Parametrized Homotopy Theory
Parametrized Homotopy Theory

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Contents

Prologue 1

Part I. Point-set topology, change functors, and proper actions 11

Introduction 13

Chapter 1. The point-set topology of parametrized spaces 15
  Introduction 15
  1.1. Convenient categories of topological spaces 15
  1.2. Topologically bicomplete categories and ex-objects 16
  1.3. Convenient categories of ex-spaces 19
  1.4. Convenient categories of ex-G-spaces 22
  1.5. Philosophical comments on the point-set topology 23
  1.6. Technical point-set topological lemmas 24
  1.7. Appendix: nonassociativity of smash products in $\mathcal{I} op_*$ 26

Chapter 2. Change functors and compatibility relations 29
  Introduction 29
  2.1. The base change functors $f_!, f^*, \text{and } f_*$ 30
  2.2. Compatibility relations 32
  2.3. Change of group and restriction to fibers 35
  2.4. Normal subgroups and quotient groups 37
  2.5. The closed symmetric monoidal category of retracts 40

Chapter 3. Proper actions, equivariant bundles and fibrations 43
  Introduction 43
  3.1. Proper actions of locally compact groups 43
  3.2. Proper actions and equivariant bundles 47
  3.3. The bundle construction 48
  3.4. Spaces of the homotopy types of $G$-CW complexes 51
  3.5. Some classical theorems about fibrations 53
  3.6. Quasifibrations 54

Part II. Model categories and parametrized spaces 57

Introduction 59

Chapter 4. Topologically bicomplete model categories 61
  Introduction 61
  4.1. Model theoretic philosophy: $h$, $q$, and $m$-model structures 62
  4.2. Strong Hurewicz cofibrations and fibrations 63
### CONTENTS

4.3. Towards classical model structures in topological categories 66
4.4. Classical model structures in general and in \( \mathcal{H} \) and \( \mathcal{U} \) 69
4.5. Compactly generated \( q \)-type model structures 72

Chapter 5. Well-grounded topological model categories 77
   Introduction 77
   5.1. Over and under model structures 78
   5.2. The specialization to over and under categories of spaces 82
   5.3. Well-grounded topologically bicomplete categories 85
   5.4. Well-grounded categories of weak equivalences 87
   5.5. Well-grounded compactly generated model structures 90
   5.6. Properties of well-grounded model categories 91

Chapter 6. The \( qf \)-model structure on \( \mathcal{H}_B \) 97
   Introduction 97
   6.1. Some of the dangers in the parametrized world 98
   6.2. The \( qf \) model structure on the category \( \mathcal{H}/B \) 100
   6.3. Statements and proofs of the thickening lemmas 102
   6.4. The compatibility condition for the \( qf \)-model structure 105
   6.5. The quasifibration and right properness properties 107

Chapter 7. Equivariant \( qf \)-type model structures 109
   Introduction 109
   7.1. Families and non-compact Lie groups 110
   7.2. The equivariant \( q \) and \( qf \)-model structures 111
   7.3. External smash product and base change adjunctions 115
   7.4. Change of group adjunctions 118
   7.5. Fiber adjunctions and Brown representability 122

Chapter 8. Ex-fibrations and ex-quasifibrations 127
   8.1. Ex-fibrations 128
   8.2. Preservation properties of ex-fibrations 129
   8.3. The ex-fibrant approximation functor 131
   8.4. Preservation properties of ex-fibrant approximation 133
   8.5. Quasifibrant ex-spaces and ex-quasifibrations 135

Chapter 9. The equivalence between \( \text{Ho} \, G\mathcal{H}_B \) and \( hG\mathcal{W}_B \) 137
   Introduction 137
   9.1. The equivalence of \( \text{Ho} \, G\mathcal{H}_B \) and \( hG\mathcal{W}_B \) 138
   9.2. Derived functors on homotopy categories 139
   9.3. The functors \( f_* \) and \( F_B \) on homotopy categories 140
   9.4. Compatibility relations for smash products and base change 142

Part III. Parametrized equivariant stable homotopy theory 147
   Introduction 149

Chapter 10. Enriched categories and \( G \)-categories 151
   Introduction 151
   10.1. Parametrized enriched categories 151
   10.2. Equivariant parametrized enriched categories 153
Chapter 11. The category of orthogonal $G$-spectra over $B$ 159

Introduction 159
11.1. The category of $\mathcal{S}_G$-spaces over $B$ 159
11.2. The category of orthogonal $G$-spectra over $B$ 163
11.3. Orthogonal $G$-spectra as diagram ex-$G$-spaces 166
11.4. The base change functors $f^*$, $f_!$, and $f_*$ 167
11.5. Change of groups and restriction to fibers 170
11.6. Some problems concerning non-compact Lie groups 172

Chapter 12. Model structures for parametrized $G$-spectra 175

Introduction 175
12.1. The level model structure on $G\mathcal{S}_B$ 176
12.2. Some Quillen adjoint pairs relating level model structures 179
12.3. The stable model structure on $G\mathcal{S}_B$ 180
12.4. Cofiber sequences and $\pi_*$-isomorphisms 183
12.5. Proofs of the model axioms 186
12.6. Some Quillen adjoint pairs relating stable model structures 190

