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Preface

The present volume follows a book, *"Why are braids orderable?"*, written by the same authors and published in 2002 by the Société Mathématique de France in the series *Panoramas et Synthèses*. We emphasize that this is *not* a new edition of that book. Although this book contains most of the material in the previous book, it also contains a considerable amount of new material. In addition, much of the original text has been completely rewritten, with a view to making it more readable and up-to-date. We have been able not only to include ideas that were unknown in 2002, but we have also benefitted from helpful comments by colleagues and students regarding the contents of the SMF book, and we have taken their advice to heart in writing this book.

The reader is assumed to have some basic background in group theory and topology. However, we have attempted to make the ideas in this volume accessible and interesting to students and seasoned professionals alike.

In fact, the question "Why are braids orderable?" has not been answered to our satisfaction, either in the book with that title or the present volume. That is, we do not understand precisely what makes the braid groups so special that they enjoy an ordering so easy to describe, so challenging to construct and with such subtle properties as are described in these pages. The best we can offer is some insight into the easier question, "How are braids orderable?"

Patrick Dehornoy, Caen

Ivan Dynnikov, Moscow

Dale Rolfsen, Vancouver

Bert Wiest, Rennes

December 2007

Key Definitions

Sigma-ordering:

• For β, β' in B_∞, the relation $\beta < \beta'$ is true if $\beta^{-1}\beta'$ is σ-positive.
• For β, β' in B_∞, the relation $\beta <^\Phi \beta'$ is true if $\beta^{-1}\beta'$ is σ^Φ-positive.

Sigma-positive braid word:

• A braid word is *σ-positive* if the σ_i with lowest index occurs positively only.
• A braid word is *σ^Φ-positive* if the σ_i with highest index occurs positively only.

Sigma-positive braid:

• A braid is *σ-positive* if it admits a σ-positive representative word.
• A braid is *σ^Φ-positive* if it admits a σ^Φ-positive representative word.

Property A (Acyclicity):

• A σ-positive braid is nontrivial.

Property C (Comparison):

• Every nontrivial braid of B_n can be represented by an n-strand braid word that is σ-positive or σ-negative.

Property S (Subword):

• Every braid of the form $\beta^{-1}\sigma_i\beta$ is σ-positive.

Complementary Definitions

• A braid word is *σ_i-positive* if it contains at least one σ_i, no σ_i^{-1}, no $\sigma_j^{\pm1}$ with $j < i$.
• ... *id.* ... *σ_i-negative* if ... at least one σ_i^{-1}, no σ_i, no $\sigma_j^{\pm1}$ with $j < i$.
• ... *id.* ... *σ_i-free* if ... no $\sigma_j^{\pm1}$ with $j \leqslant i$.

• A braid is called *σ_i-positive* if it admits a σ_i-positive expression, etc.

Property A (second, equivalent form): A σ_1-positive braid is nontrivial.

Property C (second, equivalent form): Every braid of B_n can be represented by an n-strand braid word that is σ_1-positive, σ_1-negative, or σ_1-free.

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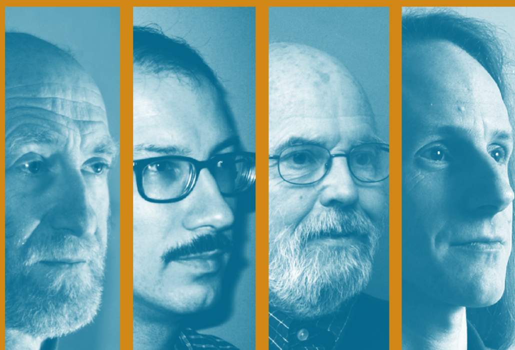

In the fifteen years since the discovery that Artin's braid groups enjoy a left-invariant linear ordering, several quite different approaches have been used to understand this phenomenon. This book is an account of those approaches, which involve such varied objects and domains as combinatorial group theory, self-distributive algebra, finite combinatorics, automata, low-dimensional topology, mapping class groups, and hyperbolic geometry. The remarkable point is that all these approaches lead to the same ordering, making the latter rather canonical.

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