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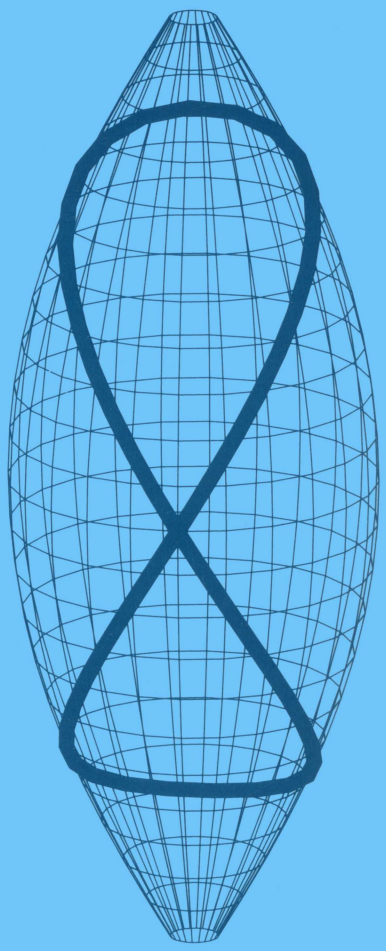
Systolic Geometry and Topology

Mikhail G. Katz

With an Appendix
by Jake P. Solomon



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Dedicated to the memory of my parents,
Tsvi Dovid ben Moshe and Chaya bas Binyomin

The photograph on the back cover of the book had been taken by the author's late mother. The author is married and lives in Bnei Braq, Israel. The author is blessed with seven children, and he strives to emulate the serene ways of his late father, in raising them.

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Preface

This text is based upon a recent course at Bar Ilan University, as well as upon numerous collaborative efforts (see the Acknowledgments Chapter following this Preface). After dealing with classical geometric preliminaries, including the *theorema egregium* of Gauss, we present new geometric inequalities of systolic type on Riemann surfaces, as well as their higher dimensional generalisations. Thus, the text can be viewed as an expanded version of a part of the survey by C. Croke and the present author [**CrK03**], which itself dealt with a small part of the material in M. Gromov's seminal text *Filling Riemannian manifolds* [**Gro83**]. Most of the results presented here have been obtained over the past four years, a reflection of the rapid progress in the field since the publication of the monograph [**Gro99**] seven years ago. M. Berger's recent monograph [**Berg03**] provides a fascinating historical perspective unburdened by proofs, *cf.* Section 2.1.

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I thank my wife for unbounded patience, and for accomodating unreasonable work schedules inherent in a task such as writing a book.

I thank Hashem Yisborach for giving me the strength to lead this work to its completion.

APPENDIX A

Period map image density (by Jake Solomon)

A.1. Introduction and outline of proof

We present a proof of the density of the image of the period map for the case of the n-fold blow-up of projective plane, $\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$. This result was first obtained by T. J. Li and A. Liu [**Li01**]. An alternative proof of the density may be found in [**GayK04**, Theorem 1, item (8)]. Our proof, similarly to that of [**Li01**], relies on the connection between Seiberg-Witten invariants and Gromov-Witten invariants, due to C. Taubes [**Ta95**], as further developed by D. McDuff [**Mc96**] and P. Biran [**Bir97**]. Our goal is to provide a non-technical exposition, accessible to a wide range of geometers and topologists.

More precisely, let M be a closed oriented four-manifold with intersection form

$$q : H^2(M,\mathbb{Z}) \otimes H^2(M,\mathbb{Z}) \to \mathbb{Z}.$$

By definition, $b_2^+(M)$ is the maximal rank of a submodule of $H^2(M,\mathbb{Z})$ on which the restriction of q is positive definite. We assume throughout that $b_2^+(M) = 1$. Given a vector space V, we denote by $\mathbb{P}(V)$ its projectivisation.

Definition A.1.1. The positive cone $\mathcal{P} = \mathcal{P}(M)$ of M is

$$\mathcal{P} := \left\{ [\alpha] \in \mathbb{P}\left(H^2(M,\mathbb{R})\right) \middle| \, q(\alpha,\alpha) > 0 \right\}.$$

Given a Riemannian metric $\mathcal{G}$ on M, its Hodge star operator $*$ defines an involution on the space of harmonic two-forms $\mathcal{H}^2(M)$ and hence, by the Hodge theorem, on $H^2(M,\mathbb{R})$. We denote by

$$\mathcal{P}_{\mathcal{G}} \subset H^2(M,\mathbb{R})$$

the $+1$ eigenspace of the operator $*$, also known as the space of self-dual harmonic two-forms on M. If $\alpha \in \mathcal{P}_{\mathcal{G}}$, $\alpha \neq 0$, is a self-dual harmonic 2-form, we have

$$q(\alpha,\alpha) = \int_M \alpha \wedge \alpha = \int_M \alpha \wedge *\alpha = \|\alpha\|_{L^2} > 0.$$

Our hypothesis $b_2^+(M) = 1$ therefore implies $\dim \mathcal{P}_{\mathcal{G}} = 1$. Let $\mathcal{M}$ denote the space of Riemannian metrics on M.

Definition A.1.2. The period map

$$\mathrm{P} : \mathcal{M} \longrightarrow \mathcal{P}(M)$$

is defined by $\mathcal{G} \mapsto [\mathcal{P}_{\mathcal{G}}]$.

