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The Ricci Flow: Techniques and Applications

Part II: Analytic Aspects

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Preface

I look at the floor and I see it needs sweeping.
– From "While My Guitar Gently Weeps" by George Harrison of The Beatles

What Part II is about

Think the time is right for a palace revolution.
– From "Street Fighting Man" by Mick Jagger and Keith Richards of The Rolling Stones

To help put the aim of this volume in perspective, we indulge in some relative volume comparison (with apologies to Bishop and Gromov). Let $g_{ij} \doteqdot$ '*The Ricci Flow: An Introduction*' by two of the authors (throughout this book we shall refer to this as '**Volume One**' [**142**]), let $\Gamma_{ij}^{k} \doteqdot$ '*Hamilton's Ricci Flow*' by three of the authors [**146**], let[1] $R_{ijk\ell} \doteqdot$ '*The Ricci Flow: Techniques and Applications, Part I: Geometric Aspects*' by The Ricci Flowers[2] (throughout this book we shall refer to this as '**Part I of this volume**' [**135**]), let $\frac{\partial}{\partial t} R_{ijk\ell} \doteqdot$ '*The Ricci Flow: Techniques and Applications, Part II: Analytic Aspects*' (i.e., this book) by The Ricci Flowers, and let $\Delta R_{ijk\ell} \doteqdot$ '*The Ricci Flow: Techniques and Applications, Part III: Geometric-Analytic Aspects*' (forthcoming) by The Ricci Flowers (we shall refer to this as '**Part III of this volume**' [**136**]).[3] Both g_{ij} and Γ_{ij}^{k} are introductory books, whereas the latter more comprehensive monographs $R_{ijk\ell} \oplus \frac{\partial}{\partial t} R_{ijk\ell} \oplus \Delta R_{ijk\ell}$ are derived from the former. Finally, **Volume Two** refers to the collection of Parts I, II, and III.

This is Part II, the sequel to Part I of this volume. In $R_{ijk\ell}$ we discussed various geometric topics in Ricci flow in more detail, such as Ricci solitons, the Kähler–Ricci flow, the compactness theorem, Perelman's energy and entropy monotonicity and the application to no local collapsing, the reduced distance function and its application to the analysis of ancient solutions, and finally a primer on 3-manifold topology.

[1]We would like to thank Andrejs Treibergs and Junfang Li for suggesting the notation $R_{ijk\ell}$.

[2]'The Ricci Flowers' is an abbreviation for the manifold/variety of authors of this volume.

[3]The originally intended two parts now comprise three parts.

Here, in $\frac{\partial}{\partial t}R_{ijk\ell}$, we discuss mostly analytic topics in Ricci flow including weak and strong maximum principles for scalar heat-type equations and systems on compact and noncompact manifolds, the classification by Böhm and Wilking of closed manifolds with 2-positive curvature operator, Bando's result that solutions to the Ricci flow are real analytic in the space variables, Shi's local derivative of curvature estimates and some variants, and differential Harnack estimates of Li–Yau-type including Hamilton's matrix estimate for the Ricci flow and Perelman's estimate for fundamental solutions of the adjoint heat equation coupled to the Ricci flow. In the appendices we review aspects of Ricci flow and related geometric analysis and tensor calculus on the frame bundle. Various topics in this part also include the works of others as well as the authors.

In Part III of this volume, i.e., in $\Delta R_{ijk\ell}$, we shall discuss aspects of Perelman's theory of ancient κ-solutions, Perelman's pseudolocality theorem, Hamilton's classification of nonsingular solutions, numerical simulations of Ricci flow, stability of the Ricci flow, the linearized Ricci flow, and the space-time formulation of the Ricci flow. In the appendices, for the convenience of the reader, we review and discuss aspects of metric and Riemannian geometry, the reduced distance function and ancient solutions, and limited aspects of Ricci flat metrics on the K3 surface.

As in previous volumes and as is perhaps typical in geometric analysis, throughout this book we apply both the techniques of the 'weak maximum detail principle' and 'exposition by parts'. To wit, we endeavor to supply the reader with as much detail as possible and we also endeavor, for the most part, to make the chapters independent of each other.

As such, $\frac{\partial}{\partial t}R_{ijk\ell}$ (and $\Delta R_{ijk\ell}$ as well) may be used either for a topics course or for self-study, where the lecturer or reader may wish to select portions from this book $\frac{\partial}{\partial t}R_{ijk\ell}$, its predecessors g_{ij}, Γ_{ij}^k, and $R_{ijk\ell}$, and its successor $\Delta R_{ijk\ell}$.

Although the intent of this series of books on the Ricci flow is expository, there is no substitute for reading the original source material in Ricci flow. In particular, the papers of Hamilton and Perelman contain a wealth of original and deep ideas. We encourage interested readers to consult these papers. We also encourage the reader to consult other sources for Ricci flow including Cao and Zhu [**78**], Chen and Zhu [**110**], Ding [**172**], Kleiner and Lott [**303**], Morgan and Tian [**363**], Müller [**371**], Tao [**461**], Topping [**475**], two of the authors [**142**], three of the authors [**146**], The Ricci Flowers [**135**] and [**136**] (Parts I and III of this volume), and sources for geometric evolution equations (e.g., the mean curvature flow) such as Chou and Zhu [**127**], Ecker [**179**], Zhu [**522**], and two of the authors [**139**]. Certain material, originally intended to appear in a successor volume to [**146**], has now been incorporated in Parts I, II, and III of this volume.

Highlights and interdependencies of Part II

Half my life is in books' written pages.
– From "Dream On" by Steven Tyler of Aerosmith

0.1. Highlights. In this book we consider the following mostly analytic topics, described in more detail in the section "Contents of Part II of Volume Two" below.

(1) Proofs are given of the weak maximum principles for scalars and systems on both compact and complete noncompact manifolds. In the noncompact case we have strived to present complete proofs of general results which are readily applicable. A proof is given of the strong maximum principle for systems with an emphasis on the evolution of the curvature operator under the Ricci flow. The application of the maximum principle to the Ricci flow was pioneered by Hamilton.
(2) We present the solution of Böhm and Wilking of the conjecture of Rauch and Hamilton that closed manifolds, in any dimension, with positive curvature operator are diffeomorphic to spherical space forms. Böhm and Wilking prove that the normalized Ricci flow evolves Riemannian metrics on closed manifolds with 2-positive curvature operator to constant positive sectional curvature metrics.[4]
(3) We discuss the following two topics: (i) nonnegative curvature conditions which are not preserved under the Ricci flow and (ii) Bando's result that solutions of the Ricci flow on closed manifolds are real analytic.
(4) We present Shi's local derivative of curvature estimates (including all higher derivatives) based on the Bernstein technique. We also present a refinement, due to one of the authors, where bounds on some higher derivatives of the initial metric are assumed and consequently improved bounds of all higher derivatives are obtained in space and time.
(5) We discuss the differential Harnack estimates of Li–Yau–Hamilton-type—giving a detailed proof of Hamilton's matrix estimate for complete solutions of the Ricci flow with bounded nonnegative curvature operator (this includes the noncompact case). An application is Hamilton's result that eternal solutions are steady gradient Ricci solitons. We also discuss a variant on Hamilton's proof of the matrix Harnack estimate.

[4]Partly based on Böhm and Wilking's work is the recent result of Brendle and Schoen [**48**] proving that positively $\frac{1}{4}$-pinched closed manifolds are diffeomorphic to spherical space forms. Related to this, Brendle and Schoen [**48**] and Nguyen [**376**] independently proved that the condition of positive isotropic curvature is preserved in all dimensions (a result previously known only in dimension 4 by the work of Hamilton).

(6) We present Perelman's differential Harnack estimate for fundamental solutions of the adjoint heat equation coupled to the Ricci flow. This result, of which we give a detailed proof, will be used in the proof of Perelman's pseudolocality discussed in Part III.
(7) We review tensor calculus on the frame bundle—a framework for proving Hamilton's matrix Harnack estimate for the Ricci flow.

The appendices are intended to make this book more self-contained.

0.2. Interdependencies. The chapters are for the most part independent. However, many of the results discussed in this book rely on various forms of the maximum principle.[5] For example we have the following reliances.

(1) Chapter 11 on manifolds with positive curvature operator, Section 1 of Chapter 13 on curvature conditions that are not preserved, and Chapter 15 on the matrix Harnack estimate all require the (time-independent) maximum principles for tensors and systems.
(2) The maximum principles on noncompact manifolds in Chapter 12 require some familiarity with the maximum principles on compact manifolds in Chapter 10.
(3) Chapter 14 on local derivative estimates, Section 2 of Chapter 13 on the real analyticity of solutions, and Chapter 16 on Perelman's differential Harnack estimate all require the maximum principle for scalars.

[5]Perhaps we may say, 'Geometric analysis is simple; just apply the maximum principle!'

Acknowledgments

Now let me welcome everybody to the wild, wild west.

– From "California Love" by Tupac Shakur featuring Dr. Dre

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Contents of Part II of Volume Two

Councillor Hamann: Almost no one comes down here, unless, of course, there's a problem.

– From the movie "The Matrix Reloaded" by the Wachowski brothers.

Chapter 10. In this chapter we discuss the general formulation of the weak maximum principle for systems on closed manifolds, which applies to bilinear forms such as curvature tensors. The maximum principle for scalars may be considered as stating that solutions to a semilinear PDE are bounded by the solutions to the associated ODE obtained by dropping the Laplacian and any gradient terms. In particular for subsolutions/supersolutions to the heat equation, the maximum/minimum is nonincreasing/nondecreasing. This last statement has a generalization to symmetric 2-tensors, considered in Chapter 4 of Volume One, which gives general sufficient conditions to prove that the nonnegativity of tensor supersolutions to heat-type equations is preserved. We had previously applied this to the Ricci tensor and also obtained pinching estimates for the curvatures this way.

To obtain various estimates for the curvatures in Volume One, we found it convenient to employ a more general formulation of the weak maximum principle. We prove this version in this chapter. More precisely we consider sections of vector bundles which satisfy a semilinear heat-type equation. The maximum principle for systems states that if the initial section lies in a subset of the vector bundle which is convex in the fibers and invariant under parallel translation and if the associated ODE obtained by dropping the Laplacian preserves this subset, then the solution to the PDE stays inside this convex set.

The idea of the proof of this maximum principle is as follows. One can prove the maximum principle for functions by considering the spatial maximum function, which is Lipschitz in time, and showing that it is nonincreasing for subsolutions to the heat equation. In the case of the maximum principle for systems, one can look at the function of time which is the maximum distance of the solution to the PDE from the subset. Using the support functions to the convex fibers, one can show that this maximum distance function $s(t)$, which is again Lipschitz in time, satisfies an ODE of the form $ds/dt \leq Cs$. Since $s(0) = 0$, we conclude that $s(t) \equiv 0$ and the maximum principle for systems follows.

Refinements of the maximum principle include the case when the subsets with convex fibers are time-dependent and also when there is a so-called avoidance set for the solutions of the PDE. In terms of applications, one of the most important special cases is when the sections of the bundle are bilinear forms. We discuss this case and applications to the curvature tensors. We also discuss the Aleksandrov–Bakelman–Pucci maximum principle for elliptic equations.

Chapter 11. In this chapter we present Böhm and Wilking's solution to the conjecture of Rauch and Hamilton on the classification of closed Riemannian manifolds with positive curvature operator.[1] The flavor of this chapter is more algebraic with an essential component of the proof being the irreducible decomposition of (algebraic) curvature tensors. Generally, one of the ideas is to study when linear transformations of convex preserved sets (with respect to the ODE corresponding to the PDE satisfied by Rm) remain preserved. More specifically, via a 1-parameter family of linear transformations the cone of 2-nonnegative curvature operators is mapped into the cone of nonnegative curvature operators. This reduces the classification problem for Riemannian manifolds with 2-positive curvature operator to that for manifolds with positive curvature operator. To study the latter problem, one would hope that the cones of 2-positive curvature operators with arbitrary Ricci pinching are preserved. Unfortunately there does not seem to be any known way to prove this. Instead, one can prove that suitable linear transformations of the cones of 2-positive curvature operators with arbitrary Ricci pinching are preserved. This is sufficient to prove the Rauch–Hamilton conjecture.

Chapter 12. This chapter comprises two main topics: (1) weak maximum principles on noncompact manifolds and (2) the strong maximum principle, which is a local result. In both cases we consider scalar parabolic equations and systems of parabolic equations.

Since singularity models are often noncompact, it is important to be able to apply the weak maximum principle on complete, noncompact manifolds. We begin with the heat equation and present the weak maximum principle of Karp and Li which applies to solutions with growth slower than exponential quadratic in distance. Using barrier functions, we then give a weak maximum principle for bounded solutions of heat-type systems. As a special case, we show that complete solutions to the Ricci flow on noncompact manifolds, with nonnegative curvature operator initially and bounded curvature on space and time, have nonnegative curvature operator for all time.

We also discuss mollifiers on Riemannian manifolds with a lower bound on the injectivity radius. One technical issue we discuss, using the mollifiers obtained above, is the construction of distance-like functions with bounds on

[1] They actually obtain the stronger result of classifying closed manifolds with 2-positive curvature operator.

their gradients and upper bounds of their Hessians on complete Riemannian manifolds with bounded curvature. These distance-like functions are used to construct the barrier functions referred to in the previous paragraph.

Moreover, this construction carries over to the case of the Ricci flow. The aforementioned mollifiers are not only applicable to the proof of the compactness theorem in regards to the center of mass (discussed in Chapter 4 of Part I of this volume), but also to constructing a barrier function used in the proof of Hamilton's matrix Harnack estimate in Chapter 15 on complete noncompact Riemannian manifolds with bounded nonnegative curvature operator.

A fundamental property of the heat equation is that a solution which is initially nonnegative immediately becomes either positive or identically zero. This property is known as the strong maximum principle. For a solution to the Ricci flow, the curvature tensor satisfies a heat-type equation. The strong maximum principle for systems, due to Hamilton based on earlier work of Weinberger and others, tells us that for solutions to the Ricci flow with nonnegative curvature operator (such as singularity models in dimension 3) the curvature operator has a special form after the initial time. In particular, the image of the curvature operator is independent of time and invariant under parallel translation in space. Moreover, using the natural Lie algebra structure on the fibers Λ_x^2, $x \in \mathcal{M}$, of the bundle of 2-forms, the image of the curvature operator is a Lie subalgebra. It is useful to observe that the strong maximum principle is a local result and does not require the solution to be complete or to have bounded curvature. We also formulate the strong maximum principle in the more general setting of Chapter 10, i.e., for sections of a vector bundle which solve a PDE.

In dimension 3, Λ_x^2 is isomorphic to $\mathfrak{so}(3)$ and its only proper nontrivial Lie subalgebras are isomorphic to $\mathfrak{so}(2)$, which is 1-dimensional. Hence, after the initial time, a solution to the Ricci flow on a 3-manifold with nonnegative sectional curvature either is flat, has positive sectional curvature, or admits a global parallel 2-form. In the last case, by taking the dual of this 2-form, we have a parallel 1-form and the solution splits locally as the product of a surface solution and a line. This classification is useful in studying the singularities which arise in dimension 3.

Chapter 13. In this chapter we discuss the following two topics in Ricci flow.

(1) Curvature conditions that are *not* preserved. The weak positivity (i.e., nonnegativity) of the following curvatures are preserved: scalar curvature, Ricci curvature in dimension 3, Riemann curvature operator, isotropic curvature, complex sectional curvature, as well as the 2-nonnegativity of the curvature operator. On the other hand, in this chapter we discuss some nonnegativity conditions which are not preserved under the Ricci flow. In particular, we consider the conditions of nonnegative Ricci curvature and nonnegative sectional curvature in dimensions at least 4 for solutions of the Ricci flow on both closed and noncompact manifolds.

(2) Bando's result that solutions to the Ricci flow on closed manifolds are real analytic in the space variables for positive time. The proof of this is based on keeping track of the constants in the higher derivatives of curvature estimates and summing these estimates.

Chapter 14. In Chapter 7 of Volume One we encountered the global derivative of curvature estimates. The idea is to assume a curvature bound K for the Riemann curvature tensor and, by applying the weak maximum principle to the appropriate quantities, obtain bounds for the m-th derivatives of the curvatures of the form $|\nabla^m \operatorname{Rm}| \leq CKt^{-m/2}$. In this chapter, we present Shi's local derivative of curvature estimates. The idea of localizing the derivative estimates is simply to multiply the quantities considered by a cutoff function.

For the global first derivative of curvature estimate we previously considered $t\left|\nabla \operatorname{Rm}\right|^2 + C\left|\operatorname{Rm}\right|^2$. This quantity does not seem to adapt well to localization. Instead we consider $\eta t\left(16K^2 + |\operatorname{Rm}|^2\right)\left|\nabla \operatorname{Rm}\right|^2$, where η is a cutoff function. The local *first* derivative estimate we prove says that if a solution is defined on a ball of radius r and time interval $[0, \tau]$ and if it has curvatures bounded by K, then we have

$$|\nabla \operatorname{Rm}| \leq CK\left(\frac{1}{r^2} + \frac{1}{\tau} + K\right)^{1/2}$$

on the concentric ball of radius $r/2$ on the time interval $[\tau/2, \tau]$.

Local *higher* derivative estimates are proven using a similar idea. For example, to bound the second derivatives, one applies the weak maximum principle to the quantity $\left(4A^2 + t\left|\nabla \operatorname{Rm}\right|^2\right) t^2 \left|\nabla^2 \operatorname{Rm}\right|^2$, where the constant A is chosen appropriately. We also discuss a version of the local derivative estimates where bounds on the higher derivatives of the curvatures of the initial metric are assumed up to some order. In this case we obtain improved bounds for all higher derivatives of the curvatures. This result is useful in an approach toward constructing Perelman's standard solution.

For complete solutions with bounded nonnegative curvature operator, the local derivative estimates, when combined with Hamilton's trace Harnack estimate, yield instantaneous local derivative bounds. Previously, in Chapters 7 and 8 of Part I of this volume, we saw applications of the local derivative estimates to the study of the reduced distance and the reduced volume.

We also briefly discuss D. Yang's local Ricci flow. Here the velocity $-2\operatorname{Rc}$ of the metric is multiplied by a nonnegative weight function with compact support. The local Ricci flow provides another approach to Shi's short time existence theorem for the Ricci flow on noncompact manifolds.

Chapter 15. This chapter discusses differential Harnack inequalities of Li–Yau–Hamilton-type for the Ricci flow. These gradient-type estimates, which are directly motivated by considering quantities which vanish on the (gradient) Ricci solitons as discussed in Chapter 1 of Part I of this volume, provide useful bounds for the solution. The general form of the main estimate, which holds for complete solutions with bounded nonnegative curvature operator and which is known as Hamilton's matrix Harnack estimate, says that a certain tensor involving two and fewer derivatives of the curvature is nonnegative definite.

The trace Harnack estimate, as the name suggests, is obtained by tracing the matrix inequality. This trace inequality has several important consequences, including the fact that scalar curvature does not decrease too fast in the sense that for a fixed point, t times the scalar curvature is a nondecreasing function of time. More generally, the trace inequality yields a lower bound for the scalar curvature at a point and time, in terms of the scalar curvature at any other point and earlier time, and the distance between the points and the time difference.

We begin the proof of the Harnack estimate with the case of surfaces (dimension 2), in which case the evolution equation for the Harnack quadratic simplifies (in comparison to higher dimensions) and one can prove the trace inequality directly. This is unlike the situation in higher dimensions, where the trace inequality apparently can only be demonstrated by proving the matrix inequality and then tracing.

In all dimensions, we recall the terms in the matrix Harnack quadratic obtained in Chapter 1 (p. 9) of Part I of this volume by differentiating the expanding gradient Ricci soliton equation. The Harnack calculations simplify when one uses the formalism, given in Appendix F, of considering tensors as vector-valued functions on the frame bundle. Using the above formalism, we present the evolution of Harnack calculations which are long but relatively straightforward. The evolution of the Harnack quantity looks formally similar to the evolution of the Riemann curvature operator and as such is amenable to the application of the weak maximum principle when the solution has nonnegative curvature operator. When the manifold is noncompact, the techniques used to enable this application are reminiscent of the techniques used to prove the maximum principle for functions.

We also give a variant on Hamilton's proof of the matrix Harnack estimate, based on reducing the problem to showing that a symmetric 2-tensor is nonnegative definite.

Chapter 16. In this chapter we give a proof of Perelman's differential Harnack-type inequality for solutions of the adjoint heat equation coupled to the Ricci flow. We begin by considering entropy and differential Harnack estimates for the heat equation. Our approach for proving Perelman's differential Harnack-type inequality is to first prove gradient estimates for positive solutions of the (adjoint) heat equation. Using this and heat kernel estimates, we give a proof that for solutions to the adjoint heat equation coupled to the Ricci flow, Perelman's Harnack quantity (or, geometrically, the modified scalar curvature) is nonpositive:

$$\tau \left(R + 2\Delta f - |\nabla f|^2 \right) + f - n \leq 0.$$

Appendix D. In this appendix we review some basic results for the Ricci flow. In particular, we recall the results on the short time and long time existence and uniqueness of the Ricci flow on closed and noncompact manifolds, convergence results on closed manifolds assuming some sort of positivity of curvature, the rotationally symmetric neckpinch, curvature pinching estimates, strong maximum principle, derivative estimates, differential Harnack estimates, Perelman's energy and entropy monotonicity and no local collapsing, compactness theorems, and the existence of singularity models.

Appendix E. In this appendix we review some basic geometric analysis related to the Ricci flow with an emphasis on the heat equation. We recall Duhamel's principle and its application to basic results for the heat kernel. We discuss the Cheeger–Yau comparison theorem for the heat kernel, the Li–Yau differential Harnack estimate, and Hamilton's gradient estimates.

Appendix F. The material in this appendix is in preparation for Chapter 15 on Hamilton's matrix Harnack estimate. Given a Riemannian manifold, we describe a formalism for considering tensors as vector-valued functions on the (orthonormal) frame bundle. In the context of the Ricci flow, where we have a 1-parameter family of metrics, we add to this a modified time derivative which may be considered as a version of Uhlenbeck's trick. We discuss tensor calculus in this setting, including commutator formulas for the heat operator and covariant derivatives. These calculations are used for the Harnack calculations in Chapter 15.

Notation and Symbols

How many times must I explain myself before I can talk to the boss?

– From "Forever Man" by Eric Clapton

The following is a list of some of the notation and symbols which we use in this book.

∇	covariant derivative
$\doteqdot$	defined to be equal to
$\cdot$	dot product or multiplication
$\nabla\nabla f$	Hessian of f
$\#$	sharp operator
$\otimes_S$	symmetric tensor product
$\alpha^\natural$	dual vector field to the 1-form α
Area	area of a surface or volume of a hypersurface
ASCR	asymptotic scalar curvature ratio
AVR	asymptotic volume ratio
$B(p,r)$	ball of radius r centered at p
b	Bianchi map
B_{abcd}	the quadratic $-R_{apbq}R_{cpdq}$
bounded curvature	bounded *sectional* curvature
$C_V\mathcal{J}$	tangent cone at V of a convex set $\mathcal{J}\subset\mathbb{R}^k$
const	constant
$\frac{d^+}{dt}$, $\frac{d^-}{dt}$, $\frac{d_+}{dt}$, $\frac{d_-}{dt}$	a Dini time-derivative
d	distance
$d\mu$	volume form
$d\sigma$	volume form on boundary or hypersurface
Δ, Δ_L, Δ_d	Laplacian, Lichnerowicz Laplacian, Hodge Laplacian

div	divergence
$\mathbb{E}^n$	$\mathbb{R}^n$ with the flat Euclidean metric
Γ_{ij}^k	Christoffel symbols
$g(X,Y) = \langle X, Y \rangle$	metric or inner product
$g(t)$	time-dependent metric, e.g., solution of the Ricci flow
h	second fundamental form
H	mean curvature
$H_V \mathcal{J}$ for $V \in \partial \mathcal{J}$	set of closed half-spaces H containing $\mathcal{J} \subset \mathbb{R}^k$ with $V \in \partial H$
$\operatorname{Hess} f$	Hessian of f (same as $\nabla\nabla f$)
id	identity
IVP	initial-value problem
L	length
LHS	left-hand side
log	natural logarithm
$\mathcal{I}$	a time interval for the Ricci flow
$\mathcal{J}$	a time interval for the backward Ricci flow
ℓ	reduced distance or ℓ-function
$\mathcal{L}$	Lie derivative or $\mathcal{L}$-length
MCF	mean curvature flow
$\mathfrak{Met}$	space of Riemannian metrics on a manifold
MVP	mean value property
$\times$	multiplication, when a formula does not fit on one line
$n\omega_n$	volume of the unit Euclidean $(n-1)$-sphere
NRF	normalized Ricci flow
ν	unit outward normal
ω_n	volume of the unit Euclidean n-ball
ODE	ordinary differential equation
PDE	partial differential equation
PIC	positive isotropic curvature
RF	Ricci flow
RHS	right-hand side
R, Rc, Rm	scalar, Ricci, and Riemann curvature tensors
$\mathrm{Rm}^{\#}$	the quadratic $\mathrm{Rm} \# \mathrm{Rm}$
$\mathbf{R}$	algebraic curvature operator
$\mathrm{Rc}(\mathbf{R})$	a trace of $\mathbf{R}$ (of two indices)

Rc_0	trace-free part of Rc
$S_B^2(\mathfrak{so}(n))$	space of algebraic curvature operators
$S\Omega_T$	side boundary $\partial\Omega \times (0, T]$
$S_V \mathcal{J}$ for $V \in \partial\mathcal{J}$	set of support functions of $\mathcal{J} \subset \mathbb{R}^k$ at V
$\mathrm{Scal}(\mathbf{R})$ or $\mathrm{trace}(\mathbf{R})$	the full trace of $\mathbf{R}$
$T_x\mathcal{M}$	tangent space of $\mathcal{M}$ at x
$T_x^*\mathcal{M}$	cotangent space of $\mathcal{M}$ at x
tr or trace	trace
tri	trilinear form on $S^2\Lambda^2 T_x\mathcal{M}$
$\mathcal{V}$	vector bundle
Vol	volume of a manifold
$W^{k,p}$	Sobolev space of functions with $\leq k$ weak derivatives in L^p
$W_{\mathrm{loc}}^{k,p}$	space of functions locally in $W^{k,p}$
WMP	weak maximum principle

APPENDIX D

An Overview of Aspects of Ricci Flow

Well you wore out your welcome with random precision.
– From "Shine On You Crazy Diamond" by Pink Floyd

In this review appendix we recall some essential results, mostly contained in the earlier volumes: Volume One [**142**], [**146**], and Part I of this volume [**135**]. The results in this appendix form only a selection of the facts we feel are among the most relevant to the discussions in this volume on Ricci flow.

1. Existence, uniqueness, convergence, and curvature evolution

1.1. Short time and long time existence and uniqueness. The basic result enabling the Ricci flow to be used to study the geometry and topology of closed manifolds is the following result of Hamilton [**244**], whose proof was simplified by DeTurck in [**169**] and [**170**] (see also Chapter 3 of Volume One for an exposition).

THEOREM D.1 (Short time existence—closed manifolds). *If $(\mathcal{M}^n, g_0)$ is a smooth closed Riemannian manifold, then there exists a unique smooth solution $g(t)$, $t \in [0,T)$, of the Ricci flow with $g(0) = g_0$.*

We also have the following short time existence result of W.-X. Shi [**444**] on noncompact manifolds.

THEOREM D.2 (Short time existence—complete with bounded curvature). *If $(\mathcal{M}^n, g_0)$ is a complete noncompact Riemannian manifold with bounded sectional curvature, then there exists a complete solution $g(t)$, $t \in [0,T)$, of the Ricci flow with $g(0) = g_0$ and sectional curvature bounded on compact time intervals.*

Uniqueness of solutions to the Ricci flow on noncompact manifolds in the class of complete solutions with bounded sectional curvature is discussed in Chen and Zhu [**111**] (see also Hsu [**277**]).

THEOREM D.3 (Uniqueness—complete with bounded curvature). *Let $\mathcal{M}^n$ be a noncompact manifold. If $g_1(t)$, $t \in [0,T_1)$, and $g_2(t)$, $t \in [0,T_2)$, are complete solutions with bounded sectional curvatures and if $g_1(0) = g_2(0)$, then $g_1(t) = g_2(t)$ for all $t \in [0, \min\{T_1, T_2\})$.*

Solutions to the Ricci flow may be continued as long as the curvature remains bounded (see Theorem 6.45 on p. 201 of Volume One).

THEOREM D.4 (Long time existence—closed manifolds). *If g_0 is a smooth Riemannian metric on a closed manifold $\mathcal{M}^n$, then the Ricci flow with $g(0) = g_0$ has a unique solution $g(t)$ on a maximal time interval $0 \leq t < T \leq \infty$. Moreover, if $T < \infty$, then*

$$\lim_{t \nearrow T} \left(\sup_{x \in \mathcal{M}} |\mathrm{Rm}(x,t)| \right) = \infty.$$

In view of the uniqueness Theorem D.3 and Shi's local derivative estimates (Theorem 14.14), we have the following.

THEOREM D.5 (Long time existence—complete with bounded curvature). *If g_0 is a complete Riemannian metric on a noncompact manifold $\mathcal{M}^n$ with bounded sectional curvature, then, in the class of complete solutions with bounded sectional curvatures, the Ricci flow with $g(0) = g_0$ has a unique solution $g(t)$ on a maximal time interval $0 \leq t < T \leq \infty$. Moreover, if $T < \infty$, then $\lim_{t \nearrow T} (\sup_{x \in \mathcal{M}} |\mathrm{Rm}(x,t)|) = \infty$.*

For finite time singular solutions, Theorem D.4 can be improved to show that the Ricci tensor is unbounded (see Šešum [**435**] or, for an exposition, Theorem 6.41 on p. 240 of [**146**]).

THEOREM D.6 (Long time existence improved). *Under the same hypotheses as Theorem D.4, we have if $T < \infty$, then*

$$\lim_{t \nearrow T} \left(\sup_{x \in \mathcal{M}} |\mathrm{Rc}(x,t)| \right) = \infty.$$

1.2. Convergence of solutions. The seminal result in Ricci flow is Hamilton's classification of closed 3-manifolds with positive Ricci curvature (see also Theorem 6.3 on p. 173 of Volume One and Theorem 2.1 on p. 96 of [**146**]).

THEOREM D.7 (Hamilton, 3-manifolds with positive Ricci curvature). *Let $(\mathcal{M}^3, g_0)$ be a closed Riemannian 3-manifold with positive Ricci curvature. Then there exists a unique solution $g(t)$ of the normalized Ricci flow with $g(0) = g_0$ for all $t \geq 0$. Furthermore, as $t \to \infty$, the metrics $g(t)$ converge exponentially fast in every C^m-norm to a C^∞ metric g_∞ with constant positive sectional curvature.*

The theorem above gives a useful criterion for when a closed 3-manifold is diffeomorphic to a spherical space form, namely it admits a metric with positive Ricci curvature.

REMARK D.8 (Ricci flow on closed surfaces). Global existence and convergence to a constant curvature metric is known for the normalized Ricci flow on closed surfaces (see Theorem 5.1 on p. 105 in Volume One).

In dimensions at least 3 we have the following result of Böhm and Wilking [**43**] generalizing Theorem D.7 (see Theorem 11.2 in this volume for an exposition).

THEOREM D.9 (Closed manifolds with 2-positive curvature operator). *If $(\mathcal{M}^n, g_0)$, where $n \geq 3$, is a closed Riemannian manifold with 2-positive curvature operator, then there exists a unique solution $(\mathcal{M}^n, g(t))$, $t \in [0, \infty)$, to the volume normalized Ricci flow with $g(0) = g_0$. This solution converges exponentially fast in each C^k-norm as $t \to \infty$ to a smooth constant positive sectional curvature metric.*

When $n = 3$, the condition of 2-positive curvature operator is the same as positive Ricci curvature, so this result is the same as Theorem D.7. Earlier, Hamilton [**245**] proved this result under the stronger hypothesis of positive curvature operator for $n = 4$ and then H. Chen [**112**] proved this result for $n = 4$. In the case of initial Riemannian metrics on closed manifolds with sufficiently pointwise-pinched positive sectional curvatures, convergence of the Ricci flow was proved independently by Huisken, Margerin, and Nishikawa (see §1 of Chapter 7 in [**146**] for an exposition of some aspects of their proofs).

