-

$$M(\lambda, x) = T^{-1}(\lambda, x)\lambda^{\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}}\Phi(\lambda, x), \tag{A.32}$$

where $\kappa_1 \geq \kappa_2$ are some integer numbers, which are the same for all $x \in D_{\delta_0}(x_0) \setminus X^0$.

The specific features of our function $M(\lambda, x)$, yield the important additional properties of the objects participating in the factorization formula (A.32).

PROPOSITION A.2. *The function $\Phi(\lambda, x)$ is an entire function of λ; indeed,*

$$\Phi(\lambda, x) \in H^0\Big(\mathbb{C} \times \big(D_{\delta_0}(x_0) \setminus X^0\big)\Big),$$

and both $\Phi(\lambda, x)$ and $\Phi^{-1}(\lambda, x)$ are meromorphic along $D_0 \times X^0$. Moreover,

$$\Phi(\lambda, x)S_1 \dots S_{k-1} \sim \lambda^{\begin{pmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{pmatrix}}B_0\Big(I + \sum_{j=1}^{\infty} \frac{\Psi_j(x)}{\lambda^j}\Big)e^{-\theta(\lambda,x)\sigma_3}, \tag{A.33}$$

$$\lambda \to \infty, \quad \lambda \in \mathcal{L}_k, \quad \det B_0 \neq 0,$$

where the coefficients $\Psi_j(x)$ are holomorphic in $D_{\delta_0}(x_0) \setminus X^0$ and meromorphic in $D_{\delta_0}(x_0)$. The asymptotics is uniform in $x \in D$, where D is an arbitrary compact subset of $D_{\delta_0}(x_0) \setminus X^0$, and it is differentiable with respect to both x and λ.

The proposition is an immediate consequence of equations (A.30) and (A.32), and of the asymptotic properties of $M(\lambda, x)$ (see (A.6)). We note also that the coefficients $\Psi_j(x)$ are finite combinations of the coefficients $\Psi_j^0(x)$ and the coefficients of the Taylor expansions of $T(\lambda, x)$ at $\lambda = \infty$.

PROPOSITION A.3. *The numbers κ_1 and κ_2 in (A.32) satisfy the equation,*

$$\kappa_1 + \kappa_2 = 0. \tag{A.34}$$

PROOF. The matrix equation (A.32) implies the following relation in the annular domain $\omega = \Omega_0 \cap \Omega_\infty$ for the corresponding determinants,

$$\det \Phi(\lambda, x) = \det T(\lambda, x) \det M(\lambda, x)\lambda^{-\kappa_1 - \kappa_2}. \tag{A.35}$$

At the same time, we observe that the asymptotic properties of the matrix function $M(\lambda, x)$ (see statement 3 of Theorem A.1) yield

$$\det M \to 1, \tag{A.36}$$

as $\lambda \to \infty$.

Assume that $\kappa_1 + \kappa_2 \neq 0$; let

$$\phi(\lambda, x) := \begin{cases} \det \Phi(\lambda, x), & \text{for} \quad \lambda \in \Omega_0, \\ \det T(\lambda, x) \det M(\lambda, x) \lambda^{-\kappa_1 - \kappa_2}, & \text{for} \quad \lambda \in \Omega_\infty \setminus \{\infty\} \end{cases}$$

($x \in D_{\delta_0}(x_0) \setminus X^0$). Equations (A.35) and (A.36) mean that, for every x, the (entire) function $\phi_x(\lambda) \equiv \phi(\lambda, x)$ is either identically zero ($\kappa_1 + \kappa_2 > 0$) or it is a polynomial of degree $-\kappa_1 - \kappa_2$ ($\kappa_1 + \kappa_2 < 0$). In both cases there are points on the finite complex plane λ where the matrix $\Phi(\lambda, x)$ is not invertible. This contradicts proposition 2 and hence the sum $\kappa_1 + \kappa_2$ must be zero. □

Equation (A.34) allows us to specify (A.32) as follows:

$$M(\lambda, x) = T^{-1}(\lambda, x) \lambda^{\kappa \sigma_3} \Phi(\lambda, x), \tag{A.37}$$

where $\kappa \geq 0$ is an integer number (the same for all $x \in D_{\delta_0}(x_0) \setminus X_s^0$), and σ_3 is defined by

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

PROPOSITION A.4. *The number κ in equation (A.37) is (identically) zero.*

PROOF. It is this statement that we use the assumption (A.5). Suppose that $\kappa > 0$. Then, the functions $T(\lambda, x)$ and $\Phi(\lambda, x)$ in (A.32) are defined up to the following transformations,

$$T(\lambda, x) \mapsto \tilde{T}(\lambda, x) = \begin{pmatrix} c_1 & 0 \\ \sum_{j=0}^{2\kappa} a_j \lambda^{j - 2\kappa} & c_2 \end{pmatrix} T(\lambda, x),$$

$$\Phi(\lambda, x) \mapsto \tilde{\Phi}(\lambda, x) = \begin{pmatrix} c_1 & 0 \\ \sum_{j=0}^{2\kappa} a_j \lambda^{j} & c_2 \end{pmatrix} \Phi(\lambda, x),$$

where $c_1 c_2 \neq 0$ and $a_0, \dots, a_{2\kappa}$, are arbitrary. Using this freedom, we can adjust the matrix functions $T(\lambda, x)$ and $\Phi(\lambda, x)$ in such a way (perhaps adding a few more points to the singular set X^0 and slightly reducing the radius δ_0) so that asymptotic relation (A.33) can be transformed into the relation,

$$\Phi(\lambda, x) S_1 \dots S_{k-1} \sim \Big(I + \sum_{j=1}^{\infty} \frac{\Psi_j(x)}{\lambda^j}\Big) e^{-\left(\theta(\lambda, x) + \kappa \ln \lambda\right)\sigma_3}, \tag{A.38}$$

$$\lambda \to \infty, \quad \lambda \in \mathcal{L}_k,$$

with the meromorphic in $D_{\delta_0}(x_0)$ and holomorphic in $D_{\delta_0}(x_0) \setminus X^0$ coefficients $\Psi_k(x)$, such that

$$(\Psi_j)_{12} \equiv 0, \quad \forall j < 2\kappa. \tag{A.39}$$

The asymptotics (A.38) is still uniform and differentiable with respect to both x and λ.