The goal of this appendix will be to prove the following theorem in the special case when $M = \mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$:

Theorem A.1.3. *The image of the period map is dense in $\mathcal{P}$.*

This result was first obtained by T. J. Li and A. Liu in [**Li01**], where it is proved for arbitrary symplectic four-manifolds with $b_2^+ = 1$. Nonetheless, the result is perhaps most difficult precisely in the case we treat.

In general outline, the proof proceeds as follows. First, we show that it suffices, for $x = [\alpha]$ belonging to a dense subset of $\mathcal{P}$, to construct a symplectic form

$$\omega \in \Omega^2(M),\ d\omega = 0,\ \omega^2 \neq 0,$$

such that $[\omega] = x$. We note that the diffeomorphism group $\mathrm{Diff}(M)$ acts by pullback both on the set of symplectic forms on M and on the positive cone $\mathcal{P}$ of M. Thus, it suffices to construct ω such that $[\omega] = x$ for x in a dense subset of a fundamental domain for the action of $\mathrm{Diff}(M)$ on $\mathcal{P}$. In fact, we will specify a subset $\mathcal{S} \subset \mathcal{P}$ that contains a fundamental domain for a suitable subgroup

$$G_{(-1,-2)} \subset \mathrm{Diff}(M),$$

cf. (A.4.6), such that the image of the map P is dense even in $\mathcal{S}$. We explicitly construct a finite set of diffeomorphisms of M which generate $G_{(-1,-2)}$, and then prove that $\mathcal{S}$ contains a fundamental domain for the action of $G_{(-1,-2)}$ on $\mathcal{P}$ by techniques of hyperbolic geometry.

On the other hand, to prove the density of the image of the map P in $\mathcal{S} \subset \mathcal{P}$, we employ powerful techniques combining gauge theory and the theory of holomorphic curves on M.

We wish to thank P. Biran for several helpful conversations, and R. Borcherds for pointing out that $\mathcal{S}$ indeed contains a fundamental domain for $G_{(-1,-2)}$.

A.2. Symplectic forms and the self-dual line

The proposition below highlights the role of symplectic structures in understanding the image of the period map.

Let M be a $2p$ dimensional smooth manifold. By definition, an almost complex structure on M is an automorphism J of the tangent bundle TM satisfying $J^2 = -1$. Now, suppose M is equipped with a symplectic form $\omega \in \Omega^2(M)$, *cf.* Section 14.2. Thus, we have $d\omega = 0, \omega^p \neq 0$.

DEFINITION A.2.1. An almost complex structure J on M is called *ω-tame* if the $(0,2)$-tensor $\omega(\cdot, J\cdot)$ defines a positive definite bilinear form on each tangent space of M. It is called *ω-compatible* if, in addition, the bilinear form is symmetric.

It is not hard to show that the space of all ω-compatible almost complex structures on M is non-empty and contractible [**McS98**].

PROPOSITION A.2.2. *Assume* $\dim M = 4$ *and* $b_2^+(M) = 1$. *An element of* $\mathcal{P}(M)$ *that can be represented by a symplectic form, necessarily lies in the image of the period map.*

PROOF. Let ω be a symplectic form on a four-manifold M. We first choose any ω-compatible almost complex structure J. Consider the metric $\mathcal{G} := \omega(\cdot, J\cdot)$. Let $* : \Omega^*(M) \to \Omega^*(M)$ denote the Hodge-star operator associated to $\mathcal{G}$. It follows from linear algebra that

$$*\omega = \omega.$$

Since $d * \omega = d\omega = 0$ by the definition of a symplectic form, we conclude that ω is harmonic with respect to the $\mathcal{G}$-Laplacian on 2-forms on M. Since ω is self-dual, it represents the line $\mathcal{P}_{\mathcal{G}}$. $\square$

A.3. A lemma from hyperbolic geometry

Let $\langle\cdot,\cdot\rangle$ denote the standard Lorentzian inner product on $\mathbb{R}^{n+1}$ given by a diagonal matrix with entries $-1,\ldots,-1,1$, on the diagonal.

Recall that the isometry group of n-dimensional hyperbolic space $\mathcal{H}^n$ is the group $O(1,n;\mathbb{R})$ of the linear transformations of $\mathbb{R}^{n+1}$ preserving $\langle\cdot,\cdot\rangle$. The action of $O(1,n)$ on $\mathcal{H}^n$ is given by restricting the action on $\mathbb{R}^{n+1}$ to the subset $\mathcal{H}^n_+ \subset \mathbb{R}^{n+1}$ defined by

$$\mathcal{H}^n_+ = \left\{ x = (x_1,\ldots,x_n;x_{n+1}) \in \mathbb{R}^{n+1} \middle| \langle x,x\rangle = +1, \quad x_{n+1} > 0 \right\}.$$

The space $\mathcal{H}^n_+$, equipped with pull back metric from $(\mathbb{R}^{n+1},\langle\cdot,\cdot\rangle)$, is one of the standard models for $\mathcal{H}^n$.

DEFINITION A.3.1. An element $r \in O(1,n)$ defines a *reflection* if for all $v \in \mathbb{R}^{n+1}$, we have

$$r(v) = v - 2\frac{\langle e,v\rangle}{\langle e,e\rangle}e$$

for a suitable $e \in \mathbb{R}^{n+1}$ with $\langle e,e\rangle < 0$.