Recently Brendle and Schoen [**48**] proved the following long-standing conjecture.

THEOREM D.10 (Pointwise 1/4-pinched manifolds are space forms). *If $(\mathcal{M}^n, g)$ is a closed Riemannian manifold with $K_{\min}(x) > \frac{1}{4} K_{\max}(x) > 0$ for all $x \in \mathcal{M}$, then $\mathcal{M}$ admits a metric with constant positive sectional curvature, i.e., $\mathcal{M}$ is diffeomorphic to a spherical space form.*

This result is a beautiful generalization of the Rauch–Klingenberg–Berger topological sphere theorem, the Micallef–Moore pointwise $\frac{1}{4}$-pinched topological sphere theorem, the differential sphere theorems of Gromoll, Calabi, Sugimoto and Shiohama, Karcher, and Ruh,[1] and the pointwise pinched differentiable sphere theorem of Huisken, Margerin, and Nishikawa.

The proof of Theorem D.10 is based on combining a new 'curvature cone preservation' result with the work of Böhm and Wilking. In particular Brendle and Schoen show that weakly positive isotropic curvature is preserved in all dimensions $n \geq 4$. This result was also independently proved by Nguyen [**376**] (see also Andrews and Nguyen [**9**] for related work).

1.3. Evolution of the Riemann curvature tensor. With respect to Uhlenbeck's trick, the evolution of the Riemann curvature tensor is given by

$$\frac{\partial}{\partial t} R_{abcd} = \Delta R_{abcd} + 2\left(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}\right),$$

where $B_{abcd} = -R_{aebf} R_{cedf}$ (see Section 2 of Chapter 6 of Volume One). The evolution of the Ricci tensor is

$$\frac{\partial}{\partial t} R_{ab} = \Delta R_{ab} + 2 R_{acdb} R_{cd}.$$

[1] See [**48**] for these references.

The facts in this paragraph are derived on pp. 185–186 of Volume One. Consider the Lie algebra $\mathfrak{g} \doteqdot \left(\Lambda^2 T_x^* \mathcal{M}^n, [\cdot,\cdot]\right)$, where the bracket is defined by

$$[U,V]_{ij} \doteqdot g^{k\ell}\left(U_{ik}V_{\ell j} - V_{ik}U_{\ell j}\right), \tag{D.1}$$

which is isomorphic to $\mathfrak{so}(n)$. Given local coordinates $\{x^i\}$, define the basis $\{\varphi^{(ij)} : 1 \leq i < j \leq n\}$ for $\mathfrak{g}$ by

$$\varphi^{(ij)} \doteqdot dx^i \wedge dx^j.$$

The structure constants $C_{(ij)}^{(pq),(rs)}$ of $\mathfrak{g}$ are defined by

$$[dx^p \wedge dx^q, dx^r \wedge dx^s] = \sum_{(ij)} C_{(ij)}^{(pq),(rs)} dx^i \wedge dx^j. \tag{D.2}$$

We have

$$\begin{aligned} C_{(ij)}^{(pq),(rs)} &= [dx^p \wedge dx^q, dx^r \wedge dx^s]_{ij} \\ &= \frac{1}{4}\left\{\begin{array}{l} g^{qr}\left(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p\right) + g^{qs}\left(\delta_i^r\delta_j^p - \delta_i^p\delta_j^r\right) \\ +g^{pr}\left(\delta_i^s\delta_j^q - \delta_i^q\delta_j^s\right) + g^{ps}\left(\delta_i^q\delta_j^r - \delta_i^r\delta_j^q\right) \end{array}\right\}. \end{aligned} \tag{D.3}$$

The components of $\operatorname{Rm}^{\#}$ are given by

$$\left(\operatorname{Rm}^{\#}\right)_{ijk\ell} = R_{pquv}R_{rswx}C_{(ij)}^{(pq),(rs)}C_{(\ell k)}^{(uv),(wx)} \tag{D.4}$$

$$= 2\left(B_{ikj\ell} - B_{i\ell jk}\right). \tag{D.5}$$

Using the first Bianchi identity, we have

$$\left(\operatorname{Rm}^2\right)_{ijk\ell} = 2\left(B_{ijk\ell} - B_{ij\ell k}\right).$$

Thus

$$\frac{\partial}{\partial t}\operatorname{Rm} = \Delta \operatorname{Rm} + \operatorname{Rm}^2 + \operatorname{Rm}^{\#}.$$

2. The rotationally symmetric neckpinch

The references for this section are the exposition in Section 5 of Chapter 2 in Volume One and the original papers [**10, 11**] by Angenent and one of the authors. Remove two points ('poles') $P_\pm$ from $\mathcal{S}^n$ and identify $\mathcal{S}^n \backslash \{P_\pm\}$ with $(-1,1) \times \mathcal{S}^{n-1}$. Consider a rotationally symmetric (i.e., $\mathrm{SO}(n)$-invariant) metric of the form

$$g = h(r)^2 dr^2 + f(r)^2 g_{\mathcal{S}^{n-1}}. \tag{D.6}$$

Parameterize by arc length s from a chosen origin, so that $ds = h(r)\,dr$. Then by equations (2.49)–(2.50) on p. 41 of Volume One, the sectional curvatures of g are

$$K_{\mathrm{rad}} = -\frac{f''}{f} \quad \text{and} \quad K_{\mathrm{sph}} = \frac{1-(f')^2}{f^2}, \tag{D.7}$$

where a prime denotes differentiation with respect to s. By equation (2.51) of Volume One, the Ricci tensor of (D.6) is

$$\operatorname{Rc} = -(n-1)\frac{f''}{f}\,ds^2 + \left((n-2)\frac{1-(f')^2}{f^2} - \frac{f''}{f}\right) f(r)^2 g_{\mathcal{S}^{n-1}}.$$

Now let (D.6) evolve by Ricci flow, $\frac{\partial}{\partial t} g = -2\operatorname{Rc}$. Regarding f and h as functions of $s(r,t)$ and t, one finds that g evolves by Ricci flow if and only if

(D.8a) $$\frac{\partial h}{\partial t} = (n-1)\frac{f''}{f}h,$$

(D.8b) $$\frac{\partial f}{\partial t} = f'' - (n-2)\frac{1-(f')^2}{f}.$$

(See equations (2.46)–(2.47) in Volume One.) Note that smoothness at the poles is equivalent to the boundary condition $\lim_{r\to\pm1} f' = \mp 1$, together with the fact that $(s_\pm - s)^{-1} f$ is a smooth, even function of $s_\pm - s$, where $s_\pm$ denotes the distance from the chosen origin to the poles $r = \pm 1$. Note too that parameterizing by arc length has the effect of fixing a gauge, thereby making system (D.8) strictly parabolic; one pays for this with the commutator

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = (n-1)K_{\mathrm{rad}}\frac{\partial}{\partial s}.$$

We now describe a set of initial data that will pinch in finite time. We call local minima of f *necks* and local maxima *bumps*. We call the region between either pole and its closest bump a *polar cap*. Consider the following assumptions.

(1) The metric has at least one neck that is 'sufficiently pinched'. To wit, the value of f at the smallest neck is sufficiently small relative to its value at either adjacent bump.
(2) The scalar curvature $R = 2(n-1)K_{\mathrm{rad}} + (n-1)(n-2)K_{\mathrm{sph}}$ is positive everywhere.
(3) The sectional curvature K_{sph} tangential to each sphere $\{r\} \times \mathcal{S}^{n-1}$ is positive.
(4) The Ricci curvature is positive on each polar cap.

Some results below also use the following additional hypothesis.

(5) The metric is reflection symmetric, i.e., $f(r) = f(-r)$, and the smallest neck occurs at the origin $r = 0$.

Note that assumption (3) is equivalent to the gradient bound $|f'| \le 1$ and hence is easily achieved by scaling h. Note too that assumption (5) implies that the antipodal map is an isometry, so that the metric descends to a solution on $\mathbb{RP}^n$.

The following result of Angenent and one of the authors yields the existence of neckpinches under assumptions (1)–(4). (See Theorem 2.16 of

Volume One.) Theorem D.12 below shows that the stated estimates are sharp.

THEOREM D.11 (Existence of rotationally symmetric neckpinch). *Assumptions* (1)–(4) *above define an open set of initial data* g_0 *in the space of rotationally symmetric metrics on* $\mathcal{S}^n$ *such that for each metric* g_0 *in this open set, there exists a unique rotationally symmetric solution* $g(t)$ *of Ricci flow on* $\mathcal{S}^n$ *with* $g(0) = g_0$. *This solution forms a Type I singularity at a finite time* $T = T(g_0)$. *Let* $\bar{s}(t)$ *denote the location of the smallest neck at time* t. *Then there exist* $C < \infty$ *and* $\delta > 0$ *such that the following estimates hold for the cylindrical radius function* f.

(1) (Interior cylinder estimate) *In the innermost region*

$$|s - \bar{s}| \leq 2\sqrt{(T-t)\log\left(\frac{1}{T-t}\right)}, \tag{D.9}$$

we have

$$1 + o(1) \leq \frac{f(r,t)}{\sqrt{2(n-2)(T-t)}} \leq 1 + C\frac{(s(r,t) - \bar{s}(t))^2}{(T-t)\log\left(\frac{1}{T-t}\right)} \tag{D.10}$$

uniformly as $t \nearrow T$.

In particular, for $s - \bar{s} = o\left(\sqrt{(T-t)\log\left(\frac{1}{T-t}\right)}\right)$, *we have*

$$\frac{f(r,t)}{\sqrt{2(n-2)(T-t)}} = 1 + o(1) \quad \text{as } t \nearrow T. \tag{D.11}$$

(2) (Intermediate cylinder estimate) *In the intermediate annular regions*

$$2\sqrt{(T-t)\log\left(\frac{1}{T-t}\right)} \leq |s - \bar{s}| \leq (T-t)^{\frac{1-\delta}{2}}, \tag{D.12}$$

we have

$$\frac{f(r,t)}{\sqrt{T-t}} \leq C\frac{s(r,t) - \bar{s}(t)}{\sqrt{(T-t)\log\left(\frac{1}{T-t}\right)}}\sqrt{\log\left(\frac{s(r,t)-\bar{s}(t)}{\sqrt{(T-t)\log\left(\frac{1}{T-t}\right)}}\right)}. \tag{D.13}$$

If one adds assumption (5), then one can derive stronger estimates. (See [**11**].) In this case, it is convenient to take $r = 0$ as the origin, so that $s(r,t) = \int_0^r h(\rho)\,d\rho$ and $\bar{s}(t) \equiv 0$. Let $T < \infty$ denote the singularity time, and let

$$u = \frac{f}{\sqrt{2(n-2)(T-t)}}$$

denote the radius of the neck rescaled by the radius of the shrinking round cylinder singularity model. Let

$$\sigma = \frac{s}{\sqrt{T-t}}$$

denote the rescaled distance to the neck, and let

$$\tau \doteqdot \log \frac{1}{T-t}$$

denote rescaled time. Note that $\tau \to \infty$ as $t \to T_-$.

THEOREM D.12 (Asymptotics of rotationally symmetric neckpinch). *Under assumptions* (1)–(5), *the singularity time T is a continuous function of the initial datum g_0. The diameter remains bounded for all $t \in [0, T)$, and the singularity occurs only on the hypersurface $\{0\} \times S^{n-1}$. As the singularity develops, the solution exhibits the following asymptotic profile.*

Inner region: *on any interval $|\sigma| \leq A$, one has*

$$u(\sigma, \tau) = 1 + \frac{\sigma^2 - 2}{8\tau} + o\left(\frac{1}{\tau}\right) \qquad \text{uniformly as } \tau \to \infty. \tag{D.14}$$

Intermediate region: *on any interval $A \leq |\sigma| \leq B\sqrt{\tau}$, one has*

$$u(\sigma, \tau) = \sqrt{1 + (1 + o(1))\frac{\sigma^2}{4\tau}} \qquad \text{uniformly as } \tau \to \infty. \tag{D.15}$$

Outer region: *for any $\varepsilon > 0$, there exist $C > 1$ and $\bar{t} < T$ such that*

(D.16)
$$\left(\frac{1}{2}\sqrt{n-2} - \varepsilon\right) \frac{s}{\sqrt{\log(1/s)}} \leq f(r,t) \leq \left(\frac{1}{2}\sqrt{n-2} + \varepsilon\right) \frac{s}{\sqrt{\log(1/s)}}$$

for all points near the neck such that $|\sigma| \geq C\sqrt{\tau}$ and $t \in (\bar{t}, T)$.

REMARK D.13. Under assumptions (1)–(5), the solution $g(t)$ descends to a solution on $\mathbb{RP}^n$, where the singularity forms along an $\mathbb{RP}^{n-1}$.

EXERCISE D.14. Show that for any $\varepsilon > 0$, there is a $t_\varepsilon < T$ such that there exists an embedded ε-neck[2] in $(\mathcal{S}^n, g(t))$ for $t \in (t_\varepsilon, T)$.

REMARK D.15. Roughly speaking, if $L(t) = o\left(\sqrt{\log(\frac{1}{T-t})}\right)$ for t near T, then for every $\varepsilon > 0$ there is a $t_\varepsilon < T$ such that there exists an embedded ε-neck with normalized length $L(t)$ in $(\mathcal{M}^n, g(t))$ for $t \in (t_\varepsilon, T)$.

For the convenience of the reader, we recall some estimates that are useful in the proof of Theorem D.11. See Propositions 2.20, 2.30, and 2.36 in Volume One. (Theorem D.12 requires weighted L^2 estimates; see [**11**] for a complete proof.)

PROPOSITION D.16 (Estimates for neckpinches). *Let $g(t)$ be a solution to Ricci flow of the form* (D.6). *If $g(0) = g_0$ satisfies assumptions* (1)–(4), *then the following hold for as long as the solution exists.*

[2]See Hamilton [**255**] or Part III of this volume for the definition of 'ε-neck'.

(1) *The gradient bound*

$$|f'| \leq 1 \tag{D.17}$$

is preserved.

(2) *The scale-invariant difference of the sectional curvatures*

$$a = f^2(K_{\mathrm{sph}} - K_{\mathrm{rad}})$$

is uniformly bounded:

$$\sup |a(\cdot, t)| \leq A \doteqdot \sup |a(\cdot, 0)|. \tag{D.18}$$

(3) *The full curvature is controlled by the radius of the $\mathcal{S}^{n-1}$ factor; that is, there exists $C = C(n, g_0)$ such that*

$$|\operatorname{Rm}| \leq \frac{C}{f^2}. \tag{D.19}$$

(4) *The radius f is strictly decreasing pointwise, and f^2 is uniformly Lipschitz in time, satisfying*

$$\left| \frac{\partial}{\partial t}(f^2) \right| \leq 2(A + n - 1). \tag{D.20}$$

(5) *The radius $f_{\min}(t)$ of the smallest neck satisfies*

$$(n-2)(T-t) \leq f_{\min}(t)^2 \leq 2(n-2)(T-t). \tag{D.21}$$

(6) *The singularity is of Type I. That is, there exists $C = C(n, g_0)$ such that*

$$|\operatorname{Rm}| \leq \frac{C}{T-t}. \tag{D.22}$$

(7) *The solution has the Sturmian property. That is, the number of zeroes of f' is finite and nonincreasing in time. If f ever has a degenerate critical point, i.e., a point such that $f' = f'' = 0$, then the number of zeroes of f' decreases.*

(8) *The solution remains concave on the polar caps, i.e., $K_{\mathrm{rad}} > 0$ there. If $\hat{r}(t)$ denotes the location of the right-most (left-most) bump, then the limit $D = \lim_{t \nearrow T} f(\hat{r}(t), t)$ exists. Also, if $D > 0$, then no singularity occurs on the polar cap.*

(9) *Let $c = 2 - \log(\inf(K_{\mathrm{sph}}(\cdot, 0)))$. The quantity*

$$\hat{F} = -\frac{K_{\mathrm{rad}}}{K_{\mathrm{sph}}} \log(K_{\mathrm{sph}} + c) \tag{D.23}$$

is bounded from above by $\max\{n-2, \sup \hat{F}(\cdot, 0)\}$.

3. Curvature pinching, derivative, and Harnack estimates

3.1. Curvature pinching estimates. The curvature pinching estimates used to prove Theorem D.7, discussed in Chapter 6 of Volume One, were reviewed on pp. 461–463 in subsection 2.4 of Appendix A of Part I of this volume.

The **Hamilton–Ivey 3-dimensional curvature estimate** was proved in [**252**] and [**286**], with the former paper proving the '*time-dependent* fiber-wise convex set' version (see also Theorem 9.4 on p. 258 of Volume One or Theorem 6.44 on p. 241 of [**146**]). The estimate says that for solutions to the Ricci flow on closed 3-manifolds, in the high curvature regions (i.e., where $|\mathrm{Rm}|$ is large) the sectional curvatures are almost nonnegative. In particular, this result applies to finite time *singular* solutions.

THEOREM D.17 (Hamilton–Ivey estimate). *Let $\left(\mathcal{M}^3, g(t)\right)$ be a complete solution of the Ricci flow with bounded curvature on a 3-manifold for $0 \leq t < T$. If $\lambda_1(\mathrm{Rm})(x,0) \geq -1$ for all $x \in \mathcal{M}^3$, then at any point $(x,t) \in \mathcal{M}^3 \times [0,T)$ where $\lambda_1(\mathrm{Rm})(x,t) < 0$, we have*

$$R \geq |\lambda_1(\mathrm{Rm})| \cdot (\log|\lambda_1(\mathrm{Rm})| + \log(1+t) - 3). \tag{D.24}$$

In particular,

$$R \geq |\lambda_1(\mathrm{Rm})| \cdot (\log|\lambda_1(\mathrm{Rm})| - 3). \tag{D.25}$$

REMARK D.18. For the justification of this result in the noncompact case, see subsection 5.1 of Chapter 12.

As a consequence of the theorem, solutions to the Ricci flow on closed 3-manifolds have ϕ-almost nonnegative curvature (see also §12.1 of [**402**], Corollary 6.47 on pp. 243–244 of [**146**], or Section 2 of Chapter 10).

Let $\psi : \left(e^3, \infty\right) \to (0,\infty)$ be defined by $\psi(x) \doteqdot x(\log x - 3)$.

COROLLARY D.19 (A 3-dimensional solution has ϕ-almost nonnegative curvature). *If $\left(\mathcal{M}^3, g(t)\right)$, $t \in [0,T)$, is a solution of the Ricci flow on a closed 3-manifold with*

$$\lambda_1(\mathrm{Rm})(x,0) \geq -1$$

for all $x \in \mathcal{M}^3$, then

$$\lambda_1(\mathrm{Rm}) \geq -\phi(R)R - 3e^6 \tag{D.26}$$

on $\mathcal{M}^3 \times [0,T)$, where $\phi : \mathbb{R} \to [0,\infty)$ is defined by

$$\phi(u) \doteqdot \begin{cases} \dfrac{\psi^{-1}(u)}{u} & \text{if } u > 3e^6, \\ 0 & \text{if } u \leq 3e^6. \end{cases}$$

The function ϕ is bounded, on the interval $[3e^6, \infty)$ it is decreasing, and $\lim_{u\to\infty}\phi(u) = 0$. Note also that $u\phi(u)$ is nonnegative for all $u \in \mathbb{R}$.

As mentioned above, another consequence of the Hamilton–Ivey estimate is that 3-dimensional ancient solutions have nonnegative sectional curvature (see also Lemma 6.50 on p. 244 of [**146**]).

LEMMA D.20 (A 3-dimensional ancient solution has $\text{Rm} \geq 0$). *Every 3-dimensional complete ancient solution with bounded curvature has nonnegative sectional curvature.*

Note that, in any dimension, complete ancient solutions with bounded curvature have nonnegative scalar curvature.

3.2. Strong maximum principle applied to Rm **and 3-dimensional rigidity.** As a consequence of Lemma D.20, we are interested in the structure of solutions on 3-manifolds with nonnegative sectional curvature. Fortunately such solutions are rather rigid. In all dimensions Hamilton's [**245**] **strong maximum principle** for Rm says the following (see also Theorem 6.60 on p. 247 of [**146**]).

THEOREM D.21 (Strong maximum principle for Rm). *Let* $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, *be a solution to the Ricci flow with nonnegative curvature operator:* $\text{Rm}[g(t)] \geq 0$. *There exists* $\delta > 0$ *such that for each* $t \in (0, \delta)$, *the set* $\text{Im}(\text{Rm}[g(t)]) \subset \Lambda^2 T^* \mathcal{M}^n$ *is a smooth subbundle which is invariant under parallel translation and constant in time. Moreover,* $\text{Im}(\text{Rm}[g(x,t)])$ *is a Lie subalgebra of* $\Lambda^2 T_x^* \mathcal{M}^n \cong \mathfrak{so}(n)$ *for all* $x \in \mathcal{M}$ *and* $t \in (0, \delta)$.

The aforementioned rigidity of 3-dimensional solutions with $\text{Rm} \geq 0$ is exhibited in the following classification of closed 3-manifolds with $\text{Rm} \geq 0$ or $\text{Rc} \geq 0$ due to Hamilton (see also Theorem 6.64 on p. 249 of [**146**]).

THEOREM D.22 (3-manifolds with $\text{Rm} \geq 0$ or $\text{Rc} \geq 0$). *If* $(\mathcal{M}^3, g(t))$, $t \in [0, T)$, *is a solution to the Ricci flow on a closed* 3-*manifold with nonnegative sectional (Ricci) curvature, then for* $t \in (0, T)$ *the universal covering solution* $\left(\tilde{\mathcal{M}}^3, \tilde{g}(t)\right)$ *is either*

(1) $\mathbb{R}^3$ *with the standard flat metric,*
(2) *the product* $(\mathcal{S}^2, h(t)) \times \mathbb{R}$, *where* $h(t)$ *is a solution to the Ricci flow with positive curvature, or*
(3) *it has positive sectional (Ricci) curvature, and hence* $\tilde{\mathcal{M}}^3$ *is diffeomorphic to* $\mathcal{S}^3$ *(and* $\mathcal{M}^3$ *is diffeomorphic to a spherical space form).*

3.3. Derivative estimates. Returning to general solutions of the Ricci flow with bounded curvature, we recall the **Bernstein–Bando–Shi global derivative of curvature estimates** (**BBS estimates**), which reflect the smoothing property of the Ricci flow (see also Theorem 6.6 on p. 215 of [**146**]).

THEOREM D.23 (Derivative estimates—global). *If* $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, *is a solution of the Ricci flow on a closed manifold, then for each* $\alpha > 0$ *and every* $m \in \mathbb{N}$, *there exists a constant* $C(m, n, \alpha)$ *depending only on* m, n, *and* $\max\{\alpha, 1\}$ *such that if*

$$|\text{Rm}(x,t)|_{g(t)} \leq K \quad \text{for all } x \in \mathcal{M}^n \text{ and } t \in [0, \frac{\alpha}{K}] \cap [0, T),$$

then

(D.27)

$$|\nabla^m \operatorname{Rm}(x,t)|_{g(t)} \leq \frac{C(m,n,\alpha)K}{t^{m/2}} \quad \text{for all } x \in \mathcal{M}^n \text{ and } t \in (0, \frac{\alpha}{K}] \cap [0,T).$$

Local versions of the above estimates are **Shi's local derivative of curvature estimates** (see also Theorem 6.9 on p. 216 of [**146**]). Such estimates form a fundamental tool in understanding singularity formation for solutions of the Ricci flow.

THEOREM D.24 (Derivative estimates—local). *For any α, K, r, n, and $m \in \mathbb{N}$, there exists $C < \infty$ depending only on α, K, r, n, and m such that if $\mathcal{M}^n$ is a manifold, $p \in \mathcal{M}$, and $g(t)$, $t \in [0, T_0]$, where $0 < T_0 \leq \alpha/K$, is a solution to the Ricci flow on an open neighborhood U of p containing $\bar{B}_{g(0)}(p,r)$ as a compact subset, and if*

$$|\operatorname{Rm}(x,t)| \leq K \text{ for all } x \in U \text{ and } t \in [0, T_0],$$

then

$$|\nabla^m \operatorname{Rm}(y,t)| \leq \frac{C(\alpha,K,r,n,m)}{t^{m/2}}$$

for all $y \in B_{g(0)}(p, r/2)$ and $t \in (0, T_0]$.

Generalizing the above estimates, we also have the following global higher derivative estimates for the curvatures *assuming bounds on some derivatives*, due to one of the authors, as stated in Theorem 6.65 on p. 251 of [**146**] (see Theorem 14.16 in this book for the proof).

THEOREM D.25 (Modified global derivative estimates). *For any constants $\alpha, K, r, \ell \geq 0$, n, and $m \in \mathbb{N}$, there exists a constant $C < \infty$ depending only on α, K, r, ℓ, n, and m such that if $\mathcal{M}^n$ is a manifold, $p \in \mathcal{M}$, and if $g(t)$, $t \in [0, T_0]$, where $0 < T_0 \leq \alpha/K$, is a solution to the Ricci flow on an open neighborhood $\mathcal{U}$ of p containing $\bar{B}_{g(0)}(p,r)$ as a compact subset, and if*

$$|\operatorname{Rm}(x,t)| \leq K \text{ for all } x \in \mathcal{U} \text{ and } t \in [0, T_0],$$
$$\left|\nabla^k \operatorname{Rm}(x,0)\right| \leq K \text{ for all } x \in \mathcal{U} \text{ and } k \leq \ell,$$

then

$$|\nabla^m \operatorname{Rm}(y,t)| \leq \frac{C}{t^{\max\{m-\ell,0\}/2}}$$

for all $y \in B_{g(0)}(p, r/2)$ and $t \in (0, T_0]$.

3.4. Differential Harnack estimates. Hamilton's **trace Harnack estimate for the Ricci flow** [**247**] says the following (see also Corollary 15.3).

THEOREM D.26 (Trace Harnack estimate for Ricci flow). *If $(\mathcal{M}^n, g(t))$, $t \in [0,T)$, is a solution to the Ricci flow with nonnegative curvature operator,* *so that* $R_{ijk\ell}U_{ij}U_{\ell k} \geq 0$ *for all* 2*-forms, and if* $(\mathcal{M}^n, g(t))$ *is either compact or complete noncompact with bounded curvature, then*

$$\frac{\partial R}{\partial t} + \frac{R}{t} + 2\langle \nabla R, V\rangle + 2\operatorname{Rc}(V,V) \geq 0 \tag{D.28}$$

for any vector field V *and* $t > 0$*. In particular, taking* $V = 0$*, one obtains*

$$\frac{\partial}{\partial t}(tR) = t\left(\frac{\partial R}{\partial t} + \frac{R}{t}\right) \geq 0. \tag{D.29}$$

Hence, for any $x \in \mathcal{M}$ *and* $0 < t_1 \leq t_2$*, we have*

$$R(x, t_2) \geq \frac{t_1}{t_2} R(x, t_1).$$

The above estimate (D.28) follows from tracing the following more complicated 'matrix Harnack estimate' (see Theorem 15.1).

THEOREM D.27 (Matrix Harnack estimate for Ricci flow). *If* $(\mathcal{M}^n, g(t))$*,* $t \in [0,T)$*, is a complete solution to the Ricci flow with bounded* $\operatorname{Rm} \geq 0$*, then for any* 1*-form* $W \in C^\infty(\Lambda^1\mathcal{M})$ *and* 2*-form* $U \in C^\infty(\Lambda^2\mathcal{M})$ *we have*

$$M_{ij}W^iW^j + 2P_{pij}U^{pi}W^j + R_{pijq}U^{pi}U^{qj} \geq 0, \tag{D.30}$$

where M_{ij} *and* P_{pij} *are defined by* (15.9) *and* (15.5)*, respectively.*

The assumption of nonnegative curvature operator is not very restrictive in dimension 3 in the sense that ancient solutions with bounded curvature satisfy this assumption. This is a consequence of Lemma D.20 above.

A generalization of Theorem D.26 is the following (see Section 2 of Chapter 20 of Part III of this volume).

THEOREM D.28 (Linear trace Harnack estimate). *Let* $(\mathcal{M}^n, g(t), h(t))$*,* $t \in [0,T)$*, be a solution to the linearized Ricci flow system*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$
$$\frac{\partial}{\partial t} h_{ij} = (\Delta_L h)_{ij} \doteqdot \Delta h_{ij} + 2R_{kij\ell}h_{k\ell} - R_{ik}h_{kj} - R_{jk}h_{ki}$$

such that $g(t)$ *is complete with bounded nonnegative curvature operator,* $h(0) \geq 0$*, and* $h(t)$ *satisfies the bounds*

$$|h(x,0)| \leq e^{A\left(1+d_{g(0)}(x,O)\right)}, \tag{D.31}$$

$$\int_0^T \int_{\mathcal{M}} e^{-Bd^2_{g(0)}(x,O)} |h(x,t)|^2 \, d\mu_{g(t)} dt < \infty \tag{D.32}$$

for all $x \in \mathcal{M}$ *and* $t \in [0,T)$*, where* $O \in \mathcal{M}$*. Then* $h(t) \geq 0$ *for* $t \in [0,t)$ *and for any vector* V *we have*

$$Z \doteqdot \operatorname{div}(\operatorname{div}(h)) + \langle \operatorname{Rc}, h\rangle + 2\langle \operatorname{div}(h), V\rangle + h(V,V) + \frac{H}{2t} \geq 0, \tag{D.33}$$

where $H = g^{ij}h_{ij}$*.*

Theorem D.26 is the special case where $h = \mathrm{Rc}$ (this choice of h is possible since $\frac{\partial}{\partial t}\mathrm{Rc} = \Delta_L \mathrm{Rc}$ under the Ricci flow).

4. Perelman's energy, entropy, and associated invariants

In [**402**] Perelman introduced new monotone invariants for arbitrary solutions of the Ricci flow on closed manifolds.

4.1. Energy and entropy monotonicity formulas. Perelman's energy functional $\mathcal{F}$ is defined, for a Riemannian metric g and function f, by (5.1) on p. 191 of Part I of this volume, i.e.,

$$\mathcal{F}(g,f) \doteqdot \int_{\mathcal{M}} \left(R + |\nabla f|^2\right) e^{-f} d\mu.$$

The first variation of $\mathcal{F}$ at (g_{ij}, f) in the direction (v_{ij}, h) is given by (Lemma 5.3 on p. 192 of Part I of this volume)

$$\text{(D.34)} \qquad \begin{aligned}\delta_{(v,h)}\mathcal{F}(g,f) = &-\int_{\mathcal{M}} v_{ij}(R_{ij} + \nabla_i\nabla_j f)e^{-f} d\mu \\ &+ \int_{\mathcal{M}} \left(\frac{V}{2} - h\right)\left(2\Delta f - |\nabla f|^2 + R\right) e^{-f} d\mu,\end{aligned}$$

where $V \doteqdot g^{ij}v_{ij}$. The energy monotonicity formula says that under the evolution equations (5.34)–(5.35) on p. 199 of Part I of this volume, i.e.,

$$\text{(D.35)} \qquad \frac{\partial g}{\partial t} = -2\,\mathrm{Rc},$$

$$\text{(D.36)} \qquad \frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R,$$

we have (see (5.41) on p. 201 of Part I of this volume)

$$\text{(D.37)} \qquad \frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_{\mathcal{M}} |\mathrm{Rc} + \nabla\nabla f|^2 e^{-f} d\mu.$$

The pre-Harnack-type quantity ((5.42) in Part I of this volume)

$$V \doteqdot (2\Delta f - |\nabla f|^2 + R)e^{-f}$$

satisfies the evolution equation ((5.43) in Part I of this volume)

$$\Box^* V \doteqdot \left(-\frac{\partial}{\partial t} - \Delta + R\right) V = -2|\,\mathrm{Rc} + \nabla\nabla f|^2 e^{-f}.$$

Integrating this formula yields (D.37) (see p. 202 of Part I of this volume).

