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Finite Dimensional Algebras and Quantum Groups

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*For all their help, encouragement, and infinite patience,
we dedicate this book to our wives and children:*

Wenlian Guo and Zhuoran Deng

Chunli Yu and Andy Du

Karen Parshall

Huiqing Xu, Xin Wang, and Yun Wang

Contents

Preface

The quantum groups investigated in this book are quantum enveloping algebras defined by their Drinfeld–Jimbo presentation once a symmetrizable (generalized) Cartan matrix is specified. This presentation is essentially a q-deformation or "quantization" of the familiar presentation (by Chevalley generators and Serre relations) of the universal enveloping algebra of a Kac–Moody Lie algebra associated with a symmetrizable Cartan matrix. Thus, one approach to quantum enveloping algebras closely follows the study of universal enveloping algebras of Lie algebras, the results often amounting to quantizations of their classical counterparts.

There is a well-known procedure for obtaining symmetrizable Cartan matrices from finite (possibly valued) graphs. About two decades before the birth of quantum groups, representations of quivers (i.e., directed graphs) were introduced and developed as part of both a new approach to the representation theory of finite dimensional algebras and a method to deal with problems in linear algebra. P. Gabriel [**118**] showed, for example, that if the underlying graph of a quiver is a (simply laced) Dynkin graph, then the indecomposable representations correspond naturally to the positive roots of the finite dimensional complex semisimple Lie algebra associated with the same Dynkin graph. Over a decade later, V. Kac [**170**] generalized Gabriel's result to an arbitrary quiver, obtaining a one-to-one correspondence between the positive real roots of the associated Lie algebra and certain indecomposable quiver representations, as well as a one-to-many correspondence from the positive imaginary roots to the remaining indecomposable representations. Thus, an essential feature of the structure of a symmetrizable Kac–Moody Lie algebra — namely, its root space decomposition — has an interpretation in terms of representations of finite dimensional algebras.

The birth of quantum groups in the 1980s provided an opportunity for quantizing and deepening the finite dimensional algebra results described above. In 1990, C. M. Ringel [**247**] introduced an algebra, which he called the Hall algebra, but which is now commonly known as the Ringel–Hall algebra, associated with the representation category of a finite dimensional algebra over the finite field $\mathbb{F}_q$. In this work, Ringel established some fundamental relations that turned out to be specializations of the modified quantum Serre relations. Ringel then proved, in the finite type case, that the structure constants of the Ringel–Hall algebra are polynomials in q; the resulting generic Ringel–Hall algebra is isomorphic to the "positive part" of the corresponding quantum enveloping algebra.

With this breakthrough in the realization of quantum enveloping algebras of finite type, the development of the theory reached a new level. First, the geometric approach (via the theory of perverse sheaves) was introduced by G. Lusztig [**206**]. He obtained not only a geometric realization of the $\pm$-parts of quantum enveloping algebras associated with symmetrizable Cartan matrices but also canonical bases for these algebras and their representations as an application. Second, J. A. Green [**136**] established a comultiplication formula for Ringel–Hall algebras of hereditary algebras and extended Ringel's algebraic realization to arbitrary types. Thus, the Gabriel–Kac work at the root system — or skeletal — level can be thought of as having been extended to an actual construction of the full quantum enveloping algebra. Beyond the theory of Ringel–Hall algebras, other developments include Nakajima's quiver varieties [**229**] and the realization of all symmetrizable Kac–Moody Lie algebras by L. Peng and J. Xiao [**238**].

At almost the same time as Ringel's work on Hall algebras, A. Beilinson, G. Lusztig, and R. MacPherson investigated a class of finite dimensional algebras, known as quantum Schur algebras, which they used to give a realization of the *entire* quantum enveloping algebras in the important case of type A, i.e., associated with the general linear Lie algebras $\mathfrak{gl}_n$. This work thus provided another finite dimensional algebra approach to quantum enveloping algebras, completely different from the theory of Ringel–Hall algebras. However, the multiplication formulas that played a key role in this approach result from an analysis of quantum Schur algebras over finite fields, using the geometry of flags on a finite dimensional vector space. A stabilization property derived from the multiplication formula permits the definition of an infinite dimensional algebra as a "limit" of all quantum Schur algebras. In turn, this algebra has a completion that naturally contains the quantum enveloping algebra as a subalgebra. As a bonus, this method leads to an explicit basis, called the BLM basis, for the entire quantum enveloping algebra, and it yields explicit multiplication formulas for any basis element by a generator. It has been proved by J. Du and B. Parshall [**104**] that a triangular part of the BLM basis coincides with the Ringel–Hall algebra basis.