Let $\Gamma \subset O(1,n;\mathbb{Z})$ be a subgroup generated by reflections. To each reflection $r \in \Gamma$ (not necessarily one of the generators), there corresponds a totally geodesic hyperplane $\mu_r \subset \mathcal{H}^n$ called the *mirror* of r, which is the locus of the fixed points of r. Consider the union of all mirrors of reflections in Γ. We call the closure of a connected component of the complement in $\mathcal{H}^n$ of this union a cell. We employ the notation $\{C_\iota \mid \iota \in \mathcal{I}\}$ for the collection of all such cells. The following result was stated, for example, in [**Vin75**, p. 324].

PROPOSITION A.3.2. *Each cell C_ι is a fundamental domain for the action of Γ on $\mathcal{H}^n$.*

PROOF. Fix a cell C_ι. Any point $x \in \mathcal{H}^n$ lies in a suitable cell C_β, $\beta \in \mathcal{I}$. It suffices to find an element of Γ carrying C_β and hence also x to C_ι.

LEMMA A.3.3. *Only a finite number of mirrors of reflections in Γ can meet any compact subset $K \subset \mathcal{H}^n$.*

PROOF. Without loss of generality, by replacing K with a ball containing it, we may assume $K = B_r(p)$, the ball of radius r centered at some point $p \in \mathcal{H}^n$. Note the following one to one correspondence between points of $B_r(p)$ and hyperplanes intersecting $B_r(p)$: By the convexity of the distance function on $\mathcal{H}^n$, any totally geodesic hyperplane $\mu \subset \mathcal{H}^n$ contains a unique point q closest to p. Clearly, if μ intersects $B_r(p)$, then $q \in B_r(p)$. On the other hand, we may recover μ from q as the totally geodesic hyperplane perpendicular to the unique length minimizing geodesic connecting q to p. Now, suppose $\{\mu_i\}$ is an infinite sequence of distinct mirrors of reflections in Γ each of which meet $B_r(p)$ and let q_i be the point of μ_i closest to p. Since $B_r(p)$ is compact, by passage to a subsequence, we may assume the sequence q_i converges. It follows that the corresponding hyperplanes μ_i converge, and so do the corresponding reflections. This contradicts the discreteness of $O(1,n;\mathbb{Z})$. $\square$

In particular, a path connecting C_β with C_ι intersects only a finite number of mirrors. So, we may choose a finite sequence of adjacent cells

$$C_\beta = C_{\beta_1}, C_{\beta_2}, \ldots, C_{\beta_m} = C_\iota.$$

Adjacency means that each pair of cells $(C_{\beta_i}, C_{\beta_{i+1}})$ shares a unique mirror μ_i, where $i = 1, \ldots, n-1$. Reflecting sequentially in the mirrors μ_i moves the cell C_β to the cell C_ι by an isometry belonging to Γ. In particular, this isometry moves x into the closure of C_ι, as required. □

A.4. Diffeomorphism group of blow-up of projective plane

Let $M_n := \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$. Thus, M_n is $\mathbb{CP}^2$ blown up at n points. We specify a basis of $H_2(M_n, \mathbb{Z})$ as follows. Let $L \in H_2(M_n, \mathbb{Z})$ be the homology class of a line in $\mathbb{CP}^2$. Let E_i denote the homology class of the ith exceptional divisor. Denote by $\mathrm{PD} : H^2(M_n, \mathbb{Z}) \to H_2(M_n, \mathbb{Z})$ the Poincaré duality isomorphism. Consider the basis in cohomology given by

$$\lambda = \mathrm{PD}^{-1}(L) \text{ and } \epsilon_i = \mathrm{PD}^{-1}(E_i) \in H^2(M_n, \mathbb{Z}).$$

We denote the intersection form on $H_2(M_n, \mathbb{Z})$ by Q or "$\cdot$", and we write $q = Q \circ \mathrm{PD}$ for the intersection form on $H^2(M_n, \mathbb{Z})$. An easy calculation yields

$$L \cdot L = 1, \qquad E_i \cdot L = 0, \qquad E_i \cdot E_j = -\delta_{ij}. \tag{A.4.1}$$

In other words, the intersection form is a Lorentzian inner product on $H_2(M_n, \mathbb{Z})$.

LEMMA A.4.1. *There is an element $r_i \in \mathrm{Diff}(M_n)$ that is the connect sum of the diffeomorphism given by complex conjugation on the ith copy of $\overline{\mathbb{CP}^2}$ with the identity diffeomorphism on the remaining $M_{n-1} = \mathbb{CP}^2 \# (n-1)\overline{\mathbb{CP}^2}$.*

PROOF. We must show that the two diffeomorphisms can be connected, to obtain a global diffeomorphism of $M_{n-1} \# \overline{\mathbb{CP}^2}$. That is, we must show that the identity map of $S^3 \subset \mathbb{C}^2$ can be smoothly isotoped through diffeomorphisms to the map arising from complex conjugation restricted to $S^3 \subset \mathbb{C}^2$. Indeed, both maps arise from matrices belonging to the special orthogonal group $SO(4)$. Since the Lie group $SO(4)$ is connected, we may connect these two matrices by a smooth path of matrices in $SO(4)$, each of which induces a diffeomorphism on S^3. □

We note that r_i^* acts on cohomology by the formula

$$r_i^*(\alpha) = \alpha - 2\frac{q(\alpha, \epsilon_i)}{q(\epsilon_i, \epsilon_i)}\epsilon_i, \qquad \alpha \in H^2(M_n, \mathbb{Z}). \tag{A.4.2}$$

Thus, r_i^* is the reflection of the Lorentzian lattice $(H^2(M_n, \mathbb{Z}), q)$, which corresponds to the vector ϵ_i. An element β of self-intersection -2 defines a reflection by a similar formula

$$r_\beta^*(\alpha) = \alpha - 2\frac{q(\alpha, \beta)}{q(\beta, \beta)}\beta. \tag{A.4.3}$$

The following lemma may be found in [**FriM94**].