The entropy functional $\mathcal{W}$ is defined, for a Riemannian metric g, function f, and number $\tau > 0$, by ((6.1) on p. 222 of Part I of this volume)

$$\mathcal{W}(g,f,\tau) \doteqdot \int_{\mathcal{M}} \left[\tau\left(R + |\nabla f|^2\right) + f - n\right](4\pi\tau)^{-n/2} e^{-f} d\mu.$$

The first variation of $\mathcal{W}$ at (g_{ij}, f, τ) in the direction (v_{ij}, h, ζ) is (Lemma 6.1 on p. 223 of Part I of this volume)

(D.38)

$$\delta\mathcal{W}_{(v,h,\zeta)}(g, f, \tau)$$
$$= \int_{\mathcal{M}} (-\tau v_{ij} + \zeta g_{ij}) \left(R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right) (4\pi\tau)^{-n/2} e^{-f} d\mu$$
$$+ \int_{\mathcal{M}} \tau \left(\frac{V}{2} - h - \frac{n\zeta}{2\tau} \right) \left(R + 2\Delta f - |\nabla f|^2 + \frac{f - n - 1}{\tau} \right) (4\pi\tau)^{-n/2} e^{-f} d\mu.$$

Perelman's entropy monotonicity formula says that under the evolution equations (6.14)–(6.16) on p. 225 of Part I of this volume, i.e.,

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{D.39}$$
$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \tag{D.40}$$
$$\frac{d\tau}{dt} = -1, \tag{D.41}$$

we have ((6.17) on p. 226 of Part I of this volume)

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t))$$
$$= \int_{\mathcal{M}} 2\tau \left| \operatorname{Rc} + \nabla\nabla f - \frac{1}{2\tau} g \right|^2 (4\pi\tau)^{-n/2} e^{-f} d\mu \geq 0. \tag{D.42}$$

Perelman's Harnack-type quantity for the adjoint heat equation ((6.20) on p. 227 of Part I of this volume), i.e.,

$$v \doteqdot \left[\tau \left(R + 2\Delta f - |\nabla f|^2 \right) + f - n \right] (4\pi\tau)^{-n/2} e^{-f},$$

satisfies the evolution equation ((6.22) in Part I of this volume)

$$\square^* v = -2\tau \left| \operatorname{Rc} + \nabla\nabla f - \frac{1}{2\tau} g \right|^2 (4\pi\tau)^{-n/2} e^{-f}.$$

Integrating this yields (D.42) (see p. 228 of Part I of this volume).

4.2. Associated monotonicity of invariants. In this subsection we recall the monotonicity of invariants associated to the energy and entropy monotonicity formulas.

Given a closed manifold $\mathcal{M}$, let $\mathfrak{Met}$ denote the space of smooth Riemannian metrics on $\mathcal{M}$. The **λ-invariant** $\lambda : \mathfrak{Met} \to \mathbb{R}$ is defined by ((5.45) on p. 204 of Part I of this volume)

$$\lambda(g) \doteqdot \inf \left\{ \mathcal{F}(g, f) : f \in C^\infty(\mathcal{M}),\ \int_{\mathcal{M}} e^{-f} d\mu = 1 \right\}. \tag{D.43}$$

Taking $w = e^{-f/2}$, we have ((5.46)–(5.47) on p. 204 of Part I of this volume)
(D.44)

$$\lambda(g) = \inf \left\{ \mathcal{G}(g,w) \doteqdot \int_{\mathcal{M}} \left(4 |\nabla w|^2 + Rw^2 \right) d\mu : \int_{\mathcal{M}} w^2 d\mu = 1, \; w > 0 \right\}.$$

One can prove the following (see (5.48) in Part I of this volume for part (1) and Lemma 5.22 on pp. 204–205 of Part I of this volume for parts (2)–(4)).

(1) The Euler–Lagrange equation for $\mathcal{G}(g,w)$ as a functional of w is

$$Lw \doteqdot -4\Delta w + Rw = \mathcal{G}(g,w)w. \tag{D.45}$$

(2) The invariant $\lambda(g)$ is equal to the lowest eigenvalue of the elliptic operator $-4\Delta + R$.
(3) There exists a unique minimizer w_0 (up to a change in sign) of $\mathcal{G}(g,w)$ under the constraint $\int_{\mathcal{M}} w^2 d\mu = 1$. This minimizer w_0 is positive and smooth.
(4) The function w_0 is the unique positive eigenfunction of

$$-4\Delta w_0 + Rw_0 = \lambda_1(g) w_0 \tag{D.46}$$

with L^2-norm equal to 1.

In terms of the functional $\mathcal{F}$, there exists a unique smooth minimizer f_0 of $\mathcal{F}(g,\cdot)$ under the constraint $\int_{\mathcal{M}} e^{-f} d\mu = 1$. The minimizer $f_0 = -2\log w_0$ of $\mathcal{F}(g,\cdot)$ is unique, C^∞, and a solution to (Lemma 5.23 on p. 206 of Part I of this volume)

$$\lambda(g) = 2\Delta f_0 - |\nabla f_0|^2 + R. \tag{D.47}$$

We have the following upper and lower bounds for λ (p. 206 of Part I of this volume):

$$R_{\min} \le \lambda(g) \le r \doteqdot \frac{1}{\operatorname{Vol}(\mathcal{M})} \int_{\mathcal{M}} R d\mu. \tag{D.48}$$

We also verify that λ depends continuously on g (Lemma 5.24 on p. 207 of Part I of this volume gives an effective estimate). In regards to the Ricci flow, we have the following (Lemma 5.25 on p. 209 of Part I of this volume).

LEMMA D.29 (λ-invariant monotonicity). *If $(\mathcal{M}^n, g(t))$, $t \in [0,T]$, is a solution to the Ricci flow, then*

$$\frac{d}{dt}\lambda(g(t)) \ge \frac{2}{n}\lambda^2(g(t)),$$

where the derivative $\frac{d}{dt}$ is in the sense of the lim inf of backward difference quotients. Hence $\lambda(g(t))$ is nondecreasing in $t \in [0,T]$.

As an application, we have the following 'no breathers' result. A steady Ricci breather is a solution where the metrics at two different times are isometric (Lemma 5.28 on p. 210 of Part I of this volume).

LEMMA D.30 (No nontrivial steady breathers on closed manifolds). *If* $(\mathcal{M}^n, g(t))$ *is a solution to the Ricci flow on a closed manifold such that there exist* $t_1 < t_2$ *with* $\lambda(g(t_1)) = \lambda(g(t_2))$, *then* $g(t)$ *is Ricci flat. In particular, a steady Ricci breather on a closed manifold is Ricci flat.*

We may also consider the **normalized λ-invariant** ((5.58) on p. 211 of Part I of this volume), i.e.,

$$\bar{\lambda}(g) \doteqdot \lambda(g) \cdot \mathrm{Vol}(\mathcal{M})^{2/n}. \tag{D.49}$$

Under the Ricci flow, we have the monotonicity of $\bar{\lambda}$ when it is nonpositive (Lemma 5.30 on p. 212 of Part I of this volume).

LEMMA D.31 ($\bar{\lambda}$-invariant monotonicity when $\bar{\lambda} \leq 0$). *Let* $g(t)$ *be a solution to the Ricci flow on a closed manifold* $\mathcal{M}^n$. *If at some time* t, $\bar{\lambda}(t) \leq 0$, *then at that time we have*

$$\begin{aligned} &\frac{d}{dt}\bar{\lambda}(g(t)) \\ &\geq 2\,\mathrm{Vol}^{2/n}(g(t)) \int_{\mathcal{M}} \left| \mathrm{Rc} + \nabla\nabla f - \frac{1}{n}(R + \Delta f)\, g \right|^2 e^{-f} d\mu \geq 0, \end{aligned} \tag{D.50}$$

where $f(t)$ *is the minimizer for* $\mathcal{F}(g(t), \cdot)$, *and the time-derivative is defined as the* lim inf *of backward difference quotients. In particular, if* $\frac{d}{dt}\bar{\lambda}(g(t)) = 0$, *then* $g(t)$ *is a gradient Ricci soliton.*

The above lemma rules out nontrivial expanding breathers on closed manifolds (Lemma 5.31 on p. 213 of Part I of this volume).

LEMMA D.32 (No nontrivial expanding breathers). *Expanding or steady breathers on closed manifolds are Einstein.*

We have the following monotonicity formula for the classical entropy

$$\mathcal{N} \doteqdot \int_{\mathcal{M}} f e^{-f} d\mu = -\int_{\mathcal{M}} u \log u \, d\mu,$$

where $u \doteqdot e^{-f}$ (Lemma 5.35 on p. 215 of Part I of this volume).

LEMMA D.33 (Monotonicity formula for the classical entropy). *Suppose* $(\mathcal{M}^n, g(t), f(t))$, $t \in [0, T)$, *is a solution of* (D.35)–(D.36) *on a closed manifold* $\mathcal{M}$. *Then*

$$\frac{d}{dt}\mathcal{F}(g(t)) \geq \frac{2}{n}\left(\int_{\mathcal{M}} e^{-f} d\mu\right)^{-1} \mathcal{F}(g(t))^2,$$

$$\mathcal{F}(g(t)) \leq \frac{n}{2(T-t)} \int_{\mathcal{M}} e^{-f} d\mu. \tag{D.51}$$

This implies the following classical entropy monotonicity formula:

$$\frac{d}{dt}\left(\mathcal{N} - \left(\frac{n}{2}\int_{\mathcal{M}} e^{-f} d\mu\right) \log(T-t)\right) \geq 0. \tag{D.52}$$

5. Compactness, no local collapsing, and singularity models

5.1. Hamilton's Cheeger–Gromov-type compactness theorem for solutions. Hamilton has proved a Cheeger–Gromov-type compactness theorem for sequences of complete solutions of the Ricci flow with bounded curvatures and injectivity radii. The following statement is Theorem 3.10 on p. 131 of Part I of this volume (an exposition of a proof, following Hamilton's [**253**], is given in Chapters 3 and 4 of Part I of this volume).

THEOREM D.34 (Compactness for solutions). *Let* $\{(\mathcal{M}_k^n, g_k(t), O_k)\}_{k\in\mathbb{N}}$, $t \in (\alpha, \omega) \ni 0$, *be a sequence of complete pointed solutions to the Ricci flow such that*

(1) ***(uniformly bounded curvatures)***

$$|\mathrm{Rm}_k|_k \leq C_0 \quad \text{on } \mathcal{M}_k \times (\alpha, \omega)$$

for some constant $C_0 < \infty$ *independent of* k *and*

(2) ***(injectivity radius estimate at*** $t = 0$***)***

$$\mathrm{inj}_{g_k(0)}(O_k) \geq \iota_0$$

for some constant $\iota_0 > 0$.

Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ *such that* $\{(\mathcal{M}_{j_k}, g_{j_k}(t), O_{j_k})\}_{k\in\mathbb{N}}$ *converges to a complete pointed solution to the Ricci flow* $(\mathcal{M}_\infty^n, g_\infty(t), O_\infty)$, $t \in (\alpha, \omega)$, *as* $k \to \infty$.

The result above is basic to understanding the formation of singularities for solutions of the Ricci flow. In particular, given a singular solution and a sequence of space-time points, one may dilate the solution about these points to obtain a sequence of solutions to the Ricci flow. The compactness theorem gives a criterion for when there exists a subsequence which converges.

REMARK D.35 (Compactness without inj bound). In [**207**] Glickenstein proved an analogue of the above theorem where the injectivity radii of the solutions in the sequence are not assumed to be bounded from below.

5.2. Perelman's no local collapsing theorem. The understanding of singularity formation was greatly advanced by Perelman's **no local collapsing** (**NLC**) theorem (Theorem 4.1 in Perelman's [**402**]; for expositions, see also Theorem 5.35 on p. 194 of [**146**] or Theorem 6.58 on p. 256 of Part I of this volume).

THEOREM D.36 (Perelman—no local collapsing). *Let* $g(t)$, $t \in [0, T)$, *be a smooth solution to the Ricci flow on a closed manifold* $\mathcal{M}^n$. *If* $T < \infty$, *then for any* $\rho \in (0, \infty)$ *there exists* $\kappa = \kappa(g(0), T, \rho) > 0$ *such that* $g(t)$ *is* κ*-noncollapsed below the scale* ρ *for all* $t \in [0, T)$.

Perelman also proved an improved version of his no local collapsing theorem (see also Theorem 6.74 on p. 267 of Part I of this volume).

THEOREM D.37 (No local collapsing theorem improved). *Let* $(\mathcal{M}^n, g(t))$, $t \in [0,T)$, *be a solution to the Ricci flow on a closed manifold with* $T < \infty$ *and let* $\rho \in (0,\infty)$. *There exists a constant* $\kappa = \kappa(n, g(0), T, \rho) > 0$ *such that if* $p \in \mathcal{M}$, $t \in [0,T)$, *and* $r \in (0,\rho]$ *are such that*

$$R \leq r^{-2} \text{ in } B_{g(t)}(p,r),$$

then

$$\frac{\operatorname{Vol}_{g(t)} B_{g(t)}(p,s)}{s^n} \geq \kappa$$

for all $0 < s \leq r$.

5.3. μ- and ν-invariants, monotonicity, and volume ratios. The **μ-invariant** is defined by ((6.49) on p. 236 of Part I of this volume)

$$\mu(g,\tau) \doteqdot \inf\left\{\mathcal{W}(g,f,\tau) : \int_{\mathcal{M}} (4\pi\tau)^{-n/2} e^{-f} d\mu = 1\right\}$$

and the **ν-invariant** is defined by ((6.50) in Part I of this volume)

$$\nu(g) \doteqdot \inf\left\{\mu(g,\tau) : \tau \in \mathbb{R}^+\right\}.$$

The Euler–Lagrange equation of $\mathcal{W}(g,f,\tau)$ as a functional of f subject to the constraint $\int_{\mathcal{M}}(4\pi\tau)^{-n/2}e^{-f}d\mu = 1$ is ((6.51) on p. 237 of Part I of this volume)

$$\tau\left(2\Delta f - |\nabla f|^2 + R\right) + f - n = \mathcal{W}(g,f,\tau).$$

For any given g and $\tau > 0$ on a closed manifold $\mathcal{M}^n$ (Lemma 6.23 in Part I of this volume),

$$\mu(g,\tau) > -\infty \tag{D.53}$$

is finite.

For any metric g on a closed manifold $\mathcal{M}^n$ and $\tau > 0$, there exists a smooth minimizer f_τ of $\mathcal{W}(g,\cdot,\tau)$ in the space

$$\left\{f \in C^\infty(\mathcal{M}) : \int_{\mathcal{M}} (4\pi\tau)^{-n/2} e^{-f} d\mu_g = 1\right\}$$

(Lemma 6.24 on p. 238 of Part I of this volume).

The entropy monotonicity formula implies the following (Lemma 6.26 on p. 239 of Part I of this volume).

LEMMA D.38 (μ-invariant monotonicity). *Let* $(\mathcal{M}^n, g(t),)$, $t \in [0,T]$, *be a solution of the Ricci flow on a closed manifold with* $\tau(t) > 0$ *satisfying* $\frac{d\tau}{dt} = -1$. *For all* $0 \leq t_1 \leq t_2 \leq T$, *we have*

$$\mu(g(t_2), \tau(t_2)) \geq \mu(g(t_1), \tau(t_1)). \tag{D.54}$$

Since we have C^∞ Cheeger–Gromov convergence by the compactness theorem when we extract a limit of a sequence of rescalings of a singular solution of the Ricci flow, we consider the μ-invariant under C^∞ Cheeger–Gromov convergence (Lemma 6.28 on p. 240 of Part I of this volume).

LEMMA D.39 (μ under Cheeger–Gromov convergence). *Suppose that we have* $(\mathcal{N}_k^n, g_k, x_k) \to (\mathcal{N}_\infty^n, g_\infty, x_\infty)$ *in the* C^∞ *Cheeger–Gromov sense. Then for any* $\tau > 0$,

$$\mu(g_\infty, \tau) \geq \limsup_{k \to \infty} \mu(g_k, \tau).$$

As an application of the μ-invariant monotonicity we have the following (Theorem 6.29 on p. 242 of Part I of this volume).

THEOREM D.40 (Shrinking breathers are gradient solitons). *A shrinking breather for the Ricci flow on a closed manifold must be a gradient shrinking Ricci soliton.*

The behavior of $\mu(g, \tau)$ for τ large has the following property (Lemma 6.30 on p. 243 of Part I of this volume).

LEMMA D.41 ($\mu \to \infty$ as $\tau \to \infty$ when $\lambda > 0$). *If* $\lambda(g) > 0$, *then*

$$\lim_{\tau \to \infty} \mu(g, \tau) = \infty.$$

The behavior of $\mu(g, \tau)$ for τ small has the following properties (Lemma 6.33 on p. 244 of Part I of this volume).

LEMMA D.42 (Behavior of $\mu(g, \tau)$ for τ small). *Suppose* $(\mathcal{M}^n, g)$ *is a closed Riemannian manifold.*

(i) *There exists* $\bar{\tau} > 0$ *such that*

$$\mu(g, \tau) < 0 \quad \text{for all } \tau \in (0, \bar{\tau}).$$

(ii)

$$\lim_{\tau \to 0_+} \mu(g, \tau) = 0.$$

The ν-invariant has the following properties (Lemma 6.35 on pp. 244–245 of Part I of this volume).

LEMMA D.43 (ν-invariant monotonicity). *Let* $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, *be a solution to the Ricci flow on a closed manifold.*

(1) *The invariant* $\nu(g(t))$ *is nondecreasing on* $[0, T)$, *as long as* $\nu(g(t))$ *is well defined and finite.*
(2) *Furthermore, if* $\lambda(g(t)) > 0$ *and if* $\nu(g(t))$ *is not strictly increasing on some interval, then* $g(t)$ *is a gradient shrinking Ricci soliton.*
(3) *If* $\nu(g(t_0)) = -\infty$ *for some* t_0, *then* $\nu(g(t)) = -\infty$ *for all* $t \in [0, t_0]$.

A link between μ-monotonicity and no local collapsing is given by the following (Proposition 6.64 on pp. 258–259 of Part I of this volume).

PROPOSITION D.44 (μ controls volume ratios). *Let* $\rho \in (0, \infty)$. *There exists a constant* $C_2 = C_2(n, \rho) < \infty$ *such that if* $\left(\hat{\mathcal{M}}^n, \hat{g}\right)$ *is a closed Riemannian manifold,* $p \in \hat{\mathcal{M}}$ *and* $r \in (0, \rho]$ *are such that* $\operatorname{Rc} \geq -c_1(n) r^{-2}$ *and* $R \leq c_1(n) r^{-2}$ *in* $B(p, r)$, *then*

$$\mu\left(\hat{g}, r^2\right) \leq \log \frac{\operatorname{Vol} B(p, r)}{r^n} + C_2(n, \rho). \tag{D.55}$$

That is,

$$\frac{\operatorname{Vol} B\left(p, r\right)}{r^{n}} \geq e^{-C_{2}\left(n, \rho\right)} e^{\mu\left(\hat{g}, r^{2}\right)}.$$

In particular, if for some $\kappa > 0$ *and* $r \in (0, \rho]$ *the metric* $\hat{g}$ *is* κ*-collapsed at the scale* r, *then*

$$\mu\left(\hat{g}, r^{2}\right) \leq \log \kappa + C_{2}\left(n, \rho\right).$$

More specifically we have the following (Proposition 6.70 on pp. 264–265 of Part I of this volume).

PROPOSITION D.45 (Bounding μ by the scalar curvature and volume ratio). *The* μ*-invariant has the following upper bound in terms of local geometric quantities. For any closed Riemannian manifold* $\left(\hat{\mathcal{M}}^{n}, \hat{g}\right)$, *point* $p \in \hat{\mathcal{M}}$, *and* $r > 0$, *we have*

(D.56)

$$\mu\left(\hat{g}, r^{2}\right) \leq \log \frac{\operatorname{Vol} B\left(p, r\right)}{r^{n}} + \left(36 + \frac{r^{2} \int_{B(p,r)} R_{+} d\mu}{\operatorname{Vol} B\left(p, r\right)}\right) \frac{\operatorname{Vol} B\left(p, r\right)}{\operatorname{Vol} B\left(p, r/2\right)},$$

where $R_{+} \doteqdot \max\left\{R, 0\right\}$ *is the positive part of the scalar curvature.*

The ν-invariant controls volume ratios (Proposition 6.72 on p. 266 of Part I of this volume).

PROPOSITION D.46 (Bounding volume ratios by ν_r). *If* $\left(\hat{\mathcal{M}}^{n}, \hat{g}\right)$ *is a closed Riemannian manifold,* $R \leq c_{1}\left(n\right) r^{-2}$ *in* $B\left(p, r\right)$, *and* $0 < s \leq r$, *then*

$$\frac{\operatorname{Vol} B\left(p, s\right)}{s^{n}} \geq e^{-3^{n}\left(36 + c_{1}\left(n\right)\right)} e^{\nu_{r}\left(\hat{g}\right)},$$

where

$$\nu_{r}\left(\hat{g}\right) \doteqdot \inf_{\tau \in \left(0, r^{2}\right]} \mu\left(\hat{g}, \tau\right) \geq \nu\left(\hat{g}\right).$$

5.4. Existence of singularity models. When combined with the compactness theorem (Theorem D.34), Theorem D.36 yields the existence of singularity models for finite time singular solutions on closed manifolds and it helps classify these singularity models. In particular, we have the following (see also Corollary 5.48 on p. 202 and Corollary 5.51 on p. 203 of [**146**] or Corollaries 3.26 and 3.29 on pp. 142–143 of Part I of this volume).

COROLLARY D.47 (Obtaining singularity models). *Let* $\left(\mathcal{M}^{n}, g\left(t\right)\right)$, $t \in [0, T)$, $T < \infty$, *be a singular solution to the Ricci flow on a closed manifold. If* $\left\{\left(x_{i}, t_{i}\right)\right\}$ *is a sequence of points and times with* $t_{i} \rightarrow T$ *and if for some* $C < \infty$ *and* $\beta > 0$

(D.57)
$$\sup_{\mathcal{M}^{n} \times \left[t_{i} - \beta K_{i}^{-1}, t_{i}\right]} \left|\operatorname{Rm}\right| \leq C K_{i},$$

where $K_i \doteqdot |\mathrm{Rm}(x_i, t_i)| \to \infty$, *then there exists a subsequence such that the dilated solutions* $(\mathcal{M}^n, g_i(t), x_i)$, *where*

$$g_i(t) \doteqdot K_i g\left(t_i + K_i^{-1} t\right),$$

converge to a complete limit solution $(\mathcal{M}^n_\infty, g_\infty(t), x_\infty)$ *of the Ricci flow defined on the time interval* $[-\beta, 0]$ *on a possibly noncompact and topologically distinct manifold* $\mathcal{M}^n_\infty$ *with* $|\mathrm{Rm}_\infty| \leq C$. *Moreover, there exists* $\kappa > 0$, *independent of* $\{(x_i, t_i)\}$, *such that* $(\mathcal{M}^n_\infty, g_\infty(t))$ *is* κ*-noncollapsed on all scales.*

Since for all $\kappa > 0$ the cigar is κ-collapsed at all large enough scales (see Exercise 5.54 on p. 204 of [**146**]), we have the following consequence (Corollary 5.55 on p. 205 of [**146**]).

COROLLARY D.48 (The cigar is not a finite time singularity model). *Let* $(\mathcal{M}^n, g(t))$, $t \in [0, T)$, *be a solution to the Ricci flow on a closed manifold with* $T < \infty$. *The cigar soliton* $\left(\Sigma^2, g_\Sigma\right)$ *or its product with a flat solution to the Ricci flow (such as the line) cannot occur as a singularity model.*

The following will appear in Chapter 20 in Part III of this volume.

THEOREM D.49 (Immortal solutions are expanding gradient solitons). *Let* $(\mathcal{M}^n, g(t))$, $t \in (\alpha, \infty)$, *be a complete immortal solution to the Ricci flow such that* $g(t)$ *has nonnegative curvature operator and* $tR(x,t)$ *achieves its space-time maximum somewhere at* (x_0, t_0). *Assume further that* $\mathcal{M}$ *is simply connected with positive Ricci curvature. Then* $(\mathcal{M}, g(t))$ *is an expanding gradient soliton.*

APPENDIX E

Aspects of Geometric Analysis Related to Ricci Flow

This is the world we live in, and these are the hands we're given.
– From "Land of Confusion" by Genesis

In this appendix we collect some basic results in geometric analysis which are related to the results and techniques in Ricci flow.

1. Green's function

In this section we discuss the Green's function first on Euclidean space and then on complete Riemannian manifolds. Two references for this section are §2.2.4 of Evans [**185**] and Aubin [**16**].

1.1. The Green's function on Euclidean space. The **Euclidean Green's function** on $\mathbb{E}^n$, where $n \geq 3$, is defined by

$$\Gamma(x,y) \doteqdot \frac{1}{(n-2)\, n\omega_n} |x-y|^{2-n}. \tag{E.1}$$

Here ω_n denotes the volume of the unit n-ball B^n, so that $n\omega_n$ is the area of the unit $(n-1)$-sphere $\mathcal{S}^{n-1}$. When $n=2$, we define $\Gamma(x,y) \doteqdot -\frac{1}{2\pi}\log|x-y|$. Henceforth we consider the case $n \geq 3$ and leave the remaining case of $n=2$ to the reader.

As we shall see in the next subsection, the Euclidean Green's function is the **fundamental solution of the Laplace equation**. That is, its Laplacian is, in the sense of distributions, the negative of the δ-function.

In preparation, we compute the partial derivatives of Γ up to second order. The first partial derivatives of the Euclidean Green's function are given by

$$\begin{aligned}\frac{\partial}{\partial x^i}\Gamma(x,y) &= \frac{1}{(n-2)\, n\omega_n}\frac{\partial}{\partial x^i}\left(\sum_{k=1}^{n}(x^k-y^k)^2\right)^{(2-n)/2} \\ &= -\frac{1}{n\omega_n}\frac{x^i-y^i}{\left(\sum_{k=1}^{n}(x^k-y^k)^2\right)^{n/2}}. \end{aligned} \tag{E.2}$$

Let $r \doteqdot |x-y|$. Taking the norm of (E.2), we have

$$\left|\frac{\partial}{\partial x^i}\Gamma(x,y)\right| \leq \frac{1}{n\omega_n}\frac{1}{r^{n-1}} \tag{E.3}$$

since $\left|x^i - y^i\right| \leq r$. Consequently, for any bounded domain Ω we have that both $\Gamma(x,y)$ and $\left|\frac{\partial}{\partial x^i}\Gamma(x,y)\right|$ are integrable over Ω. This implies we can interchange the order of differentiation and integration of Γ, so that for any C^1 function f with compact support, we have

$$\frac{\partial}{\partial x^i}\int_{\mathbb{R}^n}\Gamma(x,y)f(y)d\mu(y) = \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x^i}\Gamma(x,y)\right)f(y)d\mu(y). \tag{E.4}$$

Next we compute the Hessian of Γ:

$$\begin{aligned}\frac{\partial^2}{\partial x^i \partial x^j}\Gamma(x,y) &= -\frac{1}{n\omega_n}\frac{\partial}{\partial x^i}\left(\frac{x^j - y^j}{\left(\sum_{k=1}^n (x^k - y^k)^2\right)^{n/2}}\right) \\ &= \begin{cases} \frac{1}{\omega_n}\frac{(x^i - y^i)(x^j - y^j)}{r^{n+2}} & \text{if } i \neq j, \\ \frac{1}{\omega_n}\frac{(x^i - y^i)^2}{r^{n+2}} - \frac{1}{n\omega_n}\frac{1}{r^n} & \text{if } i = j. \end{cases}\end{aligned} \tag{E.5}$$

In particular, taking the trace of this formula, we obtain

$$\Delta_x \Gamma(x,y) = 0 \tag{E.6}$$

for $x \neq y$. By symmetry, we also have $\Delta_y \Gamma(x,y) = 0$ for $x \neq y$.

By (E.5), we have

$$\left|\frac{\partial^2}{\partial x^i \partial x^j}\Gamma(x,y)\right| \leq \frac{1}{\omega_n}\frac{1}{|x-y|^n}.$$

However for $x \in \Omega$ the function on the RHS is not integrable with respect to y over Ω, so the above estimate is not sufficient to show that we can interchange the order of differentiation and integration in computing $\Delta\left(\int_\Omega \Gamma(x,y)f(y)d\mu(y)\right)$. We shall carry out an equivalent computation by cutting off the singularity at $x = y$.

1.2. The Euclidean Green's function is the fundamental solution of the Laplace equation. In this subsection we shall show that on $\mathbb{E}^n$ we have

$$\Delta_{\text{distr}(y)}\Gamma(x,y) = -\delta_x(y),$$

i.e., for any function f which is C^2 with compact support,

$$\int_{\mathbb{R}^n}\Gamma(x,y)\Delta f(y)\; d\mu(y) = -f(x). \tag{E.7}$$

The integral on the LHS has a singularity at $y = x$, so we need to excise a ball about the singularity in order to apply the divergence theorem and integrate by parts.

Let $B_\varepsilon \doteqdot B(x,\varepsilon)$. We compute

$$\begin{aligned}
&\int_{\partial B_\varepsilon} \Gamma(x,y) \langle \nabla f(y), \nu(y)\rangle \, d\sigma(y) \\
&= \int_{\mathbb{R}^n \backslash B_\varepsilon} \operatorname{div}_y (\Gamma(x,y)\nabla f(y)) \, d\mu(y) \\
&= \int_{\mathbb{R}^n \backslash B_\varepsilon} (\langle \nabla_y \Gamma(x,y), \nabla f(y)\rangle + \Gamma(x,y)\Delta f(y)) \, d\mu(y),
\end{aligned}$$

where ν is the unit outward pointing normal to $\mathbb{R}^n \backslash B_\varepsilon$. On the other hand, we also compute

$$\begin{aligned}
&\int_{\partial B_\varepsilon} f(y) \langle \nabla_y \Gamma(x,y), \nu(y)\rangle \, d\sigma(y) \\
&= \int_{\mathbb{R}^n \backslash B_\varepsilon} \operatorname{div}_y (f(y)\nabla_y \Gamma(x,y)) \, d\mu(y) \\
&= \int_{\mathbb{R}^n \backslash B_\varepsilon} \langle \nabla_y \Gamma(x,y), \nabla f(y)\rangle \, d\mu(y),
\end{aligned}$$

where we used $\Delta_y \Gamma(x,y) = 0$. Taking the difference of the above two equations, we obtain

$$\begin{aligned}
&\int_{\partial B_\varepsilon} (\Gamma(x,y) \langle \nabla f(y), \nu(y)\rangle - f(y) \langle \nabla_y \Gamma(x,y), \nu(y)\rangle) \, d\sigma(y) \\
&= \int_{\mathbb{R}^n \backslash B_\varepsilon} \Gamma(x,y)\Delta f(y) d\mu(y).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\left| \int_{\partial B_\varepsilon} \Gamma(x,y) \langle \nabla f(y), \nu(y)\rangle \, d\sigma(y) \right| &\leq \sup_{\mathbb{R}^n} |\nabla f(y)| \operatorname{Area}(\partial B_\varepsilon) \frac{1}{(n-2)\, n\omega_n \varepsilon^{n-2}} \\
&= \frac{1}{n-2} \sup_{\mathbb{R}^n} |\nabla f(y)| \, \varepsilon,
\end{aligned}$$

which tends to zero as $\varepsilon \to 0$.