This book provides an introduction to the two algebraic approaches briefly described above, with an emphasis on the structure and realization of quantum enveloping algebras. The treatment is largely elementary and combinatorial. In so far as possible, we have written the book to be accessible to graduate students and to mathematicians who are not experts in the field. Apart from some standard material (e.g., [**BAII**], [**LAII**]), our treatment is entirely self-contained with two notable exceptions: a positivity result for Hecke algebras (in Chapter 7), which requires the use of perverse sheaves, and a theorem of Lusztig used in the proof of Green's theorem (in Chapter 12), which requires the representation theory of Kac–Moody Lie algebras. For the more advanced geometric approach using the theory of perverse sheaves, see Lusztig's book [**209**].

Although the present book centers on the finite dimensional algebra approach to quantum groups, it also takes up two other, important, related topics. First, following [**59**], we use Frobenius morphisms on algebras to link representations of a quiver directly to representations of a species (called a modulated quiver in this book) without specifically working with the species. In the language of Lie theory, a quiver determines a *symmetric* generalized Cartan matrix, while a species corresponds to a symmetrizable one. As Cartan matrices, these two cases are linked by a graph automorphism. A quiver automorphism (i.e., a graph automorphism preserving arrows) gives rise naturally to a Frobenius morphism on the path algebra of the quiver whose fixed-point algebra can be interpreted as the tensor algebra of a species. Thus, the Ringel–Hall algebras associated with the representation categories of quivers with automorphisms cover all the quantum enveloping algebras associated with symmetrizable Kac–Moody Lie algebras.

The second related topic is the Kazhdan–Lusztig theory for (Iwahori–) Hecke algebras and cells. Playing an important role in Chevalley group theory [**159**], Hecke algebras are quantum deformations of group algebras of Coxeter groups. In 1979, D. Kazhdan and G. Lusztig [**177**] discovered a remarkable basis for a Hecke algebra, known as the *Kazhdan–Lusztig* or *canonical basis*, which has important applications in the representation theory of Hecke algebras, algebraic groups, finite groups of Lie type, and quantum groups. We use the same idea in the construction of canonical bases for quantum enveloping algebras of finite type in Chapter 11. As a noteworthy crown to the whole theory, we present the modern cell approach to the representations of symmetric groups and the structure of quantum Schur algebras. The latter is fundamental in the BLM approach to the realization of the entire quantum enveloping algebra of $\mathfrak{gl}_n$.

★ ★ ★

The book consists of 14 chapters arranged in 5 parts, complemented by a leading Chapter 0 — that outlines the main features of the book — as well as three appendices. Chapter 0 begins with the two realizations of Cartan matrices: the graph realization and the root datum realization, which lead up to the theories of quiver representations and quantum enveloping algebras, respectively. The main objects discussed in the book are certain algebraic structures — Coxeter groups, associative and Lie algebras, etc. — which are often presented with generators and relations. We set down in §0.2 the relevant notations for presentations. When an algebraic structure is presented by generators and relations, the immediate question arises of a description in some concrete way. For example, Coxeter groups are defined by means of a presentation, but, as J. Tits has shown, have an elegant explicit description as "reflection groups." (See §4.1.) In general, this question is the so-called *realization problem.* In this book, our main focus will be the two beautiful realizations of quantum enveloping algebras. However, as a first taste, we discuss the problem through some relatively simple examples in §0.3 and §0.6. In §0.4, the so-called quantumization process is introduced to explain the phenomenon that counting over finite fields often leads to certain generic objects over a polynomial ring. We shall see that Hecke algebras, quantum Schur algebras, and Ringel–Hall algebras of finite type can all be produced through this process. Finally, as one of the main topics in the book, the crude model of the canonical basis theory, i.e., the elementary matrix construction of canonical bases, is discussed in §0.5.

Part 1 (Chapters 1–3) presents the theory of finite dimensional algebras, with an emphasis on representations of quivers with automorphisms. Chapter 1 begins with the basics of quiver representations and proves the theorem of Gabriel mentioned earlier using Bernstein–Gelfand–Ponomarev (BGP) reflection functors. It also lays out the relations between quivers, Euler forms, root systems, Weyl groups, and representation varieties.