LEMMA A.4.2. *Let M be a four-manifold. Let $\beta \in H^2(M, \mathbb{Z})$ be Poincaré dual to an embedded 2-sphere of self-intersection -2. Then there exists a suitable diffeomorphism $r_\beta \in \mathrm{Diff}(M)$ such that r_β^* acts on $H^2(M, \mathbb{Z})$ by reflection corresponding to β in the sense of* (A.4.3).

Now let $M = M_n$, and let $i, j, k = 1, \ldots, n$, assumed pairwise distinct. We set

$$\beta = \epsilon_{ijk} := \lambda - \epsilon_i - \epsilon_j - \epsilon_k. \tag{A.4.4}$$

Let r_β denote the diffeomorphism provided by Lemma A.4.2. In fact, in this particular case, we may choose the diffeomorphism r_β to be a biholomorphism. Using

algebraic geometry, these biholomorphisms can be constructed explicitly and are known as Cremona transformations [**GriH78**]. We define r_{ijk} to be the Cremona transformation associated to ϵ_{ijk}. We set

$$\mathcal{S}_n := \left\{ [\alpha] \in \mathcal{P}(M_n \ ) \middle| \, q(\alpha, \epsilon_i) > 0, \, q(\alpha, \epsilon_{ijk}) > 0 \quad \forall i, j, k \right\}. \tag{A.4.5}$$

Furthermore, we let

$$G_{(-1,-2)} \subset \mathrm{Diff}(M_n) \tag{A.4.6}$$

be the subgroup generated by the n diffeomorphisms r_i and the $\binom{n}{3}$ diffeomorphisms r_{ijk}. Here the indices $(-1,-2)$ help recall the nature of the generators of the group $G_{(-1,-2)}$.

PROPOSITION A.4.3. *The set $\mathcal{S} = \mathcal{S}_n$ contains a fundamental domain of the action of $G_{(-1,-2)}$ on $\mathcal{P}(M_n)$.*

PROOF. Let $\rho : \mathrm{Diff}(M_n) \to \mathrm{End}(H^2(M_n), q)$ be the natural homomorphism, and set

$$\tilde{G}_{(-1,-2)} := \rho\left(G_{(-1,-2)}\right).$$

Clearly, it suffices to show that $\mathcal{S}$ contains a fundamental domain of $\tilde{G}_{(-1,-2)}$. Let

$$\pi : H^2(M_n) \setminus \{0\} \to \mathbb{P}(H^2(M_n))$$

denote the natural projection, and let

$$H^+_{M_n} = \{\alpha \in H^2(M_n) | q(\alpha, \alpha) = 1\}.$$

Clearly, the restriction

$$\pi|_{H^+_{M_n}} : H^+_{M_n} \to \mathcal{P}$$

is a $\tilde{G}_{(-1,-2)}$-equivariant diffeomorphism. We now apply Proposition A.3.2 above with $\Gamma = \tilde{G}_{(-1,-2)}$, noting that since the inequalities (A.4.5) defining $\mathcal{S}$ arise from the mirrors of reflections in $\tilde{G}_{(-1,-2)}$, $\mathcal{S}$ must contain a cell C_ι. □

A.5. Background material from symplectic geometry

We will exploit P. Biran's criterion [**Bir97, Bir01**] for second cohomology classes of a symplectic four-manifold to be approximately represented by symplectic forms. Let (M, ω) be a closed symplectic four-manifold. A submanifold $N \subset M$ is called symplectic if the restriction of ω to N is a symplectic form. By analogy with algebraic geometry, we will call a class $E \in H_2(M)$ *exceptional* if it can be represented by an embedded symplectic 2-sphere $\Sigma \subset M$ such that $\Sigma \cdot \Sigma = -1$.

DEFINITION A.5.1. We will denote by $\mathrm{E}_\omega \subset H_2(M)$ the set of all ω-exceptional classes.

Before stating Biran's theorem, we mention that four-manifolds whose Seiberg-Witten invariants vanish when the Seiberg-Witten moduli space has non-zero dimension are said to have Seiberg-Witten simple type. For our purposes, all we need to know is that closed simply connected symplectic four-manifolds with $b_2^+ = 1$ do *not* have Seiberg-Witten simple type. As before, we denote by $q(\cdot, \cdot)$ the intersection form on $H^2(M)$.

THEOREM A.5.2 (P. Biran). *Let (M, ω) be a closed symplectic four-manifold. Assume M does not have Seiberg-Witten simple type. Let $\alpha \in H^2(M)$ belong to the positive cone of M, i.e. $q(\alpha, \alpha) > 0$. Suppose that α also satisfies*

(1) $q(\alpha, [\omega]) > 0$,
(2) $\alpha(E) > 0$ *for all* $E \in \mathrm{E}_\omega$.

Then there exist symplectic forms on M representing cohomology classes arbitrarily close to α.

This result was proved in [**Bir97**] using Taubes's Seiberg-Witten Gromov-Witten correspondence [**Ta95**], as further developed by McDuff [**Mc96**], combined with the technique of symplectic inflation due to F. Lalonde and D. McDuff [**LaM96**].