Next we calculate the term with $\langle \nabla_y \Gamma(x,y), \nu(y)\rangle$. This is easily done since $\nu(y) = -\frac{1}{\varepsilon}(y-x)$ for $y \in \partial B_\varepsilon$ (recall that ν is the outward pointing normal to $\mathbb{R}^n \backslash B_\varepsilon$). In particular, using (E.2) and noting that the derivative with respect to y is the negative of the derivative with respect to x, we have

$$\langle \nabla_y \Gamma(x,y), \nu(y)\rangle = \frac{1}{n\omega_n \varepsilon^{n-1}}$$

for $y \in B_\varepsilon$. Thus we have

$$\begin{aligned}&\int_{\partial B_\varepsilon} f(y)\left\langle \nabla_y \Gamma(x,y), \nu(y)\right\rangle d\sigma(y)\\ &= \frac{1}{n\omega_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon} f(y) d\sigma(y)\\ &\to f(x)\end{aligned}$$

as $\varepsilon \to 0$ since f is continuous at x. Thus if we let $\varepsilon \to 0$, then we obtain $\int_{\mathbb{R}^n} \Gamma(x,y)\Delta f(y) d\mu(y) = -\int_{\mathbb{R}^n} \langle \nabla_y \Gamma(x,y), \nabla f(y)\rangle d\mu(y) = -f(x)$, i.e.,

$$\Delta_{\mathrm{distr}(y)} \Gamma(x,y) = -\delta_x(y).$$

Suppose that f is a C^2 function with compact support. We now want to show that the C^2 function

$$u = Qf(x) \doteqdot \int_{\mathbb{R}^n} \Gamma(x,y) f(y) d\mu(y)$$

is a solution to the Poisson equation

$$\Delta u = f. \tag{E.8}$$

We shall show that the above equation is true in the sense of distributions. That is, we let $\phi \in C_c^\infty(\mathbb{R}^n)$ and show that

$$\int_{\mathbb{R}^n} Qf(x)\Delta\phi(x) d\mu(x) = \int_{\mathbb{R}^n} f(x)\phi(x) d\mu(x).$$

We do this as follows:

$$\begin{aligned}\int_{\mathbb{R}^n} Qf(x)\Delta\phi(x) d\mu(x) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \Gamma(x,y) f(y) d\mu(y)\right)\Delta\phi(x) d\mu(x)\\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \Gamma(x,y)\Delta\phi(x) d\mu(x)\right) f(y) d\mu(y)\\ &= \int_{\mathbb{R}^n} \phi(y) f(y) d\mu(y),\end{aligned}$$

where to obtain the last equality we used (E.7). Hence Qf is a weak solution of (E.8). Since Qf is C^2, it is actually a classical solution.

1.3. Green's function on closed manifolds. In this section we derive an expression for the Green's function for the Laplacian on a closed Riemannian manifold $(\mathcal{M}^n, g)$. We follow the exposition in Aubin [**16**]. Again we suppose $n \geq 3$ (for $n = 2$, the reader may revise the method below with appropriate changes of $\log r$ for r^{2-n}). We try to mimic the method for $\mathbb{E}^n$. The first approximation to the Green's function is $H(x,y) \doteqdot H(d_g(x,y))$, where

$$H(r) \doteqdot \frac{r^{2-n}}{(n-2)\, n\omega_n}\eta(r)$$

and $\eta(r)$ is a C^∞ function taking values in $[0,1]$ which is equal to 1 on $[0,\delta)$ for some small $\delta > 0$ and equal to 0 on $[\iota, \infty)$, where $\iota \doteqdot \min_{x\in\mathcal{M}} \mathrm{inj}_g(x)$ denotes the injectivity radius of $\mathcal{M}$.

The metric in geodesic polar coordinates centered at x can be represented as

$$g = dr^2 + r^2 g_{ij} d\theta^i d\theta^j.$$

Let $|g| \doteqdot \det(g_{ij})_{(n-1)\times(n-1)}$. The Laplacian of a radial function $f(r)$ is

$$\begin{aligned}\Delta f(r) &= \frac{1}{r^{n-1}\sqrt{|g|}}\frac{\partial}{\partial r}\left(r^{n-1}\sqrt{|g|}\frac{d}{dr}f\right)\\ &= f''(r) + \left(\frac{n-1}{r} + \frac{\partial}{\partial r}\log\sqrt{|g|}\right)f'(r).\end{aligned}$$

So

(E.9)
$$\frac{d}{dr}H(r) = -\frac{r^{1-n}}{n\omega_n}\eta(r) + \frac{r^{2-n}}{(n-2)n\omega_n}\eta'(r),$$

(E.10)
$$\frac{d^2}{dr^2}H(r) = \frac{(n-1)r^{-n}}{n\omega_n}\eta(r) - 2\frac{r^{1-n}}{n\omega_n}\eta'(r) + \frac{r^{2-n}}{(n-2)n\omega_n}\eta''(r),$$

(E.11)
$$\begin{aligned}\Delta_y H(x,y) &= \frac{(n-1)r^{-n}}{n\omega_n}\eta(r) - 2\frac{r^{1-n}}{n\omega_n}\eta'(r) + \frac{r^{2-n}}{(n-2)n\omega_n}\eta''(r)\\ &\quad + \left(\frac{n-1}{r} + \frac{\partial}{\partial r}\log\sqrt{|g|}\right)\left(-\frac{r^{1-n}}{n\omega_n}\eta(r) + \frac{r^{2-n}}{(n-2)n\omega_n}\eta'(r)\right)\\ &= \left(\left(\frac{n-1}{n-2} - 2\right)\eta'(r) - \left(\frac{\partial}{\partial r}\log\sqrt{|g|}\right)\eta(r)\right)\frac{r^{1-n}}{n\omega_n}\\ &\quad + \left(\eta'(r)\frac{\partial}{\partial r}\log\sqrt{|g|} + \eta''(r)\right)\frac{r^{2-n}}{(n-2)n\omega_n}.\end{aligned}$$

Once again, we remove a small ball from a neighborhood of the singularity. Let $f : \mathcal{M} \to \mathbb{R}$ be C^2 with compact support inside $B(x,\iota)$ and let $B_\varepsilon \doteqdot B(x,\varepsilon)$ with $\varepsilon > 0$ smaller than the injectivity radius. Now,

$$\begin{aligned}&\int_{\mathcal{M}\setminus B_\varepsilon} H(x,y)\Delta f(y)\,d\mu(y) - \int_{\mathcal{M}\setminus B_\varepsilon} \Delta_y H(x,y) f(y)\,d\mu(y)\\ &= \int_{\mathcal{M}\setminus B_\varepsilon} \mathrm{div}_y\left[H(x,y)\nabla f(y)\right]d\mu(y) - \int_{\mathcal{M}\setminus B_\varepsilon} \mathrm{div}_y\left[\nabla_y H(x,y) f(y)\right]d\mu(y)\\ &= \int_{\partial B_\varepsilon} H(x,y)\langle\nabla f(y),\nu(y)\rangle\,d\sigma(y) - \int_{\partial B_\varepsilon} f(y)\langle\nabla_y H(x,y),\nu(y)\rangle\,d\sigma(y).\end{aligned}$$

Now, since ε is smaller than the injectivity radius, the definition of $H(x,y)$ implies that there is a constant $C < \infty$ depending only on the geometry and dimension such that

$$\left| \int_{\partial B_\varepsilon} H(x,y) \langle \nabla f(y), \nu(y) \rangle \, d\sigma(y) \right| \leq C \varepsilon^{2-n} \sup_{\mathcal{M}} |\nabla f| \, \varepsilon^{n-1} = C \sup_{\mathcal{M}} |\nabla f| \, \varepsilon.$$

This goes to zero as $\varepsilon \to 0$. The other term looks like

$$\int_{\partial B_\varepsilon} f(y) \langle \nabla_y H(x,y), \nu(y) \rangle \, d\sigma(y)$$
$$= -\left(-\frac{\varepsilon^{1-n}}{n\omega_n} \eta(\varepsilon) + \frac{\varepsilon^{2-n}}{(n-2)\, n\omega_n} \eta'(\varepsilon) \right) \int_{\partial B_\varepsilon} f(y) \, d\sigma(y),$$

recalling that $\nu(y)$ is the outward pointing normal for $\mathcal{M} \backslash B_\varepsilon$. If $\varepsilon < \delta$, then this is equal to

$$\frac{1}{n\omega_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon} f(y) \, d\sigma(y).$$

Since the local geometry as $\varepsilon \to 0$ is Euclidean, this converges to $f(x)$. We thus see that

$$\int_{\mathcal{M}} H(x,y) \triangle f(y) \, d\mu(y) = -f(x) + \int_{\mathcal{M}} \triangle_y H(x,y) f(y) \, d\mu(y).$$

If we could solve

$$\int_{\mathcal{M}} F(x,y) \triangle f(y) \, d\mu(y) = \int_{\mathcal{M}} \triangle_y H(x,y) f(y) \, d\mu(y),$$

then the Green's function would be $H(x,y) - F(x,y)$. This comes down to solving

$$\triangle_y F(x,y) = \tilde{H}(x,y)$$

in the weak sense (for each x), where $\tilde{H}(x,y) \doteqdot \triangle_y H(x,y)$ is a known function. If $\tilde{H}(x,y)$ were continuous in y for all x, then we could solve this equation in the classical sense and we would be done. However, $\triangle_y H(x,y)$ is not necessarily continuous. Note that by (E.11) we only know that $|\triangle_y H(x,y)| \leq C r^{1-n}$.

Instead, we use $H(x,y)$ again to come close to solving this problem, taking the second approximation to the Green's function to be

$$G_2(x,y) = H(x,y) + H_2(x,y),$$

where

$$H_2(x,y) \doteqdot \int_{\mathcal{M}} H(x,z) \triangle_z H(z,y) \, d\mu(z).$$

We see that $G_2(x,y)$ satisfies

$$\begin{aligned}
&\int_{\mathcal{M}} G_2(x,y)\,\triangle f(y)\,d\mu(y) \\
&= -f(y) + \int_{\mathcal{M}} \triangle_y H(x,y)\, f(y)\,d\mu(y) \\
&+ \int_{\mathcal{M}} \left(\int_{\mathcal{M}} H(x,z)\,\triangle_z H(z,y)\,d\mu(z) \right) \triangle f(y)\,d\mu(y) \\
&= -f(y) + \int_{\mathcal{M}} \triangle_y H(x,y)\, f(y)\,d\mu(y) \\
&+ \int_{\mathcal{M}} \left(-f(z) + \int_{\mathcal{M}} \triangle_y H(z,y)\, f(y)\,d\mu(y) \right) \triangle_z H(z,x)\,d\mu(z) \\
&= -f(y) + \int_{\mathcal{M}} \left(\int_{\mathcal{M}} \triangle_y H(z,y)\,\triangle_z H(z,x)\,d\mu(z) \right) f(y)\,d\mu(y),
\end{aligned}$$

where we have used the symmetry of $H(x,y)$ and $H_2(x,y)$.

For $k \geq 2$ we inductively define

$$\begin{aligned}
&H_{k+1}(x,y) \\
&\quad \doteqdot \int_{\mathcal{M}} \cdots \int_{\mathcal{M}} H(x,z_1)\,\triangle_{z_1} H(z_1,z_2) \cdots \triangle_{z_k} H(z_k,y)\,d\mu(z_1)\cdots d\mu(z_k),
\end{aligned}$$

$$G_{k+1}(x,y) \doteqdot H(x,y) + H_2(x,y) + \cdots + H_{k+1}(x,y).$$

Let k_0 be the smallest integer such that the multiple integral

$$\begin{aligned}
&\tilde{H}(x,y) \\
&\doteqdot \int_{\mathcal{M}} \cdots \int_{\mathcal{M}} \triangle_x H(x,z_1)\,\triangle_{z_1} H(z_1,z_2) \cdots \triangle_{z_{k_0}} H(z_{k_0},y)\,d\mu(z_1)\cdots d\mu(z_{k_0})
\end{aligned}$$

is a continuous function of x and y. We see that

$$\int_{\mathcal{M}} G_{k_0+1}(x,y)\,\triangle f(y)\,d\mu(y) = -f(y) + \int_{\mathcal{M}} \tilde{H}(y,x)\, f(y)\,d\mu(y).$$

The existence of $k_0 \in \mathbb{N}$ is guaranteed by the following.

LEMMA E.1. *Let $\Phi(x,y) = \int_{\mathcal{M}} \Psi(x,z)\,\Theta(z,y)\,d\mu(z)$ and suppose that $|\Psi(x,z)| \leq C d(x,z)^{a-n}$ and $|\Theta(z,y)| \leq C d(y,z)^{b-n}$, where $0 < a,b < n$. Then there exists a constant $C' < \infty$ such that*

$$|\Phi(x,y)| \leq \begin{cases} C' d(x,y)^{a+b-n} & \text{if } a+b<n, \\ C'(1+|\log d(x,y)|) & \text{if } a+b=n, \\ C' & \text{if } a+b>n. \end{cases}$$

PROOF. We may assume that $\frac{3}{2}d(x,y) < \iota$, since otherwise $\Phi(x,y)$ is bounded by a constant. Let $d \doteqdot d(x,y)/2$. In order to avoid singularities in one of the functions, we split $\Phi(x,y)$ into

$$\begin{aligned}\Phi(x,y) = &\int_{B(x,d)} \Psi(x,z)\,\Theta(z,y)\,d\mu(z) \\ &+ \int_{B(y,3d)\setminus B(x,d)} \Psi(x,z)\,\Theta(z,y)\,d\mu(z) \\ &+ \int_{\mathcal{M}\setminus B(y,3d)} \Psi(x,z)\,\Theta(z,y)\,d\mu(z).\end{aligned} \tag{E.12}$$

There is no singularity for Θ in the first integral in (E.12), so

$$\begin{aligned}\left|\int_{B(x,d)} \Psi(x,z)\,\Theta(z,y)\,d\mu(z)\right| &\leq \int_{B(x,d)} |\Psi(x,z)\,\Theta(z,y)|\,d\mu(z) \\ &\leq Cd^{b-n}\int_{B(x,d)} |\Psi(x,z)|\,d\mu(z) \\ &\leq C^2 d^{b-n}\int_{B(x,d)} d(x,z)^{a-n}\,d\mu(z) \\ &\leq C' d^{a+b-n}.\end{aligned}$$

The second integral in (E.12) has no singularity in Ψ, so

$$\begin{aligned}&\left|\int_{B(y,3d)\setminus B(x,d)} \Psi(x,z)\,\Theta(z,y)\,d\mu(z)\right| \\ &\leq Cd^{a-n}\int_{B(y,3d)} |\Theta(z,y)|\,d\mu(z) \\ &\leq C^2 d^{a-n}\int_{B(y,3d)} d(y,z)^{b-n}\,d\mu(z) \\ &\leq C' d^{a+b-n}.\end{aligned}$$

Finally, we compute the third integral in (E.12):

$$\begin{aligned}&\left|\int_{\mathcal{M}\setminus B(y,3d)} \Psi(x,z)\,\Theta(z,y)\,d\mu(z)\right| \\ &\leq C_1 + C^2\left|\int_{B(y,\iota)\setminus B(y,3d)} d(x,z)^{a-n}\,d(z,y)^{b-n}\,d\mu(z)\right| \\ &\leq C_1 + C^2\left|\int_{B(y,\iota)\setminus B(y,3d)} (d(y,z)-2d)^{a-n}\,d(y,z)^{b-n}\,d\mu(z)\right| \\ &\leq C_2\left(1 + \left|\int_{S^{n-1}}\int_{3d}^{\iota}\left(\frac{1}{3}r\right)^{a-n} r^{b-1}dr\,d\sigma\right|\right)\end{aligned}$$

since $d(y,z) \leq d(x,z) + 2d$ and $\frac{2}{3}r \geq 2d$ on the domain. Hence

$$\left| \int_{\mathcal{M}\setminus B(y,3d)} \Psi(x,z)\,\Theta(z,y)\,d\mu(z) \right| \leq \begin{cases} C'\left(1 + d^{a+b-n}\right) & \text{if } a+b \neq n, \\ C'\left(1 + |\log d|\right) & \text{if } a+b = n. \end{cases}$$

□

Since $|\triangle_x H(x,y)| \leq C d(x,y)^{1-n}$, Lemma E.1 can be applied inductively to get the following:

$$\left| \int_{\mathcal{M}} \cdots \int_{\mathcal{M}} \triangle_x H(x,z_1)\,\triangle_{z_1} H(z_1,z_2) \cdots \triangle_{z_{k_0}} H(z_{k_0},y)\,d\mu(z_1) \cdots d\mu(z_{k_0}) \right| \leq C d(x,y)^{k_0+1-n}.$$

Hence, there exists $k_0 \leq n$ such that $\left|\tilde{H}(x,y)\right| \leq C$. Thus, we can solve

$$\triangle_y F(x,y) = \tilde{H}(x,y)$$

and the Green's function is

$$G(x,y) \doteqdot G_{k_0}(x,y) - F(x,y).$$

We have the following result of Colding and Minicozzi, Theorem 0.1 in [**154**] (see also Theorem 1.1 in [**334**]).

THEOREM E.2 (Asymptotics of the Green's function when $\mathrm{Rc} \geq 0$ and $\theta_\infty > 0$). *If $(\mathcal{M}^n, g)$, $n \geq 3$, is a complete noncompact Riemannian manifold with nonnegative Ricci curvature and maximum volume growth, then*

$$\lim_{y\to\infty} \frac{n(n-2)\,\theta_\infty G(x,y)}{d^{2-n}(x,y)} = 1,$$

where θ_∞ is defined by (16.72).

As a consequence of Theorem 16.38, in particular using the estimates in (16.74) and the relation $G(x,y) = \int_0^\infty H(x,y,\tau)\,d\tau$, Li, Tam, and Wang (Corollary 2.2 in [**334**]) obtained the following sharp estimates for the Green's function.

THEOREM E.3. *If $(\mathcal{M}^n, g)$ is a complete noncompact Riemannian manifold with nonnegative Ricci curvature and maximum volume growth, then for any $\delta > 0$ the Green's function of $(\mathcal{M}, g)$ has the following upper and lower bounds:*

$$\begin{aligned} &(1+9\delta)^{1-\frac{n}{2}} \frac{d^{2-n}(x,y)}{n(n-2)\,\theta_x(\delta d(x,y))} \\ &\leq G(x,y) \\ &\leq (1 + C(n,\theta_\infty)(\delta+\beta))\,(1-\delta)^{1-\frac{n}{2}} \frac{d^{2-n}(x,y)}{n(n-2)\,\theta_\infty}, \end{aligned}$$

where $\beta(\delta, d(x,y))$ is defined in (16.75).

1.4. Hopf boundary point lemma. Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold. We say that a closed domain Ω satisfies the **interior sphere condition** at a point $x_0 \in \partial\Omega$ if there exists a geodesic ball $B(p,r) \subset \operatorname{int}(\Omega)$ with $x_0 \in \partial B(p,r)$.

The Hopf boundary point lemma says the following (see Lemma 3.4 on p. 34 of Gilbarg and Trudinger [**206**]).

THEOREM E.4 (Hopf boundary point lemma, I). *Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold and let $\Omega \subset \mathcal{M}$ be a closed domain. If $u : \Omega \to \mathbb{R}$ is a function with $u \in C^2(\operatorname{int}(\Omega))$ satisfying*

$$\Delta u + \langle X, \nabla u \rangle \geq 0, \tag{E.13}$$

where X is a bounded vector field, $x_0 \in \partial\Omega$ is a point where

$$u(x) < u(x_0) \quad \text{for all } x \in \Omega,$$

u is continuous at x_0, and Ω satisfies the interior sphere condition at x_0, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

if this unit outward normal derivative exists.

We also have the following variants.

THEOREM E.5 (Hopf boundary point lemma, II). *Same hypotheses as above except that instead of* (E.13) *we have*

$$\Delta u + \langle X, \nabla u \rangle + cu \geq 0$$

in Ω, where $c : \Omega \to \mathbb{R}$ is a bounded function. Then the same conclusion holds in either of the following cases:

(1) *$c \leq 0$ and $u(x_0) \geq 0$,*
(2) *arbitrary c and $u(x_0) = 0$.*

REMARK E.6. The Hopf boundary point lemma is proved by the weak maximum principle using a radial barrier (comparison) function (this is where the interior sphere condition is used).

2. Positive and fundamental solutions to the heat equation

The proofs of results quoted in this section regarding fundamental solutions may be found in Berger, Gauduchon, and Mazet [**36**] and Chavel [**92**]. The original work on the heat kernel parametrix is due to Minakshisundaram and Pleijel [**356**]. The proofs of the differential Harnack estimates may be found in the original [**335**], the book [**429**], or the expository [**139**]. We shall use the terminologies 'heat kernel' and 'minimal positive fundamental solution of the heat equation' interchangeably.

A **fundamental solution to the heat equation** is defined as follows. See Definition E.I.1 on p. 204 of [**36**] (or Definition 1 on p. 135 of [**92**]).

DEFINITION E.7 (Fundamental solution to heat equation). Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold. We say a function $h : \mathcal{M} \times \mathcal{M} \times (0, \infty) \to \mathbb{R}$ is a fundamental solution to the heat equation if

(1) *(regularity)* h is continuous, C^2 in the first variable, and C^1 in the third variable,

(2) *(solution to the heat equation)*
$$\left(\frac{\partial}{\partial t} - \Delta_x \right) h(x, y, t) = 0,$$

(3) *(tends to the delta function)*
$$\lim_{t \to 0_+} h(\cdot, y, t) = \delta_y$$
for any $y \in \mathcal{M}$, i.e., for any continuous function f on $\mathcal{M}$ with compact support we have
$$\lim_{t \to 0_+} \int_{\mathcal{M}} h(x, y, t) f(x) \, d\mu(x) = f(y).$$

As we shall see below, there exists a C^∞ fundamental solution $h(x, y, t)$ which is symmetric in x and y, and, in the case where $\mathcal{M}$ is closed, which is unique and positive. When $(\mathcal{M}, g)$ is complete and *noncompact*, the *minimal positive* fundamental solution (i.e., heat kernel) is unique and symmetric in x and y; see subsection 2.2 below.

First note that if u is a solution of the heat equation on a *closed* manifold, then
$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{M}} u(x, t) \, d\mu(x) &= \int_{\mathcal{M}} \frac{\partial u}{\partial t}(x, t) \, d\mu(x) \\ &= \int_{\mathcal{M}} \Delta u(x, t) \, d\mu(x) = 0. \end{aligned} \tag{E.14}$$

We also have that if $u > 0$, then
$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{M}} u(x, t) \log u(x, t) \, d\mu(x) &= \int_{\mathcal{M}} (\Delta u \cdot \log u + \Delta u)(x, t) \, d\mu(x) \\ &= -\int_{\mathcal{M}} \frac{|\nabla u|^2}{u}(x, t) \, d\mu(x) \leq 0. \end{aligned} \tag{E.15}$$

In particular, for the fundamental solution h we have
$$\int_{\mathcal{M}} h(x, y, t) \, d\mu(x) \equiv 1 \tag{E.16}$$
for all $y \in \mathcal{M}$ and $t \in (0, \infty)$ (since $\lim_{t \to 0_+} \int_{\mathcal{M}} h(x, y, t) \, d\mu(x) = 1$).

2.1. Uniqueness, symmetry, and existence of the fundamental solution on closed manifolds. A basic tool in the study of the heat equation is Duhamel's principle, which is based on integration by parts and may be considered as a space-time version of Green's formula (see p. 137 of Chavel [**92**]).[1] Let $\square = \frac{\partial}{\partial t} - \Delta$ be the heat operator.

LEMMA E.8 (Duhamel's principle, I). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold. If A and B are functions on $\mathcal{M} \times (0,t)$ which are both C^2 in space and C^1 in time, then for any $0 < t_1 < t_2 < t$ we have*

(E.17)
$$\int_{t_1}^{t_2} ds \int_{\mathcal{M}} \left(-\left(\frac{\partial A}{\partial t}\right)(w, t-s)\, B(w,s) + A(w,t-s)\left(\frac{\partial B}{\partial t}\right)(w,s)\right) d\mu(w)$$
$$= \int_{\mathcal{M}} A(w, t-t_2)\, B(w,t_2)\, d\mu(w) - \int_{\mathcal{M}} A(w,t-t_1)\, B(w,t_1)\, d\mu(w),$$

and

(E.18)
$$\int_{t_1}^{t_2} ds \int_{\mathcal{M}} \left(-(\square A)(w,t-s)\, B(w,s) + A(w,t-s)(\square B)(w,s)\right) d\mu(w)$$
$$= \int_{\mathcal{M}} \left(A(w,t-t_2)\, B(w,t_2) - A(w,t-t_1)\, B(w,t_1)\right) d\mu(w).$$

PROOF. The LHS of (E.18) and the LHS of (E.17) are both equal to[2]

$$\int_{t_1}^{t_2} ds \int_{\mathcal{M}} \left(-\left(\frac{\partial A}{\partial t}\right)(w, t-s)\, B(w,s) + A(w,t-s)\left(\frac{\partial B}{\partial t}\right)(w,s)\right) d\mu(w)$$
$$+ \int_{t_1}^{t_2} ds \int_{\mathcal{M}} \left((\Delta A)(w,t-s)\, B(w,s) - A(w,t-s)(\Delta B)(w,s)\right) d\mu(w)$$
$$= \int_{t_1}^{t_2} ds \frac{\partial}{\partial s} \int_{\mathcal{M}} A(w,t-s)\, B(w,s)\, d\mu(w)$$
$$= \int_{\mathcal{M}} A(w, t-t_2)\, B(w,t_2)\, d\mu(w) - \int_{\mathcal{M}} A(w,t-t_1)\, B(w,t_1)\, d\mu(w),$$

which is equal to the RHS of (E.18). ☐

The following two theorems, both consequences of Duhamel's principle, comprise Theorem 1 on p. 138 of [**92**].[3]

THEOREM E.9 (Uniqueness and symmetry of fundamental solution). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold. The fundamental solution $h(x,y,t)$ to the heat equation is unique and symmetric, i.e., $h(y,x,t) = h(x,y,t)$.*

[1] This statement applies to (E.18); equation (E.17) follows simply from the fundamental theorem of calculus.

[2] We used $\int_{\mathcal{M}} ((\Delta A)\, B - A(\Delta B))\, d\mu = 0$ to obtain the first equality.

[3] Compare with the symmetry of the Green's function: $G(y,x) = G(x,y)$.

PROOF. Suppose that $h_1(x,y,t)$ and $h_2(x,y,t)$ are fundamental solutions of the heat equation. Taking $t_1 \to 0_+$, $t_2 \to t_-$, $A(w,\tau) = h_1(w,x,\tau)$, and $B(w,s) = h_2(w,y,s)$ in (E.18), we have

$$\begin{aligned} 0 &= \lim_{t_2 \to t_-} \int_{\mathcal{M}} h_1(w,x,t-t_2)\, h_2(w,y,t_2)\, d\mu(w) \\ &\quad - \lim_{t_1 \to 0_+} \int_{\mathcal{M}} h_1(w,x,t-t_1)\, h_2(w,y,t_1)\, d\mu(w) \\ &= h_2(x,y,t) - h_1(y,x,t). \end{aligned}$$

In particular, if h is a fundamental solution, then by taking $h_1 = h_2 = h$ we have

$$h(x,y,t) = h(y,x,t).$$

Thus, for any fundamental solutions h_1 and h_2,

$$h_1(x,y,t) = h_2(x,y,t).$$

□

Via convolution, the fundamental solution may be used to solve the initial-value problem (**IVP**) for the heat equation.

THEOREM E.10 (Representation for solution to IVP). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold. If $f : \mathcal{M} \to \mathbb{R}$ is a continuous function and $F : \mathcal{M} \times (0,\infty) \to \mathbb{R}$ is a bounded continuous function, then the solution $u : \mathcal{M} \times [0,\infty) \to \mathbb{R}$ to the initial-value problem*

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = F \quad \text{on } \mathcal{M} \times (0,\infty), \tag{E.19a}$$

$$u(\cdot,0) = f(\cdot) \quad \text{on } \mathcal{M}, \tag{E.19b}$$

if it exists, is given by

(E.20)

$$u(x,t) = \int_{\mathcal{M}} h(x,y,t)\, f(y)\, d\mu(y) + \int_0^t ds \int_{\mathcal{M}} h(x,y,t-s)\, F(y,s)\, d\mu(y).$$

If $F \equiv 0$, then the solution does exist and is given by

$$u(x,t) = \int_{\mathcal{M}} h(x,y,t)\, f(y)\, d\mu(y). \tag{E.21}$$

PROOF. By taking $t_1 \to 0_+$, $t_2 \to t_-$,

$$A(w,\tau) = h(x,w,\tau), \quad \text{and} \quad B(w,s) = u(w,s)$$

all in (E.18), we have

$$\begin{aligned}&\int_0^t ds \int_{\mathcal{M}} h(x,w,t-s) F(w,s)\, d\mu(w)\\&= \lim_{\varepsilon\to 0_+} \int_{\mathcal{M}} \left(h(x,w,\varepsilon) u(w,t) - h(x,w,t) u(w,\varepsilon)\right) d\mu(w)\\&= u(x,t) - \int_{\mathcal{M}} h(x,w,t) f(w)\, d\mu(w).\end{aligned}$$

This shows that if the solution to the IVP (E.19) exists, then it is of the form (E.20).

Now if $F \equiv 0$, then we can differentiate (E.21) under the integral sign:

$$\left(\frac{\partial u}{\partial t} - \Delta u\right)(x,t) = \int_{\mathcal{M}} \left(\frac{\partial h}{\partial t} - \Delta h\right)(x,y,t) f(y)\, d\mu(y) = 0.$$

On the other hand, for each $x \in \mathcal{M}$ we have

$$\lim_{t\to 0_+} \int_{\mathcal{M}} h(x,y,t) f(y)\, d\mu(y) = f(x)$$

so that $\lim_{t\to 0_+} u(x,t) = f(x)$. □

For $t \in (0,\infty)$ define

$$\mathfrak{H}_t : C^0(\mathcal{M}) \to C^\infty(\mathcal{M})$$

by

$$(\mathfrak{H}_t(f))(x) = \int_{\mathcal{M}} h(x,y,t) f(y)\, d\mu(y). \tag{E.22}$$

That is, $\mathfrak{H}_t(f)$ is the solution at time t to the heat equation with initial value f. By the uniqueness of solutions to the heat equation on closed manifolds we have the semi-group property:

$$\mathfrak{H}_{t_2} \circ \mathfrak{H}_{t_1} = \mathfrak{H}_{t_1+t_2}. \tag{E.23}$$

By definition this says

$$\begin{aligned}&\int_{\mathcal{M}} h(x,y,t_1+t_2) f(y)\, d\mu(y)\\&= \int_{\mathcal{M}} h(x,z,t_2) (\mathfrak{H}_{t_1}(f))(z)\, d\mu(z)\\&= \int_{\mathcal{M}} h(x,z,t_2) \left(\int_{\mathcal{M}} h(z,y,t_1) f(y)\, d\mu(y)\right) d\mu(z)\\&= \int_{\mathcal{M}} \left(\int_{\mathcal{M}} h(x,z,t_2) h(z,y,t_1)\, d\mu(z)\right) f(y)\, d\mu(y)\end{aligned}$$

for all continuous functions f. Hence

$$h(x,y,t_1+t_2) = \int_{\mathcal{M}} h(x,z,t_2) h(z,y,t_1)\, d\mu(z). \tag{E.24}$$

For any continuous function f we have

$$\langle \mathfrak{H}_t(f), f\rangle_{L^2} = \left\langle \mathfrak{H}_{t/2} \circ \mathfrak{H}_{t/2}(f), f\right\rangle_{L^2} = \left\|\mathfrak{H}_{t/2}(f)\right\|_{L^2}^2 \geq 0.$$

Hence $\mathfrak{H}_t$ is a nonnegative operator. Another consequence is $h(x,x,t) \geq 0$. In fact, we have the following lemma which is a consequence of the strong maximum principle (see Theorem 1 on p. 181 of [**92**]).