Applying to the logarithmic derivatives $\Phi_\lambda \Phi^{-1}$ and $\Phi_x \Phi^{-1}$ the arguments based on the Liouville theorem, which we have already used in Chapters 3 and 4 when deriving the isomonodromy deformation equations (4.2.28), we conclude that the function $\Phi(\lambda, x)$ satisfies the isomonodromy Lax pair (4.2.24). The functional parameters $u(x)$, $v(x)$, $w(x)$, and $z(x)$ are meromorphic in $D_{\delta_0}(x_0)$ and holomorphic in $D_{\delta_0}(x_0) \setminus X^0$; furthermore, they satisfy the general relations,

$$w = u_x, \quad y = v_x, \tag{A.40}$$

$$\nu + \kappa = vw - uy. \tag{A.41}$$

Moreover,

$$u = 2(\Psi_1)_{12}, \quad \text{and} \quad v = 2(\Psi_1)_{21}.$$

¿From the first equation and (A.39) we conclude that

$$u(x) \equiv 0,$$

which in virtue of (A.40) means that

$$w(x) \equiv 0$$

as well, and hence the right-hand side of (A.41) is identically zero, but this cannot be, since ν is not an integer. This contradiction proves that κ must be zero[3]. □

Propositions A.2–A.4 yield the following corollary of Theorem A.2 which constitutes the *local* (with respect to x) solution of the original RH problem.

THEOREM A.3. *Let $x_0 \in \mathbb{C}$. Then, there exists a number $\delta_0 > 0$, the finite set $X_s^0 \equiv \{x_{0j}\} \subset D_{\delta_0}(x_0)$, and the unique matrix function $\Psi_0(\lambda, x)$, such that:*

- **(i)** *$\Psi_0(\lambda, x)$ and $\Psi_0^{-1}(\lambda, x)$ are holomorphic in $\mathbb{C} \times \left(D_{\delta_0}(x_0) \setminus X_s^0\right)$ and meromorphic along $\mathbb{C} \times X_s^0$.*
- **(ii)** $\det \Psi_0(\lambda, x) \equiv 1$.
- **(iii)** *For every $x \in D_{\delta_0}(x_0) \setminus X_s^0$,*

$$\Psi_0(\lambda, x) S_1 \ldots S_{k-1} \sim \Big(I + \sum_{j=1}^{\infty} \frac{\Psi_j(x)}{\lambda^j}\Big) e^{-\theta(\lambda, x)\sigma_3},$$

$$\lambda \to \infty, \quad \lambda \in \mathcal{L}_k,$$

where the coefficients $\Psi_j(x)$ are holomorphic in $D_{\delta_0}(x_0) \setminus X_s^0$ and meromorphic in $D_{\delta_0}(x_0)$. These asymptotics are uniform in $x \in D_{\delta_0}(x_0) \setminus X_s^{0,\epsilon}$ and differentiable with respect to both x and λ. $X_s^{0,\epsilon}$ denotes an arbitrary ϵ-neighborhood of the set X_s^0.

PROOF. Let $\Phi(\lambda, x)$ and $T(\lambda, x)$ be the functions from Theorem A.2. Set

$$\Psi_0(\lambda, x) := T^{-1}(\infty, x)\Phi(\lambda, x).$$

We will argue that this function satisfies the conditions **(i)**–**(iii)** above. Indeed, condition **(i)** follows directly from Proposition A.2 and the analytic properties of $T(\lambda, x)$. By virtue of (A.37) and Proposition A.4, the following equation holds:

$$\Psi_0(\lambda, x) = T^{-1}(\infty, x) T(\lambda, x) M(\lambda, x).$$

Note that

$$T^{-1}(\infty, x) T(\lambda, x) = I + \sum_{j=1}^{\infty} T_j(x) \lambda^{-j},$$

[3]Strictly speaking, in the proof we have excluded the case where $T_{11}(\infty, x) \equiv 0$. If $T_{11}(\infty, x) \equiv 0$, then κ must be replaced by $-\kappa$ in (A.38), and "12" by "21" in (A.39). The rest of the proof will go along the same lines as before.

where the series converges in Ω_∞ uniformly for $x \in D$, $\overline{D} \subset D_{\delta_0}(x_0) \setminus \Theta_0$, and the coefficients $T_j(x)$ are holomorphic in $D_{\delta_0}(x_0) \setminus X_s^0$ and meromorphic in $D_{\delta_0}(x_0)$. Hence, taking into account the asymptotic properties of the Sibuya function $M(\lambda, x)$ (see statement 2 of Theorem A.4), we arrive to the asymptotic relation (**iii**). To obtain (**ii**), we note that for every fixed x,

$$\det \Psi_0(\lambda, x) \to 1, \quad \lambda \to \infty,$$

and hence (**ii**) follows via the Liouville theorem. Finally, to prove the uniqueness statement of the theorem, assume that $\tilde{\Psi}_0(\lambda, x)$ is another function satisfying (**i**)–(**iii**) and consider the ratio,

$$R(\lambda, x) := \tilde{\Psi}_0(\lambda, x)\Psi_0^{-1}(\lambda, x).$$

For every fixed x, we have that $R(\lambda, x)$ is an entire function of λ. Moreover, as $\lambda \to \infty$ and $\lambda \in \mathcal{L}_k$, we have from (**iii**),

$$R(\lambda, x) = \tilde{\Psi}_0(\lambda, x)\Psi_0^{-1}(\lambda, x)$$

$$= \left[\tilde{\Psi}_0(\lambda, x)S_1 \dots S_{k-1}e^{\theta(\lambda,x)\sigma_3}\right]\left[\Psi_0(\lambda, x)S_1 \dots S_{k-1}e^{\theta(\lambda,x)\sigma_3}\right]^{-1}$$

$$= I + O\left(\frac{1}{\lambda}\right).$$

In other words, in the whole neighborhood of infinity we have

$$R(\lambda, x) \to I, \quad \lambda \to \infty,$$

and therefore

$$R(\lambda, x) \equiv I.$$

The uniqueness, and hence the theorem are proven.