REMARK A.5.3. Although in general, when $b_2^+ = 1$, the Seiberg-Witten invariants depend on the choice of a chamber, there is a canonical choice of chamber for symplectic manifolds. So, in the case under consideration, we may refer to the Seiberg-Witten invariants unambiguously.

REMARK A.5.4. The main idea behind the proof Theorem A.5.2 is to use the non-vanishing of the Seiberg-Witten invariants to construct a symplectically embedded surface representing a multiple of the Poincare dual of the chosen class α. Taubes did this in a remarkable string of papers outlined in [**Ta95**]. Symplectic inflation uses the embedded symplectic surface to construct a symplectic form representing a cohomology class arbitrarily close to α.

In very broad outline, Taubes's argument runs as follows: A solution to the Seiberg-Witten equations consists of a section of a certain vector bundle as well as a connection on that bundle. The non-vanishing of the Seiberg-Witten invariant implies the existence of such a solution to the equations. Moreover, this solution persists under a large variety of perturbations. In the limit of a specially chosen large perturbation, the vanishing sets of the sections become symplectically embedded surfaces.

In order to apply Theorem A.5.2, we will need to analyze which classes in $H_2(M)$ can be ω-exceptional. This will rely heavily on the theory of J-holomorphic curves, which we now outline. Let M be a smooth manifold admitting an almost complex structure J and let Σ be a Riemann surface with complex structure j.

DEFINITION A.5.5. A J-holomorphic Σ-curve in M is a map $u : \Sigma \to M$ satisfying the Cauchy-Riemann equation

$$J \circ du - du \circ j = 0.$$

Here, we will be interested exclusively in J-holomorphic spheres, *i.e.* in the case $\Sigma \simeq S^2$. Now we formulate the connection between ω-exceptional classes, on the one hand, and J-holomorphic curves, on the other. If (M, ω) is a symplectic manifold, we denote by $\mathcal{J}_\omega(M)$ the set of ω-tame complex structures on M, *cf.* Definition A.2.1. The following result was proved by McDuff [**Mc90**].

THEOREM A.5.6. *For J belonging to a residual subset of $\mathcal{J}_\omega(M)$, there exists a unique embedded J-holomorphic sphere representing each element $E \in \mathrm{E}_\omega$.*

Our main tool will be the property of positivity of intersections for J-holomorphic curves, first stated by M. Gromov [**Gro85**]. For a detailed history and exposition of the proof of this theorem, see [**McS04**, Appendix E].

A J-holomorphic curve $u : \Sigma \to M$ is called *simple* if it does not factor as the composition $u' \circ \phi$ of a map of Riemann surfaces $\phi : \Sigma \to \Sigma'$ with a J-holomorphic curve $u' : \Sigma' \to M$ such that the degree of ϕ is is greater than one.

THEOREM A.5.7. *Each pair A_0, $A_1 \in H_2(M)$ of homology classes represented by distinct, connected, simple J-holomorphic curves, satisfies $A_0 \cdot A_1 \geq 0$.*

Since embedded J-holomorphic spheres are always simple, we can combine Theorem A.5.6 with Theorem A.5.7 to deduce constraints on E_ω.

Finally, we note that if (M, ω) is a symplectic manifold, then each almost complex structure $J \in \mathcal{J}_\omega(M)$ makes TM into a complex vector bundle. Recall that the Chern classes of a complex vector bundle belong to integral cohomology. Since $\mathcal{J}_\omega(M)$ is contractible, it follows that the Chern classes of TM do not depend on the choice of $J \in \mathcal{J}_\omega(M)$. We will use the following topological lemma relating intersection numbers in a symplectic four-manifold (M, ω) with the first Chern class of its tangent bundle $c_1(TM)$.

LEMMA A.5.8 (The adjunction formula). *Let (M, ω) be a symplectic four-manifold and let $\Sigma \subset M$ be a symplectic submanifold of* $\dim \Sigma = 2$. *Then*

$$c_1(TM)([\Sigma]) = [\Sigma] \cdot [\Sigma] + \chi(\Sigma).$$

PROOF. Choosing $J \in \mathcal{J}_\omega(M)$, we may consider TM as a complex vector bundle. Since Σ is embedded, choosing a Hermitian metric on TM, we have the splitting

$$TM|_\Sigma \simeq T\Sigma \oplus \nu_\Sigma,$$

where ν_Σ denotes the normal bundle of Σ. Since the first Chern class of a line bundle is its Euler class, we have

$$c_1(\nu_\Sigma) = \Sigma \cdot \Sigma, \qquad c_1(T\Sigma) = \chi(\Sigma).$$

The lemma follows immediately from the Whitney sum formula. □

A.6. Proof of density of image of period map

As before, let

$$\pi : H^2(M_n) \setminus \{0\} \to \mathbb{P}(H^2(M_n))$$

denote the natural projection. Define

$$\tilde{\mathcal{S}} := \{\alpha \in \pi^{-1}(\mathcal{S}) \,|\, q(\alpha, \lambda) > 0\}.$$

By Proposition A.4.3, it suffices to prove that if $\alpha \in \tilde{\mathcal{S}}$, we can find cohomology classes arbitrarily close to α represented by symplectic forms. To this end, we will need to apply Theorem A.5.2.

Theorem A.5.2 starts with a symplectic manifold (M, ω). So, to apply Theorem A.5.2 in the case $M = M_n$, we first need to construct a symplectic form ω on M. In fact, in order to facilitate the study of E_ω, we will work in the more rigid Kähler category. Recall that a Kähler form on M is a symplectic form ω such that $\mathcal{J}_\omega$ contains an integrable complex structure induced by holomorphic coordinate charts on M. We use the fact that M_n arises as the complex analytic blowup of $\mathbb{CP}^2$ at n points. As such, it admits a canonical complex structure, which we denote by J_n.