Chapter 13. Adjunctions and compatibility relations 195

Introduction 195
13.1. Brown representability and the functors $f_*$ and $F_B$ 196
13.2. The category $G\mathcal{S}_B$ of excellent prespectra over $B$ 200
13.3. The level ex-fibrant approximation functor $P$ on prespectra 202
13.4. The auxiliary approximation functors $K$ and $E$ 205
13.5. The equivalence between $Ho G\mathcal{P}_B$ and $hG\mathcal{S}_B$ 207
13.6. Derived functors on homotopy categories 208
13.7. Compatibility relations for smash products and base change 209

Chapter 14. Module categories, change of universe, and change of groups 215

Introduction 215
14.1. Parametrized module $G$-spectra 215
14.2. Change of universe 219
14.3. Restriction to subgroups 223
14.4. Normal subgroups and quotient groups 226

Part IV. Parametrized duality theory 229

Introduction 231

Chapter 15. Fiberwise duality and transfer maps 233

Introduction 233
15.1. The fiberwise duality theorem 234
15.2. Duality and trace maps in symmetric monoidal categories 236
15.3. Transfer maps of Hurewicz fibrations 238
15.4. The bundle construction on parametrized spectra 240
15.5. II-free parametrized $\Gamma$-spectra 242
15.6. The fiberwise transfer for $\left(I\Pi; \Gamma\right)$-bundles 244

Chapter 16. Closed symmetric bicategories 247
### CONTENTS

Introduction 247
16.1. Recollections about bicategories 248
16.2. The definition of symmetric bicategories 249
16.3. The definition of closed symmetric bicategories 252
16.4. Duality in closed symmetric bicategories 255
16.5. Composites and naturality of dualities 259
16.6. A quick review of triangulated categories 261
16.7. Compatibly triangulated symmetric bicategories 262
16.8. Duality in triangulated symmetric bicategories 266

**Chapter 17. The closed symmetric bicategory of parametrized spectra** 269
   Introduction 269
   17.1. The definition of the bicategory $\mathcal{B}x$ 269
   17.2. Base change spectra 273
   17.3. Duality of base change spectra 277
   17.4. Using $\mathcal{B}x$ to encode relations between $\text{Ho}G\mathcal {I}_B$ and $\text{Ho}G\mathcal {I}$ 278
   17.5. Sketch proofs of the compatible triangulation axioms 280

**Chapter 18. Costenoble-Waner duality** 285
   Introduction 285
   18.1. The two notions of duality in $\text{Ho}G\mathcal {I}_B$ 286
   18.2. Costenoble-Waner dualizability of finite cell spectra 288
   18.3. Costenoble-Waner $V$-duality 290
   18.4. Preliminaries on unreduced relative mapping cones 292
   18.5. $V$-duality of $G$-ENRs 295
   18.6. Parametrized Atiyah duality for closed manifolds 296
   18.7. Parametrized Atiyah duality for manifolds with boundary 300
   18.8. The proof of the Costenoble-Waner duality theorem 302

**Chapter 19. Fiberwise Costenoble-Waner duality** 311
   Introduction 311
   19.1. Costenoble-Waner duality and homotopical Poincaré duality 312
   19.2. The bicategories $\mathcal{B}x_B$ 314
   19.3. Comparisons of bicategories 316
   19.4. The bundle construction pseudo-functor 319
   19.5. The fiberwise Costenoble-Waner duality theorem 320
   19.6. Fiberwise Poincaré duality 324
   19.7. The Adams isomorphism 326
   19.8. Some background and comparisons 328

**Part V. Homology and cohomology, Thom spectra, and addenda** 333

Introduction 335

**Chapter 20. Parametrized homology and cohomology theories** 337
   Introduction 337
   20.1. Axioms for parametrized homology and cohomology theories 338
   20.2. Represented homology and cohomology theories 341
   20.3. Coefficient systems and restriction maps 343
   20.4. The Serre spectral sequence 344
20.5. Poincaré duality and the Thom isomorphism 347
20.6. Relative Poincaré duality 350
20.7. Products in parametrized homology and cohomology 350
20.8. The representability of homology theories 353

Chapter 21. Equivariant parametrized homology and cohomology 357

Introduction 357
21.1. Equivariant homology and cohomology theories 358
21.2. Represented equivariant theories 360
21.3. Change of base and equivariant coefficient systems 361
21.4. Duality theorems and orientations 363
21.5. Products and the representability of homology 366
21.6. Fiberwise parametrized homology and cohomology 367
21.7. Fiberwise Poincaré duality and orientations 369

Chapter 22. Twisted theories and spectral sequences 373

Introduction 373
22.1. Twisted homology and cohomology theories 374
22.2. Automorphism monoids of spectra and $GL_1(k)$ 375
22.3. Twisted $K$-theory 378
22.4. The simplicial spectral sequence 380
22.5. Čech type spectral sequences 384
22.6. The twisted Rothenberg–Steenrod spectral sequence 386
22.7. The parametrized Künneth spectral sequence 388

Chapter 23. Parametrized FSP’s and generalized Thom spectra 393

Introduction 393
23.1. $\mathcal{D}$-functors with products in symmetric monoidal categories 395
23.2. The specialization of $\mathcal{D}$-FP’s to spaces and ex-spaces 397
23.3. Group, monoid, and module FCP’s; examples 399
23.4. The two-sided bar construction on FCP’s 402
23.5. Examples: iterated Thom spectra 403
23.6. $\mathcal{I}_c$-FCP’s and $\mathcal{L}$-spaces 405
23.7. Universal spherical fibration spectra 407
23.8. Some historical background 408