LEMMA E.11 (Strict positivity of the heat kernel). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold. The heat kernel is positive, i.e., $h(x,y,t) > 0$.*

PROOF. We first show that $h(x,y,t) \geq 0$. Given a nonnegative function $f : \mathcal{M} \to \mathbb{R}$, let $u : \mathcal{M} \times [0,\infty) \to \mathbb{R}$ be the solution to the heat equation with $u(x,0) = f(x)$ for all $x \in \mathcal{M}$. By the maximum principle we have $u \geq 0$. On the other hand, by (E.21) we have

$$u(x,t) = \int_{\mathcal{M}} h(x,z,t) f(z)\, d\mu(z).$$

Now given $y \in \mathcal{M}$, take f_k to be a sequence of nonnegative functions limiting to δ_y. Since $\int_{\mathcal{M}} h(x,z,t) f_k(z)\, d\mu(z) \geq 0$, we conclude $h(x,y,t) \geq 0$.

The strict positivity $h(x,y,t) > 0$ follows from the strong maximum principle and the simple fact that h is not identically zero; note that

$$\lim_{t\to 0_+} (4\pi t)^{n/2} h(x,x,t) = 1.$$

□

As a consequence of the positivity of h and (E.21) we have the following.

COROLLARY E.12 (Heat kernel has largest sup norm). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold. We have*

$$|u(x,t)| \leq \max_{y\in\mathcal{M}} h(x,y,t) \int_{\mathcal{M}} |f(y)|\, d\mu(y). \tag{E.25}$$

Thus, at any positive time (a multiple of) the fundamental solution has the largest L^∞-norm among all solutions of the heat equation with fixed initial L^1-norm.

For closed manifolds, the fundamental solution exists. See §E.III of Chapter III in [**36**] (or §4 of Chapter VI, in particular (45)–(46) on p. 154, in [**92**]).

THEOREM E.13 (Existence of the heat kernel). *If $(\mathcal{M}^n, g)$ is a closed Riemannian manifold, then there exists a unique fundamental solution of the heat equation.*

The rough idea of the proof of the existence of the heat kernel on $\mathcal{M}$ is to transplant the Euclidean heat kernel to $\mathcal{M}$ and use it as an approximate solution by multiplying it by a finite power series in t with coefficients which are functions of x and y.

2.2. Uniqueness and existence of the minimal positive fundamental solution on complete noncompact manifolds. The idea for proving the existence of the minimal positive fundamental solution of the heat equation on complete noncompact manifolds is to choose an exhaustion $\{\mathcal{U}_i\}$ of $\mathcal{M}$ by compact regular domains and to take the limit of the associated Dirichlet heat kernels. The following is on p. 158 of [**92**].

DEFINITION E.14 (Dirichlet heat kernel). Let $(\mathcal{M}^n, g)$ be a smooth compact Riemannian manifold with interior $\operatorname{int}(\mathcal{M})$ and nonempty boundary $\partial\mathcal{M}$. We say that $h_{\mathrm{D}} : \mathcal{M} \times \mathcal{M} \times (0, \infty) \to \mathbb{R}$ is a **Dirichlet heat kernel** of $(\mathcal{M}, g)$ if h_{D} is continuous and $h_{\mathrm{D}}|_{\operatorname{int}(\mathcal{M})\times\operatorname{int}(\mathcal{M})\times(0,\infty)}$ is a fundamental solution[4] and satisfies the Dirichlet boundary condition:

$$h_{\mathrm{D}}(x, y, t) = 0$$

for all $x \in \partial\mathcal{M}$, $y \in \operatorname{int}(\mathcal{M})$, and $t \in (0, \infty)$.

Clearly

$$\lim_{t\to 0_+} \int_{\mathcal{M}} h_{\mathrm{D}}(x, y, t)\, d\mu(x) = 1. \tag{E.26}$$

Note that in comparison to (E.16) we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}} h_{\mathrm{D}}(x, y, t)\, d\mu(x) &= \int_{\mathcal{M}} \Delta h_{\mathrm{D}}(x, y, t)\, d\mu(x) \\ &= \int_{\partial\mathcal{M}} \frac{\partial h_{\mathrm{D}}}{\partial\nu}(x, y, t)\, d\mu(x) < 0,\end{aligned}$$

where we used the Hopf boundary point lemma (Theorem E.4). Hence

$$\int_{\mathcal{M}} h_{\mathrm{D}}(x, y, t)\, d\mu(x) < 1 \tag{E.27}$$

for all $y \in \operatorname{int}(\mathcal{M})$ and $t \in (0, \infty)$.

The following is on p. 164 of [**92**].

PROPOSITION E.15 (Existence of Dirichlet heat kernel). *If $(\mathcal{M}^n, g)$ is a smooth compact Riemannian manifold with interior* $\operatorname{int}(\mathcal{M})$ *and nonempty boundary $\partial\mathcal{M}$, then there exists a Dirichlet heat kernel of $(\mathcal{M}, g)$.*

PROOF. (*Sketch.*) There exists an extension of $(\mathcal{M}^n, g)$ to a smooth closed Riemannian manifold $\left(\hat{\mathcal{M}}^n, \hat{g}\right)$. Let

$$\hat{h} : \hat{\mathcal{M}} \times \hat{\mathcal{M}} \times (0, \infty) \to \mathbb{R}$$

be the heat kernel of $\left(\hat{\mathcal{M}}, \hat{g}\right)$, which exists by Theorem E.13. There exists a solution

$$u : \operatorname{int}(\mathcal{M}) \times \operatorname{int}(\mathcal{M}) \times (0, \infty) \to \mathbb{R}$$

---

[4]That is, $\left(\frac{\partial}{\partial t} - \Delta_x\right) h(x, y, t) = 0$ and for any continuous function f on $\operatorname{int}(\mathcal{M})$ with compact support we have $\lim_{t\to 0_+} \int_{\operatorname{int}(\mathcal{M})} h(x, y, t) f(x)\, d\mu(x) = f(y)$.

to $\left(\frac{\partial}{\partial t} - \Delta_x\right) u(x,y,t) = 0$ with $u(x,y,0) = 0$ for $x, y \in \operatorname{int}(\mathcal{M})$ and with

$$u(x,y,t) = -\hat{h}(x,y,t)$$

for $x \in \partial\mathcal{M}$, $y \in \operatorname{int}(\mathcal{M})$, and $t \in (0,\infty)$ (see pp. 161–164 of [**92**]). Define

$$h_{\mathrm{D}} : \operatorname{int}(\mathcal{M}) \times \operatorname{int}(\mathcal{M}) \times (0,\infty) \to \mathbb{R}$$

by

$$h_{\mathrm{D}}(x,y,t) \doteqdot \hat{h}(x,y,t) + u(x,y,t).$$

We leave it as an exercise for the reader to check that h_{D} is a Dirichlet heat kernel of $(\mathcal{M}, g)$. □

The following is Duhamel's principle for manifolds with boundary (see p. 145 of [**92**]).

LEMMA E.16 (Duhamel's principle, II). *Let $(\mathcal{M}^n, g)$ be a smooth compact Riemannian manifold with boundary $\partial\mathcal{M}$ and let $\square = \frac{\partial}{\partial t} - \Delta$. If A and B are functions on $\mathcal{M} \times (0,t)$ which are both C^2 in space and C^1 in time, then for any $0 < t_1 < t_2 < t$ we have*

$$\begin{aligned}
&\int_{\mathcal{M}} \left(A(w,t-t_2)B(w,t_2) - A(w,t-t_1)B(w,t_1)\right) d\mu(w) \\
&= \int_{t_1}^{t_2} ds \int_{\mathcal{M}} \left(-(\square A)(w,t-s)B(w,s) + A(w,t-s)(\square B)(w,s)\right) d\mu(w) \\
&+ \int_{t_1}^{t_2} ds \int_{\partial\mathcal{M}} \left(-\left(\frac{\partial A}{\partial\nu}\right)(w,t-s)B(w,s) + A(w,t-s)\left(\frac{\partial B}{\partial\nu}\right)(w,s)\right) d\sigma(w),
\end{aligned}$$

where $\frac{\partial}{\partial\nu}$ and $d\sigma$ are the unit outward normal derivative and volume form on $\partial\mathcal{M}$, respectively.

The proof of this lemma is the same as for Lemma E.8 except that now Green's formula says that

$$\int_{\mathcal{M}} \left((\Delta A)B - A(\Delta B)\right) d\mu = \int_{\partial\mathcal{M}} \left(B\frac{\partial A}{\partial\nu} - A\frac{\partial B}{\partial\nu}\right) d\sigma.$$

DEFINITION E.17 (Minimal positive fundamental solution). We say that a positive fundamental solution $h(x,y,t)$ is the **minimal positive fundamental solution** if for every positive fundamental solution $\tilde{h}(x,y,t)$ we have $\tilde{h}(x,y,t) \geq h(x,y,t)$. Clearly, if h exists, then it is unique.

Let $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ be an exhaustion of $\mathcal{M}$ by compact regular domains, so that $\mathcal{U}_i \subset \operatorname{int}(\mathcal{U}_{i+1})$ for all $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} \mathcal{U}_i = \mathcal{M}$. Let $h_i : \mathcal{U}_i \times \mathcal{U}_i \times (0,\infty) \to \mathbb{R}$ be the Dirichlet heat kernel of $\left(\mathcal{U}_i, g|_{\mathcal{U}_i}\right)$. By the maximum principle we have $h_{i+1} > h_i$ on $\mathcal{U}_i \times \mathcal{U}_i \times (0,\infty)$. We define

$$h : \mathcal{M} \times \mathcal{M} \times (0,\infty) \to \mathbb{R}$$

by

$$h \doteqdot \lim_{i\to\infty} h_i = \sup_{i\in\mathbb{N}} h_i. \tag{E.28}$$

EXERCISE E.18. Show that

$$h = \sup_{\mathcal{U}} h_{\mathcal{U}},$$

where $h_{\mathcal{U}}$ denotes the Dirichlet heat kernel of $(\mathcal{U}, g|_{\mathcal{U}})$ and the supremum is taken over all compact regular domains $\mathcal{U}$ (with nonempty interior).

HINT: Apply the maximum principle. Note that given any compact regular domain $\mathcal{U}$, there exists $i \in \mathbb{N}$ such that $\mathcal{U} \subset \operatorname{int}(\mathcal{U}_i)$.

We now show that h is the minimal positive fundamental solution of $(\mathcal{M}^n, g)$.

THEOREM E.19 (Existence of minimal positive fundamental solution). *If $(\mathcal{M}^n, g)$ is a complete noncompact Riemannian manifold, then there exists a minimal positive fundamental solution of $(\mathcal{M}^n, g)$.*

The main tool we use is the following (see Lemma 3 on p. 187 of [**92**]).

LEMMA E.20 (Local bound for solutions of the heat equation). *Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold and suppose $u : \mathcal{M} \times (0, T) \to \mathbb{R}$ is a solution to the heat equation with*

$$\int_{\mathcal{M}} |u(x,t)| \, d\mu(x) \leq C$$

for all $t \in (0, T)$. Then for any compact subset $K \subset \mathcal{M}$ and $t_0 \in (0, T)$,

$$|u(x,t)| \leq C$$

for all $x \in K$ and $t \in [t_0, T)$, where $C < \infty$ depends only on $(\mathcal{M}^n, g)$, K, T, and t_0. Moreover, for each $k \in \mathbb{N}$ we have

$$\left|\nabla^k u\right| \leq C_k \quad \text{on } K \times [t_0, T),$$

where $C_k < \infty$ depends only on $(\mathcal{M}^n, g)$, k, K, T, and t_0.

PROOF. The idea of the proof is similar to that for obtaining inequality (E.25). In the present case we localize the application of Duhamel's principle as follows. Given any compact subset $K \subset \mathcal{M}$, let $\varphi : \mathcal{M} \to [0, 1]$ be a smooth function with compact support and with $\varphi \equiv 1$ on K. Let $u : \mathcal{M} \times (0, T) \to \mathbb{R}$ be a solution to the heat equation with

$$\int_{\mathcal{M}} |u(x,t)| \, d\mu(x) \leq C$$

for all $t \in (0, T)$. Let Ω be a compact regular domain with $K \subset \operatorname{supp}(\varphi) \subset \Omega \subset \mathcal{M}$ and let h_Ω denote the Dirichlet heat kernel of $(\Omega, g|_\Omega)$, which exists by Proposition E.15. Given any $t_0 \in (0, T)$, if $x \in \mathcal{M}$ and $t \in \left(\frac{t_0}{2}, T\right)$, then by taking $t_1 = \frac{t_0}{2}$, $t_2 \nearrow t$, $A(w, \tau) = h_\Omega(w, x, \tau)$, and $B(w, s) =$

$\varphi(w)\,u(w,s)$ all in (E.18), we have

$$\varphi(x)\,u(x,t) = \int_{\mathcal{M}} h_\Omega\left(w,x,t-\frac{t_0}{2}\right)\varphi(w)\,u\left(w,\frac{t_0}{2}\right)d\mu(w) - \int_{\frac{t_0}{2}}^{t} ds \int_{\mathcal{M}} h_\Omega(\cdot,x,t-s)\,(u(s)\,\Delta\varphi + 2\langle\nabla u(s),\nabla\varphi\rangle)\,d\mu$$

since

$$\square\,(\varphi(\cdot)\,u(\cdot,s)) = -u(s)\,\Delta\varphi - 2\langle\nabla u(s),\nabla\varphi\rangle\,.$$

Integrating by parts, we have

$$\int_{\mathcal{M}} h_\Omega(\cdot,x,t-s)\,(u(s)\,\Delta\varphi + 2\langle\nabla u(s),\nabla\varphi\rangle)\,d\mu = \int_{\mathcal{M}} (-h_\Omega(\cdot,x,t-s)\,u(s)\,\Delta\varphi - 2u(s)\,\langle\nabla h_\Omega(\cdot,x,t-s),\nabla\varphi\rangle)\,d\mu.$$

Hence, if $x \in K$ and $t \in \left(\frac{t_0}{2},T\right)$, then

(E.29)

$$u(x,t) = \int_{\Omega} h_\Omega\left(w,x,t-\frac{t_0}{2}\right)\varphi(w)\,u\left(w,\frac{t_0}{2}\right)d\mu(w) + \int_{\frac{t_0}{2}}^{t} ds \int_{\Omega-K} u(s)\,(h_\Omega(\cdot,x,t-s)\,\Delta\varphi + 2\langle\nabla h_\Omega(\cdot,x,t-s),\nabla\varphi\rangle)\,d\mu,$$

where we could change the domains of integration from $\mathcal{M}$ to Ω and $\Omega - K$ since $\operatorname{supp}(\varphi) \subset \Omega$ and $\varphi \equiv 1$ on K, respectively.

We have

$$0 \le h_\Omega\left(w,x,t-\frac{t_0}{2}\right) \le C$$

for all $x \in K$, $w \in \Omega$, and $t \in [t_0,T)$. Hence

$$\left|\int_{\Omega} h_\Omega\left(w,x,t-\frac{t_0}{2}\right)\varphi(w)\,u\left(w,\frac{t_0}{2}\right)d\mu(w)\right| \le C\int_{\Omega}\left|u\left(w,\frac{t_0}{2}\right)\right|d\mu(w) \le C\int_{\mathcal{M}}\left|u\left(w,\frac{t_0}{2}\right)\right|d\mu(w) \le C. \tag{E.30}$$

Moreover,

$$|h_\Omega(w,x,t-s)\,\Delta\varphi(w) + 2\langle\nabla h_\Omega(w,x,t-s),\nabla\varphi(w)\rangle| \le C$$

for $x \in K$, $w \in \Omega - K$, $t \in [t_0,T)$, and $s \in \left[\frac{t_0}{2},t\right]$. Hence

(E.31)

$$\left|\int_{\frac{t_0}{2}}^{t} ds \int_{\Omega-K} u(s)\,(h_\Omega(\cdot,x,t-s)\,\Delta\varphi + 2\langle\nabla h_\Omega(\cdot,x,t-s),\nabla\varphi\rangle)\,d\mu\right| \le C\left|\int_{\frac{t_0}{2}}^{t} ds \int_{\Omega-K} u(s)\,d\mu\right| \le CT.$$

We conclude from (E.29), (E.30), and (E.31) that for $x \in K$, $t_0 \in (0,T)$, and $t \in [t_0, T)$,

$$|u(x,t)| \leq C.$$

Taking covariant derivatives of (E.29), we obtain bounds of all spatial derivatives of u in $K \times [t_0, T)$. □

With this local bound we may give the

PROOF OF THEOREM E.19. Let $h : \mathcal{M} \times \mathcal{M} \times (0,\infty) \to \mathbb{R}$ be the function defined by (E.28) as the limit of increasing Dirichlet heat kernels associated to an exhaustion $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ of $\mathcal{M}$ by compact regular domains. By Lemma E.20, for any compact subset $K \subset \mathcal{M}$ and $t_0 \in (0,T)$ we have

$$|h_i(x,y,t)| \leq C_0$$

for all $x, y \in K$ and $t \in [t_0, T)$, where $C_0 < \infty$ is independent of i. Hence

$$h(x,y,t) \doteqdot \lim_{i\to\infty} h_i(x,y,t) \tag{E.32}$$

is finite on $K \times K \times [t_0, T)$. By Lemma E.20, for all $k \in \mathbb{N}$ we have $\left|\nabla^k h_i(x,y,t)\right| \leq C_k < \infty$ independent of $i \in \mathbb{N}$ in $K \times K \times [t_0, T)$. Hence the limit function h is C^∞ and after passing to a subsequence the convergence of $\{h_i\}$ to h in (E.32) is uniform in any C^ℓ on compact subsets of $\mathcal{M}$. Thus h is a solution of the heat equation, i.e.,

$$\left(\frac{\partial}{\partial t} - \Delta_x\right) h(x,y,t) = 0.$$

Since $h_i(x,y,t) = h_i(y,x,t)$, we have that h is symmetric, i.e.,

$$h(x,y,t) = h(y,x,t).$$

Since $h_i > 0$ in $\operatorname{int}(\mathcal{U}_i)$ and $h \geq h_i$, we have $h > 0$ on $\mathcal{M} \times \mathcal{M} \times (0,\infty)$.

Now we show that h is a fundamental solution of the heat equation. By (E.27) we have

$$\int_{\mathcal{M}} h(x,y,t)\, d\mu(x) \leq 1 \tag{E.33}$$

for all $y \in \mathcal{M}$ and $t \in (0,\infty)$. If $\mathcal{U} \subset \mathcal{M}$ is a compact regular domain with nonempty interior, then by (E.33) and (E.26) and since $h \geq h_{\mathcal{U}}$, where $h_{\mathcal{U}}$ is the Dirichlet heat kernel of $(\mathcal{U}, g|_{\mathcal{U}})$, we have

$$\lim_{t\to 0_+} \int_{\mathcal{U}} h(x,y,t)\, d\mu(x) = 1$$

for any $y \in \mathcal{M}$. Finally, if φ is continuous, then

$$\lim_{t\to 0_+} \int_{\mathcal{M}} h(x,y,t)\, \varphi(y)\, d\mu(y) = \varphi(x).$$

(See p. 190 of [**92**] for a proof of this.) □

We have the following uniqueness result of Dodziuk [**174**] (see also Theorem 3 on p. 183 and Theorem 4 on pp. 188–189 of [**92**]).

THEOREM E.21 (Uniqueness of fundamental solution when $\mathrm{Rc} \geq -k$). *If $(\mathcal{M}^n, g)$ is a complete noncompact Riemannian manifold with Ricci curvature bounded from below, then the minimal positive fundamental solution (i.e., the heat kernel) of $(\mathcal{M}^n, g)$ is the unique positive fundamental solution of $(\mathcal{M}^n, g)$.*

2.3. Comparison theorem for fundamental solutions. Let $h_{n,K}$ denote the heat kernel of the simply-connected, complete, n-dimensional manifold $\left(\tilde{\mathcal{M}}^n, \tilde{g}\right)$ of constant sectional curvature K. At each time $h_{n,K}$ is a radial function so that we may define $\mathrm{H}_{n,K}$ by

$$h_{n,K}(\tilde{x}, \tilde{y}, t) \doteqdot \mathrm{H}_{n,K}(d_{\tilde{g}}(\tilde{x}, \tilde{y}), t).$$

We have the following comparison theorem for positive fundamental solutions [**105**].

THEOREM E.22 (Cheeger and Yau). *If $(\mathcal{M}^n, g)$ is a complete Riemannian manifold with $\mathrm{Rc} \geq (n-1) K$ for some $K \in \mathbb{R}$, then the positive fundamental solution $h(x, y, t)$ of the heat equation has the lower bound*

$$h(x, y, t) \geq \mathrm{H}_{n,K}(d(x, y), t). \tag{E.34}$$

REMARK E.23. When $K = 0$, we obtain (16.12).

PROOF. (*Sketch.*) The idea of the proof is to apply Duhamel's principle and the Laplacian comparison theorem. By Duhamel's principle, in particular, taking $t_1 \to 0_+$, $t_2 \to t_-$, $A(w, \tau) = \mathrm{H}_{n,K}(d(x, w), \tau)$, and $B(w, s) = h(w, y, s)$ all in (E.17), we have

$$\begin{aligned} & h(x, y, t) - \mathrm{H}_{n,K}(d(x, y), t) \\ & = -\int_0^t ds \int_{\mathcal{M}} \left(\frac{\partial \mathrm{H}_{n,K}}{\partial t}\right)(d(x, w), t-s) \cdot h(w, y, s)\, d\mu(w) \\ & \quad + \int_0^t ds \int_{\mathcal{M}} \mathrm{H}_{n,K}(d(x, w), t-s) \cdot \left(\frac{\partial h}{\partial t}\right)(w, y, s)\, d\mu(w). \end{aligned} \tag{E.35}$$

Now $\frac{\partial h}{\partial t} = \Delta_g h$, whereas we have

$$\begin{aligned} & -\left(\frac{\partial \mathrm{H}_{n,K}}{\partial t}\right)(d(x, w), t-s) \\ & = -\left(\tilde{\Delta}\, \mathrm{H}_{n,K}\right)(d(x, w), t-s) \\ & = -\left(\frac{\partial^2}{\partial r^2} + H_K(d(x, w)) \frac{\partial}{\partial r}\right) \mathrm{H}_{n,K}(d(x, w), t-s), \end{aligned}$$

where $\left(\tilde{\Delta}\, \mathrm{H}_{n,K}\right)(d_{\tilde{g}}(\tilde{x}, \tilde{y}), t) \doteqdot \left(\tilde{\Delta} h_{n,K}\right)(\tilde{x}, \tilde{y}, t)$ is well-defined since the latter is a radial function and where $H_K(r)$ denotes the mean curvature of the sphere of radius r in the simply-connected space form of constant sectional curvature K.

Given a point $p \in \mathcal{M}$, let $\operatorname{Cut}(p)$ denote the cut locus of p. Recall that the Laplacian on $\mathcal{M}$ may be written in geodesic spherical coordinates centered at p as

$$(\Delta f)(w) = \frac{\partial^2 f}{\partial r^2}(w) + H(w,p)\frac{\partial f}{\partial r}(w) + \left(\Delta_{S(p,r)} f\right)(w) \tag{E.36}$$

for $w \in \mathcal{M} - (\{p\} \cup \operatorname{Cut}(p))$, where $H(w,p)$ denotes the mean curvature at w of the geodesic sphere $S(p,r) \doteqdot \{y \in \mathcal{M} : d(y,p) = r\}$, where $r = d(w,p)$, and where $\Delta_{S(p,r)}$ denotes the intrinsic Laplacian of $S(p,r)$.

By the Laplacian comparison theorem (i.e., by the comparisons for the mean curvatures of geodesic spheres) we have

$$H(w,x) \leq H_K(d(x,w)).$$

This inequality and (see Lemma 2.3 of [**105**])

$$\left(\frac{\partial}{\partial r} \mathrm{H}_{n,K}\right)(d(x,w), t-s) < 0$$

imply

$$\begin{aligned} &- \left(\frac{\partial^2}{\partial r^2} + H_K(d(x,w))\frac{\partial}{\partial r}\right) \mathrm{H}_{n,K}(d(x,w), t-s) \\ &\geq - \left(\frac{\partial^2}{\partial r^2} + H(w,x)\frac{\partial}{\partial r}\right) \tilde{h}(x,w,t-s), \end{aligned}$$

where $\tilde{h}(x,w,t-s) \doteqdot \mathrm{H}_{n,K}(d(x,w), t-s)$. Hence

$$-\frac{\partial \mathrm{H}_{n,K}}{\partial t}(d(x,w), t-s) \geq -\left(\Delta_{g,w}\tilde{h}\right)(x,w,t-s),$$

where $\Delta_{g,w}$ denotes the Laplacian with respect to the metric g and the variable w. Applying this to (E.35), we conclude that

$$\begin{aligned} &h(x,y,t) - \mathrm{H}_{n,K}(d(x,y),t) \\ &\geq -\int_0^t ds \int_{\mathcal{M}} \Delta_{g,w}\tilde{h}(x,w,t-s) \cdot h(w,y,s)\, d\mu(w) \\ &\quad + \int_0^t ds \int_{\mathcal{M}} \tilde{h}(x,w,t-s) \cdot \Delta_{g,w} h(w,y,s)\, d\mu(w) \\ &= 0. \end{aligned}$$

□

2.4. Fundamental solutions on Riemannian products.

EXERCISE E.24 (Fundamental solution on a product). For $i = 1, 2$ let $(\mathcal{M}_i^{n_i}, g_i)$ be complete Riemannian manifolds and let $h_i : \mathcal{M}_i \times (0, \infty) \to \mathbb{R}$

be fundamental solutions to the heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta_{g_i}\right) h_i = 0,$$
$$\lim_{t\to 0^+} h_i = \delta_{y_i},$$

where $y_i \in \mathcal{M}_i$. Show that the function

$$h(x_1, x_2, t) \doteqdot h_1(x_1, t) \cdot h_2(x_2, t)$$

is a fundamental solution to the heat equation on $\mathcal{M}_1 \times \mathcal{M}_2$, in particular,

$$\left(\frac{\partial}{\partial t} - \Delta_{g_1+g_2}\right) h = 0,$$
$$\lim_{t\to 0^+} h = \delta_{(y_1,y_2)}.$$

SOLUTION TO EXERCISE E.24. We compute

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{g_1+g_2}\right)(h_1(x_1,t)\cdot h_2(x_2,t)) \\
&= \left(\frac{\partial}{\partial t} - \Delta_{g_1}\right) h_1(x_1,t)\cdot h_2(x_2,t) + h_1(x_1,t)\cdot\left(\frac{\partial}{\partial t} - \Delta_{g_2}\right) h_2(x_2,t) \\
&= 0
\end{aligned}$$

and for any smooth function with compact support f on $\mathcal{M}_1 \times \mathcal{M}_2$

$$\begin{aligned}
&\lim_{t\to 0^+}\int_{\mathcal{M}_1\times\mathcal{M}_2} h_1(x_1,t)\, h_2(x_2,t)\, f(x_1,x_2)\, d\mu_{g_1+g_2}(x_1,x_2) \\
&= \lim_{t\to 0^+}\int_{\mathcal{M}_1} h_1(x_1,t)\left(\int_{\mathcal{M}_2} h_2(x_2,t)\, f(x_1,x_2)\, d\mu_{g_2}(x_2)\right) d\mu_{g_1}(x_1) \\
&= \lim_{t\to 0^+}\int_{\mathcal{M}_1} h_1(x_1,t)\, f(x_1,y_2)\, d\mu_{g_1}(x_1) \\
&= f(y_1,y_2).
\end{aligned}$$

For the second equality we used

$$\left|\int_{\mathcal{M}_2} h_2(x_2,t)\, f(x_1,x_2)\, d\mu_{g_2}(x_2) - f(x_1,y_2)\right| \leq Ct,$$

where $C < \infty$ is independent of x_1, which holds since f is smooth with compact support.

EXAMPLE E.25 (Flat plane bundles over $\mathcal{S}^1$). Given $A \in \mathrm{O}(2)$, consider the flat rank 2 real vector bundle $\mathcal{V}$ over the circle $\mathcal{S}^1 \doteqdot \mathbb{R}/\mathbb{Z} = [0,1]/\tilde{}$, where $0\tilde{}1$, defined by

$$\mathcal{V} = \left([0,1]\times\mathbb{R}^2\right)/\approx,$$

where $(0, A(x)) \approx (1, x)$ for $x \in \mathbb{R}^2$. Note that $\mathcal{V}$ is orientable if $A \in \mathrm{SO}(2)$ and $\mathcal{V}$ is nonorientable if $A \in \mathrm{O}(2) - \mathrm{SO}(2)$. The standard flat metric on $[0,1]\times\mathbb{R}^2$ induces a flat Riemannian metric on $\mathcal{V}$.

Generalizing the above example, consider the following situation. Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold and suppose that $\pi : \mathcal{V} \to \mathcal{M}$ is a flat rank k real vector bundle over $\mathcal{M}$, where locally $(\mathcal{V}, g_\mathcal{V})$ is isometric to the product $\mathcal{M} \times \mathbb{E}^k$ and parallel translation defines local isometries from $\left(\pi^{-1}(\mathcal{U}), g_\mathcal{V}\right)$ to $(\mathcal{U}, g) \times \mathbb{E}^k$ for sufficiently small neighborhoods $\mathcal{U}$ in $\mathcal{M}$. Note that if $(\mathcal{M}, g)$ is flat, then $(\mathcal{V}, g_\mathcal{V})$ is also flat (as a Riemannian manifold).

Let $h : \mathcal{M} \times (0, \infty) \to \mathbb{R}$ be a fundamental solution to the heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) h = 0,$$
$$\lim_{t \to 0^+} h = \delta_y,$$

where $y \in \mathcal{M}$. The fundamental solution on Euclidean space $\mathbb{E}^k = \left(\mathbb{R}^k, g_{\mathbb{R}^k}\right)$, centered at the origin, is given by

$$h_{\mathbb{R}^k}(w) = (4\pi t)^{-k/2} e^{-|w|^2/4t},$$

where $w = \left(w^1, \ldots, w^k\right)$ and $|w|^2 = \sum_{i=1}^k \left(w^i\right)^2$. Given $z \in \mathcal{V}$, let $|z|$ be the norm of z considered as a vector in the fiber $\mathcal{V}_{\pi(z)}$ (this is the same as the distance of z to the 0-section of $\mathcal{V}$). Given $y \in \mathcal{M}$, let $0_y \in \mathcal{V}_y$ denote the zero vector in the fiber over y. We define the function

$$h_\mathcal{V}(z, t) \doteqdot h(\pi(z), t) \cdot (4\pi t)^{-k/2} e^{-|z|^2/4t}.$$

Then $h_\mathcal{V} : \mathcal{V} \times (0, \infty) \to \mathbb{R}$ is a fundamental solution centered at $0_y \in \mathcal{V}$. That is, a fundamental solution of $\mathcal{V}$, when centered at a point on the 0-section, can be well-defined as a 'product' of fundamental solutions of $(\mathcal{M}, g)$ and Euclidean space.

Consider the circle of length 1, i.e., $\mathcal{S}^1 \doteqdot \mathbb{R}/\mathbb{Z}$.[5] Its universal cover is $\mathbb{Z} \to \mathbb{R} \to \mathcal{S}^1$ with the deck transformations given by the translations by the integers. The fundamental solution on $\mathbb{R}$ centered at 0 is

$$h_\mathbb{R}(x, t) = (4\pi t)^{-1/2} e^{-x^2/4t}.$$

This induces the fundamental solution on $\mathcal{S}^1$, which as a $\mathbb{Z}$-invariant function on $\mathbb{R} \times (0, \infty)$, is defined by

$$h_{\mathcal{S}^1}(x, t) = (4\pi t)^{-1/2} \sum_{m \in \mathbb{N}} e^{-(x+m)^2/4t}.$$

By $\mathbb{Z}$-invariant we mean $h_{\mathcal{S}^1}(x + \ell, t) = h_{\mathcal{S}^1}(x, t)$ for all $\ell \in \mathbb{N}$, $x \in \mathbb{R}$, and $t > 0$.