Chapter 2 treats the general theory of representations of algebras with Frobenius morphisms. A Frobenius morphism F on a finite dimensional algebra A defined over the algebraic closure $\mathcal{K}$ of the finite field $\mathbb{F}_q$ is a ring automorphism satisfying $F(\lambda a) = \lambda^q a$, for all $\lambda \in \mathcal{K}$ and $a \in A$. It induces a functor on the category of finite dimensional A-modules, called the Frobenius twist functor. If the Frobenius twist of a module is isomorphic to itself, then the module is called an F-stable module. We show that the subcategory of F-stable modules with morphisms compatible with F-stability is equivalent to the module category of the F-fixed point algebra A^F. Thus, the determination of indecomposable A^F-modules is equivalent to that of indecomposable F-stable modules. Additionally, this method provides a relation between almost split sequences for A^F-modules and A-modules in the Auslander–Reiten theory. In preparation for those results, Chapter 2 contains a brief and self-contained introduction to almost split sequences and irreducible morphisms.

In Chapter 3, we apply the general theory to the path algebra A of a quiver Q with automorphism σ. If F is the Frobenius morphism on A induced from σ, the F-fixed point algebra A^F is a hereditary algebra over the finite field $\mathbb{F}_q$ and is the tensor algebra of the species associated with (Q, σ). Up to Morita equivalence, every finite hereditary algebra arises in this way. We further extend the folding relation associated with the quiver automorphism to a folding relation between the Auslander–Reiten quivers of A and A^F. Finally, we study representations of affine quivers with automorphisms and describe their Frobenius twists explicitly as an example of the applications of the theory. The formulas for the number of indecomposable representations of the associated F-fixed point algebra are also presented.

Part 2 (Chapters 4–6) constructs, via generators and relations, the algebras that play an important role throughout the book. It opens in Chapter 4 with the basic theory of Coxeter groups. Symmetric groups and affine Weyl groups provide important examples, which we look at in some detail. A modification of the defining relations for a Coxeter group leads naturally to the construction of the associated Hecke algebra, the properties of which are also rather fully explored. Chapter 4 concludes with a further example showing that Hecke algebras for the symmetric groups arise in a quantumization process that starts with the endomorphism algebra of the complete flag variety of a finite general linear group.

Chapter 5 begins with a brief tour of the basics of Hopf algebras. It continues with the fundamental example of universal enveloping algebras, emphasizing Kac–Moody Lie algebras and their symmetry structure. These results serve as a template for quantum enveloping algebras. The chapter ends with a discussion of the simplest quantum enveloping algebra, quantum $\mathfrak{sl}_2$.

Chapter 6 is devoted to quantum enveloping algebras — defined by means of the Drinfeld–Jimbo presentation — associated with symmetrizable Cartan matrices. There, we first show that these algebras are infinite dimensional and carry Hopf algebra structures. Actions of suitable braid groups on these algebras lead to the definition of root vectors for arbitrary roots as well as to the construction of PBW-type bases in the finite type case.

Part 3 (Chapters 7–9) presents a modern approach to the ordinary representation theory of symmetric groups and the associated Hecke algebras. Chapter 7 is concerned with the combinatorial part of Kazhdan–Lusztig theory — the calculus of Hecke algebras and cells. After introducing the canonical bases for Hecke algebras, we develop Kazhdan–Lusztig polynomials, dual bases, inverse Kazhdan–Lusztig polynomials, and Knuth, cell, and Vogan equivalence relations. We prove that the Knuth equivalence is finer than the left cell equivalence which is, in turn, finer than Vogan equivalence. We conclude with a brief explanation of the geometric meaning of the Kazhdan–Lusztig polynomials, including the positivity property and its applications.

Chapter 8 explicitly determines the cells for the symmetric groups and constructs the simple representations of symmetric groups and their associated Hecke algebras. A main tool is the Robinson–Schensted algorithm. For later application to quantum Schur algebras, we adopt a generalized version, known as the Robinson–Schensted–Knuth (RSK) correspondence which associates with each square matrix over $\mathbb{N}$ a pair of semistandard tableaux — the insertion tableau and the recording tableau. Given two elements in a symmetric group, if they are Vogan equivalent, then they have the same recording tableau; hence, they are Knuth equivalent. This completes the decomposition of a symmetric group into left (or right) cells. As a further application of the positivity property, we introduce the asymptotic Hecke algebras and an Artin–Wedderburn decomposition for the type A Hecke algebras.