Chapter 24. Epilogue: cellular philosophy and alternative approaches 411

Introduction 411
24.1. CW spaces over $B$ 412
24.2. CW spectra and stable homotopy categories 415
24.3. Structured spectra and well-grounded model categories 418
24.4. The stable category of parametrized spectra 420
24.5. Towards parametrized $S_G$-modules 423

Bibliography 425

Index 433

Index of notation 439
Bibliography


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Index

Čech spectral sequence, 385
2-category, 248
acyclicity condition, 73, 91
Adams isomorphism, 326
algebraic stable homotopy category, 267
approximation
ex-fibrant —, 133
excellent prespectrum —, 207
h-fibrant —, 132
Ω-prespectrum —, 205
Σ-cofibrant —, 205
well-sectioned —, 132
Atiyah-Hirzebruch spectral sequence, 347, 375
base change
spectrum, 273
duality of —, 277
bicategory, 248
biequivalence of —s, 249
closed —, 252
of bimodules, 250
of parametrized spectra, 270
opposite —, 249
symmetric —, 250
triangulated —, 262
Brown representability, 123, 125, 197
of parametrized cohomology, 342
of parametrized homology, 353
bundle
construction, 48, 241
pseudo-functor, 319
principal —, 38, 47
category
algebraic stable homotopy —, 267
based bicocomplete —, 17
based topologically bicocomplete —, 17
classical homotopy —, 62
derived homotopy —, 63
G- —, 22
G-topological
bicocomplete over B, 153
closed symmetric monoidal over B, 154
over B, 153
of R-module spectra, 216
of based spaces, 16
of compactly generated spaces, 16
of ex-objects, 18, 78
of ex-spaces, 19
of ex-G-spaces, 22
of G-spaces, 22
of k-spaces, 16
of objects over B, 17, 78
of parametrized spectra, 164
of retracts, 40, 81
of spaces, 16
over B, 19
of weak Hausdorff spaces, 16
sub— of cofibrations, 73
sub— of weak equivalences, 73
sub— of well-grounded objects, 85
topological
bicocomplete —, 16, 151
bicocomplete over B, 152
closed symmetric monoidal over B, 153
over B, 152
triangulated —, 261
unital stable homotopy —, 267
cell complex
relative —, 72
cellular approximation theorem, 415
CHP, 64
relative —, 64
classical Thom spectra, 403
closed
bicategory, 252
⊙ — bicategory, 252
cocylinder object, 64
coefficient
groups, 344
system, 344
cofiber, 262
sequence lemma, 92
cofibrant
Σ — approximation, 205
Σ — prespectrum, 200
cofibration
bi—, 86
cyl —, 85, 176
f —, 80
fp —, 81
ex-space, 19
   ex-fibrant —, 83
   quasi-fibrant —, 135
   well-fibered —, 128
   well-grounded —, 86
   well-sectioned —, 83, 128
exact functor, 262
excellent
   prespectrum, 200
   prespectrum approximation, 207
extension
   of scalars, 218
   of universe, 220
external product
   of \( \mathcal{D} \)-objects, 396
family
   of subgroups, 110
fiberwise
   parametrized Poincaré duality, 369
   product, 20
   smash product, 20
fibrant
   \( h \) — approximation, 132
   quasi— ex-space, 135
fibration
   ex—, 83, 128
   ex-quasi—, 135
   \( f \) —, 80
   \( f_p \) —, 81
   \( \mathcal{D} \)-quasi—, 113
   \( h \) —, 64, 80
Hurewicz —, 64
   \( k \)-oriented \( V \)-sphere —, 364
   \( k \)-oriented spherical —, 348
level type of —, 176
mapping path —, 64
mixed —, 63
Moore mapping path —, 132
\( qf \) —, 101
\( qf(\epsilon) \) —, 113
quasi—, 54, 102
\( r \) —, 82
\( s \) —, 182
Serre —, 100
spherical \( G \) —, 365
spherical —, 347
   strong Hurewicz —, 64
   \( V \)-sphere —, 364
fixed point functor
   of spaces, 37
   of spectra, 226
functor
   change of
      base —s, 30, 168
      groups —s, 35, 37, 170
   coinduction —, 35, 170
\( \mathcal{D} \)-FCP, 397
\( \mathcal{D} \)-FP, 395
\( \mathcal{D} \)-FSP, 397
evaluation —, 167
ex-fibrant approximation —, 133
exact —, 262
excellent prespectrum approximation —, 207
fixed point —, 225
   of spaces, 37
   of spectra, 226
good —, 139, 208
   \( \mathcal{E}_\text{-FCP} \), 405
induction —, 35, 170
\( k \)-ification, 16
lax —, 248
Moore mapping path fibration —, 132
   \( \Omega \)-prespectrum approximation —, 205
   oplax —, 248
orbit —, 37
   of spaces, 37
   of spectra, 226
parametrized — with smash products, 398
pseudo—, 248
restriction of groups —s, 223
restriction/extension of universe —s, 220
shift desuspension —, 167
\( \Sigma \)-cofibrant approximation —, 205
spectrification, 421
spherical —, 395
strict —, 248
weak Hausdorffification —, 16
whiskering —, 131