0.3. The solution of the RH problem. By virtue of Theorem A.3, and the standard topological properties of the complex plane, there exists a countable covering,

$$\mathbb{C} = \bigcup_{n=1}^{\infty} D_{\delta_n}(x_n),$$

of the complex plane x by the disks $D_{\delta_n}(x_n)$, such that every disc $D_{\delta_n}(x_n)$ has nonempty intersection only with finitely many other discs from the collection, and such that for each $D_{\delta_n}(x_n)$ the statement of Theorem A.3 is valid. In other words, for every n there exists the finite set $X_s^n \equiv \{x_{nj}\} \subset D_{\delta_n}(x_n)$, and the unique matrix function $\Psi_n(\lambda, x)$ such that:

- (**i**) $\Psi_n(\lambda, x)$ and $\Psi_n^{-1}(\lambda, x)$ are holomorphic in $\mathbb{C} \times \left(D_{\delta_n}(x_n) \setminus X_s^n\right)$ and meromorphic along $\mathbb{C} \times X_s^n$.
- (**ii**) $\det \Psi_n(\lambda, x) \equiv 1$.
- (**iii**) For every $x \in D_{\delta_n}(x_n) \setminus X_s^n$,

$$\Psi_n(\lambda, x)S_1 \dots S_{k-1} \sim \left(I + \sum_{j=1}^{\infty} \frac{\Psi_j(x)}{\lambda^j}\right)e^{-\theta(\lambda,x)\sigma_3},$$

$$\lambda \to \infty, \quad \lambda \in \mathcal{L}_k,$$

where the coefficients $\Psi_j(x)$ are holomorphic in $D_{\delta_n}(x_n) \setminus X_s^n$ and meromorphic in $D_{\delta_n}(x_n)$. The asymptotics is uniform in $x \in D$, where D is an arbitrary compact subset of $D_{\delta_n}(x_n) \setminus X_s^n$, and it is differentiable with respect to both x and λ.

Our aim now is to glue this collection of the local (with respect to x) solutions of the original RH problem, into the global function $\Psi(\lambda, x)$ which would solve this problem for all complex $x \in \mathbb{C} \setminus X_s$, where

$$X_s := \bigcup_{n=1}^{\infty} X_s^n.$$

(Note that the set X_s is countable and has ∞ as its only limit point.) The following lemma plays a central role in this construction.

LEMMA A.3. *Suppose that*

$$\overset{o}{D}_{\delta_n}(x_n) \cap \overset{o}{D}_{\delta_m}(x_m) \neq \emptyset.$$

Then

$$\Psi_n(\lambda, x) = \Psi_m(\lambda, x), \quad \forall (\lambda, x) \in \mathbb{C} \times \Big(\big(D_{\delta_n}(x_n) \setminus X_s^n\big) \cap \big(D_{\delta_m}(x_m) \setminus X_s^m\big) \Big).$$

PROOF. The proof is identical with the proof used for the uniqueness part of Theorem A.3. The key point is that both functions has exactly the same asymptotic behaviour (with exactly the same Stokes matrices) as $\lambda \to \infty$.

Let us now set

$$\Psi(\lambda, x) := \Psi_n(\lambda, x), \quad (\lambda, x) \in \mathbb{C} \times \big(D_{\delta_n}(x_n) \setminus \Theta_n\big). \tag{A.42}$$

Because of Lemma A.3, this equation defines $\Psi(\lambda, x)$ as a holomorphically invertible function on the whole product $\mathbb{C} \times \big(\mathbb{C} \setminus X_s\big)$. Moreover, it inherits all the common analytic properties of the local functions $\Psi_n(\lambda, x)$, including uniqueness. In other words, the function $\Psi(\lambda, x)$ satisfies all the properties stated in Theorem A.0. $\square$

REMARK A.2. The poles of the functions $u(x; s)$ and $v(x; s)$ are the values of the parameter x for which the generalized inverse monodromy problem with the given monodromy data,

$$s = \{S_1, \dots, S_6; \nu\},$$

is not solvable. At the same time, by using the Schlesinger transformations indicated in Lemma A.1, we see that if x is a pole of the functions $u(x; s)$ and $v(x; s)$, then the inverse monodromy problem is solvable for each of the following choices of the monodromy data

$$\hat{s} = \{S_1, \dots, S_6; \nu - 1\}$$

and

$$\tilde{s} = \{S_1, \dots, S_6; \nu + 1\}.$$

REMARK A.3. The Sibuya function $M(\lambda, x)$ defines a holomorphic vector bundle over $\mathbb{C}P^1$. The points x_j of the set X_s, i.e., the poles of the functions $u(x; s)$ and $v(x; s)$, are the points where this bundle is not trivial. In this case, the index κ in the representation (A.37) equals 1.

APPENDIX B

The Birkhoff-Grothendieck Theorem with a Parameter

The Appendix is based on the work of A. A. Bolibruch [**Bol4**].

Denote by Ω_0, Ω_∞, and ω the following subsets of the Riemann sphere $\mathbb{C}P^1$:

$$\Omega_0 = \{\lambda \in \mathbb{C} : |\lambda| \le R\},$$
$$\Omega_\infty = \{\lambda \in \mathbb{C} \cup \infty : |\lambda| \ge r\},$$
$$\omega = \Omega_0 \cap \Omega_\infty,$$

where $R > r$. Also, as in Appendix A, we denote by $D_\delta(x_0)$ the closed disk in the x - plane of radius δ and centered at x_0, i.e.

$$D_\delta(x_0) = \{x \in \mathbb{C} : |x - x_0| \le \delta\}.$$

Our goal is to present an elementary proof of the following (local) version of the classical Birkhoff-Grothendieck theorem with the parameter.

THEOREM B.1. *Let $M(\lambda, x)$ belongs to the set*[1] *$H^0(\omega \times D_\delta(x_0))$. Then there exists a number $0 < \delta_0 \le \delta$, the finite set $X^0 \equiv \{x_{0j}\} \subset D_{\delta_0}(x_0)$, and the matrix functions $T(\lambda, x)$ and $\Phi(\lambda, x)$ such that*

- *$T(\lambda, x)$ and $T^{-1}(\lambda, x)$ are holomorphic in $\Omega_\infty \times \left(D_{\delta_0}(x_0) \setminus X^0\right)$ and meromorphic along $\Omega_\infty \times X^0$ (i.e., they have poles at $x_{0j} \in X^0$ and the coefficients of the corresponding Laurent series are holomorphic in D_∞).*
- *$\Phi(\lambda, x)$ and $\Phi^{-1}(\lambda, x)$ are holomorphic in $\Omega_0 \times \left(D_{\delta_0}(x_0) \setminus \Theta_0\right)$ and meromorphic along $\Omega_0 \times X^0$.*
- $$M(\lambda, x) = T^{-1}(\lambda, x)\lambda^{\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}}\Phi(\lambda, x), \tag{B.1}$$

  where $\kappa_1 \ge \kappa_2$ are some integer numbers, which are the same for all $x \in D_{\delta_0}(x_0) \setminus X^0$.