Chapter 9 takes up the Kazhdan–Lusztig calculus for quantum Schur algebras, or q-Schur algebras, as a natural extension of the theory of Hecke algebras. Beginning with the Dipper–James definition of a quantum Schur algebra as the endomorphism algebra of tensor space, we immediately establish its integral quasi-heredity by showing the existence of a Specht datum in the sense of [**106**]. We then construct canonical bases for these algebras as a natural extension of the counterpart for Hecke algebras. These bases are, in fact, cellular bases in the sense of Graham–Lehrer [**134**] and can be used to establish the integral quasi-hereditary property for quantum Schur algebras. In addition, the duality between Specht and Δ-filtrations is discussed, and tilting module theory is developed. As an application, we establish the integral double centralizer property which will be further extended in Chapter 14 to the integral quantum Schur–Weyl reciprocity.

Part 4 (Chapters 10–12) presents Ringel's Hall algebra approach to quantum enveloping algebras. The story begins in Chapter 10 with the basic definition of the (integral) Hall algebra of a finitely generated algebra over a finite field. We establish that Hall algebras satisfy certain fundamental relations. These become the quantum Serre relations in a Ringel–Hall algebra which is defined in this book as the twisted Hall algebra associated with a quiver with automorphism (and a finite field). It turns out that there is a surjective algebra homomorphism from a triangular part of a quantum enveloping algebra to the generic composition algebra associated with a quiver with automorphism. In the Dynkin quiver case, the existence of Hall polynomials provides a direct definition of the generic Ringel–Hall algebra. In this case, a dimension comparison shows that the algebra homomorphism above is an isomorphism.

Chapter 11 focuses on Ringel–Hall algebras of Dynkin quivers with automorphisms and the construction of bases for the corresponding quantum enveloping algebras of finite type. Starting from a monoid structure and a poset structure on the set of isomorphism classes of representations, we first obtain a systematic construction of monomial bases for quantum enveloping algebras. We then show that BGP reflection functors induce certain isomorphisms of the subalgebras of Ringel–Hall algebras, which are the restrictions of the Lusztig symmetries defined in Chapter 6. This gives a construction of PBW-type bases, which was mentioned in Chapter 6 without proof. Finally, by relating monomial and PBW-type bases, we present an elementary algebraic construction of Lusztig canonical bases for quantum enveloping algebras of finite type.

Chapter 12 deals with a comultiplication defined by Green [**136**] on the Ringel–Hall algebras. The compatibility of multiplication and comultiplication is based on what is called Green's formula. This result, together with a theorem of Lusztig, shows that the surjective algebra homomorphism defined in Chapter 10 is actually an isomorphism. Hence, the Ringel–Hall algebras provide a realization of the triangular parts of all quantum enveloping algebras.

Part 5 (Chapters 13–14) gives a full account of the Beilinson–Lusztig–MacPherson (BLM) construction for the quantum enveloping algebra associated with $\mathfrak{gl}_n$. Chapter 13 derives in an elementary geometric setting some fundamental multiplication formulas for the natural basis elements in quantum Schur algebras. This leads to a new basis for a quantum Schur algebra — the BLM basis — and to the derivation of some multiplication formulas among the new basis elements. The quantum Serre relations in a quantum Schur algebra result from these multiplication formulas. As a byproduct, a certain monomial basis, which is triangularly related to the natural basis, is constructed in order to give a presentation of a quantum Schur algebra.

Finally, in Chapter 14, a further analysis of the fundamental multiplication formulas gives a stabilization property. This prompts the definition of the BLM algebra $\mathbf{K}$ — an infinite dimensional algebra without identity — and some modified fundamental multiplication formulas. By taking a completion of $\mathbf{K}$, we obtain an algebra $\widehat{\mathbf{K}}$ with identity and derive some multiplication formulas from the modified ones. With these formulas, we prove that a certain subspace $\mathbf{V}$ of $\widehat{\mathbf{K}}$ is a subalgebra with quantum Serre relations. We then prove the isomorphism between $\mathbf{V}$ and the entire quantum $\mathfrak{gl}_n$ before closing with the establishment of the integral Schur–Weyl reciprocity. This basic result is obtained by combining the double centralizer property with the surjection from a type of integral Lusztig form to the integral quantum Schur algebras.