\( G \)-connected space, 365
\( G \)-topological, 155
generalized Thom spectra, 403
generating
   cofibration, 72
   set, 112, 123, 124
Gluing lemma, 87
good functor, 208
ground cofibrations, 85
ground structure, 85
   level — on spectra over \( B \), 176
   on \( G.X/B \) and \( G.X_B \), 86
   on \( R \)-modules, 216
group
   FCP, 400
handicrafted smash products, 210
HELP, 412
HEP, 64
   relative —, 64
\( h \)-model structure, 70
homology theory
   \( RO(G) \)-graded parametrized —, 358
classical —, 341
fiberwise parametrized —, 367
ordinary parametrized —, 344
parametrized —, 338
reduced parametrized —, 339
representability of —, 353
represented —, 341, 360
homomorphism
of bicategories, 248
homotopy
coequalizer, 262
colimit, 262
extension and lifting property, 412
extension property, 64
relative —, 64
groups
of parametrized spectra, 181
\(\mathcal{X}\)-FCP, 405
induction, 35, 170
internal equivalence, 248
involution
on a bicategory, 249
\(k\)-orientation, 348, 364
\(k\)-trivialization, 348, 364
\(k\)-ification functor, 16
\(k\)-space, 16
\(\mathcal{L}\)-space, 405
ladder lemma, 87
latching maps, 381
lax functor, 248
Lefschetz constant, 236
left lifting property, 64
lemma
cofiber sequence —, 92
colimit —, 87
gluing —, 87
ladder —, 87
\(\lim^1\) lemma, 94
wedge —, 93
level model structure, 177
lifting property, 155
relative \(f\)-disk —, 101
\(\lim^1\) lemma, 94
linear isometries operad, 405
LLP, 64
long exact sequence
of homotopy groups, 184
mapping
cylinder, 64
path fibration, 64
space
external —, 40
of ex-spaces, 21
of spaces over \(B\), 21
model category
\(G\)-topological —, 155
model structure

compactly generated
on \(\mathcal{C}_B\), 79
compactly generated —, 72
\(f\)-
on \(\mathcal{C}/B\) and \(\mathcal{C}_B\), 80
on \(\mathcal{X}/B\), \(\mathcal{V}/B\) and \(\mathcal{V}_B\), 84
\(f_\mathcal{P}\)-
on \(\mathcal{C}_B\), 81
general philosophy, 62
generic \(h\)-structures, 66
\(h\)-
on \(\mathcal{C}\), 69
on \(\mathcal{C}/B\) and \(\mathcal{C}_B\), 80
on spaces, 70
level \(qf\)- —, 177
mixed —, 63
on \(R\)-modules, 217
over and under —, 79
positive —, 178
\(q\)-
on \(G\mathcal{X}/B\) and \(G\mathcal{C}_B\), 111
on spaces, 74
\(qf\)-
on \(\mathcal{X}/B\) and \(\mathcal{C}_B\), 101
on \(G\mathcal{X}/B\) and \(G\mathcal{C}_B\), 113
Reedy —, 380
stable —, 183
well-grounded —, 91
modification, 249
module of units, 401
monoid
FCP, 400
of based homotopy equivalences, 400
Moore
mapping path fibration, 132
paths, 131
morphism
of bicategories, 248
natural transformation
lax —, 249
NDR, 83
neighborhood deformation retract
fiberwise —, 83
numerable cover, 44
operad
linear isometries —, 405
oplax functor, 248
opposite
bicategory, 249
orbit functor
of spaces, 37
of spectra, 226
\(\pi_*\)-isomorphism, 181
parametrized functor with protuchs, 398
parametrized Küneth spectral sequence, 390
INDEX

Eilenberg-Moore —, 391
parametrized Künneth —, 390
Serre —, 345
simplicial —, 381
Twisted Rothenberg–Steenrod —, 387
spectrification functor, 421
spectrum
$R$-module —, 216
base change —a, 273
category of parametrized —a, 164
compact —, 196
Costenoble-Waner dual —, 287
duality of base change —a, 277
dualizable —, 235
EKMM —a, 416
excellent pre—, 200
excellent pre— approximation, 207
homotopy groups of —a, 181
invertible —, 235
level type of —, 176
Lewis-May —a, 416
$\Omega$—, 182
$\Omega$-pre— approximation, 205
over $B$, 163
$\Pi$-free —, 243
pre— over $B$, 165
$\Sigma$-cofibrant pre—, 200
sphere —, 163
suspension —, 163
Thom — of a map, 407
universal spherical fibration —, 407
weak map of pre—a, 203
well-grounded —, 176
well-sectioned —, 176
well-structured pre—, 200
spherical fibration, 347
$k$-oriented —, 348, 364
spherical $G$-fibration, 365
$V$—, 364
stable
equivalence, 183
model structure, 183
strict functor, 248
structure group, 47
symmetric
bicategory, 250
telescope, 94
tensor
for ex-spaces, 20
for spaces over $B$, 20
of $F$-spaces over $B$, 161
with based spaces, 17
with spaces, 16
theorem
cellular approximation —, 415
Costenoble-Waner duality —, 295