We shall prove the theorem in a series of lemmas, following closely the proof of [**Bol4**] where the case of $N \times N$ matrix and an arbitrary finite number of parameters is considered. As in [**Bol4**], the first two lemmas are just specification of the corresponding statements proven in [**AnoBol**] for the 2×2 case, as well as the modification of this result for the case that the matrix depends of a parameter x. Lemma 3 is a particular case of the Sauvage lemma (see [**H**]).

The Banach spaces $H(\omega \times D_\delta(x_0))$, $H(\Omega_\infty \times D_\delta(x_0))$, and $H(\Omega_0 \times D_\delta(x_0))$ are equipped by the standard norm $\|f\|_H = \max_{(\lambda,x)} \|f(\lambda, x)\|$. In addition to the subsets $H^0(\Omega_{\infty,0} \times D_\delta(x_0))$, we also introduce the subspace, $\hat{H}(\Omega_\infty \times D_\delta(x_0))$, of

[1] We recall that for any set $\mathbb{M} \subset \mathbb{C}^N$, with nonempty interior, we denote $H(\mathbb{M})$ the Banach space of the 2×2 matrix functions holomorphic in the interior of $\mathbb{M}$ and continuous in $\overline{\mathbb{M}}$. We also denote by $H^0(\mathbb{M})$ the subset of $H(\mathbb{M})$ consisting of holomorphically invertible matrix functions.

the space $H(\Omega_\infty \times D_\delta(x_0))$, consisting of all $f(\lambda, x) \in H(\Omega_\infty \times D_\delta(x_0))$ such that $f(\infty, x) \equiv 0$. We also note that the spaces $H(\Omega_{\infty,0} \times D_\delta(x_0))$ are the subspaces of $H(\omega \times D_\delta(x_0))$.

LEMMA B.1. *If under the conditions of the theorem, $\|M - I\|_H < \epsilon$ for sufficiently small $\epsilon > 0$, then there exists unique functions $T(\lambda, x) \in H^0(\Omega_\infty \times D_\delta(x_0))$ and $\Phi(\lambda, x) \in H^0(\Omega_0 \times D_\delta(x_0))$, such that $T(\infty, x) \equiv I$ and*

$$M(\lambda, x) = T^{-1}(\lambda, x)\Phi(\lambda, x). \tag{B.2}$$

PROOF. Similar to the proof of the basic factorization lemma in Appendix A (and, once again, following [**AnoBol**]), we introduce the Cauchy operators $P_\pm$: $H(\omega \times D_\delta(x_0)) \mapsto H(\Omega_{0,\infty} \times D_\delta(x_0))$, by the equations

$$\big(P_+ f\big)(\lambda, x) = \frac{1}{2\pi i}\oint_{|\mu|=R} \frac{f(\mu, x)}{\mu - \lambda} d\mu, \quad \big(P_- f\big)(\lambda, x) = -\frac{1}{2\pi i}\oint_{|\mu|=r} \frac{f(\mu, x)}{\mu - \lambda} d\mu,$$

where the orientation, in both integrals, is counterclockwise. Let us now show that the operators $P_\pm$ are bounded. Once again, the crucial role is played by the Cauchy formula

$$f(\lambda, x) = \big(P_+ f\big)(\lambda, x) + \big(P_- f\big)(\lambda, x), \tag{B.3}$$

which holds for any $f \in H(\omega \times D_\delta(x_0))$. Note also that

$$\operatorname{Im} P_- = \hat{H}(\Omega_\infty \times D_\delta(x_0)).$$

Denote by s the number $(R - r)/2$. If $0 \le |\lambda| < s + r$, then

$$\|\big(P_+ f\big)(\lambda, x)\| \le \frac{1}{2\pi}\oint_{|\mu|=R} \left\| \frac{f(\mu, x)}{\mu - \lambda} \right\| |d\mu| \le ||f||_H \frac{1}{2\pi}\oint_{|\mu|=R} \frac{1}{s}|d\mu| \le \frac{R}{s}||f||_H.$$

If $s + r \le |\lambda| \le R$, using (B.3), we obtain

$$\|\big(P_+ f\big)(\lambda, x)\| \le ||f||_H + \|\big(P_- f\big)(\lambda, x)\| \le$$

$$||f||_H + \frac{1}{2\pi}\oint_{|\mu|=r} \left\| \frac{f(\mu, x)}{\mu - \lambda} \right\| |d\mu| \le ||f||_H + ||f||_H \frac{1}{2\pi}\oint_{|\mu|=r} \frac{1}{s}|d\mu| \le \frac{s + r}{s}||f||_H.$$

Thus,

$$|\big(P_+ f\big)(\lambda, x)| \le \max\left\{\frac{R}{s}, \frac{s + r}{s}\right\} ||f||_H,$$

for all $(\lambda, x) \in \Omega_0 \times D_\delta(x_0)$, and hence

$$||P_+||_H \le C := \max\left\{\frac{R}{s}, \frac{s + r}{s}\right\}.$$

Similar arguments show that the same estimate is also true for the norm of the operator P_-.

Let $N(\lambda, x) := M(\lambda, x) - I$, then $||N||_H < \epsilon$. To prove the existence statement of the lemma, it is sufficient to show the existence of a matrix function $X(\lambda, x) \in \hat{H}(\Omega_\infty \times D_\delta(x_0))$, such that $||X||_H < 1/2$ and

$$P_- \left((I + X)M\right) = 0. \tag{B.4}$$

Indeed, in this case we can put

$$T(\lambda, x) := I + X(\lambda, x) \in H^0(\Omega_\infty \times D_\delta(x_0))$$

and

$$\Phi(\lambda, x) := T(\lambda, x)M(\lambda, x) \in H^0(\Omega_0 \times D_\delta(x_0)).$$

Since (B.3), (B.4), it is clear that the function $\Phi(\lambda, x)$ is in $H(\Omega_0 \times D_\delta(x_0))$, and we only need to establish the holomorphic invertibility of $\Phi(\lambda, x)$. We have,

$$\Phi = I + X + (I + X)N,$$

or, using $\Phi = P_+(\Phi)$,

$$\Phi = I + P_+\big((I + X)N\big),$$

and the norm $||P_+\big((I + X)N\big)||_H < ||P_+||_H \frac{3\epsilon}{2}$, can be made smaller than $1/2$ for sufficiently small ϵ.