In addition to the chapters described above, this book contains three chapter-long appendices. Appendix A outlines basic ideas from algebraic geometry and algebraic group theory that are required in the book and concludes with a brief discussion of some more advanced topics in the representation theory of semisimple groups. Appendix B gives a largely self-contained discussion of quantum matrix spaces and quantum general linear groups — both including standard and multiparametered — and ties them with the theory of quantum Schur algebras given in Chapter 9. Finally, Appendix C provides a short and self-contained account of the theories of quasi-hereditary algebras and cellular algebras which are needed in Part 3. Making use of the results in Appendices B and C, we discuss several of the standard examples of quasi-hereditary algebras and highest weight categories that arise in representation theory.

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As evidenced by the bibliography, this book clearly could not have been written without the work of the many mathematicians who have contributed over the years to this evolving theory. It also draws, at a number of critical points, from previous book-length treatments. For example, Chapters 4 and 6 reflect the influence of Humphreys [**157**], Carter [**35, 36**], Jantzen [**165**], and Lusztig [**209, 213**], while Chapter 8 incorporates and builds on Stanley's development of the Robinson–Schensted–Knuth correspondence in [**281**]. The Notes at the end of each chapter record our indebtedness to this and other work and form a critical part of our exposition.

Each chapter also closes with a series of exercises, some of which are routine, some of which serve to fill in steps in the various arguments, and some of which call attention to the literature by sketching proofs of results to be found there.

Despite its length, there are many important topics that have not been included in this book. The chosen material reflects our own interests and forms what we hope is a coherent whole. Other topics are sometimes briefly mentioned in the Notes (and references).

★ ★ ★

Historical notes and acknowledgments: Although the theoretical interrelations between the representation theory of finite dimensional algebras and Lie theory date at least to the 1970s, the interrelations between

finite dimensional algebras and the representation theory of Lie algebras and algebraic groups became truly apparent at the Ottawa–Moosonee Algebra Workshop in 1987. First, Claus Ringel presented his ideas, which led to the development of Ringel–Hall algebras as realizations of the $\pm$-parts of quantum enveloping algebras. Second, the third author of the book and Leonard Scott presented their discovery (with Edward Cline) of quasi-hereditary algebras and highest weight categories. Since then a number of conferences touching on the same theme have been held in Ottawa (1992), Shanghai (1998), Kunming (2001), Toronto (2002), Chengdu and Banff (2004), and Lhasa (2007). The authors would like to thank the organizers of these conferences for the opportunity to observe and participate in the exciting developments in this area. The idea to write a book originated with the second and third authors 15 years ago at the Ottawa meeting. Then the last three authors made an early effort in this direction, but as the project has evolved and the subject matter developed, the first author became a member of the team.

In writing this book, we have profited from comments on several chapters by a number of graduate students, among them, Liping Wang at Chinese Academy of Sciences; Jiangrong Chen, Lingyan Guo, Baolin Xiong, Guiyu Yang, Jie Zhang, and Zhonghua Zhao at Beijing Normal University; and Wes Cramer, Chris Drupieski, and Jill Tysse at the University of Virginia. We are also grateful to colleagues Qiang Fu (Tongji University), Terrell Hodge (Western Michigan University), Paramasamy Karuppuchamy (University of Virginia), Chi Mak (University of New South Wales), and Hebing Rui (East China Normal University) for critiquing parts of the book. We thank Karen Parshall (University of Virginia) for reading the entire manuscript and making many helpful suggestions. Ms. Jing Yu provided secretarial support on an early version. We extend special thanks to Xiaoming Yu (University of Virginia) for special assistance over the years. Last, but certainly not least, we thank Leonard Scott for his friendship and collaboration over many years. Some of the results in Part 3 were discovered in collaboration by the second and third authors with him.