partial map classifier, 20
path-lifting function, 128
PPSP, 398
Poincaré duality, 347, 364
relative —, 350
presheaf of parametrized cohomology groups, 384
principal
bundle, 38, 47
space, 38
projection formula, 34
projective unitary group, 378
proper simplicial object, 381
proper space, action or map, 44
pseudo-functor, 248
quasi-fibration, 54
$R$-module spectrum, 216
Reedy model structure, 380
representation
generalized homotopy —, 235
stable homotopy —, 235
restriction, 218
of groups functors, 223
of universe, 220
right lifting property, 64
RLP, 64
Serre spectral sequence, 345
shift desuspension functor, 167
simplicial spectral sequence, 381
slice, 44
small object argument, 72
smash product
external —, 40
external — of spectra, 162
handicrafted —, 210
non-associative in $F_{op}$, 26
space
compactly closed —, 16
compactly generated —, 16
ex—, 19
external function —, 162
$F$—, 110
finite domination —, 235
$G$-connected, 365
$F_G$—, 160
$F_{G,B}$—, 166
$k$—, 16
$L$—, 405
principal —, 38
weak Hausdorff —, 16
well-based or nondegenerately based —, 82
well-grounded — over $B$, 86
spectral sequence
Čech —, 385
Atiyah-Hirzebruch —, 347, 375
fiberwise Costenoble-Waner duality —, 320
fiberwise duality —, 234
fiberwise parametrized Poincaré duality, 369
homotopical Poincaré duality, 313
for bundles, 325
of Brown, 123, 125, 197
of Dold, 53
of Milnor, Waner, 52
of Palais, 46
of Staffeif, Schön, 53
of Steinberger-West, Cauty, 54
of Strøm, 70
of Whitehead, 414
pairing — of Schwänzl and Vogt, 67
parametrized Atiyah duality —, 296, 300
Poincaré duality, 347, 364
relative Poincaré duality, 350
Thom isomorphism —, 349
triangulation — of Illman, 51
Wirthmüller isomorphism, 312
Thom
complex, 31
diagonal, 31
isomorphism, 349
object, 423
spectrum
classical — a, 403
generalized — a, 403
of a map, 407
topologically bicomplete category, 16
trace, 236
additivity of — s, 237
transfer
fiberwise — for bundles, 244
map, 236
of fibrations, 239
triangle
distinguished —, 261
triangulated category, 261
tube, 44
twisted
function spectrum functor, 423
half smash product, 423
twisted $K$-theory, 378
twisted Rothenberg–Steenrod spectral sequence, 387
unital stable homotopy category, 267
weak
compatibility, 263
Hausdorff space, 16
Hausdorffification functor, 16
map of spectra, 203
weak equivalence
mixed —, 63
subcategory of — s, 73
well-grounded — s, 87
in $G\mathcal{X}$, 90
in $G\mathcal{X}/B$ and $G\mathcal{X}_B$, 90
wedge lemma, 93
well-fibered
ex-space, 128
well-grounded
ex-space, 86
level $q$-equivalences, 176
model structure, 91
object, 85
$R$-modules, 216
$S$-modules, 420
space, 85
space over $B$, 86
spectrum, 176
stable equivalences, 185
weak equivalences, 87
well-sectioned
approximation, 132
ex-space, 83, 128
spectrum, 176
well-structured
prespectrum, 200
whiskering functor, 131
Wirthmüller
isomorphism, 312
Wirthmüller
context, 32
## Index of notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>/</td>
<td>351</td>
</tr>
<tr>
<td>\</td>
<td>351</td>
</tr>
<tr>
<td>∪</td>
<td>351</td>
</tr>
<tr>
<td>⊂</td>
<td>253, 270</td>
</tr>
<tr>
<td>⊗</td>
<td>248, 270</td>
</tr>
<tr>
<td>△</td>
<td>253, 270</td>
</tr>
<tr>
<td>∧</td>
<td>351</td>
</tr>
<tr>
<td>*&lt;B</td>
<td>80</td>
</tr>
<tr>
<td>A × c B</td>
<td>25</td>
</tr>
<tr>
<td>Aut(k)</td>
<td>376</td>
</tr>
<tr>
<td>Aut(X)</td>
<td>376</td>
</tr>
<tr>
<td>α × X</td>
<td>423</td>
</tr>
<tr>
<td>δ</td>
<td>31</td>
</tr>
<tr>
<td>B(D, Π, E)</td>
<td>402</td>
</tr>
<tr>
<td>B(Π; Γ)</td>
<td>243</td>
</tr>
<tr>
<td>$\mathcal{B}$</td>
<td>250</td>
</tr>
<tr>
<td>$\mathcal{G}_B$</td>
<td>18, 78, 153</td>
</tr>
<tr>
<td>$\mathcal{G}_F$</td>
<td>40, 81</td>
</tr>
<tr>
<td>$\mathcal{G}/B$</td>
<td>17, 78</td>
</tr>
<tr>
<td>C(f)</td>
<td>262</td>
</tr>
<tr>
<td>C(f, g)</td>
<td>262</td>
</tr>
<tr>
<td>χ(f)</td>
<td>236</td>
</tr>
<tr>
<td>$\psi^\beta(i, p)$</td>
<td>66</td>
</tr>
<tr>
<td>Ccyl(X)</td>