[5]A nice treatment of the discussion in this paragraph is given in Mumford [**372**].

3. Li–Yau differential Harnack estimate

We now recall the seminal differential Harnack estimate of Li and Yau [**335**].

THEOREM E.26 (Li–Yau estimate—$\mathrm{Rc} \geq -kg$). *If $(\mathcal{M}^n, g)$ is a complete Riemannian manifold with* $\mathrm{Rc} \geq -k$ *and if u is a positive solution to the heat equation, then for any $\alpha > 1$,*

$$\frac{\partial}{\partial t} \log u - \frac{1}{\alpha} |\nabla \log u|^2 + \alpha \left(\frac{n}{2t} + \frac{nk}{2(\alpha - 1)} \right) \geq 0. \tag{E.37}$$

When $k = 0$, by taking $\alpha \to 1$, we obtain (16.8):

$$\frac{\partial}{\partial t} \log u - |\nabla \log u|^2 + \frac{n}{2t} \geq 0.$$

Theorem E.26 is an immediate consequence of taking $R \to \infty$ in the following.

PROPOSITION E.27 (Li–Yau estimate on balls for $\mathrm{Rc} \geq -k$). *If $(\mathcal{M}^n, g)$ is a complete Riemannian manifold with* $\mathrm{Rc} \geq -k$ *and if u is a positive solution to the heat equation, then for any $\alpha > 1$ we have*

(E.38)
$$\sup_{B(p,R)} \left(\frac{\partial L}{\partial t} - \frac{1}{\alpha} |\nabla L|^2 \right) + \alpha \left(\frac{n}{2t} + \frac{nk}{2(\alpha - 1)} \right) + \frac{C\alpha}{R^2} \left(\frac{\alpha^2}{\alpha - 1} + \sqrt{k}R \right) \geq 0,$$

where $B(p, R) \doteqdot \{x \in \mathcal{M} : d(x, p) < R\}$ and $C < \infty$.

The idea of the proof of the proposition is as follows. Define a cutoff function by[6]

$$\phi(x) \doteqdot \psi \left(\frac{d(x,p)}{R} \right),$$

where the C^∞ function $\psi : [0, \infty) \to [0, 1]$ is defined so that

(1) $\psi = 1$ for $t \in [0, 1]$ and $\psi = 0$ for $t \geq 2$,
(2) $\psi' \leq 0$, $\frac{|\psi'|^2}{\psi} \leq C$, and $|\psi''| \leq C$.

Define the gradient quantity

$$P \doteqdot \frac{\partial}{\partial t} \log u - (1 - \varepsilon) |\nabla \log u|^2 = \Delta \log u + \varepsilon |\nabla \log u|^2, \tag{E.39}$$

where $\varepsilon \geq 0$ is to be chosen below (when $k = 0$, it is optimal to take $\varepsilon = 0$).[7] One may compute that

$$\frac{\partial P}{\partial t} \geq \Delta P + 2 \langle \nabla \log u, \nabla P \rangle + \frac{2}{n}(1-\varepsilon) (\Delta \log u)^2 - 2(1-\varepsilon) k |\nabla \log u|^2 \tag{E.40}$$

[6]The cutoff function is introduced to handle the case where $\mathcal{M}$ is *noncompact*; it is also useful in that it localizes the estimate.

[7]Note that, in the case where u is the Euclidean heat kernel, P vanishes everywhere on $\mathbb{R}^n \times (0, \infty)$ when $\varepsilon = 0$.

(note that, when $k = \varepsilon = 0$, this says $\frac{\partial P}{\partial t} \geq \triangle P + 2 \langle \nabla \log u, \nabla P \rangle + \frac{2}{n} P^2$). Consider any time interval $[0, T]$, where $T \in (0, \infty)$, and suppose $(x_0, t_0) \in B(p, 2R) \times [0, T]$ is a point at which ϕP attains a *negative* minimum. Using that at (x_0, t_0) we have $\nabla(\phi P) = 0$, $\Delta(\phi P) \geq 0$, and $\frac{\partial}{\partial t}(\phi P) \leq 0$, one obtains after some calculation

$$0 \leq -\frac{2t}{n\alpha^2}(\phi P)^2 - (\phi P)\left(1 + \frac{C}{R^2}\left(\frac{\alpha^2}{\alpha - 1} + R\sqrt{k}\right)t\right) + \frac{nk^2\alpha^2 t}{2(\alpha - 1)^2}.$$

This then implies for any $x \in B(p, 2R)$

$$(\phi P)(x, T) \geq (\phi P)(x_0, t_0) \geq -\frac{n\alpha^2}{2T} - \frac{Cn\alpha^2}{2R^2}\left(\frac{\alpha^2}{\alpha - 1} + R\sqrt{k}\right) - \frac{n\alpha^2 k}{2(\alpha - 1)}.$$

The result follows since $\phi \equiv 1$ in $B(p, R)$ and T is arbitrary.

Integrating the inequality in Theorem E.26 along space-time paths yields the following.

THEOREM E.28 (Path-integrated Harnack estimate for $\mathrm{Rc} \geq -k$). *If $(\mathcal{M}^n, g)$ is a complete Riemannian manifold with $\mathrm{Rc} \geq -k$ and if u is a positive solution to the heat equation, then for any $\alpha > 1$ we have*

$$\text{(E.41)} \qquad \frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-\alpha\frac{n}{2}} \exp\left\{-\frac{\alpha}{4}\frac{d(x_1, x_2)^2}{t_2 - t_1} - \frac{n\alpha k}{2(\alpha - 1)}(t_2 - t_1)\right\}$$

for any $x_1, x_2 \in \mathcal{M}$ and $0 \leq t_1 < t_2$.

Note that, taking $k = 0$ and $\alpha \to 1$, we obtain (16.10):

$$\text{(E.42)} \qquad \frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-n/2} e^{-\frac{d(x_1, x_2)^2}{4(t_2 - t_1)}}.$$

In the case of a fundamental solution for $k = 0$ we have

$$H(x, y, t) \geq H(x, x, t_1)\left(\frac{t}{t_1}\right)^{-n/2} e^{-\frac{d(x, y)^2}{4(t - t_1)}}.$$

Since $\lim_{t_1 \to 0_+} (4\pi t_1)^{n/2} H(x, x, t_1) = 1$, we obtain (16.12):

$$H(x, y, t) \geq (4\pi t)^{-n/2} e^{-\frac{d(x, y)^2}{4t}}.$$

EXERCISE E.29 (Heat equation under Cheeger–Gromov convergence). Suppose that $\{(\mathcal{M}_i^n, g_i, x_i)\}$ is a sequence of pointed complete Riemannian manifolds with $\mathrm{Rc}(g_i) \geq 0$ which converges in the pointed C^∞ Cheeger–Gromov sense to a complete Riemannian manifold $(\mathcal{M}_\infty^n, g_\infty, x_\infty)$ (note that this limit has $\mathrm{Rc}(g_\infty) \geq 0$ but may not have bounded curvature even if each manifold in the sequence does). That is, there exist an exhaustion $\{U_i\}$ of $\mathcal{M}_\infty$ by open sets with $x_\infty \in U_i$ and a sequence of diffeomorphisms $\Phi_i : U_i \to V_i \doteqdot \Phi_i(U_i) \subset \mathcal{M}_i$ with $\Phi_i(x_\infty) = x_i$ such that $\left(U_i, \Phi_i^*\left[g_i|_{V_i}\right]\right)$ converges in C^∞ to $(\mathcal{M}_\infty, g_\infty)$ uniformly on compact sets in $\mathcal{M}_\infty$. Under what conditions does a sequence $\{u_i : \mathcal{M}_i \times (0, T) \to (0, \infty)\}$ of positive solutions of the heat equation admit a subsequence for which the functions $u_i \circ \Phi_i : U_i \to (0, \infty)$

converge to a positive solution $u_\infty : \mathcal{M}_\infty \to (0, \infty)$ of the heat equation? One may ask the same question assuming a uniform lower bound for the Ricci curvatures.

4. Gradient estimates for the heat equation

The following is Hamilton's Theorem 1.1 of [**249**].

THEOREM E.30 (Gradient estimate for bounded solutions). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold with $\mathrm{Rc} \geq -k_1$, where $k_1 \geq 0$. If u is a positive solution to the heat equation with $u \leq A$, then*

$$t \frac{|\nabla u|^2}{u^2} \leq (1 + 2k_1 t) \log\left(\frac{A}{u}\right). \tag{E.43}$$

PROOF. Since $\frac{\partial u}{\partial t} = \Delta u$, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log u = |\nabla \log u|^2$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla \log u|^2$$
$$= -2 |\nabla\nabla \log u|^2 - 2\,\mathrm{Rc}\,(\nabla \log u, \nabla \log u) + 2 \left\langle \nabla \log u, \nabla |\nabla \log u|^2 \right\rangle.$$

Hence, using $\mathrm{Rc} \geq -k_1$, we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right) \left(t |\nabla \log u|^2\right) \tag{E.44}$$
$$\leq (1 + 2k_1 t) |\nabla \log u|^2 + 2 \left\langle \nabla \log u, \nabla \left(t |\nabla \log u|^2\right) \right\rangle.$$

On the other hand,

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log\left(\frac{A}{u}\right) = - |\nabla \log u|^2$$
$$= 2 \left\langle \nabla \log u, \nabla \log\left(\frac{A}{u}\right) \right\rangle + |\nabla \log u|^2,$$

where we rewrote the equation so as to match the gradient term in (E.44), and then

(E.45)
$$\left(\frac{\partial}{\partial t} - \Delta\right) \left((1 + 2k_1 t) \log\left(\frac{A}{u}\right)\right) = 2 \left\langle \nabla \log u, \nabla \left((1 + 2k_1 t) \log\left(\frac{A}{u}\right)\right) \right\rangle$$
$$+ (1 + 2k_1 t) |\nabla \log u|^2 + 2k_1 \log\left(\frac{A}{u}\right).$$

Since $\log\left(\frac{A}{u}\right) \geq 0$ and $k_1 \geq 0$, we conclude from (E.44) and (E.45) that

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right) \left(t |\nabla \log u|^2 - (1 + 2k_1 t) \log\left(\frac{A}{u}\right)\right) \leq 0.$$

Now $t\left|\nabla \log u\right|^2-(1+2k_1t)\log\left(\frac{A}{u}\right)\leq 0$ at $t=0$. Hence the weak maximum principle implies

$$t\left|\nabla \log u\right|^2-(1+2k_1t)\log\left(\frac{A}{u}\right)\leq 0$$

for all $t\geq 0$. □

EXERCISE E.31 (An extension of the gradient estimate). Let $(\mathcal{M}^n, g(t))$ be a closed Riemannian manifold evolving by

$$\frac{\partial}{\partial t}g_{ij}=-2h_{ij}$$

with $R_{ij}\geq -k_1g_{ij}$, $h_{ij}\leq k_1g_{ij}$, $V\leq k_1$, and $\left|\nabla V\right|^2\leq k_1$, where $k_1\geq 0$. Show that if u is a positive solution to the heat-type equation

$$\frac{\partial u}{\partial t}=\Delta u+Vu$$

with $u\leq A$, then

$$t\left|\nabla \log u\right|^2\leq 2(1+4k_1t)\log\left(\frac{A}{u}\right)+\left(\frac{T}{4}+2(1+4k_1t)k_1\right)t \tag{E.46}$$

for $t\in[0,T]$.

SOLUTION TO EXERCISE E.31. Since $\frac{\partial u}{\partial t}=\Delta u+Vu$,

$$\left(\frac{\partial}{\partial t}-\Delta\right)\log u=\left|\nabla \log u\right|^2+V$$

and, using $\frac{\partial}{\partial t}g_{ij}=-2h_{ij}$, we compute

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\left|\nabla \log u\right|^2&=-2\left|\nabla\nabla \log u\right|^2-2(\operatorname{Rc}-h)(\nabla \log u,\nabla \log u)\\&\quad+2\left\langle\nabla \log u,\nabla\left|\nabla \log u\right|^2\right\rangle+2\left\langle\nabla \log u,\nabla V\right\rangle.\end{aligned}$$

Since $R_{ij}\geq -k_1g_{ij}$ and $h_{ij}\leq k_1g_{ij}$, this implies

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\left(t\left|\nabla \log u\right|^2\right)&\leq(1+4k_1t)\left|\nabla \log u\right|^2\\&\quad+2\left\langle\nabla \log u,\nabla\left(t\left|\nabla \log u\right|^2\right)\right\rangle\\&\quad+2t\left\langle\nabla \log u,\nabla V\right\rangle.\end{aligned}\tag{E.47}$$

Hence, using $\left|\nabla V\right|^2\leq k_1$, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\Delta-2\nabla\log u\cdot\nabla\right)\left(t\left|\nabla \log u\right|^2\right)\\&\leq 2(1+4k_1t)\left|\nabla \log u\right|^2+\frac{t^2}{1+4k_1t}\left|\nabla V\right|^2\\&\leq 2(1+4k_1T)\left|\nabla \log u\right|^2+\frac{T}{4}\end{aligned}$$

provided $t \leq T$. On the other hand,

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log\left(\frac{A}{u}\right) = - |\nabla \log u|^2 - V$$
$$\geq 2\left\langle \nabla \log u, \nabla \log\left(\frac{A}{u}\right)\right\rangle + |\nabla \log u|^2 - k_1,$$

where we used $V \leq k_1$, and then

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right)\left(2(1+4k_1 t)\log\left(\frac{A}{u}\right)\right) \geq 2(1+4k_1 t)|\nabla \log u|^2 - 2(1+4k_1 t)k_1. \tag{E.48}$$

Combining (E.47) and (E.48), we obtain

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right)\left(t|\nabla \log u|^2 - 2(1+4k_1 t)\log\left(\frac{A}{u}\right)\right)$$
$$\leq \frac{T}{4} + 2(1+4k_1 t)k_1 \doteqdot C_1.$$

The weak maximum principle implies

$$t|\nabla \log u|^2 \leq 2(1+4k_1 t)\log\left(\frac{A}{u}\right) + C_1 t$$

for all $t \geq 0$.

As an application of the gradient estimate we may bound the maximum of a positive solution by its integral (where the bound gets worse as $t \to 0$). The following is Corollary 1.2 of [**249**].

COROLLARY E.32. *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold. There exists a constant $C < \infty$ depending only on $(\mathcal{M}^n, g)$ and $T < \infty$ such that if u is a positive solution to the heat equation, then*

$$u(y,t) \leq Ct^{-n/2} \int_{\mathcal{M}} u(x,t)\, d\mu(x)$$

for all $y \in \mathcal{M}$ and $t \in [0,T]$.

REMARK E.33. Qualitatively, this estimate is sharp as can be seen by considering the fundamental solution to the heat equation.

PROOF. Let $k_1 = \max\{0, -\min_{\mathcal{M}} \mathrm{Rc}\}$ so that $\mathrm{Rc} \geq -k_1$ and $k_1 \geq 0$. Since $\mathcal{M}$ is compact, there exists $(\bar{x}, \bar{t}) \in \mathcal{M} \times [0,T]$ such that

$$\bar{t}^{n/2} u(\bar{x}, \bar{t}) = \max_{\mathcal{M}\times[0,T]} t^{n/2} u(x,t) \doteqdot J.$$

For $t \in [\bar{t}/2, \bar{t}]$, we have

$$u(x,t) \leq t^{-n/2} J \leq \left(\frac{\bar{t}}{2}\right)^{-n/2} J \doteqdot A.$$

Note that

$$u(\bar{x}, \bar{t}) = \bar{t}^{-n/2} J = 2^{-n/2} A. \tag{E.49}$$

Let $L(x,t) \doteqdot \log\left(\frac{A}{u(x,t)}\right)$. We may apply Theorem E.30 to the solution $u\left(t+\frac{\bar{t}}{2}\right)$ for $t \in [0, \bar{t}/2]$ to obtain

$$\frac{\bar{t}}{2}\left|\nabla L(x,\bar{t})\right|^2 \leq (1+k_1\bar{t})\, L(x,\bar{t})$$

for all $x \in \mathcal{M}$. In other words,

$$\left|\nabla\sqrt{L}(x,\bar{t})\right| \leq \sqrt{\frac{1+k_1\bar{t}}{2\bar{t}}} \leq \frac{C_1}{\sqrt{\bar{t}}},$$

where $C_1 \doteqdot \frac{\sqrt{1+k_1T}}{2}$. By integrating this estimate along minimal geodesics, we obtain for any $x, \bar{x} \in \mathcal{M}$ and $\bar{t} \in (0,T]$

$$\sqrt{L}(x,\bar{t}) - \sqrt{L}(\bar{x},\bar{t}) \leq \frac{C_1}{\sqrt{\bar{t}}} d(x,\bar{x}).$$

Hence, if $d(x,\bar{x}) \leq \sqrt{\bar{t}}$, then

$$\sqrt{L}(x,\bar{t}) \leq \sqrt{L}(\bar{x},\bar{t}) + C_1 = \sqrt{\frac{n}{2}\log 2} + C_1 \doteqdot C_2,$$

where we used (E.49).

In terms of the original solution u, this says that if $d(x,\bar{x}) \leq \sqrt{\bar{t}}$, then

$$u(x,\bar{t}) = Ae^{-L(x,\bar{t})} \geq Ae^{-C_2^2} \doteqdot c_3 A,$$

where $c_3 > 0$ depends only on $(\mathcal{M}^n, g)$ and T. Since $\bar{t} \leq T$, we have

$$\operatorname{Vol} B\left(\bar{x}, \sqrt{\bar{t}}\right) \geq c_4 \bar{t}^{n/2},$$

where $c_4 > 0$ depends only on $(\mathcal{M}^n, g)$ and T. From all of this we conclude that

$$\begin{aligned}\int_{\mathcal{M}} u(x,\bar{t})\, d\mu(x) &\geq \int_{B(\bar{x},\sqrt{\bar{t}})} u(x,\bar{t})\, d\mu(x) \\ &\geq c_3 A \operatorname{Vol} B\left(\bar{x},\sqrt{\bar{t}}\right) \\ &\geq c_3 c_4 A \bar{t}^{n/2},\end{aligned}$$

that is,

$$\begin{aligned}t^{n/2} u(y,t) \leq \bar{t}^{n/2} u(\bar{x},\bar{t}) &\leq 2^{-n/2} A \bar{t}^{n/2} \\ &\leq C_5 \int_{\mathcal{M}} u(x,\bar{t})\, d\mu(x) \\ &= C_5 \int_{\mathcal{M}} u(x,t)\, d\mu(x)\end{aligned}$$

for all $y \in \mathcal{M}$ and $t \in [0,T]$, where we used $\int_{\mathcal{M}} u(x,t)\, d\mu(x) = \text{const}$. This completes the proof of the corollary. □

By Corollary E.32 we may prove a version of Theorem E.30 which does not assume a pointwise bound on u. The following is Corollary 1.3 of [**249**].

COROLLARY E.34. *If u is a positive solution to the heat equation on $\mathcal{M} \times [0,T]$ with*

$$\int_{\mathcal{M}} u(y,t)\, d\mu(y) = 1,$$

then

$$t\,|\nabla \log u|^2 \leq C \log\left(\frac{B}{t^{n/2}u}\right)$$

on $\mathcal{M} \times (0,T]$, where C and B depend only on $(\mathcal{M}, g)$ and T.

PROOF. Given any $t_0 \in (0,T]$, consider the time interval $[\frac{t_0}{2}, t_0]$. By Corollary E.32 we have for every $x \in \mathcal{M}$ and $t \in [\frac{t_0}{2}, t_0]$,

$$u(x,t) \leq Ct^{-n/2} \leq 2^{n/2} C t_0^{-n/2} \doteqdot A.$$

We now can apply Theorem E.30 on this same time interval $[\frac{t_0}{2}, t_0]$ to obtain that for all $x_0 \in \mathcal{M}$

$$\frac{t_0}{2}\,|\nabla \log u|^2 (x_0, t_0) \leq (1 + k_1 t_0) \log\left(\frac{2^{n/2} C t_0^{-n/2}}{u(x_0,t_0)}\right).$$

□

EXERCISE E.35 (Heat operator of ratios). Show that for any function F and positive function G, we have

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log G \cdot \nabla\right)\left(\frac{F}{G}\right) = \frac{1}{G}\left(\frac{\partial}{\partial t} - \Delta\right) F - \frac{F}{G^2}\left(\frac{\partial}{\partial t} - \Delta\right) G.$$

THEOREM E.36 (Laplacian estimate for bounded solutions). *Let $(\mathcal{M}^n, g)$ be a closed Riemannian manifold with $\mathrm{Rc} \geq -k_1$, where $k_1 \geq 0$. If u is a positive solution to the heat equation with $u \leq A$, then*

$$\Delta \log u + 2\,|\nabla \log u|^2 \leq \frac{k_1 e^{k_1 t}}{e^{k_1 t} - 1}\left(n + 4 \log\left(\frac{A}{u}\right)\right).$$

PROOF. By the exercise we have

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right)\left(\frac{\Delta u}{u}\right) = 0$$

and

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right)|\nabla \log u|^2 \leq -\frac{2}{n}(\Delta \log u)^2 + 2k_1\,|\nabla \log u|^2,$$

where we used $|\nabla\nabla \log u|^2 \geq \frac{1}{n}(\Delta \log u)^2$. Hence for any function $\phi(t)$ we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right)\left(\phi(t)\left(\frac{\Delta u}{u} + |\nabla \log u|^2\right)\right) \\
&\leq -\frac{2}{n}\phi(t)(\Delta \log u)^2 + 2k_1\phi(t)\,|\nabla \log u|^2 + \phi'(t)\left(\Delta \log u + 2\,|\nabla \log u|^2\right) \\
&= -\frac{2}{n}\phi(t)(\Delta \log u)^2 + \phi'(t)\,\Delta \log u + 2\left(\phi'(t) + k_1\phi(t)\right)|\nabla \log u|^2,
\end{aligned}$$

where we used $\frac{\Delta u}{u} + |\nabla \log u|^2 = \Delta \log u + 2 |\nabla \log u|^2$. On the other hand,

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right) \log\left(\frac{A}{u}\right) = |\nabla \log u|^2 .$$

Hence

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right)\left(\phi\left(\Delta \log u + 2|\nabla \log u|^2\right) - B - 4\log\left(\frac{A}{u}\right)\right)$$
$$\leq -\frac{2}{n}\phi(\Delta \log u)^2 + \phi' \Delta \log u + 2\left(\phi' + k_1\phi - 2\right)|\nabla \log u|^2 .$$

Now choose $\phi(t) = \frac{e^{k_1 t}-1}{k_1 e^{k_1 t}}$ so that $\phi'(t) = \frac{1}{e^{k_1 t}}$ and $\phi' + k_1\phi = 1$; note also that $\phi(t) \in [0, 1/k_1)$. Then

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right)\left(\phi\left(\Delta \log u + 2|\nabla \log u|^2\right) - n - 4\log\left(\frac{A}{u}\right)\right)$$
$$\leq -\frac{2}{n}\phi(\Delta \log u)^2 + \phi' \Delta \log u - 2|\nabla \log u|^2 \doteqdot J.$$

At any point where

$$\phi\left(\Delta \log u + 2|\nabla \log u|^2\right) \geq n + 4\log\left(\frac{A}{u}\right),$$

we have either $\phi\Delta \log u \geq n/2$ or $2\phi|\nabla \log u|^2 \geq n/2$. If $\phi\Delta \log u \geq n/2$, then

$$J \leq \left(\phi' - 1\right)\Delta \log u - 2|\nabla \log u|^2 \leq 0.$$

On the other hand, if $2\phi|\nabla \log u|^2 \geq n/2$, then

$$J \leq -\frac{2}{n}\phi(\Delta \log u)^2 + \phi' \Delta \log u - \frac{n}{2\phi}$$
$$\leq \frac{n}{2\phi}\left(\frac{(\phi')^2}{4} - 1\right) \leq 0.$$

Let $\Phi \doteqdot \phi\left(\Delta \log u + 2|\nabla \log u|^2\right) - n - 4\log\left(\frac{A}{u}\right)$. We have shown that at any point where $\Phi \geq 0$ we have

$$\left(\frac{\partial}{\partial t} - \Delta - 2\nabla \log u \cdot \nabla\right)\Phi \leq 0.$$

On the other hand, $\Phi \leq 0$ at $t = 0$. Hence by the maximum principle we conclude that $\Phi \leq 0$ for $t \geq 0$. □

APPENDIX F

Tensor Calculus on the Frame Bundle

Tell it to me slowly.

– From "Time of the Season" by The Zombies

1. Introduction

In this appendix we describe a formalism for performing tensor calculations on both fixed and evolving (by Ricci flow) Riemannian manifolds. We use this formalism in computing the evolution equation satisfied by the differential Harnack quadratic for the Ricci flow in Chapter 15.

In particular we shall view tensors on a differentiable manifold $\mathcal{M}^n$ as vector-valued ($\mathbb{R}^k$) functions on the frame bundle by evaluating the tensors on the vectors in the frames. We define natural, globally defined frame fields on the frame bundle $F\mathcal{M}$ of a Riemannian manifold $(\mathcal{M}^n, g)$ consisting of vertical and horizontal vector fields. These frame fields are orthonormal when restricted/projected to the orthonormal frame bundle $O\mathcal{M}$.[1] Via the correspondence between tensors and vector-valued functions, the covariant derivatives of tensors on $(\mathcal{M}^n, g)$ become the directional derivatives of the corresponding functions on $F\mathcal{M}$ with respect to the horizontal vectors in the globally defined frame fields.[2]

The directional derivatives with respect to the vertical vectors act algebraically on the functions on $F\mathcal{M}$ corresponding to tensors on $\mathcal{M}$ because the functions are invariant under the action of $\mathrm{GL}(n, \mathbb{R})$.

We find it convenient to confirm these formulas by using local coordinates $\{x^i\}$ on $\mathcal{M}$, which induce local coordinates on $F\mathcal{M}$. The brackets of the horizontal vector fields in the frame fields on $F\mathcal{M}$ are curvatures contracted with the vertical vector fields; this corresponds to commuting covariant derivatives. The brackets of the vertical vector fields are again vertical, and the bracket of a horizontal and a vertical vector field is horizontal; in both of these cases the bracket operation is algebraic.

The above considerations also apply to the subbundle of orthonormal frames $O\mathcal{M}$. Since we are most interested in a time-dependent metric $g(t)$, which is a solution to Ricci flow on some interval $\mathcal{I}$, we consider the corresponding time-dependent bundles $O\mathcal{M}(t)$. In general, for solutions to the

[1] See Lemma F.12 below.

[2] See Lemma F.14 below.

Ricci flow, we consider the time-derivatives of various tensors, usually tensors in the curvature and its covariant derivatives. These time derivatives correspond to directional derivatives with respect to a 'time' vector field defined on $F\mathcal{M} \times \mathcal{I}$. This time vector field is tangent to the submanifold

$$O\tilde{\mathcal{M}} \doteqdot \bigcup_{t \in \mathcal{I}} O\mathcal{M}(t) \times \{t\} \subset F\mathcal{M} \times \mathcal{I}.$$

With this setup, we can compute the commutator formulas for the heat operator and the covariant derivatives, which we use to carry out the calculations for the proof of the Harnack estimate in Section 4 of Chapter 15 of this volume. Calculations in these frames on $O\tilde{\mathcal{M}}$ are equivalent to those with respect to a moving orthonormal frame (or more generally, using Uhlenbeck's trick), as described in Section 2 of Chapter 6 in Volume One.

The interested reader may wish to keep the following in mind while reading this appendix.

PROBLEM F.1. Derive the main formulas (F.2), (F.8), (F.9), (F.12), (F.13), and (F.14) on $F\mathcal{M}$ invariantly (without using local coordinates). Do the same for $O\mathcal{M}$ and the time-dependent case.

2. Tensors as vector-valued functions on the frame bundle

Let $\mathcal{M}^n$ be a differentiable manifold and let $\pi : F\mathcal{M} \to \mathcal{M}^n$ denote the general linear frame bundle, which is a $\mathrm{GL}(n, \mathbb{R})$-principal bundle over $\mathcal{M}^n$ with fibers $F\mathcal{M}_x = \pi^{-1}(x)$ consisting of ordered bases (frames) for the tangent space $T\mathcal{M}_x$. Our convention shall be that $\mathrm{GL}(n, \mathbb{R})$ acts on the left on $F\mathcal{M}$. As described below, a tensor of degree r on $\mathcal{M}^n$ is equivalent to a $\mathrm{GL}(n, \mathbb{R})$-equivariant function on $F\mathcal{M}$ with values in $\mathbb{R}^{n^r} \cong \bigotimes^r \mathbb{R}^n$.

Given a tensor and a frame at a point in $\mathcal{M}^n$, we obtain a Euclidean vector whose components are the coefficients of the tensor with respect to that frame. Let U be a tensor of type (p, q), and consider a frame $Y = \{Y_a\}_{a=1}^n$ at a point $x \in \mathcal{M}^n$. Let $Y^* = \left\{(Y^*)^b\right\}_{b=1}^n$ denote the dual coframe of 1-forms at x defined by

$$(Y^*)^b (Y_a) \doteqdot \delta_a^b.$$

Then $\left\{Y_{a_1} \otimes \cdots \otimes Y_{a_q} \otimes (Y^*)^{b_1} \otimes \cdots \otimes (Y^*)^{b_p}\right\}_{a_1,\ldots,a_q,b_1,\ldots,b_p=1}^n$ is a basis for $(\bigotimes^q T\mathcal{M}_x) \otimes (\bigotimes^p T^*\mathcal{M}_x)$. Using this basis, the components

$$U_{b_1 \cdots b_p}^{a_1 \cdots a_q} : F\mathcal{M} \to \mathbb{R}$$

are defined by

$$U_{b_1 \cdots b_p}^{a_1 \cdots a_q} (Y) \doteqdot U\left((Y^*)^{a_1}, \ldots, (Y^*)^{a_q}, Y_{b_1}, \ldots, Y_{b_p}\right).$$

Note that U can be written

$$U = \sum_{a_1,\ldots,a_q,b_1,\ldots,b_p=1}^{n} U_{b_1\cdots b_p}^{a_1\cdots a_q}(Y) \cdot Y_{a_1} \otimes \cdots \otimes Y_{a_q} \otimes (Y^*)^{b_1} \otimes \cdots \otimes (Y^*)^{b_p}$$
$$\in \left(\bigotimes{}^q T\mathcal{M}_x\right) \otimes \left(\bigotimes{}^p T^*\mathcal{M}_x\right)$$

for any $Y \in F\mathcal{M}$ and where at a point $x \in \mathcal{M}$ the LHS (and hence the RHS) is independent of $Y \in F\mathcal{M}_x$.

DEFINITION F.2. Every tensor U of type (p,q) has a corresponding vector-valued function $U^\dagger$ on the frame bundle

$$U^\dagger : F\mathcal{M} \to \mathbb{R}^{n^{p+q}}$$

defined by

$$U^\dagger(Y) \doteqdot \left(U_{b_1\cdots b_p}^{a_1\cdots a_q}(Y)\right)_{a_1,\ldots,a_q,b_1,\ldots,b_p=1}^{n}.$$

We shall often abuse notation and write $U^\dagger$ as U.

Note that if $Y = \{Y_a\}_{a=1}^n$ is a frame, then

$$U\left(Y_{b_1},\ldots,Y_{b_p}\right) = \sum_{a_1,\ldots,a_q=1}^{n} U_{b_1\cdots b_p}^{a_1\cdots a_q}(Y) \cdot Y_{a_1} \otimes \cdots \otimes Y_{a_q}.$$

(Here we are not abusing notation since we are considering U as a tensor on $\mathcal{M}$.) For example, if V is a $(p,0)$-tensor, the components $V_{b_1\cdots b_p} : F\mathcal{M} \to \mathbb{R}$ satisfy

$$V_{b_1\cdots b_p}(Y) \doteqdot \left[V^\dagger(Y)\right]_{b_1\cdots b_p} = V\left(Y_{b_1},\ldots,Y_{b_p}\right).$$

We emphasize that $V_{b_1\cdots b_p}$ is a real-valued function on $F\mathcal{M}$ (we have one for each choice of $b_1,\ldots,b_p$) while V is a tensor on $\mathcal{M}$.