LEMMA A.6.1. *There exists a Kähler form ω_δ on M_n representing the class*

$$\lambda - \delta(\epsilon_1 + \ldots + \epsilon_k) \in H^2(M_n)$$

for all sufficiently small $\delta > 0$. The canonical complex structure J_n on M_n arising from the complex analytic blow-up construction is compatible with ω_δ for all δ.

PROOF. The Fubini-Study form on $\mathbb{CP}^2$ is well known to be Kähler [**GriH78**]. The lemma then follows from the general construction of Kähler forms on blow-ups [**GriH78**, p. 192]. We note that if we were not interested in the existence of a compatible integrable complex structure, this argument could be carried through entirely in the symplectic category using McDuff's symplectic blow-up construction [**McS98**]. □

It is important to emphasize that in order to apply Theorem A.5.2, we need first to *fix* a symplectic structure ω on M. Theorem A.5.2 then allows us to construct new symplectic forms representing cohomology classes α that satisfy conditions (1) and (2). However, Lemma A.6.1 allows us some freedom as to which symplectic structure we fix. So, for any fixed class $\alpha \in \tilde{\mathcal{S}}$, we can choose a symplectic form ω such that condition (1) holds. Indeed, since by assumption $q(\alpha, \lambda) > 0$, for δ_0 sufficiently small we have $q(\alpha, \omega_{\delta_0}) > 0$. At this point, we fix $\omega = \omega_{\delta_0}$.

In order to verify condition (2) of Theorem A.5.2, we need to derive constraints on the set of exceptional classes E_ω. The following lemma, which shows that certain classes actually are exceptional, via the combination of Theorems A.5.6 and A.5.7 will help constrain which other classes might be exceptional. We define

$$E_{ij} := L - E_i - E_j$$

for $i \neq j$.

LEMMA A.6.2. *The classes E_i and E_{ij} are exceptional for any symplectic form ω with respect to which J_n is compatible. In particular, we have $E_i, E_{ij} \in \mathrm{E}_\omega$ for $\omega = \omega_{\delta_0}$.*

PROOF. It suffices to show that E_i, E_{ij}, can be represented by spheres holomorpically embedded with respect to J_n. Then since J_n is ω compatible, such spheres will automatically be symplectic.

Indeed, let $\Sigma \subset M$ be a holomorphically embedded sphere, let $x \in \Sigma$ and let $X \in T_x\Sigma$. Then the vectors X, J_nX, form a basis for $T_x\Sigma$, and by compatibility $\omega(X, J_nX) > 0$. Hence, $\omega|_{T\Sigma}$ is non-degenerate, *i.e.* a symplectic form.

In particular, E_i is represented by the exceptional divisor $\mathfrak{E}_i$ of the ith blowup. This is well known to be an embedded holomorphic sphere with respect to J_n, *cf.* [**GriH78**, p. 182]. Furthermore, using the fact that the Cremona transformation r_{ijk} is a biholomorphism, and noting that

$$r^*_{ijk}(\epsilon_i) = \lambda - \epsilon_j - \epsilon_k,$$

we see that E_{jk} is represented by the embedded holomorphic sphere $r_{ijk}(\mathfrak{E}_i)$. □

Given $E \in \mathrm{E}_\omega$, we write $E = e_0L - \sum_i e_iE_i$. Also, we use the abbreviated notation $c_1 = c_1(TM) \in H^2(M)$.

LEMMA A.6.3. *Every exceptional class E satisfies the following 4 conditions:*

(i) $E \cdot E = -1$, *i.e.* $e_0^2 - \sum_i e_i^2 = -1$.
(ii) $c_1(E) = 1$, *i.e.* $3e_0 - \sum_i e_i = 1$.

(iii) *Either* $E = E_i$, *or* $e_0 > 0$ *and* $e_i \geq 0$, $\forall i$.
(iv) *Either* $E = E_{ij}$ *for some* $i \neq j$, *or* $E \cdot E_{ij} \geq 0$, *i.e.* $e_0 - e_i - e_j \geq 0$ *for all* $i \neq j$.

PROOF. Item (i) is part of the definition of an exceptional class. To prove (ii), we apply Lemma A.5.8 to obtain

$$c_1(E) = -1 + 2 = 1, \quad c_1(L) = 1 + 2 = 3, \quad c_1(E_i) = -1 + 2 = 1.$$

To prove item (iii), assume $E \neq E_i$. Combining Lemma A.6.2 with Theorems A.5.6 and A.5.7, we deduce that $E \cdot E_i \geq 0$. Hence $e_i \geq 0$ for all $i \geq 1$. It then follows from (ii) that $e_0 > 0$. Finally, (iv) follows immediately from Lemma A.6.2 combined with Theorems A.5.6 and A.5.7, as before. □

We now apply the preceding lemma to verify condition (2). By definition, if $\alpha \in \tilde{\mathcal{S}}$, we have $\alpha(E_i) = q(\alpha, \epsilon_i) > 0$. In particular, condition (2) is satisfied when $E = E_i$. To deal with the case $E \neq E_i$, we introduce the following lemma.