<td>64</td>
</tr>
<tr>
<td>$\mathcal{C}(X, Y)$</td>
<td>16</td>
</tr>
<tr>
<td>Cyl(X)</td>
<td>64</td>
</tr>
<tr>
<td>$\mathcal{D}[\mathcal{C}]$</td>
<td>395</td>
</tr>
<tr>
<td>$\mathcal{D}B$</td>
<td>124, 196</td>
</tr>
<tr>
<td>$D^W X$</td>
<td>287</td>
</tr>
<tr>
<td>Df(X)</td>
<td>258</td>
</tr>
<tr>
<td>Dn</td>
<td>100</td>
</tr>
<tr>
<td>Dr(X)</td>
<td>258</td>
</tr>
<tr>
<td>E</td>
<td>205</td>
</tr>
<tr>
<td>End(k)</td>
<td>376</td>
</tr>
<tr>
<td>E(Π; Γ)</td>
<td>243</td>
</tr>
<tr>
<td>ϵ*</td>
<td>37</td>
</tr>
<tr>
<td>η#</td>
<td>256</td>
</tr>
<tr>
<td>ε#</td>
<td>256</td>
</tr>
<tr>
<td>Env</td>
<td>167</td>
</tr>
<tr>
<td>$\delta_x$</td>
<td>270</td>
</tr>
<tr>
<td>dX</td>
<td>315</td>
</tr>
<tr>
<td>F</td>
<td>378</td>
</tr>
<tr>
<td>SF</td>
<td>378</td>
</tr>
<tr>
<td>F(X)</td>
<td>400</td>
</tr>
<tr>
<td>$F[\alpha, X]$</td>
<td>423</td>
</tr>
<tr>
<td>$\bar{F}_B(K, X)$</td>
<td>20, 152</td>
</tr>
<tr>
<td>$\bar{F}_B(X, Y)$</td>
<td>21</td>
</tr>
<tr>
<td>$F_B(Y, Z)$</td>
<td>162</td>
</tr>
<tr>
<td>$FCP(\mathcal{C})$</td>
<td>399</td>
</tr>
<tr>
<td>$F_B^I$</td>
<td>177</td>
</tr>
<tr>
<td>$F(I_x, X)$</td>
<td>63</td>
</tr>
<tr>
<td>$F_B^I$</td>
<td>177</td>
</tr>
<tr>
<td>$F(K, X)$</td>
<td>17</td>
</tr>
<tr>
<td>$f^*Y$</td>
<td>30, 168</td>
</tr>
<tr>
<td>$f_1X$</td>
<td>30, 168</td>
</tr>
<tr>
<td>$f_1X$</td>
<td>30, 168</td>
</tr>
<tr>
<td>Fred(θ)</td>
<td>378</td>
</tr>
<tr>
<td>$F_RF_B^I$</td>
<td>217</td>
</tr>
<tr>
<td>$F_RF_B^I$</td>
<td>217</td>
</tr>
<tr>
<td>$F_RFK_B^I$</td>
<td>217</td>
</tr>
<tr>
<td>$F_R(N, L)$</td>
<td>218</td>
</tr>
<tr>
<td>$F_V$</td>
<td>167</td>
</tr>
<tr>
<td>$F(X, Y)$</td>
<td>16</td>
</tr>
<tr>
<td>$\tilde{F}(Y, Z)$</td>
<td>40</td>
</tr>
<tr>
<td>$\mathcal{G}$</td>
<td>110</td>
</tr>
<tr>
<td>Γ(U; j* X)</td>
<td>384</td>
</tr>
<tr>
<td>$G^\mathcal{G}$</td>
<td>153</td>
</tr>
<tr>
<td>$G\mathcal{G}_B$</td>
<td>200</td>
</tr>
<tr>
<td>$G\mathcal{G}$</td>
<td>22</td>
</tr>
<tr>
<td>$G\mathcal{G}_B$</td>
<td>22</td>
</tr>
<tr>
<td>GL1(R)</td>
<td>401, 406</td>
</tr>
<tr>
<td>GL1(k)</td>
<td>377, 378</td>
</tr>
<tr>
<td>$G\mathcal{G}_B$</td>
<td>165</td>
</tr>
<tr>
<td>$G\mathcal{G}_B$</td>
<td>163</td>
</tr>
<tr>
<td>$G\mathcal{G}_B(U'; U)$</td>
<td>423</td>
</tr>
<tr>
<td>$GY_B$</td>
<td>138</td>
</tr>
<tr>
<td>$G\mathcal{G}_B$</td>
<td>138</td>
</tr>
<tr>
<td>hθ*</td>
<td>62</td>
</tr>
<tr>
<td>h$G\mathcal{G}_B$</td>
<td>138</td>
</tr>
<tr>
<td>Hoθ*</td>
<td>63</td>
</tr>
<tr>
<td>hocolim Xi</td>
<td>262</td>
</tr>
<tr>
<td>HX</td>
<td>376</td>
</tr>
<tr>
<td>I</td>
<td>111</td>
</tr>
<tr>
<td>$I_B$</td>
<td>98, 111</td>
</tr>
<tr>
<td>Symbol</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>$I_{B}$</td>
<td>101</td>
</tr>
<tr>
<td>$i_{B}$</td>
<td>156</td>
</tr>
<tr>
<td>$J_{c}$</td>
<td>405</td>
</tr>
<tr>
<td>$J_{c}$: FCP(μ),</td>
<td>405</td>
</tr>
<tr>
<td>$J_{B}^f(\mathcal{E})$,</td>
<td>112</td>
</tr>
<tr>
<td>$(\mathcal{E}, G, \mathcal{F})$,</td>
<td>159</td>
</tr>
<tr>
<td>$(\mathcal{E}, \mathcal{B}, G, \mathcal{F}, \mathcal{X}_B)$,</td>
<td>161</td>
</tr>
<tr>
<td>$\iota^*$,</td>
<td>35, 170</td>
</tr>
<tr>
<td>$\iota$,</td>
<td>35</td>
</tr>
<tr>
<td>$\iota^*$,</td>
<td>35</td>
</tr>
<tr>
<td>Iso(k),</td>
<td>375</td>
</tr>
<tr>
<td>$J$,</td>
<td>111</td>
</tr>
<tr>
<td>$J_{B}$,</td>
<td>98, 111</td>
</tr>
<tr>
<td>$J_{B}^f$,</td>
<td>101</td>
</tr>
<tr>
<td>$J_{B}^f(\mathcal{E})$,</td>
<td>113</td>
</tr>
<tr>
<td>$J_{G,B}$,</td>
<td>166</td>
</tr>
<tr>
<td>$J_{V}^{(K,p)}(X)$,</td>
<td>368</td>
</tr>
<tr>
<td>$J_{V}^{(K,p)}(X)$,</td>
<td>368</td>
</tr>
<tr>
<td>$J_{s}$,</td>
<td>338</td>
</tr>
<tr>
<td>$J_{s}^*$,</td>
<td>338</td>
</tr>
<tr>
<td>$J_{G}$,</td>
<td>359</td>
</tr>
<tr>
<td>$J_{G}^*$,</td>
<td>358</td>
</tr>
<tr>
<td>$J_{s}$,</td>
<td>339</td>
</tr>
<tr>
<td>$J_{s}^*$,</td>
<td>339</td>
</tr>
<tr>
<td>$K$,</td>
<td>205</td>