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Notational conventions

We adopt the following conventional notation:

$\mathbb{C}$	field of complex numbers
$\mathbb{F}_q$	finite field with q elements, q ($\neq 1$) being a prime power (thus, by a prime power we always mean a prime to a *positive* integer power)
$\mathbb{N}$	set of nonnegative integers $0, 1, 2, \ldots$
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{Z}$	ring of integers
$\mathbb{Z}^+$	set of positive integers $1, 2, \ldots$

A ground field or (commutative) ring over which algebras and representations are defined is usually written in Zapf Chancery fonts. In particular, we use the following notation throughout the book:

$\mathcal{k}$, $\mathcal{K}$	ground fields, with $\mathcal{K}$ often algebraically closed
$\mathcal{R}$	a (commutative) ground ring
$\mathcal{Z}$	$:= \mathbb{Z}[v, v^{-1}]$, the ring of Laurent polynomials over $\mathbb{Z}$ in an *indeterminate* v
$\mathcal{A}$	$:= \mathbb{Z}[q]$, where $q = v^2$

We also make the following conventions:

- $A \subseteq B$ means A is a subset of B, while $A \subset B$ means A is a *proper* subset of B; and

- for a module M over a ring, nM, where $n \in \mathbb{N}$, stands for the direct sum of n copies of M, i.e.,
$$nM = \underbrace{M \oplus \cdots \oplus M}_{n},$$
with the exceptions that
  - a free module of rank n over the ground ring $\mathcal{R}$ is denoted by $\mathcal{R}^n$, e.g., $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{Z}^n$, etc.; and
  - $\mathrm{M}_{m\times n}(\mathcal{k})$ denotes the space of $m \times n$ matrices over $\mathcal{k}$, and, if $m = n$, this space is simply denoted $\mathrm{M}_n(\mathcal{k})$. (The notation $\mathrm{M}_n(-)$ is also used over other rings or their subsets, e.g., $\mathrm{M}_n(\mathcal{D})$, for a division ring $\mathcal{D}$, or $\mathrm{M}_n(\mathbb{N})$.)

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Chapter 0
Cartan matrices, etc.

Part 1
Quivers with automorphisms,
algebras with Frobenius morphisms,
their representations

Part 2
Hecke algebras,
Hopf algebras,
quantum enveloping algebras (QEAs)

Part 3
Symmetric groups,
cells and cell modules,
quantum Schur algebras

Part 4
Ringel–Hall algebras,
bases of QEAs of finite type,
realization of positive parts of QEAs

Part 5
Geometric approach to quantum Schur algebras,
presentations of quantum Schur algebras,
BLM realization of quantum $\mathfrak{gl}_n$

Appendices

Varieties and affine algebraic groups

This appendix provides a brief outline of some material from algebraic geometry and algebraic groups needed in this book. Occasionally, we sketch (sometimes in exercises) details, but the reader is referred to the Notes at the end of the appendix for references to complete proofs in the literature. The final four sections (§§A.5–A.8) are more sophisticated, sketching several basic ideas from the representation theory of semisimple groups. These results are used in Appendix C to construct some important highest weight categories.

A topological space X is called *irreducible* if it cannot be written as a union of two *proper* closed subspaces. A subspace Z of X is irreducible if it is irreducible in its subspace topology. The *dimension*, $\dim X$, of X is the maximal length (possibly ∞) t of a proper chain $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_t$ of closed, nonempty, irreducible subsets X_i of X. Recall that $Z \subseteq X$ is locally closed if it is the intersection of an open subset with a closed subset of X or, equivalently, if Z is open in its closure $\overline{Z}$.

Let $\Bbbk$ be a fixed field. If X is a set, the set $\mathrm{Map}(X, \Bbbk)$ of functions $f\colon X \to \Bbbk$ is a commutative algebra over $\Bbbk$, using pointwise addition and multiplication of functions. If $X = \emptyset$, $\mathrm{Map}(X, \Bbbk)$ is the "zero algebra" consisting of just one element $0 = 1$. If $\varphi\colon X \to Y$ is a map, let $\varphi^*\colon \mathrm{Map}(Y, \Bbbk) \to \mathrm{Map}(X, \Bbbk)$ be the corresponding *comorphism*, defined by putting $\varphi^*(f) = f \circ \varphi$.

We now introduce a category $\mathsf{Shv}_{\Bbbk}$. An object in this category is a topological space together with a certain sheaf of $\Bbbk$-algebras. Explicitly, an object in $\mathsf{Shv}_{\mathcal{k}}$ is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a rule $\mathcal{O}_X$ which assigns to each open subset U of X a subalgebra $\mathcal{O}_X(U) \subseteq \mathrm{Map}(U, \mathcal{k})$ satisfying the following two conditions:

(1) Let $V \overset{\iota}{\hookrightarrow} U$ be open subsets of X, and, for $f \in \mathrm{Map}(U, \mathcal{k})$, let $f|_V = \iota^*(f) \in \mathrm{Map}(V, \mathcal{k})$ be the restriction of f to V. Then if $f \in \mathcal{O}_X(U)$, we require that $f|_V \in \mathcal{O}_X(V)$. Thus, $|_V \colon \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is an algebra homomorphism.