Equation (B.4) can be rewritten as follows,

$$P_-\left((I + X)M\right) = P_-\left((I + X)(I + N)\right)$$

$$= P_-\left((I + X + XN + N)\right) = X + P_-(XN) + P_-(N) = 0,$$

where we have used the obvious facts that

$$P_-(X) = X, \quad \text{and} \quad P_-(I) = 0.$$

Therefore, the equation we need to solve, can be written as the integral equation,

$$X + P_-(XN) = -P_-(N),$$

in the space $\hat{H}(\Omega_\infty \times D_\delta(x_0))$. Since the operator P_- is bounded and $||N||_H < \epsilon$, for sufficiently small ϵ the operator $I + P_-[\cdot\, N]$ is invertible and the norm of the solution $X = \big(I + P_-[\cdot\, N]\big)^{-1}\big(-P_-(N)\big)$ can be made as small as needed. This proves the existence part of the lemma.

Suppose $\{\tilde{T}(\lambda, x), \tilde{\Phi}(\lambda, x)\}$ is another pair of functions with the same properties. Then, $M = T^{-1}\Phi = \tilde{T}^{-1}\tilde{\Phi}$, and the function R defined by

$$R(\lambda, x) := \begin{cases} \tilde{T}(\lambda, x)T^{-1}(\lambda, x), & \text{for} \quad \lambda \in \Omega_\infty, \\ \tilde{\Phi}(\lambda, x)\Phi^{-1}(\lambda, x), & \text{for} \quad \lambda \in \Omega_0, \end{cases}$$

is a holomorphically invertible matrix function in $\overline{\mathbb{C}} \times D_\delta(x_0)$. Thus, by Liouville theorem, $R(\lambda, x)$ does not depend on λ. On the other hand, from the above construction one has $R(\infty, x) \equiv I$, therefore $R(\lambda, x) \equiv I$ and hence, $\tilde{T}(\lambda, x) \equiv T(\lambda, x)$, $\tilde{\Phi}(\lambda, x) \equiv \Phi(\lambda, x)$.

The second step of the proof of the theorem is to replace the function $M(\lambda, x)$, by some matrix function rational in λ. □

LEMMA B.2. *For every matrix function $M(\lambda, x) \in H^0(\omega \times D_\delta(x_0))$, there exists a number $0 < \delta'' \leq \delta$, a matrix function $F(\lambda, x)$ rational in λ with holomorphic coefficients in x and matrix functions $T(\lambda, x)$, $\Phi(\lambda, x)$ such that:*

(1) *The only possible poles of $F(\lambda, x)$ are $\lambda = 0$ and $\lambda = \infty$. Moreover,*

$$\alpha_{0,\infty} := \sup_{x \in D_\delta(x_0)} (\textit{order of the pole at } 0, \infty) < \infty.$$

(2) *$F(\lambda, x)$ is holomorphically invertible in $\omega \times D_{\delta''}(x_0)$, and, furthermore, the positions of the zeros of* $\det F(\lambda, x)$ *in $\big(\Omega_0 \setminus \omega\big) \times D_{\delta''}(x_0)$, do not depend on x.*

(3) *$T(\lambda, x)$ and $T^{-1}(\lambda, x)$ are holomorphic in $\Omega_\infty \times D_{\delta''}(x_0)$.*

(4) *$\Phi(\lambda, x)$ and $\Phi^{-1}(\lambda, x)$ are holomorphic in $\Omega_0 \times D_{\delta''}(x_0)$.*

(5) *$M(\lambda, x) = T^{-1}(\lambda, x)F(\lambda, x)\Phi(\lambda, x)$.*

PROOF. The function $M(\lambda, x_0)$ can be uniformly approximated in ω by a matrix function rational in λ with possible poles in 0 and ∞ only. For this one can just take the parts of the Laurent expansion of $M(\lambda, x_0)$ in the annular domain ω, cutting the infinite tails of the expansion in an appropriate way .

On the other hand, for every $\epsilon > 0$, one can choose a number $\delta' > 0$, such that $\|M^{-1}(\lambda, x_0)M(\lambda, x) - I\| < \epsilon$ in $\omega \times D_{\delta'}(x_0)$.

Thus, one can always find a number $0 < \delta' \leq \delta$, and a holomorphically invertible in ω matrix function $B(\lambda)$ rational in λ with poles at 0 and ∞ only, such that $\|B^{-1}(\lambda)M(\lambda, x) - I\|$ is uniformly small in $\omega \times D_{\delta'}(x_0)$. Applying to the product $B^{-1}(\lambda)M(\lambda, x)$ lemma B.1, we obtain

$$B^{-1}M = T_1^{-1}\Phi_1, \quad M = BT_1^{-1}\Phi_1,$$

where $T_1(\lambda, x) \in H^0(\Omega_\infty \times D_{\delta'}(x_0))$ and $\Phi_1(\lambda, x) \in H^0(\Omega_0 \times D_{\delta'}(x_0))$. Consider the function $B(\lambda)T_1^{-1}(\lambda, x)$, and approximate it from the right in a similar way by a function $H(\lambda)$ with the same properties as B. Applying again lemma B.1, we find

$$BT_1^{-1}H^{-1} = T_2^{-1}\Phi_2,$$

where $T_2(\lambda, x)$ and $\Phi_2(\lambda, x)$ have the same properties as $T_1(\lambda, x)$ and $\Phi_1(\lambda, x)$ (for some $0 < \delta'' \leq \delta'$). Consequently,

$$\Phi_2 = T_2BT_1^{-1}H^{-1}, \quad M = T_2^{-1}\Phi_2H\Phi_1 = T_2^{-1}F\Phi_1, \quad F := \Phi_2H.$$

From the first relation above it follows that for every $x \in D_{\delta''}(x_0)$, the matrix function $\Phi_2(\lambda, x)$ is meromorphic in Ω_∞ with poles independent of x, and with Laurent series coefficients holomorphic in x. On the other hand, Φ_2, by the construction, is holomorphically invertible in $\Omega_0 \times D_{\delta''}(x_0)$. Thus, Φ_2 is rational in λ with holomorphic coefficients in x. This implies that F possesses exactly the same properties. Moreover, since $\Phi_2(\lambda, x)$ and $H(\lambda)$ are holomorphically invertible in $\omega \times D_{\delta''}(x_0)$, $F(\lambda, x)$ has the same property.