</tr>
<tr>
<td>$\mathcal{K}$,</td>
<td>16</td>
</tr>
<tr>
<td>$\mathcal{K}/B$,</td>
<td>19</td>
</tr>
<tr>
<td>$\mathcal{K}_B$,</td>
<td>19</td>
</tr>
<tr>
<td>$k_{B}^p$,</td>
<td>341</td>
</tr>
<tr>
<td>$k_{B}^p$,</td>
<td>31, 36</td>
</tr>
<tr>
<td>$k_{B}$,</td>
<td>31, 36</td>
</tr>
<tr>
<td>$k_{B}^p$,</td>
<td>341</td>
</tr>
<tr>
<td>$\mathcal{K}_{G,B}(X,Y)$,</td>
<td>22</td>
</tr>
<tr>
<td>$k_{p}^p(X)$,</td>
<td>374</td>
</tr>
<tr>
<td>$k_{p}^p(X)$,</td>
<td>374</td>
</tr>
<tr>
<td>$k_{p}$,</td>
<td>374</td>
</tr>
<tr>
<td>$\mathcal{K}_{s}$,</td>
<td>16</td>
</tr>
<tr>
<td>$k_{s}(X,p)$,</td>
<td>374</td>
</tr>
<tr>
<td>$k_{s}^*(X,p)$,</td>
<td>374</td>
</tr>
<tr>
<td>$L$,</td>
<td>421</td>
</tr>
<tr>
<td>$\mathcal{L}$,</td>
<td>405</td>
</tr>
<tr>
<td>$\mathcal{L}^s(J)$,</td>
<td>344</td>
</tr>
<tr>
<td>$\Lambda(T)$,</td>
<td>401</td>
</tr>
<tr>
<td>$\mathcal{L}^s(X,J)$,</td>
<td>344</td>
</tr>
<tr>
<td>$\mathcal{L}^s(X,J)$,</td>
<td>344</td>
</tr>
<tr>
<td>$\mathcal{L}[\mathcal{F}]$,</td>
<td>405</td>
</tr>
<tr>
<td>$\lambda_{B}^{W}$,</td>
<td>187</td>
</tr>
<tr>
<td>$\Lambda(R)$,</td>
<td>401</td>
</tr>
<tr>
<td>$\Lambda_B$,</td>
<td>131</td>
</tr>
<tr>
<td>$L_X$,</td>
<td>132</td>
</tr>
<tr>
<td>$\mathcal{M}(\alpha)$,</td>
<td>423</td>
</tr>
<tr>
<td>Map$_B(K,X)$,</td>
<td>20</td>
</tr>
<tr>
<td>Map$_B(X,Y)$,</td>
<td>21</td>
</tr>
<tr>
<td>Map$(I,X)$,</td>
<td>63</td>
</tr>
<tr>
<td>Map$(K,X)$,</td>
<td>16</td>
</tr>
<tr>
<td>Map$(X,Y)$,</td>
<td>16</td>
</tr>
<tr>
<td>$M_{f}$,</td>
<td>64</td>
</tr>
<tr>
<td>$M \wedge_R N$,</td>
<td>218</td>
</tr>
<tr>
<td>$N_{f}$,</td>
<td>64</td>
</tr>
<tr>
<td>$(-)^{N}$,</td>
<td>37, 226</td>
</tr>
<tr>
<td>$(-)/N$,</td>
<td>37, 226</td>
</tr>
<tr>
<td>$\mathcal{O}_{s}(B)$,</td>
<td>24</td>
</tr>
<tr>
<td>$\mathcal{O}(B)$,</td>
<td>24</td>
</tr>
<tr>
<td>$\mathcal{O}_{G}$,</td>
<td>112</td>
</tr>
<tr>
<td>$\Omega_{S}^{G}$,</td>
<td>163</td>
</tr>
<tr>
<td>$\text{Orb}(P)$,</td>
<td>38</td>
</tr>
<tr>
<td>$\Omega_{B}^{G}X$,</td>
<td>163</td>
</tr>
<tr>
<td>$P$,</td>
<td>133</td>
</tr>
<tr>
<td>$P$,</td>
<td>319</td>
</tr>
<tr>
<td>$P$,</td>
<td>179</td>
</tr>
<tr>
<td>$P_{B}^{s}(i,p)$,</td>
<td>155</td>
</tr>
<tr>
<td>$P_{B}(X,Y)$,</td>
<td>161</td>
</tr>
<tr>
<td>$P_{B}(X,Y)$,</td>
<td>152</td>
</tr>
<tr>
<td>$P_{F}$,</td>
<td>48</td>
</tr>
<tr>
<td>$P_{F}$,</td>
<td>240</td>
</tr>
<tr>
<td>$\mathcal{O}_{G,B}$,</td>
<td>165</td>
</tr>
<tr>
<td>$\pi(X,Y)$,</td>
<td>92</td>
</tr>
<tr>
<td>$P(U)(\mathcal{K})$,</td>
<td>379</td>
</tr>
<tr>
<td>$P \times_{\Pi} F$,</td>
<td>47</td>
</tr>
<tr>
<td>$Q^{S}$,</td>
<td>378</td>
</tr>
<tr>
<td>$\mathbb{R}^2$,</td>
<td>100</td>
</tr>
<tr>
<td>$\mathcal{R}_{O}(G)$,</td>
<td>358</td>
</tr>
<tr>
<td>$S_{B}$,</td>
<td>163</td>
</tr>
<tr>
<td>Sec,</td>
<td>31</td>
</tr>
<tr>
<td>$S_{f}$,</td>
<td>273</td>
</tr>
<tr>
<td>$S_{F}(X)$,</td>
<td>400</td>
</tr>
<tr>
<td>$\mathcal{K}_{G,B}$,</td>
<td>163</td>
</tr>
<tr>
<td>$SGL_{1}(R)$,</td>
<td>401</td>
</tr>
<tr>
<td>$S_{H}^{b}$,</td>
<td>124</td>
</tr>
<tr>
<td>$\Sigma_{H}^{b}$,</td>
<td>163</td>
</tr>
<tr>
<td>$SL_{1}(R)$,</td>
<td>406</td>
</tr>
<tr>
<td>$SL_{1}(k)$,</td>
<td>377, 378</td>
</tr>
<tr>
<td>$S^{n-1}$,</td>
<td>100</td>
</tr>
<tr>
<td>$S_{H}^{n}$,</td>
<td>196</td>
</tr>
<tr>
<td>$S_{Top}(X)$,</td>
<td>400</td>
</tr>
<tr>
<td>$\Sigma_{B}^{n}X$,</td>
<td>163</td>
</tr>
<tr>
<td>$T$,</td>
<td>207</td>
</tr>
<tr>
<td>$\mathcal{O}$,</td>
<td>16</td>
</tr>
<tr>
<td>$t$,</td>
<td>249, 272</td>
</tr>
<tr>
<td>$\tau(f)$,</td>
<td>236</td>
</tr>
<tr>
<td>$\text{Tel}X$,</td>
<td>94</td>
</tr>
<tr>
<td>$\mathcal{O}_{op}$,</td>
<td>16</td>
</tr>
<tr>
<td>Top$(X)$,</td>
<td>400</td>
</tr>
<tr>
<td>$t_{S_{f}}$,</td>
<td>273</td>
</tr>
<tr>
<td>$U$,</td>
<td>78</td>
</tr>
</tbody>
</table>
\(\Psi, 16\)
\(\mathcal{W}/B, 19\)
\(\mathcal{W}_B, 19\)
\(\mathbb{U}, 179\)
\(\mathfrak{A}_*, 16\)