LEMMA A.6.4. *Let* $\alpha \in \tilde{\mathcal{S}}$. *Write* $\alpha = a_0\lambda - \sum_i a_i\epsilon_i$. *Let* $a_{i_1} \geq a_{i_2} \geq a_{i_3}$ *be the three largest coefficients among the* $a_1, \ldots, a_n$. *Let*

$$E = e_0 L - \sum_i e_i E_i$$

be an exceptional class, and assume $E \neq E_i$. *Then*

$$\alpha(E) \geq e_0 q(\alpha, \epsilon_{i_1 i_2 i_3}), \tag{A.6.1}$$

where $\epsilon_{i_1 i_2 i_3}$ *is the class defining the appropriate Cremona map as in* (A.4.4).

PROOF. First we prove that if $E \neq E_i$ then

$$0 \leq e_i \leq e_0 \tag{A.6.2}$$

for all i. Indeed, if $E = E_{ij}$ then inequality (A.6.2) is immediate. Otherwise, by rearranging the inequality of (iv), we obtain

$$e_i \leq e_0 - e_j.$$

Then we use the fact that by (iii) we have $e_j \geq 0$ and $e_i \geq 0$ to conclude equation (A.6.2).

Combining (A.6.2) and (ii), we know that the coefficients e_i of any exceptional divisor $E \neq E_i \in \mathrm{E}_\omega$ must satisfy the inequalities

$$0 \leq e_i \leq e_0 \text{ and } \sum_i e_i = 3e_0 - 1. \tag{A.6.3}$$

Now, by the formula for the intersection form,

$$q(\alpha, E) = a_0 e_0 - \sum_i a_i e_i. \tag{A.6.4}$$

We think of (A.6.1) as a lower bound for expression (A.6.4) as we hold α fixed and allow E to vary over all exceptional classes with fixed non-zero e_0. By (A.6.3), for a fixed value of e_0, the number of possible choices for the coefficients e_i are finite. So, for one particular choice of the e_i, the expression (A.6.4) must be minimized. Clearly it suffices to prove the lower bound (A.6.1) for this minimizing choice. It is not important whether this choice of the e_i can actually be realized by an exceptional class.

Recall that i_1, i_2, i_3, are the indices such that $a_{i_1} \geq a_{i_2} \geq a_{i_3}$ are the largest of the coefficients a_i of α. We assert that the choice of the coefficients e_i that minimizes the expression (A.6.4) is given by

$$e_{i_1} = e_{i_2} = e_0, \qquad e_{i_3} = e_0 - 1, \qquad e_i = 0,\ i \neq i_1,\ i_2,\ i_3. \tag{A.6.5}$$

Indeed, we prove by induction that for an arbitrary choice of the coefficients e_i the value of expression (A.6.4) cannot be less than the value given the choice (A.6.5). Indeed, starting with an arbitrary choice of the coefficients e_i satisfying (A.6.3), we execute the following algorithm repeatedly until it terminates:

(1) If e_{i_1} or $e_{i_2} < e_0$ and $e_j > 0$ for some $j \neq i_1$, i_2, i_3, then increment e_{i_1} or e_{i_2} by 1 and decrement e_j by 1.
(2) Otherwise, if e_{i_1} or $e_{i_2} < e_0$ and $e_{i_3} = e_0$ then increment e_{i_1} or e_{i_2} by 1 and decrement e_{i_3} by 1.
(3) Otherwise, if $e_{i_3} < e_0 - 1$ and $e_j > 0$ for some $j \neq i_1$, i_2, i_3, increment e_{i_3} by 1 and decrement e_j by 1.
(4) Otherwise, terminate.

It is clear that executing this algorithm can only decrease the expression (A.6.4). On the other hand, using equations (A.6.3), it is easy to see that this algorithm terminates only when e_i are as in (A.6.5). Finally, since e_{i_1}, e_{i_2}, e_{i_3}, are bounded by e_0, the process must terminate after a finite number of steps. Indeed, the value of one of the coefficients e_{i_1}, e_{i_2}, e_{i_3}, must increase at each step. Furthermore, the value of e_{i_3} can only decrease once, wherease the values of e_{i_1} and e_{i_2} can never decrease.

Now, substituting choice (A.6.5) into expression (A.6.4), we obtain

$$\begin{aligned} \alpha(E) &= a_0 e_0 - \sum_i a_i e_i \\ &\geq a_{i_3} + a_0 e_0 - (a_{i_1} + a_{i_2} + a_{i_3}) e_0 \\ &= a_{i_3} + e_0 q(\alpha, \epsilon_{123}), \end{aligned}$$

Since $\alpha \in \widetilde{\mathcal{S}}$ we know that $q(\alpha, \epsilon_{i_3}) > 0$ and hence $a_{i_3} > 0$. Equation (A.6.1) follows immediately. □

Finally, to check condition (2) of Theorem A.5.2, we apply Lemma A.6.4 to conclude $\alpha(E) \geq e_0 q(\alpha, \epsilon_{i_1 i_2 i_3}) > 0$ by the defining equation of the fundamental domain which corresponds to the appropriate Cremona transformation.

APPENDIX B

Open problems

We conclude with a list of open problems in systolic topology, geometry, and arithmetic.

B.1. Topology

B.1.1. In Subsection 1.3.3 above, we described an application of a map between classifying spaces to the (still open) problem of the determination of the optimal systolic ratio of projective spaces, see [**BaKSS06**]. The argument in [**BaKSS06**] exploits the inclusion of classifying spaces, $BS^1 \to BS^3$, resulting from the Lie group inclusion $S^1 \to S^3$. There is a natural way to try to generalize such an argument.