The effect of a change of frame on the components of a tensor U is given by the following:

LEMMA F.3 ($\mathrm{GL}(n,\mathbb{R})$-invariance of $U^\dagger$). *If Y and $Z = \{Z_b\}_{b=1}^n$ are frames with*

$$Z_b = G_b^a Y_a, \qquad \textit{where } G = (G_b^a)_{a,b=1}^n \in \mathrm{GL}(n,\mathbb{R}),$$

then

$$U_{b_1\cdots b_p}^{a_1\cdots a_q}(Z) = \left(G^{-1}\right)_{c_1}^{a_1} \cdots \left(G^{-1}\right)_{c_q}^{a_q} G_{b_1}^{d_1} \cdots G_{b_p}^{d_p} \cdot U_{d_1\cdots d_p}^{c_1\cdots c_q}(Y),$$

where we sum over repeated indices.

For example, if V is a 1-form on $\mathcal{M}$, then $V_b(Z) = G_b^a V_a(Y)$. On the other hand, if W is a vector field on $\mathcal{M}$, then $W^b(Z) = \left(G^{-1}\right)_a^b W^a(Y)$.

3. Local coordinates on the frame bundle

In carrying out the various computations, we shall find it convenient to use local coordinates. Let $\{x^i\}_{i=1}^n$ be a local coordinate system defined on an open set $\mathcal{O} \subset \mathcal{M}$. Then $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ is a frame field on $\mathcal{O}$, so it is a section of $\pi_{\mathcal{O}} : F\mathcal{M}|_{\mathcal{O}} \to \mathcal{O}$, where $\pi_{\mathcal{O}}$ is the restriction of the natural projection $\pi : F\mathcal{M} \to \mathcal{M}$. We define functions

$$y_a^i : F\mathcal{M}|_{\mathcal{O}} \to \mathbb{R}$$

by

$$Y_a \doteqdot \sum_{i=1}^n y_a^i (Y) \frac{\partial}{\partial x^i}.$$

The function y_a^i assigns to a frame Y the i-th component of the a-th vector. The vector- (or matrix-) valued function

$$y \doteqdot \left(y_a^i\right)_{i,a=1}^n : F\mathcal{M}|_{\mathcal{O}} \to \mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$$

describes the transition of the frame $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ to the frame $Y = \{Y_a\}_{a=1}^n$. That is, $\{y_a^i\}$ are the components of the frame $\left\{\frac{\partial}{\partial x^i}\right\}$ with respect to the frame $\{Y_a\}$.

REMARK F.4. Note that if $Z_b = G_b^a Y_a$, then $y_b^i(Z) = G_b^a y_a^i(Y)$. The image is in $\mathrm{GL}(n,\mathbb{R})$ because the transition is nondegenerate. For the dual coframes, we have

$$(Y^*)^b = \sum_{i=1}^n \left[y^{-1}\right]_i^b (Y) \; dx^i,$$

where $\left[y^{-1}\right]_i^b : F\mathcal{M}|_{\mathcal{O}} \to \mathbb{R}$ are the entries of the inverse matrix of $\left(y_a^i\right)_{i,a=1}^n$.

Let $\tilde{x}^i = x^i \circ \pi : F\mathcal{M}|_{\mathcal{O}} \to \mathbb{R}$. The collection $\left(\{\tilde{x}^j\}_{j=1}^n, \left(y_a^i\right)_{i,a=1}^n\right)$ is a coordinate system defined on the open set $F\mathcal{M}|_{\mathcal{O}} \subset F\mathcal{M}$. In particular, $\left\{\left\{\frac{\partial}{\partial \tilde{x}^j}\right\}_{j=1}^n, \left\{\frac{\partial}{\partial y_a^i}\right\}_{i,a=1}^n\right\}$ is a basis for the tangent spaces of $F\mathcal{M}$ at points in $F\mathcal{M}|_{\mathcal{O}}$. Note that $\left\{\frac{\partial}{\partial y_a^i}\right\}_{i,a=1}^n$ is a basis for the tangent spaces of the fibers $F\mathcal{M}_x$ for $x \in \mathcal{O}$ (the functions $\tilde{x}^j$ are constant on each $F\mathcal{M}_x$), i.e., $\frac{\partial}{\partial y_a^i}$ are vertical, whereas the $\left\{\frac{\partial}{\partial \tilde{x}^j}\right\}_{j=1}^n$ are transverse to the fibers. Also, we have $\pi_*\left(\frac{\partial}{\partial \tilde{x}^j}\right) = \frac{\partial}{\partial x^j}$.

Let V be a 1-form on $\mathcal{M}$. In local coordinates, $V = V_j dx^j$, where $V_j : \mathcal{O} \to \mathbb{R}$ are functions. It is easy to compute that as functions on $F\mathcal{M}|_{\mathcal{O}}$

$$V_c(Y) \doteqdot \left[V^\dagger(Y)\right]_c = \sum_{j=1}^n y_c^j(Y) \; \tilde{V}_j(Y),$$

where $\tilde{V}_j \doteqdot V_j \circ \pi : F\mathcal{M}|_{\mathcal{O}} \to \mathbb{R}$. Thus the functions $\tilde{V}_j$ and V_a on $F\mathcal{M}|_{\mathcal{O}}$ are related by

$$\tilde{V}_j(Y)\ y_a^j(Y) = V_a(Y).$$

Note that $\tilde{V}_j$ is constant on each fiber $F\mathcal{M}_x$. Later, we shall sometimes denote $\tilde{V}_j$ simply by V_j.

NOTATION F.5. *Throughout this appendix we shall use the indices i, j, k for coordinates on $\mathcal{M}^n$ and a, b, c for components of the frame; for instance the metric in local coordinates is $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, whereas $g_{ab}(Y) = g(Y_a, Y_b)$ for a given frame Y. If $Y_a = y_a^i \frac{\partial}{\partial x^i}$, then $g_{ab} = g_{ij} y_a^i y_b^j$. We shall also use the extended Einstein convention where repeated indices are summed independently of whether they are up or down.*

4. The metric on the frame bundle

Now let g be a Riemannian metric on $\mathcal{M}^n$. We proceed to define the induced Riemannian metric on the principal $\mathrm{GL}(n,\mathbb{R})$-bundle $F\mathcal{M}$. Let $\rho : \mathrm{GL}(n,\mathbb{R}) \times F\mathcal{M} \to F\mathcal{M}$ denote the left action. Given $Y \in F\mathcal{M}$, define

$$\rho_Y : \mathrm{GL}(n,\mathbb{R}) \to F\mathcal{M}$$

by

$$\rho_Y(G) \doteqdot \rho(G, Y) \quad \text{for } G \in \mathrm{GL}(n,\mathbb{R}).$$

We have

$$\rho_Y(G)_a = \sum_{b=1}^n G_a^b Y_b.$$

The derivative

$$(\rho_Y)_* : \mathfrak{gl}(n,\mathbb{R}) \to T_Y(F\mathcal{M}_x)$$

is a vector space isomorphism, where $x = \pi(Y)$. Given $G \in \mathrm{GL}(n,\mathbb{R})$, left multiplication $L_G : F\mathcal{M} \to F\mathcal{M}$ is defined by $L_G(Y) \doteqdot \rho(G, Y)$. The vertical spaces are the tangent spaces to the fibers:

$$V_Y \doteqdot T_Y(F\mathcal{M}_x) = \mathrm{image}\left[(\rho_Y)_*\right],$$

where $x = \pi(Y)$.

The Levi-Civita connection ∇ on the associated tangent bundle $T\mathcal{M}$ induces a connection on the principal $\mathrm{GL}(n,\mathbb{R})$-bundle $F\mathcal{M}$, which is a $\mathrm{GL}(n,\mathbb{R})$-invariant 1-form ω on $F\mathcal{M}$ with values in $\mathfrak{gl}(n,\mathbb{R})$, called the connection 1-form. Defining ω is equivalent to defining $\mathrm{GL}(n,\mathbb{R})$-invariant horizontal subspaces of the tangent spaces of $F\mathcal{M}$ (the null spaces of ω). The horizontal spaces in $F\mathcal{M}$ have the following property. Given a path $\gamma(t)$ in $\mathcal{M}$, a parallel frame $Y(t)$ along $\gamma(t)$ is tangent to the horizontal spaces. The horizontal spaces $H_Y = \ker\omega$ are the kernels of $\omega : T_Y(F\mathcal{M}_x) \to \mathbb{R}$, which are $\mathrm{GL}(n,\mathbb{R})$-invariant:

$$(L_G)_*(H_Y) = H_{L_G(Y)}$$

for all $G \in \mathrm{GL}(n,\mathbb{R})$ and $Y \in F\mathcal{M}$.

Given an inner product h on the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$, the frame bundle has an induced Riemannian metric g^F defined uniquely by the following conditions:

(1) $(\rho_Y)_* : (\mathfrak{gl}(n,\mathbb{R}), h) \to \left(T_Y(F\mathcal{M}_x), g^F\right)$ is an isometry, i.e.,

$$(\rho_Y)^* \left(g^F|_{T_Y(F\mathcal{M}_x)}\right) = h$$

for all $Y \in F\mathcal{M}_x$ and $x \in \mathcal{M}$;

(2) the horizontal subspaces induced by the Levi-Civita connection are the orthogonal complements of the tangent spaces to the fibers with respect to the metric g^F; and

(3) $\pi : \left(F\mathcal{M}, g^F\right) \to (\mathcal{M}, g)$ is a Riemannian submersion, i.e.,

$$(\pi^* g)(V,W) = g^F(V,W)$$

for all horizontal vectors $V, W \in H_Y \subset T_Y F\mathcal{M}$.

Consider the standard basis $\{e(a,b)\}_{a,b=1}^n$ of $\mathfrak{gl}(n,\mathbb{R})$ defined by

$$e(a,b)_c^d \doteqdot \delta_c^a \delta_b^d.$$

We choose the metric h on $\mathfrak{gl}(n,\mathbb{R})$ by taking this basis to be orthonormal with respect to $h = \langle\ ,\ \rangle$, namely

$$\langle e(a,b), e(c,d)\rangle = \delta_f^a \delta_b^e \delta_e^c \delta_d^f = \delta_d^a \delta_b^c.$$

The Lie brackets of these basis elements are given by

$$[e(a,b), e(c,d)] = \delta_b^c e(a,d) - \delta_d^a e(c,b).$$

5. A natural frame field on $F\mathcal{M}$

5.1. Vertical vector fields on $F\mathcal{M}$. We now define linearly independent vertical tangent vector fields on $F\mathcal{M}$ which, together with the horizontal vector fields ∇_a (defined in the next subsection), will form a global frame field (see Lemma F.12) which is orthonormal when restricted to $O\mathcal{M}$.

Definition F.6. For each a and b, the vertical vector field Λ_b^a on $F\mathcal{M}$ is defined by

$$\Lambda_b^a(Y) \doteqdot (\rho_Y)_* e(a,b) \qquad \text{for all } Y \in F\mathcal{M}. \tag{F.1}$$

The $\{\Lambda_b^a\}_{a,b=1}^n$ are linearly independent and infinitesimally represent the left action of $\mathrm{GL}(n,\mathbb{R})$ on $F\mathcal{M}$.

Lemma F.7. *At each point, the vectors $\{\Lambda_b^a\}_{a,b=1}^n$ are orthonormal with respect to g^F. In local coordinates, they are given by*

$$\Lambda_b^a(Y) = \sum_{i=1}^n y_b^i(Y) \left.\frac{\partial}{\partial y_a^i}\right|_Y.$$

PROOF. The $\{\Lambda_b^a\}_{a,b=1}^n$ are orthonormal since they are the push forwards of $\{e(a,b)\}_{a,b=1}^n$ by the isometry ρ_Y and $\{e(a,b)\}_{a,b=1}^n$ are orthonormal with respect to h on $\mathfrak{gl}(n,\mathbb{R})$. Since $\{y_a^i(Y), \tilde{x}^j(Y)\}_{a,i,j=1}^n$ are the local coordinates of Y, if we let

$$\gamma(t) = \exp[te(a,b)] \in \mathrm{GL}(n,\mathbb{R})$$

and $\gamma(t)Y \doteqdot \rho_Y(\gamma(t))$, we can easily compute

$$\begin{aligned}
[(\rho_Y)_*(e(a,b))] &= \left(\frac{d}{dt}\bigg|_{t=0} y_d^i(\gamma(t)Y)\right)\frac{\partial}{\partial y_d^i}\bigg|_Y \\
&\quad + \left(\frac{d}{dt}\bigg|_{t=0} \tilde{x}^j(\gamma(t)Y)\right)\frac{\partial}{\partial \tilde{x}^j}\bigg|_Y \\
&= \delta_d^a\delta_b^e y_e^i(Y)\frac{\partial}{\partial y_d^i}\bigg|_Y \\
&= y_b^i(Y)\frac{\partial}{\partial y_a^i}\bigg|_Y .
\end{aligned}$$

□

The action of the frame bundle vector fields Λ_b^a on tensors by directional differentiation is given by the next lemma. Since Λ_b^a are vertical and tensors on $\mathcal{M}$ correspond to $\mathrm{GL}(n,\mathbb{R})$-invariant functions on $F\mathcal{M}$ (Lemma F.3), their directional derivatives act algebraically.

LEMMA F.8. *If U is a (p,q)-tensor, then*

$$\text{(F.2)}\qquad \begin{aligned}\left(\Lambda_d^c U_{b_1\cdots b_p}^{a_1\cdots a_q}\right)(Y) &= \sum_{\ell=1}^p \delta_{b_\ell}^c U_{b_1\cdots b_{\ell-1}db_{\ell+1}\cdots b_p}^{a_1\cdots a_q}(Y) \\ &\quad - \sum_{k=1}^q \delta_d^{a_k} U_{b_1\cdots b_p}^{a_1\cdots a_{k-1}ca_{k+1}\cdots a_q}(Y).\end{aligned}$$

In particular, if $V_c = y_c^j V_j$ is a 1-form, then

$$(\Lambda_b^a V_c)(Y) = \delta_c^a V_b(Y);$$

if $V_{ab} = y_a^i y_b^j V_{ij}$ is a covariant 2-tensor, then

$$\text{(F.3)}\qquad (\Lambda_b^a V_{cd})(Y) = \delta_c^a V_{bd}(Y) + \delta_d^a V_{cb}(Y).$$

PROOF. If V is a 1-form, then we compute

$$(\Lambda_b^a V_c)(Y) = y_b^i(Y)\frac{\partial}{\partial y_a^i}\bigg|_Y (y_c^j V_j) = y_b^i(Y)\delta_i^j\delta_c^a V_j = \delta_c^a V_b(Y)$$

since $V_j = \tilde{V}_j$ is constant on $F\mathcal{M}_{\pi(Y)}$. In general, if U is a (p,q)-tensor, then we use

$$\frac{\partial}{\partial y_c^m}(y^{-1})_{i_k}^{a_k} = -(y^{-1})_r^{a_k}\left(\frac{\partial}{\partial y_c^m}y_s^r\right)(y^{-1})_{i_k}^s = -(y^{-1})_m^{a_k}(y^{-1})_{i_k}^c$$

and $y_d^m \left(y^{-1}\right)_m^{a_k} = \delta_d^{a_k}$ to compute

$$\left(\Lambda_d^c U_{b_1 \cdots b_p}^{a_1 \cdots a_q}\right)(Y) = y_d^m (Y) \left.\frac{\partial}{\partial y_c^m}\right|_Y \left[\left(y^{-1}\right)_{i_1}^{a_1} \cdots \left(y^{-1}\right)_{i_q}^{a_q} y_{b_1}^{j_1} \ldots y_{b_p}^{j_p}\right] U_{j_1 \cdots j_p}^{i_1 \cdots i_q}(Y)$$
$$= \delta_{b_\ell}^c U_{b_1 \cdots b_{\ell-1} d b_{\ell+1} \cdots b_p}^{a_1 \cdots a_q}(Y) - \delta_d^{a_k} U_{b_1 \cdots b_p}^{a_1 \ldots a_{k-1} c a_{k+1} \cdots a_q}(Y).$$

□

5.2. Horizontal vector fields on $F\mathcal{M}$—the connection. We shall describe the components of the covariant derivatives of tensors on $\mathcal{M}$ with respect to arbitrary frames $\{Y_a\}$ as directional derivatives of the corresponding vector-valued functions with respect to a natural basis $\{\nabla_a\}_{a=1}^n$ of horizontal vector fields on $F\mathcal{M}$. The vector field ∇_a at Y corresponds to parallel translating a frame along a path in $\mathcal{M}^n$ with initial velocity Y_a. Let $x \in \mathcal{M}$, $X \in T_x\mathcal{M}$, and $Y = \{Y_a\} \in F\mathcal{M}_x$. We start with any path $\sigma(t)$ in $\mathcal{M}$ such that $\sigma(0) = x$ and $\sigma'(0) = X$. Now we choose the path

$$\gamma_X(t) = (P_{\sigma(t)}Y_1, \ldots, P_{\sigma(t)}Y_n), \tag{F.4}$$

where $P_{\sigma(t)}$ is parallel transport along $\sigma(t)$. Any such path $\gamma(t)$ is tangent to the horizontal spaces; this property characterizes the horizontal spaces.

DEFINITION F.9. The vector field ∇_a on $F\mathcal{M}$ is defined by

$$\nabla_a|_Y \doteqdot \left.\frac{d}{dt}\right|_{t=0} \gamma_{Y_a}(t)$$

at $Y \in F\mathcal{M}$.

The following lemma is immediate.

LEMMA F.10. *The vector fields ∇_a on $F\mathcal{M}$ are horizontal. In particular, $\nabla_a|_Y \in T_Y(F\mathcal{M})$ is the horizontal lift of $Y_a \in T\mathcal{M}_x$ at $Y \in F\mathcal{M}_x$.*

Next we express the vector fields ∇_a on $F\mathcal{M}$ in terms of local coordinates, recalling that σ is a path on $\mathcal{M}$ starting at x in the direction of Y_a. First note that $P_{\sigma(t)}(Y_b) = y_b^k(Y) P_{\sigma(t)}\left(\frac{\partial}{\partial x^k}\big|_x\right)$. So we need to compute the parallel transport,

$$P_{\sigma(t)}\left(\left.\frac{\partial}{\partial x^k}\right|_x\right) \doteqdot f_k^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\sigma(t)},$$

which is the solution to the differential equation

$$0 = \nabla_{\sigma'(t)}\left(f_k^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\sigma(t)}\right) = \frac{df_k^i}{dt}(t) \left.\frac{\partial}{\partial x^i}\right|_{\sigma(t)} + f_k^i(t)\nabla_{\sigma'(t)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\sigma(t)}\right),$$

$$f_k^i(0) = \delta_k^i.$$

Taking $t = 0$, we have

$$0 = \frac{df_k^i}{dt}(0) \left.\frac{\partial}{\partial x^i}\right|_x + \nabla_{Y_a}\left(\left.\frac{\partial}{\partial x^k}\right|_x\right)$$
$$= \left(\frac{df_k^i}{dt}(0) + y_a^j(Y)\,\Gamma_{jk}^i(x)\right)\left.\frac{\partial}{\partial x^i}\right|_x,$$

which implies

$$\frac{df_k^i}{dt}(0) = -y_a^j(Y)\,\Gamma_{jk}^i(x).$$

The path

$$\gamma_{Y_a}(t) = \left\{P_{\sigma(t)}(Y_b)\right\}_{b=1}^n = \left\{y_b^k(Y)\,f_k^i(t)\left.\frac{\partial}{\partial x^i}\right|_{\sigma(t)}\right\}_{b=1}^n$$

in $F\mathcal{M}$ is tangent to ∇_a at $t = 0$. Since $P_{\sigma(t)}(Y_b) = y_b^i(\gamma_{Ya}(t))\left.\frac{\partial}{\partial x^i}\right|_{\sigma(t)}$, we have

$$y_b^i(\gamma_{Y_a}(t)) = y_b^k(Y)\,f_k^i(t).$$

Hence

$$\left.\frac{d}{dt}\right|_{t=0} y_b^i(\gamma_{Y_a}(t)) = y_b^k(Y)\frac{df_k^i}{dt}(0) = -y_b^k(Y)\,y_a^j(Y)\,\Gamma_{kj}^i(x).$$

Since $\left.\frac{d}{dt}\right|_{t=0}\tilde{x}^j(\gamma_{Ya}(t)) = y_a^j(Y)$, we have

$$\nabla_a|_Y = \left.\frac{d}{dt}\right|_{t=0}\gamma_{Y_a}(t)$$
$$= \left.\frac{d}{dt}\right|_{t=0}\tilde{x}^j(\gamma_{Y_a}(t))\left.\frac{\partial}{\partial\tilde{x}^j}\right|_Y + \left.\frac{d}{dt}\right|_{t=0} y_b^i(\gamma_{Y_a}(t))\left.\frac{\partial}{\partial y_b^i}\right|_Y$$
$$= y_a^j(Y)\left.\frac{\partial}{\partial\tilde{x}^j}\right|_Y - y_b^k(Y)\,y_a^j(Y)\,\Gamma_{kj}^i(x)\left.\frac{\partial}{\partial y_b^i}\right|_Y.$$

That is,

LEMMA F.11. *The horizontal vector fields ∇_a on $F\mathcal{M}$ are given in local coordinates by the formula*

$$\nabla_a|_Y = y_a^j(Y)\left(\left.\frac{\partial}{\partial\tilde{x}^j}\right|_Y - y_b^k(Y)\,\Gamma_{kj}^i(x)\left.\frac{\partial}{\partial y_b^i}\right|_Y\right),$$

or more succinctly as

$$\nabla_a = y_a^j\left(\frac{\partial}{\partial\tilde{x}^j} - y_b^k\Gamma_{kj}^i\frac{\partial}{\partial y_b^i}\right). \tag{F.5}$$

The frame field on $F\mathcal{M}$ that we use to carry out our tensor calculations is given by the following.

LEMMA F.12. *The globally defined vector fields $\{\nabla_a\}$and $\{\Lambda_c^b\}$ form a basis on $F\mathcal{M}$ which, when restricted to the subbundle $O\mathcal{M}$, is orthonormal with respect to the metric g^F.*

PROOF. The ∇_a are the horizontal lifts of Y_a and the Λ_c^b are vertical. Since $\pi : \left(F\mathcal{M}, g^F\right) \to (\mathcal{M}, g)$ is a Riemannian submersion, this implies that $g^F(\nabla_a, \nabla_c) = g(Y_a, Y_c)$ and $g^F\left(\nabla_a, \Lambda_c^b\right) = 0$. We have already observed that $\left\{\Lambda_c^b\right\}$ is orthonormal since $e(a,b)$ is orthonormal in $\mathfrak{gl}(n, \mathbb{R})$. If $Y \in O\mathcal{M}$, then $g(Y_a, Y_c) = \delta_{ac}$ implies $g^F(\nabla_a, \nabla_c) = \delta_{ac}$. □

6. Covariant differentiation

We now describe covariant differentiation from the point of view of tensors as Euclidean vector-valued functions on the frame bundle. We start with the case of the covariant derivative ∇V of a 1-form V. Now, ∇V is a 2-tensor which we consider as a function

$$\nabla V : F\mathcal{M} \to \mathbb{R}^{n^2}.$$

Given $Y = \{Y_a\}_{a=1}^n \in F\mathcal{M}$, let

$$\left(\nabla_a V\right)_c = \left(\nabla V\right)_{ac}(Y) = \left(\nabla V\right)(Y_a, Y_c),$$

where the first term is the c-component of the function $\nabla_a V : F\mathcal{M} \to \mathbb{R}^n$, the second term is the ac-component of the function $\nabla V : F\mathcal{M} \to \mathbb{R}^{n^2}$, and the third term is the 2-tensor ∇V applied to the vectors Y_a and Y_c. In local coordinates, we compute

$$\left(\nabla_a V\right)_c = y_a^i y_c^j \left(\nabla_i V\right)_j,$$

where $\left(\nabla_i V\right)_j \doteqdot (\nabla V)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ is the ij-th component of the 2-tensor ∇V on $\mathcal{M}$ with respect to the coordinate system $\left\{x^i\right\}$. Note that $\left(\nabla_a V\right)_c$ is a function on $F\mathcal{M}$, y_a^i and y_c^j are functions on $F\mathcal{M}|_{\mathcal{O}}$, and, by abuse of notation, we are considering $\left(\nabla_i V\right)_j$ as a function on $F\mathcal{M}|_{\mathcal{O}}$ by composition with $\pi : F\mathcal{M}|_{\mathcal{O}} \to \mathcal{O}$ (so that $\left(\nabla_i V\right)_j$ is constant on the fibers).

Since

$$\left(\nabla_i V\right)_j = \frac{\partial}{\partial x^i} V_j - \Gamma_{ij}^k V_k,$$

we have

$$\left(\nabla_a V\right)_c = y_a^i y_c^j \left(\frac{\partial}{\partial x^i} V_j - \Gamma_{ij}^k V_k\right). \tag{F.6}$$

Differentiating $V_c = y_c^j V_j$, which is a function on $F\mathcal{M}|_{\mathcal{O}}$, we obtain

$$\frac{\partial}{\partial \tilde{x}^i} V_c = \frac{\partial}{\partial \tilde{x}^i}\left(y_c^j V_j\right) = y_c^j \frac{\partial}{\partial \tilde{x}^i} V_j = y_c^j \frac{\partial}{\partial x^i} V_j.$$

We also compute

$$\frac{\partial}{\partial y_b^k} V_c = \frac{\partial}{\partial y_b^k}\left(y_c^j V_j\right) = \left(\frac{\partial}{\partial y_b^k} y_c^j\right) V_j = \delta_c^b \delta_k^j V_j = \delta_c^b V_k.$$

Thus (F.6) implies that

$$\begin{aligned}(\nabla_a V)_c &= y_a^i \frac{\partial}{\partial \tilde{x}^i} V_c - y_a^i y_b^j \Gamma_{ij}^k \delta_c^b V_k \\ &= y_a^i \frac{\partial}{\partial \tilde{x}^i} V_c - y_a^i y_b^j \Gamma_{ij}^k \frac{\partial}{\partial y_b^k} V_c \\ &= y_a^i \left(\frac{\partial}{\partial \tilde{x}^i} - \Gamma_{ij}^k y_b^j \frac{\partial}{\partial y_b^k} \right) V_c.\end{aligned}$$

Recall from (F.5) that for each $a = 1, \ldots, n$, using a local coordinate system $(x^i, \mathcal{O})$, the vector field ∇_a on $F\mathcal{M}$ is given in $F\mathcal{M}|_{\mathcal{O}}$ by

$$\nabla_a = y_a^i \left\{ \frac{\partial}{\partial \tilde{x}^i} - \Gamma_{ij}^k (x) y_b^j \frac{\partial}{\partial y_b^k} \right\}.$$

Hence $(\nabla_a V)_c = \nabla_a (V_c)$. Since this is a coordinate-independent formula, we have the following.

LEMMA F.13. *The globally defined vector fields ∇_a on $F\mathcal{M}$ acting on a 1-form V by directional differentiation are given by the formula*

$$\nabla_a (V_c) = (\nabla V)_{ac}. \tag{F.7}$$

The left-hand side is interpreted as a directional derivative of the function V_c on $F\mathcal{M}$ in the direction ∇_a, whereas the right-hand side is interpreted as the ac-component of the $\mathbb{R}^{n^2}$-valued function ∇V on $F\mathcal{M}$. One can think of ∇_a as telling us at each frame $\{Y_b\}_{b=1}^n \in F\mathcal{M}$ at $x \in \mathcal{M}^n$ how to (infinitesimally) parallel translate the frame $Y = \{Y_b\}_{b=1}^n$ in the direction of $Y_a \in T\mathcal{M}_x$. We define $\nabla \doteqdot (\nabla_1, \ldots, \nabla_n)$.

Taking the covariant derivative ∇U of a (p,q)-tensor U is similar, albeit more tedious to write down, so we just state the results. The covariant derivative ∇U is a $(p+1,q)$-tensor which we consider as a function

$$\nabla U : F\mathcal{M} \to \mathbb{R}^{n^{(p+1)q}}.$$

We can compute

$$(\nabla U)_{cb_1 \cdots b_p}^{a_1 \cdots a_q} = (\nabla_c U)_{b_1 \cdots b_p}^{a_1 \cdots a_q} = y_c^k y_{b_1}^{j_1} \cdots y_{b_p}^{j_p} (y^{-1})_{i_1}^{a_1} \cdots (y^{-1})_{i_q}^{a_q} (\nabla_k U)_{j_1 \cdots j_p}^{i_1 \cdots i_q},$$

and so using

$$\begin{aligned}(\nabla_k U)_{j_1 \cdots j_p}^{i_1 \cdots i_q} = \frac{\partial}{\partial x^k} U_{j_1 \cdots j_p}^{i_1 \cdots i_q} &- \sum_{r=1}^{p} \sum_{\ell=1}^{n} \Gamma_{kj_r}^{\ell} U_{j_1 \cdots j_{r-1} \ell j_{r+1} \cdots j_p}^{i_1 \cdots i_q} \\ &+ \sum_{s=1}^{q} \sum_{m=1}^{n} \Gamma_{km}^{i_s} U_{j_1 \cdots j_p}^{i_1 \cdots i_{s-1} m i_{s+1} \cdots i_q},\end{aligned}$$

we have

$$(\nabla_c U)_{b_1 \cdots b_p}^{a_1 \cdots a_q} = y_c^k \left(\frac{\partial}{\partial \tilde{x}^k} - \Gamma_{km}^{\ell} y_e^m \frac{\partial}{\partial y_e^{\ell}} \right) U_{b_1 \cdots b_p}^{a_1 \cdots a_q},$$

where $\Gamma_{km}^{\ell}(Y) \doteqdot \Gamma_{km}^{\ell}(\pi(Y))$ in a slight abuse of notation. Hence

LEMMA F.14. *The vector fields* ∇_a *on* $F\mathcal{M}$ *acting on a* (p,q)*-tensor* U *by directional differentiation are given by*

$$\nabla_c(U^{a_1\ldots a_q}_{b_1\ldots b_p}) = (\nabla_c U)^{a_1\ldots a_q}_{b_1\ldots b_p}. \tag{F.8}$$

EXERCISE F.15. Complete the proof of Lemma F.14 started in the previous paragraph.