\(V, 78\)
\(V^*, 167\)

\(WX, 131\)

\(X \wedge_B K, 20, 152\)
\(X_b, 31, 36\)
\(X \wedge_B Y, 162\)
\(X \wedge_B Y, 20\)
\(X \times I, 63\)
\(X \wedge I_+, 63\)
\(X \times K, 16\)
\(X \wedge K, 17\)
\(X \otimes Y, 396\)

\(X \times_B K, 20\)
\(X \times_B Y, 20\)

\([X, Y], 92\)

\(X \wedge Y, 40\)

\(\check{Y}, 20\)
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This book develops rigorous foundations for parametrized homotopy theory, which is the algebraic topology of spaces and spectra that are continuously parametrized by the points of a base space. It also begins the systematic study of parametrized homology and cohomology theories.

The parametrized world provides the natural home for many classical notions and results, such as orientation theory, the Thom isomorphism, Atiyah and Poincaré duality, transfer maps, the Adams and Wirthmüller isomorphisms, and the Serre and Eilenberg–Moore spectral sequences. But in addition to providing a clearer conceptual outlook on these classical notions, it also provides powerful methods to study new phenomena, such as twisted $K$-theory, and to make new constructions, such as iterated Thom spectra.

Duality theory in the parametrized setting is particularly illuminating and comes in two flavors. One allows the construction and analysis of transfer maps, and a quite different one relates parametrized homology to parametrized cohomology. The latter is based formally on a new theory of duality in symmetric bicategories that is of considerable independent interest.

The text brings together many recent developments in homotopy theory. It provides a highly structured theory of parametrized spectra, and it extends parametrized homotopy theory to the equivariant setting. The theory of topological model categories is given a more thorough treatment than is available in the literature. This is used, together with an interesting blend of classical methods, to resolve basic foundational problems that have no nonparametrized counterparts.