Finally, we have

$$F = \begin{cases} T_2BT_1^{-1}, & \text{for} \quad \lambda \in \Omega_\infty \\ \Phi_2H, & \text{for} \quad \lambda \in \Omega_0, \end{cases}$$

and hence the only possible poles of F are 0 and ∞. Moreover, the corresponding maximal orders of the poles, α_0 and α_∞, do not exceed the orders of the corresponding poles of the (independent of x) matrix functions $H(\lambda)$ and $B(\lambda)$, respectively. Also, the position of zeros of the $\det F(\lambda, x)$ in $(\Omega_0 \setminus \omega) \times D_{\delta''}(x_0)$, do not depend on x, because $\det \Phi_2$ has no zeros in Ω_0 and H does not depend on x.

Now, denoting $T = T_2$ and $\Phi = \Phi_1$ we get the decomposition (5). □

The following lemma plays a crucial role in proving the theorem. In fact, it proves the theorem for a particular class of the rational matrices M without any small norm assumptions.

It is convenient to introduce one more notation: Let Ω be a subset of the extended complex λ plane, and let X be a finite subset of the interior of the disk $D_\delta(x_0)$ on the complex x plane. We will denote

$$H_m^0\Big(\Omega \times \big(D_\delta(x_0) \setminus X\big)\Big)$$

the set of matrix valued functions $f(\lambda, x)$ such that $f(\lambda, x)$ and $f^{-1}(\lambda, x)$ are holomorphic in $\Omega \times \big(D_\delta(x_0) \setminus X\big)$ and meromorphic along $\Omega \times X$.

LEMMA B.3. *Let $M(\lambda, x) \in H^0_m\Big((\Omega_0 \setminus \{0\}) \times (D_\delta(x_0) \setminus X)\Big)$ where X is a finite subset of the interior of $D_\delta(x_0)$. Moreover, we assume that $M(\lambda, x)$ is rational in λ with the coefficients meromorphic in $D_\delta(x_0)$ and holomorphic in $D_\delta(x_0) \setminus X$. Suppose in addition that the only possible poles of M as a function of λ are $\lambda = 0$ and $\lambda = \infty$, and that the corresponding maximal orders of these poles, α_0 and α_∞, are finite. Then, there exists a number $\delta_0 \leq \delta$, the finite set $X^0 \subset D_{\delta_0}(x_0)$, $X \subset X^0$, and the matrix functions $T(\lambda, x)$ and $\Phi(\lambda, x)$, such that:*

- *$T(\lambda, x)$ and $T^{-1}(\lambda, x)$ are holomorphic in $(\overline{\mathbb{C}} \setminus \{0\}) \times (D_{\delta_0}(x_0) \setminus X^0)$ and meromorphic along $\overline{\mathbb{C}} \setminus \{0\} \times X^0$. In other words,*

$$T(\lambda, x) \in H^0_m\Big((\overline{\mathbb{C}} \setminus \{0\}) \times (D_{\delta_0}(x_0) \setminus X^0)\Big).$$

- *$\Phi(\lambda, x)$ and $\Phi^{-1}(\lambda, x)$ are holomorphic in $\Omega_0 \times (D_{\delta_0}(x_0) \setminus X^0)$ and meromorphic along $\Omega_0 \times X^0$. In other words,*

$$\Phi(\lambda, x) \in H^0_m\Big(\Omega_0 \times (D_{\delta_0}(x_0) \setminus X^0)\Big).$$

  Moreover, $\Phi(\lambda, x)$ is a polynomial in λ with the coefficients meromorphic in $D_{\delta_0}(x_0)$ and holomorphic in $D_{\delta_0}(x_0) \setminus X^0$, and also $\deg \Phi(\lambda, x) \leq \alpha_0 + \alpha_\infty \quad \forall x$.

-
$$M(\lambda, x) = T^{-1}(\lambda, x)\lambda^{\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}}\Phi(\lambda, x), \tag{B.5}$$

  where $\kappa_1 \geq \kappa_2$ are some integer numbers, which are the same for all $x \in D_{\delta_0}(x_0) \setminus X^0$.

PROOF. Since the only poles of M in the λ-plane are 0 and ∞, the matrix function M can be assumed to be of the form

$$M(\lambda, x) = \lambda^{\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}} L(\lambda, x), \tag{B.6}$$

for some integers $\kappa_1 \geq \kappa_2$, where $L(\lambda, x)$ is a polynomial matrix,

$$L(\lambda, x) = L_0(x) + L_1(x)\lambda + \cdots + L_m(x)\lambda^m.$$

(One can just take $\kappa_1 = \kappa_2 = -\alpha_0$.) We note that

$$L(\lambda, x) \in H^0_m\Big((\Omega_0 \setminus \{0\}) \times (D_\delta(x_0) \setminus X)\Big). \tag{B.7}$$

The degree m of the polynomial $L(\lambda, x)$ can be assumed to be the same for all x (equals, e.g., $\alpha_0 + \alpha_\infty$), although for some x the coefficient for the highest degree term might be zero. We also note that all the coefficients $L_j(x)$ are meromorphic in $D_\delta(x_0)$ with possible poles at the points of the set X.

If for all $x \in D_\delta(x_0) \setminus X$ the inequality, $\det L(0, x) \neq 0$, holds, then one has nothing to prove: just put $\delta_0 \equiv \delta$, $X^0 \equiv X$, $T \equiv I$, and $\Phi \equiv L$.