(2) Let U be an open subset of X, written as a union $U = \bigcup_i U_i$ of open subsets U_i. Let $f \in \mathrm{Map}(U, \mathcal{k})$ be such that $f|_{U_i} \in \mathcal{O}_X(U_i)$, for each i. Then we require that $f \in \mathcal{O}_X(U)$.

A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of objects in $\mathsf{Shv}_{\mathcal{k}}$ is a continuous map $\varphi \colon X \to Y$ such that, for any open subset $V \subseteq Y$, $\varphi^*\mathcal{O}_Y(V) \subseteq \mathcal{O}_X(\varphi^{-1}(V))$. In this way, $\mathsf{Shv}_{\mathcal{k}}$ becomes a category.

If $(X, \mathcal{O}_X)$ belongs to $\mathsf{Shv}_{\mathcal{k}}$ and if U is an open subset of X, let $\mathcal{O}_U$ be the "restriction" of $\mathcal{O}_X$ to U; if $V \subseteq U$ is open, put $\mathcal{O}_U(V) := \mathcal{O}_X(V)$. Then $(U, \mathcal{O}_U)$ also belongs to $\mathsf{Shv}_{\mathcal{k}}$.

All the varieties considered below will be objects in $\mathsf{Shv}_{\mathcal{k}}$ which satisfy additional conditions. If $(X, \mathcal{O}_X)$ is a variety and if U is an open subset of X, then $(U, \mathcal{O}_U)$ will also be a variety. Similarly, if Y is a closed subset of X, we will see there is a naturally induced variety $(Y, \mathcal{O}_Y)$. In this way, any locally closed subspace of a variety carries a natural variety structure. For our purposes, the varieties of interest all arise as subvarieties of affine space $\mathbb{A}^n$ or projective space $\mathbb{P}^n$.

From now on, we will work over a fixed algebraically closed field $\mathcal{K}$ instead of an arbitrary field $\mathcal{k}$. (Some remarks about the nonalgebraically closed case are given in Remarks A.10 and A.16.) At a primitive level, a variety is a set consisting of the solutions of a collection of polynomial equations $f(x_1, \ldots, x_n) = 0$ in some set of fixed variables $x_1, \ldots, x_n$. These solutions can be taken in affine n-space $\mathbb{A}^n$ (which we view as $\mathcal{K}^n$ stripped of its vector space structure) or, if the polynomials f are homogeneous, the solutions can be taken in projective $(n-1)$-space $\mathbb{P}^{n-1}$. Part of our task is to provide these "varieties" with additional structure, so that notions of "dimension," "morphism," "local structure," etc., make sense.

A.1. Affine varieties

We begin with the following definition of an affine variety. Later, affine varieties will be interpreted as objects in $\mathsf{Shv}_{\mathcal{K}}$.

Definition A.1. An *affine variety* over $\mathcal{K}$ is a pair (X, A) consisting of a set X and a subalgebra A of $\mathrm{Map}(X, \mathcal{K})$ which satisfies the following conditions:

(1) A is a finitely generated $\mathcal{K}$-algebra.

(2) Given distinct $x, y \in X$, there exists $f \in A$ such that $f(x) \neq f(y)$.

(3) Given any $\mathcal{K}$-algebra homomorphism $\alpha\colon A \to \mathcal{K}$, there exists $x \in X$ such that $\alpha(f) = f(x)$, for all $f \in A$.

Usually, we relax the formality and refer to X as an affine variety, keeping A in the background. The algebra A is called the *coordinate algebra* of X and is often denoted $\mathcal{O}(X)$. (After identifying an affine variety X as an object of $\mathsf{Shv}_{\mathcal{K}}$, the notation $\mathcal{O}(X)$ will become reasonable.)