Since the group S^3 is the standard building block in Lie group theory, in principle the field is wide open for generalisation, by exploiting the inclusion of classifying spaces corresponding to the subgroup inclusion $S^3 \hookrightarrow G$, where G is a compact Lie group. It remains to be seen whether geometrically meaningful results can be obtained in this fashion.

More fundamentally, while the link of homotopy theory to systolic geometry tends to involve integral homotopy theory, many of the arguments, *e.g.* for general systolic freedom [**Katz95a, CrK03**], and the above results on quaternionic projective space, tend to be rational (see, *e.g.* [**Su74**]). It would be desirable to forge a direct connection between rational homotopy theory and systolic geometry. This would constitute a considerable simplification, and requires clarification.

B.1.2. The equality of the systolic category and the Lusternik-Schnirelmann category for 3-manifolds stems from the fact both are determined via the simple "free *versus* non-free" dichotomy for the fundamental group. In this dimension, the only intermediate value of either category is 2, and the result for this value follows from the appropriate classification in terms of sphere bundles over the circle. The main mathematical content on the systolic side is provided by Gromov's 1-systolic inequality for essential manifolds, which characterizes the maximal value of the category.

The situation is more complex for 4-manifolds, where there is a pair intermediate values, namely, 2 and 3. Thus, in the non-essential case (not covered by Gromov's inequality), there is a potential for a deeper invariant. The distinction between the two intermediate values may depend on a study of Massey products in 4-manifolds.

Given a (non-essential) 4-manifold M with $b_1(M) \geq 1$, we note the following. If the cup product on $H^1(M, \mathbb{R})$ does not vanish, both categories must be at least 3 (by comparison with real cup-length). Otherwise, one expects the existence of a nontrivial Massey product in this case to be the deciding factor in distinguishing

between the two intermediate values of systolic category, similarly to the Lescop invariant for 3-manifolds. The issue is related to the surjectivity of the Abel-Jacobi map $M \to J_1(M)$ to the Jacobi torus $J_1(M) = T^{b_1(M)}$.

This surjectivity can be studied at the level of lifts to suitable covers, such as the universal free abelian cover. Here the nontriviality of the homology fiber of the lift at this level, is a sufficient condition.

Alternatively, one can study the map from M (with $b_1(M) = 2$) to a more general space, *e.g.* exploiting the step 2 nilpotent completion and the map to the corresponding 3-manifold, instead of the map to the 2-torus, *cf.* Remark 12.7.5.

B.2. Geometry

B.2.1. It was shown in [**BaKSS06**] that, contrary to the *complex* case clarified by M. Gromov, the symmetric metric on the *quaternionic* projective plane turns out not to be its systolically optimal metric, contrary to expectation expressed in a number of publications in Riemannian geometry, see Subsection 1.3.3 above.

The technique used in [**BaKSS06**] involves the exceptional Lie algebra E_7. Can this technique be refined to determine the precise value of the optimal systolic ratio of the two-point homogeneous spaces of non-complex type?

B.2.2. It was proved in [**KatzS06a**] that a genus 2 surface satisfies the Loewner inequality, and in [**KatzS05**], that surfaces of genus at least 20 are Loewner. The cases $3 \leq g \leq 19$ are open. One approach would be to use the capacity of annuli, *cf.* Remark 6.2.2.

Can efficient pair of pants decompositions for hyperbolic surfaces, combined with capacity of cylinders, be used to obtain efficient lower bounds for systoles of surfaces? By using the results by P. Buser and M. Seppälä [**BusSe02**] on "short" pair of pant decompositions of hyperbolic surfaces, one may be able to obtain suitable bounds for the capacity of the corresponding cylinders (*i.e.* annuli obtained by cutting open each pair of pants), and derive inequalities for higher genus surfaces.

B.3. Arithmetic

B.3.1. Consider the Weeks 3-manifold M_W. It is well known that its orbifold fundamental group $\pi \subset \mathrm{PSL}(2, \mathbb{C})$ is arithmetic. Consider principal congruence subgroups of π and the corresponding covers $M(p)$ of M_W. Do such covers satisfy a bound similar to the 4/3-bound (1.2.3) for the principal congruence tower of Hurwitz surfaces, *i.e.* $\mathrm{sys}\pi_1(M(p)) \geq \frac{2}{3} \log \|M(p)\|$ (without an additive constant)?

B.3.2. For hyperbolic 3-manifolds, the orthogonal group $O(3,1)$ is identified with the complex 2 by 2 matrices, hence admits trace estimates for congruence towers similar to the $\mathrm{SL}(2, \mathbb{R})$ case, dealt with in [**KatzSV07**]. For n-manifolds, no such identification is available for $n \geq 4$. What kind of lower bounds can one obtain for congruence towers of arithmetic hyperbolic n-manifolds?

B.3.3. The most "symmetric" genus 2 surface, namely the Bolza surface, is not a Hurwitz surface (which is, by definition, a $(2,3,7)$ triangle surface), but rather a (2,3,8) triangle surface, *cf.* Section 10.2. Do (2,3,8) triangle surfaces satisfy a bound similar to the bound (1.2.3) in the Hurwitz case?

B.3.4. In the case of arithmetic Fuchsian groups defined over $\mathbb{Q}$, P. Buser and P. Sarnak [**BusS94**] use the existence of a non-congruence subgroup of index 2, to prove a lower bound for the conformal systole via an eigenvalue estimate. Does this technique extend to arbitrary arithmetic groups? Does it extend to Kleinian groups?

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