Let $\det L(0, x_1) = 0$ for some $x_1 \in D_\delta(x_0) \setminus X$. We will then prove that

$$\det L(0, x) \equiv 0. \tag{B.8}$$

Indeed, we first observe that

$$\det L(\lambda, x) = l_0(x) + l_1(x)\lambda + \cdots + l_n(x)\lambda^n,$$

for some integer n (which can be choosen the same for all x). All the coefficients $l_k(x)$ are meromorphic functions of $x \in D_\delta(x_0)$ with possible poles belonging to the set X. According to our assumption, $l_0(x_1) = 0$. At the same time, for at least one

$0 < k \leq n$ we must have that $l_k(x_1) \neq 0$, otherwise, $\det L(\lambda, x_1) \equiv \det L(0, x_1) = 0$ which contradicts (B.7). Denote by k_0 the maximal of such k's, and let

$$P_0(\lambda, x) := l_0(x) + l_1(x)\lambda + \cdots + l_{k_0}(x)\lambda^{k_0}.$$

Take $R_0 < R$ so that the circle $C_0 := \{\lambda : |\lambda| = R_0\}$ does not pass through the zeros of $P_0(\lambda, x_1)$. The choice of k_0 implies that $\det L(\lambda, x_1) = P_0(\lambda, x_1)$. Therefore,

$$\det L(\lambda, x) - P_0(\lambda, x) \to 0, \quad x \to x_1,$$

uniformly with respect to $\lambda \in C_0$. At the same time,

$$P_0(\lambda, x) \to P_0(\lambda, x_1), \quad x \to x_1,$$

uniformly with respect to $\lambda \in C_0$. The choice of R_0 implies that $|P_0(\lambda, x_1)| \geq c_0 > 0$, for all $\lambda \in C_0$. Hence, there exists $\delta' > 0$, such that

$$|\det L(\lambda, x) - P_0(\lambda, x)| < |P_0(\lambda, x)|, \tag{B.9}$$

for all $\lambda \in C_0$, and for all $x \in D_{\delta'}(x_1) \subset D_\delta(x_0) \setminus X$.

Suppose now that (B.8) is not true. Then we can find $0 < \delta'' \leq \delta'$ such that

$$l_0(x) \neq 0, \quad \forall x \in D_{\delta''}(x_1) \setminus \{x_1\}. \tag{B.10}$$

Denote by $\lambda_j(x)$, $j = 1, \ldots, k_0$, the zeros of the polynomial $P_x(\lambda) := P_0(\lambda, x)$. These zeros satisfy the relation

$$\prod_j \lambda_j(x) = \frac{l_0(x)}{l_{k_0}(x)}(-1)^{k_0}. \tag{B.11}$$

Note that

$$|l_{k_0}(x)| \geq c_0' > 0, \quad \forall x \in D_{\delta'''}(x_1),$$

for some $0 < \delta''' \leq \delta''$. Taking into account this inequality, the fact that $l_0(x) \to 0$ as $x \to x_1$, and equation (B.11), we conclude that in any neighborhood of $\lambda = 0$ we can find at least one $\lambda_j(x)$ for some $x \in D_{\delta'''}(x_1) \setminus \{x_1\}$. In particular, we have that for some $y \in D_{\delta'''}(x_1) \setminus \{x_1\}$, at least one zero of $P_0(\lambda, y)$ is inside the circle C_0. In virtue of (B.9), using the Rouche theorem, we conclude that inside the circle C_0 the polynomial $Q_y(\lambda) := \det L(\lambda, y)$ has at least one zero. Denote this zero as μ. Since (B.10), $Q_y(0) \neq 0$ and therefore $\mu \neq 0$. Thus, there exist at least one pair (λ, x) in $(\Omega_0 \setminus \{0\}) \times D_\delta(x_0)$, namely (μ, y), for which $\det L(\lambda, x) = 0$. This contradicts (B.7), and hence proves (B.8).

Denote by $m_1(x)$ and $m_2(x)$ the rows of the matrix $L_0(x) \equiv L(0, x)$, and assume that

$$m_1(x) \neq 0 \quad \text{for some } x. \tag{B.12}$$

Then, the identity (B.8) implies that there exists a scalar function $s(x)$ rational in entries of $L_0(x)$ such that

$$m_2(x) = s(x)m_1(x) \tag{B.13}$$

(e.g., $s(x) = m_{21}(x)/m_{11}(x)$ if $m_{11}(x) \neq 0$ for some x). Define T_1 by

$$T_1(\lambda, x) := \begin{pmatrix} 1 & 0 \\ s(x)\lambda^{k_2 - k_1} & 1 \end{pmatrix}.$$

It is clear that $T_1(\lambda,x) \in H_m^0\Big((\overline{\mathbb{C}} \setminus \{0\}) \times (D_\delta(x_0) \setminus X_1)\Big)$, where X_1 is the set of poles of $s(x)$, i.e., the set X extended by the zeros of the relevant entry of $L_0(x)$). Restricting, if needed, the x-domain to $D_{\delta_1}(x_0)$ with $\delta_1 < \delta$, we can always make X_1 finite (and still containing X).

It follows immediately from (B.13) that

$$T_1(\lambda,x)M(\lambda,x) = \lambda^{\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2+1 \end{pmatrix}} L_1(\lambda,x), \tag{B.14}$$

where

$$L_1(\lambda,x) = \begin{pmatrix} 1 & 0 \\ -s(x)\lambda^{-1} & \lambda^{-1} \end{pmatrix} L(\lambda,x).$$

Because of (B.13) the function $L_1(\lambda,x)$ is still a matrix polynomial in λ. Its coefficients are meromorphic in $D_{\delta_1}(x_0)$ and holomorphic in $D_{\delta_1}(x_0) \setminus X_1$, and furthermore, $\deg L_1 \le \deg L \le \alpha_0 + \alpha_\infty$. Also, equation (B.14) implies that L_1 inherits the basic analytic properties of M. Indeed,

$$L_1(\lambda,x) \in H_m^0\Big((\Omega_0 \setminus \{0\}) \times (D_{\delta_1}(x_0) \setminus X_1)\Big). \tag{B.15}$$

If (B.12) is not true, i.e., if

$$m_1(x) \equiv 0,$$

then we can directly increase the number κ_1 without the procedure presented above. In fact, in this case,

$$M(\lambda,x) = \lambda^{\begin{pmatrix} \kappa_1+1 & 0 \\ 0 & \kappa_2 \end{pmatrix}} L_1(\lambda,x), \tag{B.16}$$

where

$$L_1(\lambda,x) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} L(\lambda,x)$$

is again a polynomial satisfying (B.15).