7. Curvature and commuting covariant derivatives

We shall next compute the commutators of the global horizontal vector fields $\{\nabla_a\}$ and vertical vector fields $\{\Lambda^b_c\}$ on $F\mathcal{M}$. Let

$$[\nabla_a, \nabla_b] \doteqdot \nabla_a\nabla_b - \nabla_b\nabla_a.$$

LEMMA F.16. *On* $F\mathcal{M}$,

$$[\nabla_a, \nabla_b] = -R^d_{abc}\Lambda^c_d, \tag{F.9}$$

where

$$R(Y_a, Y_b)Y_c \doteqdot \sum_{d=1}^{n} R^d_{abc}(Y)\cdot Y_d.$$

PROOF. We have

$$\nabla_a = y^i_a\left\{\frac{\partial}{\partial\tilde{x}^i} - \Gamma^k_{ij}y^j_c\frac{\partial}{\partial y^k_c}\right\}$$

and

$$y^q_d\frac{\partial}{\partial y^k_d} = y^q_d\left(y^{-1}\right)^c_k\Lambda^d_c,$$

(where in the sequel we shall use $\{p,q,r\}$ in the same manner as $\{i,j,k\}$) which follows from

$$\left(y^{-1}\right)^c_k\Lambda^d_c = \frac{\partial}{\partial y^k_d}.$$

Using

$$\frac{\partial}{\partial\tilde{x}^i}y^b_j = 0 \qquad \text{for all } i,j,b,$$

we compute

$$\begin{aligned}
\nabla_a\nabla_b - \nabla_b\nabla_a &= y^i_a\left\{\frac{\partial}{\partial\tilde{x}^i} - \Gamma^k_{ij}y^j_c\frac{\partial}{\partial y^k_c}\right\}\left(y^p_b\left\{\frac{\partial}{\partial\tilde{x}^p} - \Gamma^r_{pq}y^q_d\frac{\partial}{\partial y^r_d}\right\}\right)\\
&\quad - y^p_b\left\{\frac{\partial}{\partial\tilde{x}^p} - \Gamma^r_{pq}y^q_d\frac{\partial}{\partial y^r_d}\right\}\left(y^i_a\left\{\frac{\partial}{\partial\tilde{x}^i} - \Gamma^k_{ij}y^j_c\frac{\partial}{\partial y^k_c}\right\}\right)\\
&= y^i_a y^p_b\left(-\frac{\partial}{\partial x^i}\Gamma^k_{pq} + \frac{\partial}{\partial x^p}\Gamma^k_{iq} + \Gamma^j_{iq}\Gamma^k_{pj} - \Gamma^j_{pq}\Gamma^k_{ij}\right)y^q_d\frac{\partial}{\partial y^k_d}\\
&= y^i_a y^p_b y^q_d\left(y^{-1}\right)^c_k\left(-\frac{\partial}{\partial x^i}\Gamma^k_{pq} + \frac{\partial}{\partial x^p}\Gamma^k_{iq} + \Gamma^j_{iq}\Gamma^k_{pj} - \Gamma^j_{pq}\Gamma^k_{ij}\right)\Lambda^d_c\\
&= -y^i_a y^p_b y^q_d\left(y^{-1}\right)^c_k R^k_{ipq}\Lambda^d_c\\
&= -R^c_{abd}\Lambda^d_c.
\end{aligned}$$

□

Together with Lemma F.8, this describes how commuting covariant derivatives act on tensors. For example, if V_c is a 1-form, then

$$[\nabla_a, \nabla_b] V_c = -R_{abe}^d \Lambda_d^e V_c = -R_{abc}^d V_d, \tag{F.10}$$

and if V_{ab} is a covariant 2-tensor, then

$$[\nabla_a, \nabla_b] V_{cd} = -R_{abf}^e \Lambda_e^f V_{cd} = -R_{abc}^e V_{ed} - R_{abd}^e V_{ce}. \tag{F.11}$$

More generally, if U is a (p,q)-tensor, then

$$[\nabla_c, \nabla_d] U_{b_1\cdots b_p}^{a_1\cdots a_q} = -\sum_{\ell=1}^{p} R_{cdb_\ell}^f U_{b_1\cdots b_{\ell-1} f b_{\ell+1}\cdots b_p}^{a_1\cdots a_q} + \sum_{k=1}^{q} R_{cde}^{a_k} U_{b_1\cdots b_p}^{a_1\cdots a_{k-1} e a_{k+1}\cdots a_q}. \tag{F.12}$$

One proves the following lemma similarly:

LEMMA F.17. *On $F\mathcal{M}$,*

$$[\Lambda_b^a, \nabla_c] = \Lambda_b^a \nabla_c - \nabla_c \Lambda_b^a = \delta_c^a \nabla_b \tag{F.13}$$

and

$$[\Lambda_b^a, \Lambda_d^c] = \Lambda_b^a \Lambda_d^c - \Lambda_d^c \Lambda_b^a = \delta_d^a \Lambda_b^c - \delta_b^c \Lambda_d^a. \tag{F.14}$$

EXERCISE F.18. Prove Lemma F.17.

Note that there is a sign difference between this and the Lie brackets for $e(a,b) \in \mathfrak{gl}(n, \mathbb{R})$. It is a general fact for homogeneous spaces that if G acts on $\mathcal{M}$, $\mathfrak{g}$ is the Lie algebra of G, and the vector field X^* on $\mathcal{M}$ is defined by $X^*(p) \doteqdot \frac{d}{dt}\big|_{t=0} [(\exp tX) p]$ for $X \in \mathfrak{g}$, $p \in \mathcal{M}$, then

$$[X, Y]^* = -[X^*, Y^*]$$

for any $X, Y \in \mathfrak{g}$, which can be seen by simple computation.

8. Reduction to the orthonormal frame bundle

It will simplify our calculations considerably to work on the orthonormal frame bundle $O\mathcal{M}$, which is the subbundle of $F\mathcal{M}$ with $g_{ab} = \delta_{ab}$; i.e.,

$$O\mathcal{M} \doteqdot \{Y \in F\mathcal{M} : g_{ab}(Y) = \delta_{ab} \text{ for all } a, b\}.$$

The metric g may be considered as a function

$$g = (g_{ab})_{a,b=1}^n : F\mathcal{M} \to \mathbb{S}_+^{n\times n} \subset \mathbb{R}^{n(n+1)/2},$$

where $\mathbb{S}_+^{n\times n}$ denotes the set of symmetric, positive-definite $n \times n$ matrices. In particular, $O\mathcal{M} = g^{-1}(I)$, where $I \in \mathbb{S}_+^{n\times n}$ is the identity matrix. Note that given a point $x \in \mathcal{M}$, a tensor U at x, as a function on $F\mathcal{M}_x$, is determined by its value $U(Y)$ at one frame $Y \in F\mathcal{M}_x$ because $GL(n, \mathbb{R})$ acts transitively on the frames. In particular, tensors, as functions on $F\mathcal{M}$, are determined by their restrictions to $O\mathcal{M}$.

Since $\mathcal{OM}$ is the set of frames with $g_{ab} = \delta_{ab}$, to be tangent to $\mathcal{OM}$ a vector field V on $F\mathcal{M}$ must satisfy

$$V(g_{ab}) = 0$$

on $\mathcal{OM}$. We compute this condition in local coordinates. Let $\tilde{g}_{ij} = g_{ij} \circ \pi : F\mathcal{M} \to \mathbb{R}$. If $V = \alpha_c^i \frac{\partial}{\partial y_c^i} + \beta^j \frac{\partial}{\partial \tilde{x}^j}$, then

$$\begin{aligned} V(g_{ab}) &= \left(\alpha_c^i \frac{\partial}{\partial y_c^i} + \beta^j \frac{\partial}{\partial \tilde{x}^j}\right) g_{ab} \\ &= \left(\alpha_c^i \frac{\partial}{\partial y_c^i} + \beta^j \frac{\partial}{\partial \tilde{x}^j}\right) \left(y_a^k y_b^\ell \tilde{g}_{k\ell}\right) \\ &= \alpha_c^i \delta_i^k \delta_a^c y_b^\ell \tilde{g}_{k\ell} + \alpha_c^i \delta_i^\ell \delta_b^c y_a^k \tilde{g}_{k\ell} + \beta^j \frac{\partial}{\partial \tilde{x}^j} \tilde{g}_{k\ell} \\ &= \alpha_a^k y_b^\ell g_{k\ell} + \alpha_b^\ell y_a^k g_{k\ell} + \beta^j \frac{\partial}{\partial x^j} g_{k\ell}, \end{aligned}$$

where in the last line we have dropped the tildes from the notation. Thus the tangency condition is

$$\alpha_a^k y_b^\ell g_{k\ell} + \alpha_b^\ell y_a^k g_{k\ell} + \beta^j \frac{\partial}{\partial x^j} g_{k\ell} = 0. \tag{F.15}$$

In the case of vertical vectors fields, we shall use $V = \alpha_c^d \Lambda_d^c$ and from (F.3) we find the tangency condition is

$$\alpha_a^d g_{bd} + \alpha_b^c g_{ac} = 0. \tag{F.16}$$

The vertical vector fields Λ_b^a on $F\mathcal{M}$ are not tangent to $\mathcal{OM}$. Therefore we project by antisymmetrizing to obtain vector fields tangent to $\mathcal{OM}$:

DEFINITION F.19. Let

$$\rho_{ab} \doteqdot \sum_{c=1}^{n} \left(g_{ac} \Lambda_b^c - g_{bc} \Lambda_a^c\right), \tag{F.17}$$

which is a vertical vector field on $F\mathcal{M}$. Restricting to $\mathcal{OM}$, we have

$$\rho_{ab} = \Lambda_b^a - \Lambda_a^b.$$

REMARK F.20. The reader who prefers to be consistent with up and down indices may wish to write this as $\rho_{ab} = \delta_{ac}\Lambda_b^c - \delta_{cb}\Lambda_a^c$ and similarly for some of the formulas below.

Note that on $\mathcal{OM}$ we have $\rho_{ba} = -\rho_{ab}$. We will show in Corollary F.23 that the restriction of ρ_{ab} to $\mathcal{OM}$ is tangent to $\mathcal{OM}$.

REMARK F.21. As a vector field on $\mathcal{OM}$, ρ_{ab} represents an infinitesimal counterclockwise rotation of the ab-plane. In particular, consider the path

$$\gamma(t) \doteqdot (x, Y_1, \ldots, Y_{a-1}, \cos t Y_a + \sin t Y_b, Y_{a+1}, \ldots, Y_{b-1}, -\sin t Y_a + \cos t Y_b, Y_{b+1}, \ldots, Y_n)$$

in $\mathcal{OM}$. We have

$$\frac{d\gamma}{dt}(0) = \rho_{ab}|_Y.$$

Rotating counterclockwise in the ab-plane skews the a-axis into the b-axis and the b-axis into the negative a-axis.

LEMMA F.22. *If V is a 1-form, then*

$$\rho_{ab}(V_c) = g_{ac}V_b - g_{bc}V_a$$

and if V is a 2-tensor, then

$$\rho_{ab}(V_{cd}) = g_{ac}V_{bd} + g_{ad}V_{cb} - g_{cb}V_{ad} - g_{db}V_{ca}.$$

More generally, if U is a (p,q)-tensor, then

(F.18)

$$\begin{aligned}\rho_{cd}\left(U_{b_1\cdots b_p}^{a_1\cdots a_q}\right) = &\sum_{\ell=1}^{p} g_{cb_\ell}U_{b_1\cdots b_{\ell-1}db_{\ell+1}\dots b_p}^{a_1\cdots a_q} - \sum_{k=1}^{q} g_{ce}\delta_d^{a_k}U_{b_1\dots b_p}^{a_1\cdots a_{k-1}ea_{k+1}\cdots a_q}\\ &- \sum_{\ell=1}^{p} g_{b_\ell d}U_{b_1\cdots b_{\ell-1}cb_{\ell+1}\dots b_p}^{a_1\cdots a_q} + \sum_{k=1}^{q} g_{fd}\delta_c^{a_k}U_{b_1\cdots b_p}^{a_1\cdots a_{k-1}fa_{k+1}\cdots a_q}.\end{aligned}$$

PROOF. This follows from the definition of ρ_{ab} and Lemma F.8. □

COROLLARY F.23. *We have $\rho_{ab}(g_{cd}) = 0$; hence, ρ_{ab} is tangent to the subbundle $A\mathcal{M} \doteqdot g^{-1}(A)$ for each $A \in \mathrm{GL}(n,\mathbb{R})$. In particular, ρ_{ab} is tangent to $O\mathcal{M}$.*

PROOF. Using (F.16) along with the definition of Λ_d^c, we get

$$\rho_{ab}(g_{cd}) = g_{ac}g_{bd} + g_{ad}g_{cb} - g_{bc}g_{ad} - g_{bd}g_{ca} = 0.$$

□

LEMMA F.24. *The vector fields $\{\nabla_a\}$and $\{\rho_{bc}\}_{(b<c)}$ form an orthogonal basis on $O\mathcal{M}$ with $|\nabla_a| = 1$ and $|\rho_{bc}| = \sqrt{2}$.*

PROOF. Since $\nabla g = 0$, we have

$$\nabla_a(g_{bc}) = 0.$$

Thus ∇_a restricts to a vector field on $O\mathcal{M}$. We also calculate

$$\langle\rho_{ab},\rho_{cd}\rangle = \left\langle\Lambda_b^a - \Lambda_a^b, \Lambda_d^c - \Lambda_c^d\right\rangle = 2\delta_{ac}\delta_{bd}$$

for $a < b$ and $c < d$. □

We saw in Lemma F.16 that $[\nabla_a, \nabla_b]$ is a linear combination of $\{\Lambda_d^c\}$, and by the symmetry of the curvature tensor it is also a linear combination of $\{\rho_{ab}\}$. We leave to the reader the proof of the following, which follows from Lemma F.16 and the definition of ρ and the symmetries of Rm. As usual, let $R_{abcd} = R_{abc}^e g_{de}$.

LEMMA F.25. *On $F\mathcal{M}$, we have*

$$[\nabla_a, \nabla_b] = \frac{1}{2}R_{abc}^d g^{ce}\rho_{de}.$$

In particular, on $O\mathcal{M}$, we have $[\nabla_a, \nabla_b] = \frac{1}{2}R_{abcd}\rho_{dc}$.

Lowering the indices on Rm in formula (F.12), we have

COROLLARY F.26. *On $\mathcal{OM}$*

$$[\nabla_e, \nabla_f] U^{a_1\cdots a_q}_{b_1\cdots b_p} = -\sum_{\ell=1}^{p} R_{efb_\ell d} U^{a_1\cdots a_q}_{b_1\cdots b_{\ell-1} d b_{\ell+1}\cdots b_p} + \sum_{k=1}^{q} R_{efca_k} U^{a_1\cdots a_{k-1} c a_{k+1}\cdots a_q}_{b_1\cdots b_p}.$$

For example, if V is a 1-form, then on $\mathcal{OM}$

$$[\nabla_a, \nabla_b] V_c = -R_{abcd} V_d .$$

The other commutators are given in the following lemma.

LEMMA F.27. *On $F\mathcal{M}$, we have*

$$[\rho_{ab}, \nabla_c] = \rho_{ab}\nabla_c - \nabla_c\rho_{ab} = g_{ac}\nabla_b - g_{bc}\nabla_a$$

and

$$\begin{aligned} [\rho_{ab}, \rho_{cd}] &= \rho_{ab}\rho_{cd} - \rho_{cd}\rho_{ab} \\ &= g_{ac}\rho_{bd} + g_{bd}\rho_{ac} - g_{ad}\rho_{bc} - g_{bc}\rho_{ad} . \end{aligned}$$

PROOF. These equations follow from the commutators $\left[\Lambda^d_b, \nabla_c\right]$ and $\left[\Lambda^e_b, \Lambda^f_d\right]$ (equations (F.13) and (F.14)) and the definition of ρ_{ab}. □

In dimension 3, we see that the Lie algebra structure on $\mathfrak{so}(3,\mathbb{R})$ is given by the cross product.

COROLLARY F.28. *If $n = 3$, then*

$$[\rho_{12}, \rho_{13}] = \rho_{23}, \qquad [\rho_{13}, \rho_{23}] = \rho_{12}, \qquad [\rho_{23}, \rho_{12}] = \rho_{13}.$$

9. Time dependent orthonormal frame bundle for solutions to the Ricci flow

Now we consider a solution to the Ricci flow $(\mathcal{M}^n, g(t))$, $t \in [0,T)$. Let $\mathcal{M}^n \times [0,T)$ be the space-time manifold. We have the principal $\mathrm{GL}(n,\mathbb{R})$-bundle

$$\tilde{\pi} : F\mathcal{M} \times [0,T) \to \mathcal{M} \times [0,T)$$

defined by $\tilde{\pi} \doteqdot \pi \times \mathrm{id}_{[0,T)}$. Recall that the time-dependent metric $g(t)$ may be considered as a function

$$g : F\mathcal{M} \times [0,T) \to \mathbb{S}^{n\times n}$$

($\mathbb{S}^{n\times n}$ are the symmetric $n \times n$ matrices). Consider the degenerate metric $\tilde{g}^F$ on $F\mathcal{M} \times [0,T)$ given by $\tilde{g}^F|_{F\mathcal{M}\times\{t\}} = g^F(t)$, the metric on $F\mathcal{M}$ induced by $g(t)$. That is,

$$\tilde{g}^F|_{F\mathcal{M}\times\{t\}}((X,u),(Y,v)) = g^F(t)(X,Y)$$

for $X, Y \in T(F\mathcal{M})$ and $u, v \in \mathbb{R}$. We have the vector fields $\{\nabla_a\}$, $\{\Lambda^b_a\}$, and $\{\rho_{ab}\}$, which are all tangent to $F\mathcal{M} \times \{t\}$ for each $t \in [0,T)$.

Consider the subbundle

$$O\tilde{\mathcal{M}} \doteqdot \bigcup_{t\in[0,T)} O\mathcal{M}_{g(t)} \times \{t\} = g^{-1}(I) \subset F\mathcal{M} \times [0,T).$$

The time-like vector field $\frac{\partial}{\partial t}$ is not always tangent to $O\tilde{\mathcal{M}}$, unless $g(t)$ is independent of time; however, there is a canonical way to modify it so that it becomes tangent to $O\tilde{\mathcal{M}}$.

10. The time vector field and its action on tensors

Given a solution to the Ricci flow $(\mathcal{M}^n, g(t))$, when considering time-dependent tensors on $\mathcal{M}$, it is natural to consider their time derivatives not as the usual time derivative $\frac{\partial}{\partial t}$ but rather the directional derivative with respect to the following vector field.

DEFINITION F.29. The vector field ∇_t on $F\mathcal{M} \times [0,T)$ is defined by

$$\nabla_t \doteqdot \frac{\partial}{\partial t} + R_{ab} g^{bc} \Lambda_c^a. \tag{F.19}$$

LEMMA F.30. *The vector field ∇_t restricted to $O\tilde{\mathcal{M}} \subset F\mathcal{M} \times [0,T)$ is the unique vector field that is tangent to the subbundle $O\tilde{\mathcal{M}}$ and such that $\nabla_t - \frac{\partial}{\partial t}$ is space-like (tangent to $F\mathcal{M} \times \{t\}$), vertical, and perpendicular to each $O\mathcal{M}_{g(t)} \subset F\mathcal{M}$.*

PROOF. On $O\tilde{\mathcal{M}}$ we have $g_{ab} = \delta_{ab}$, which we shall use a few times below. Now let's look carefully at the conditions.

(1) Since $\nabla_t - \frac{\partial}{\partial t}$ is space-like and vertical, it is of the form

$$\nabla_t - \frac{\partial}{\partial t} = \alpha_c^d \Lambda_d^c,$$

where α_c^d are functions on $O\tilde{\mathcal{M}}$.

(2) Since $\nabla_t - \frac{\partial}{\partial t}$ is perpendicular to the subbundles $O\mathcal{M}_{g(t)}$, we need

$$0 = \left\langle \rho_{ab}, \alpha_c^d \Lambda_d^c \right\rangle = \alpha_a^b - \alpha_b^a$$

and hence α is symmetric.

(3) Now since ∇_t is tangent to $O\tilde{\mathcal{M}}$, we have $\nabla_t g_{ab} = 0$. Thus

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} g_{ab} + \alpha_a^d g_{bd} + \alpha_b^c g_{ac} \\ &= -2R_{ab} + \alpha_a^b + \alpha_b^a. \end{aligned}$$

Hence $\alpha_a^b = R_{ab}$ since α is symmetric.

Thus we must have that $\nabla_t - \frac{\partial}{\partial t} = R_{ab}\Lambda_b^a$. Conversely, if ∇_t is defined this way, then it satisfies all of the above properties. □

EXERCISE F.31. Formulate the analogue of the above lemma for $A\mathcal{M} \doteqdot g^{-1}(A)$, where $A \in \mathrm{GL}(n, \mathbb{R})$.

REMARK F.32. As we shall see when we consider the space-time approach to the Harnack estimate in Part III of this volume, the time covariant derivative of a time-dependent tensor with respect to the evolving metric $g(t)$ is the directional derivative of the tensor in the direction ∇_t. The formula $\nabla_t g_{ab} = 0$ above is part of the equation that shows that the space-time covariant derivative of the space-time degenerate metric is zero.

LEMMA F.33. *If V is a time-dependent 1-form, then*

$$\nabla_t V_a = \frac{\partial}{\partial t} V_a + R_{ab} g^{bc} V_c;$$

and if V is a 2-tensor, then

$$\nabla_t V_{ab} = \frac{\partial}{\partial t} V_{ab} + R_{ac} g^{cd} V_{db} + R_{bc} g^{cd} V_{ad}.$$

In general, if U is a time-dependent (p,q)-tensor, then

$$\nabla_t U_{b_1 \cdots b_p}^{a_1 \cdots a_q} = \frac{\partial}{\partial t} U_{b_1 \cdots b_p}^{a_1 \cdots a_q} + \sum_{j=1}^{p} \sum_{c,d=1}^{n} R_{b_j c} g^{cd} U_{b_1 \cdots b_{j-1} d b_{j+1} \cdots b_p}^{a_1 \cdots a_q}$$

$$- \sum_{i=1}^{q} \sum_{c,d=1}^{n} g^{a_i c} R_d^c U_{b_1 \cdots b_p}^{a_1 \cdots a_{i-1} d a_{i+1} \cdots a_q}. \tag{F.20}$$

PROOF. By Lemma F.8,

$$\Lambda_c^d V_a = y_c^i \delta_a^d V_i = \delta_a^d V_c$$

and so the first equation follows immediately:

$$\nabla_t V_a = \frac{\partial}{\partial t} V_a + R_{db} g^{bc} \Lambda_c^d V_a = \frac{\partial}{\partial t} V_a + R_{ab} g^{bc} V_c.$$

Again by Lemma F.8 we have

$$\Lambda_c^d V_{ab} = \delta_a^d V_{cb} + \delta_b^d V_{ac},$$

and so

$$\nabla_t V_{ab} = \frac{\partial}{\partial t} V_{ab} + R_{de} g^{ec} \Lambda_c^d V_{ab} = \frac{\partial}{\partial t} V_{ab} + R_{ae} g^{ec} V_{cb} + R_{be} g^{ec} V_{ac}.$$

The equation for general tensors follows similarly. □

In particular,

$$\nabla_t g_{ab} = \frac{\partial}{\partial t} g_{ab} + R_{ac} g^{cd} g_{db} + R_{bc} g^{cd} g_{ad} = 0. \tag{F.21}$$

This agrees with the fact that ∇_t is tangent to $O\tilde{\mathcal{M}}$.

11. The heat operator and commutation formulas

We now define the Laplacian (or Laplace operator)

$$\Delta \doteqdot \sum_{e=1}^{n} \nabla_e \nabla_e \tag{F.22}$$

acting on vector-valued functions on $O\tilde{\mathcal{M}}$. This corresponds to the rough Laplacian $\Delta = \sum_{i,j=1}^{n} g^{ij}\nabla_i\nabla_j$ acting on tensors on $\mathcal{M}$. Using (see Lemma 6.15 on p. 179 in Volume One)

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk\ell} &= \Delta R_{ijk\ell} + 2\left(B_{ijk\ell} - B_{ij\ell k} + B_{ikj\ell} - B_{i\ell jk}\right) \\ &\quad - \left(R_i^p R_{pjk\ell} + R_j^p R_{ipk\ell} + R_k^p R_{ijp\ell} + R_\ell^p R_{ijkp}\right),\end{aligned} \tag{F.23}$$

(F.20), and (F.19), we can calculate the evolution equations of R_{abcd} and R_{ab}, obtaining

$$(\nabla_t - \Delta)R_{abcd} = 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}), \tag{F.24}$$

where $B_{abcd} = -R_{aebf}R_{cedf}$, and

$$(\nabla_t - \Delta)R_{ab} = 2R_{acdb}R_{cd}. \tag{F.25}$$

REMARK F.34. Note that equation (F.24) matches with equation (6.21) of Volume One, which was obtained by Uhlenbeck's trick. In Uhlenbeck's trick one evolves a frame by (equation (6.18a) of Volume One)

$$\frac{d}{dt}Y_a = \mathrm{Rc}\left(Y_a\right) = R_{ab}g^{bc}Y_c. \tag{F.26}$$

Note that in coordinates on the frame bundle, this says

$$\frac{d}{dt}y_a^k\left(Y\right) = R_{ab}g^{bc}y_c^k\left(Y\right).$$

This is equivalent to equation (6.19a) of Volume One: $\frac{\partial}{\partial t}\iota = \mathrm{Rc}\circ\iota$ for a bundle isomorphism $\iota : V \to TM$. One computes that under the Ricci flow $\frac{\partial}{\partial t}\left[g\left(Y_a, Y_b\right)\right] = 0$. In particular, if a frame is initially orthonormal, then it remains so. The path $\gamma\left(t\right) \doteqdot \left(Y\left(t\right), t\right)$ lies in $O\tilde{\mathcal{M}}$ so that $\frac{d}{dt}\gamma\left(t\right) \in T\left(O\tilde{\mathcal{M}}\right)$. We claim $\frac{d}{dt}\gamma\left(t\right) = \nabla_t$. To see this, we first note that

$$\frac{d}{dt}\gamma\left(t\right) = \mathrm{Rc}\left(Y\right) + \frac{\partial}{\partial t}.$$

In coordinates this is

$$\begin{aligned}\frac{d}{dt}\gamma\left(t\right) &= R_{ab}g^{bc}y_c^k\left(Y\right)\left.\frac{\partial}{\partial y_a^k}\right|_{\gamma(t)} + \left.\frac{\partial}{\partial t}\right|_{\gamma(t)} \\ &= R_{ab}g^{bc}\left.\Lambda_c^a\right|_{\gamma(t)} + \left.\frac{\partial}{\partial t}\right|_{\gamma(t)} \\ &= \nabla_t.\end{aligned}$$

Thus the time-derivative ∇_t defined by (F.19) corresponds to taking the time-derivative $\frac{\partial}{\partial t}\left[U\left(Y_{a_1},\ldots,Y_{a_p}\right)\right]$ of a tensor where Y_{a_i} satisfy (F.26). For example, the terms in $\left(\nabla_t - \frac{\partial}{\partial t}\right)\operatorname{Rm}$ match with those in $\frac{\partial}{\partial t}\left(\iota^* \operatorname{Rm}\right) - \iota^*\left(\frac{\partial}{\partial t}\operatorname{Rm}\right)$ as in the proof of Lemma 6.22 of Volume One.

We conclude this section with the following lemma:

LEMMA F.35. *On $O\tilde{\mathcal{M}}$ we have the commutators*

$$[\nabla_t, \nabla_a] = \nabla_b R_{ac}\rho_{bc} + R_{ac}\nabla_c$$

and

$$[\Delta, \nabla_a] = R_{eadc}\nabla_e\rho_{cd} + R_{ad}\nabla_d + \nabla_b R_{ac}\rho_{bc},$$

which imply

$$(\nabla_t - \Delta)\nabla_a - \nabla_a(\nabla_t - \Delta) = R_{abcd}\nabla_b\rho_{cd}\ . \tag{F.27}$$

PROOF. Since

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = g^{k\ell}\left(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij}\right)$$

for Ricci flow, we first calculate

$$\begin{aligned}\left[\frac{\partial}{\partial t}, \nabla_a\right] &= -\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)y_a^i y_b^j \frac{\partial}{\partial y_b^k} \\ &= g^{k\ell}(\nabla_i R_{j\ell} - \nabla_j R_{i\ell} - \nabla_\ell R_{ij})y_a^i y_b^j \frac{\partial}{\partial y_b^k} \\ &= y_c^\ell y_d^k g^{cd}(\nabla_i R_{j\ell} + \nabla_j R_{i\ell} - \nabla_\ell R_{ij})y_a^i y_b^j \frac{\partial}{\partial y_b^k} \\ &= g^{cd}(\nabla_a R_{bc} + \nabla_b R_{ac} - \nabla_c R_{ab})\Lambda_d^b.\end{aligned}$$

Together with the commutator for ∇_a and Λ_b^c (equation (F.13)), this implies that

$$\begin{aligned}[\nabla_t, \nabla_a] &= g^{cd}(\nabla_b R_{ac} - \nabla_c R_{ab})\Lambda_d^b + R_{bc}g^{cd}\delta_a^b\nabla_d \\ &= \nabla_b R_{ac}(g^{cd}\Lambda_d^b - g^{bd}\Lambda_d^c) + R_{ac}g^{cd}\nabla_d \\ &= \nabla_b R_{ac}\rho_{bc} + R_{ac}\nabla_c.\end{aligned}$$

We next compute $[\Delta, \nabla_a]$. Using

$$\begin{aligned}\nabla_e\nabla_e\nabla_a &= \nabla_e\left(\frac{1}{2}R_{eadc}\rho_{cd} + \nabla_a\nabla_e\right) \\ &= \frac{1}{2}\nabla_e R_{eadc}\rho_{cd} + \frac{1}{2}R_{eadc}\nabla_e\rho_{cd} + \left(\frac{1}{2}R_{eadc}\rho_{cd} + \nabla_a\nabla_e\right)\nabla_e,\end{aligned}$$

we obtain

$$\begin{aligned}[\Delta, \nabla_a] &= \frac{1}{2}\nabla_e R_{eadc}\rho_{cd} + \frac{1}{2}R_{eadc}\nabla_e\rho_{cd} + \frac{1}{2}R_{eadc}\rho_{cd} \\ &= \frac{1}{2}R_{eadc}\nabla_e\rho_{cd} + \frac{1}{2}R_{eadc}(g_{ce}\nabla_d - g_{de}\nabla_c) \\ &\quad + \frac{1}{2}\nabla_e R_{eadc}\rho_{cd} + \frac{1}{2}R_{eadc}\nabla_e\rho_{cd} \\ &= R_{eadc}\nabla_e\rho_{cd} + R_{ad}\nabla_d + \frac{1}{2}\nabla_e R_{eadc}\rho_{cd}.\end{aligned}$$

By the second Bianchi identity,

$$\nabla_e R_{eadc} = \nabla_e R_{dcea} = \nabla_c R_{da} - \nabla_d R_{ca}.$$

Since $\rho_{cd} = -\rho_{dc}$, this implies

$$\Delta\nabla_a - \nabla_a\Delta = R_{eadc}\nabla_e\rho_{cd} + R_{ad}\nabla_d + \nabla_b R_{ac}\rho_{bc}\,; \tag{F.28}$$

hence,

$$(\nabla_t - \Delta)\nabla_a - \nabla_a(\nabla_t - \Delta) = R_{abdc}\nabla_b\rho_{cd}. \tag{F.29}$$

□

We may use (F.27) to obtain the following formulas which are used in the proof of the Harnack estimate. First,

$$\begin{aligned}(\nabla_t - \Delta)\nabla_a V_b - \nabla_a(\nabla_t - \Delta)V_b &= R_{abdc}\nabla_b\rho_{cd}V_b \\ &= R_{aedc}\nabla_e(g_{cb}V_d - g_{db}V_c) \\ &= R_{acdb}\nabla_c V_d - R_{acbd}\nabla_c V_d \\ &= 2R_{acdb}\nabla_c V_d. \qquad \text{(F.30)}\end{aligned}$$

Similarly,

$$(\nabla_t - \Delta)\nabla_a V_{bc} - \nabla_a(\nabla_t - \Delta)V_{bc} = 2R_{adeb}\nabla_d V_{ec} + 2R_{adec}\nabla_d V_{be}, \tag{F.31}$$

and

(F.32)

$$(\nabla_t - \Delta)\nabla_a V_{bcd} - \nabla_a(\nabla_t - \Delta)V_{bcd} = R_{aefb}\nabla_e V_{fcd} + R_{aefc}\nabla_e V_{bfd} + R_{aefd}\nabla_e V_{bcf}.$$

Analogous formulas hold for higher tensors. We leave it as an exercise for the reader to derive these from (F.27).

12. Notes and commentary

A nice reference for the theory of principal bundles and their associated vector bundles is Kobayashi and Nomizu [**311**]; see also Poor [**413**]. The exposition in this appendix follows §2 of Hamilton [**247**].

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