Multiplying the right hand side of (B.14) or of (B.16) by either the identity or by σ_1, we can rewrite these equations in the form

$$T'(\lambda,x)M(\lambda,x) = \lambda^{\begin{pmatrix} \kappa_1' & 0 \\ 0 & \kappa_2' \end{pmatrix}} L'(\lambda,x),$$

where $\kappa_1' \ge \kappa_2'$ and the matrices $T'(\lambda,x)$ and $L'(\lambda,x)$ have the same properties as the matrices $T_1(\lambda,x)$ and $L_1(\lambda,x)$, respectively. Since $\kappa_1'+\kappa_2' > \kappa_1+\kappa_2$, iterating if necessary the procedure presented above, we obtain after a finite[2] number of steps that $\det L'(0,x) \ne 0$ (for all x), and hence the decomposition (B.6) of the lemma is proven. The final set X^0 is obtained as the finite union of the X-sets generated at every step of this procedure. This completes the proof of the lemma.

We are now in a position to prove the theorem.

[2]The process is finite because of the obvious relation, $\det M = \pm\lambda^{\kappa_1+\kappa_2} \det L'$, and the fact that $\det M$ is a rational function, and hence has a finite order pole at $\lambda = \infty$.

PROOF OF THE THEOREM. Because of Lemma B.2, we only need to consider the case of a matrix function $M(\lambda, x)$, rational in λ with coefficients holomorphic in $D_\delta(x_0)$ and such that:

(1) The only possible poles of $M(\lambda, x)$ are $\lambda = 0$ and $\lambda = \infty$. Moreover,

$$\alpha_{0,\infty} := \sup_{x \in D_\delta(x_0)} (\text{order of the pole at } 0, \infty) < \infty.$$

(2) The positions of the zeros of the $\det M(\lambda, x)$ in $\big(\Omega_0 \setminus \omega\big) \times D_\delta(x_0)$ do not depend on x.

If in addition $M(\lambda, x)$ is holomorphically invertible in $\big(\Omega_0 \setminus \{0\}\big) \times D_\delta(x_0)$, then the theorem follows immediately from lemma B.3 (with $X = \emptyset$). Suppose then, that for some $0 \neq \lambda_0 \in \Omega_0 \setminus \omega$, we have $\det M(\lambda_0, x_0) = 0$.

Consider new local coordinates $\zeta = \lambda - \lambda_0$ and define the *polynomial* matrix function

$$\tilde{M}(\zeta, x) := (\zeta + \lambda_0)^{\alpha_0} M(\zeta + \lambda_0).$$

Denote by $\tilde{\Omega}_0$ a small disk in the ζ-plane with center at $\zeta = 0$ such that the matrix function $\tilde{M}(\zeta, x)$ is holomorphically invertible in $\big(\tilde{\Omega}_0 \setminus \{0\}\big) \times D_\delta(x_0)$. Applying lemma B.3 to $\tilde{M}(\zeta, x)$ (again with $X = \emptyset$), we obtain the representation

$$M(\lambda, x) = T^{-1}(\lambda, x)(\lambda - \lambda_0)^C \Phi(\lambda, x), \tag{B.17}$$

where $C = \operatorname{diag}(\kappa_1, \kappa_2)$, $T(\lambda, x) \in H^0\Big(\big(\overline{\mathbb{C}} \backslash \{\lambda_0\}\big) \times \big(D_{\delta_0}(x_0) \backslash X^0\big)\Big)$, and the matrix function $\Phi(\lambda, x)$ is a polynomial in λ whose coefficients are holomorphic in $D_{\delta_0}(x_0) \backslash X^0$ and meromorphic in $D_{\delta_0}(x_0)$. Moreover, by construction, $\det \Phi(\lambda_0, x) \neq 0$, while from the equation (B.17) it follows that $\Phi(\lambda, x)$ is holomorphically invertible in the annular domain ω and its determinant may have zeros in the set $\Omega_0 \setminus \{0\}$ only at the possible zeros of $\det M$ different from λ_0. Noticing that the function $(\lambda/\lambda - \lambda_0)^C$ is holomorphically invertible in $\Omega_\infty \times \big(D_{\delta_0}(x_0) \setminus X^0\big)$, we derive from (B.17) the equation

$$M(\lambda, x) = T_1^{-1}(\lambda, x) M_1(\lambda, x),$$

where $M_1(\lambda, x) := \lambda^C \Phi(\lambda, x)$. The function $M_1(\lambda, x)$ has all the properties of the original function $M(\lambda, x)$ except its determinant has one zero less in $\Omega_0 \setminus \omega$. The function $T_1^{-1}(\lambda, x)$ has exactly the properties stated in the theorem. Repeating, if necessary, the above construction for the function $M_1(\lambda, x)$ we can eliminate, after a finite number of steps, all the possible zeros of the $\det M$ in $\Omega_0 \setminus \omega$ and hence we arrive to the function $M'(\lambda, x)$ which satisfies the conditions of lemma B.3 (with, generally, a nonempty singularity set X'). Then the theorem again follows from the lemma. $\square$

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At the turn of the twentieth century, the French mathematician Paul Painlevé and his students classified second order nonlinear ordinary differential equations with the property that the location of possible branch points and essential singularities of their solutions does not depend on initial conditions. It turned out that there are only six such equations (up to natural equivalence), which later became known as Painlevé I–VI.

Although these equations were initially obtained answering a strictly mathematical question, they appeared later in an astonishing (and growing) range of applications, including, e.g., statistical physics, fluid mechanics, random matrices, and orthogonal polynomials. Actually, it is now becoming clear that the Painlevé transcendents (i.e., the solutions of the Painlevé equations) play the same role in nonlinear mathematical physics that the classical special functions, such as Airy and Bessel functions, play in linear physics.

The explicit formulas relating the asymptotic behaviour of the classical special functions at different critical points play a crucial role in the applications of these functions. It is shown in this book that even though the six Painlevé equations are nonlinear, it is still possible, using a new technique called the Riemann-Hilbert formalism, to obtain analogous explicit formulas for the Painlevé transcendents. This striking fact, apparently unknown to Painlevé and his contemporaries, is the key ingredient for the remarkable applicability of these "nonlinear special functions".

The book describes in detail the Riemann-Hilbert method and emphasizes its close connection to classical monodromy theory of linear equations as well as to modern theory of integrable systems. In addition, the book contains an ample collection of material concerning the asymptotics of the Painlevé functions and their various applications, which makes it a good reference source for everyone working in the theory and applications of Painlevé equations and related areas.

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