Condition (1) simply means that there exist a positive integer n and a surjective algebra homomorphism $\theta\colon \mathcal{K}[x_1, \dots, x_n] \twoheadrightarrow A$ from the algebra $\mathcal{K}[x_1, \dots, x_n]$ of polynomials over $\mathcal{K}$ in indeterminates $x_1, \dots, x_n$ to the algebra A. If $\mathfrak{a} = \mathrm{Ker}(\theta)$, then $A \cong \mathcal{K}[x_1, \dots, x_n]/\mathfrak{a}$. Of course, the polynomial algebra $\mathcal{K}[x_1, \dots, x_n]$ is *not* uniquely determined by (X, A). Because $A \subseteq \mathrm{Map}(X, \mathcal{K})$, A contains no nonzero nilpotent elements, i.e., A is *reduced.* Thus, the ideal $\mathfrak{a}$ above is a radical ideal in the sense that

$$\mathfrak{a} = \sqrt{\mathfrak{a}} := \{f \in \mathcal{K}[x_1, \dots, x_n] \mid f^m \in \mathfrak{a}, \text{ for some } m\}.$$

Also, because $\mathcal{K}[x_1, \dots, x_n]$ is noetherian (by the Hilbert basis theorem [**CA**, Th. 7.4]), A is itself noetherian.

Next, define $\varepsilon\colon X \to \mathrm{Hom}_{\mathcal{K}\text{-alg}}(A, \mathcal{K})$ by setting $\varepsilon(x)(f) = f(x)$, $x \in X$, $f \in A$. Then conditions (2) and (3) in Definition A.1 are equivalent to the requirement that the "evaluation" map ε is a *bijection of sets.* Thus, the variety X can be recovered, as a set, from its coordinate algebra A. A converse result also holds, as we now describe.

Given an ideal $\mathfrak{a}$ in the polynomial algebra $\mathcal{K}[x_1, \dots, x_n]$, its *zero set* $\mathscr{V}(\mathfrak{a}) \subseteq \mathbb{A}^n$ is defined by

$$\mathscr{V}(\mathfrak{a}) = \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid f(x_1, \dots, x_n) = 0, \text{ for all } f \in \mathfrak{a}\}.$$

Conversely, given a subset $X \subseteq \mathbb{A}^n$, let

$$\mathscr{I}(X) = \{f \in \mathcal{K}[x_1, \dots, x_n] \mid f(x_1, \dots, x_n) = 0, \text{ for all } (x_1, \dots, x_n) \in X\}.$$

Clearly, $\mathscr{I}(X)$ is an ideal in $\mathcal{K}[x_1, \dots, x_n]$.

Theorem A.2 (Hilbert's Nullstellensatz). *Let $\mathfrak{a}$ be an ideal in $\mathcal{K}[x_1, \dots, x_n]$.*

(1) (Weak form) *If $\mathfrak{a}$ is a maximal ideal, then $\mathfrak{a} = (x_1 - x_1, \dots, x_n - x_n)$, for some $(x_1, \dots, x_n) \in \mathbb{A}^n$.*

(2) (Strong form) *Suppose that $f \in \mathcal{K}[x_1, \dots, x_n]$ vanishes on all points in $\mathscr{V}(\mathfrak{a})$. Then $f \in \sqrt{\mathfrak{a}}$. In other words, $\mathscr{I}(\mathscr{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

For a proof of (1), see [**CA**, Ex. 17, p. 69]; (2) is also proved in [**CA**, Ex. 14, p. 85].

Using this result, we can conclude that any commutative $\mathcal{K}$-algebra A which is reduced and finitely generated as a $\mathcal{K}$-algebra has the form $A = \mathcal{O}(X)$, for some affine variety X. Let $\mathsf{Affalg}_{\mathcal{K}}$ denote the full subcategory of the category of commutative $\mathcal{K}$-algebras whose objects are finitely generated reduced $\mathcal{K}$-algebras.

Corollary A.3. *If $A \in \mathsf{Affalg}_{\mathcal{K}}$, and if $X = \mathrm{Hom}_{\mathcal{K}\text{-alg}}(A, \mathcal{K})$ is the set of algebra homomorphisms $A \to \mathcal{K}$, then (X, A) is an affine variety.*

Proof. We can assume that A is not the zero algebra. Write A in the form $\mathcal{K}[x_1, \ldots, x_n]/\mathfrak{a}$, for some integer n and ideal $\mathfrak{a}$. Since the kernel of any homomorphism $A \to \mathcal{K}$ is a maximal ideal, Theorem A.2(1) shows that X is not empty and identifies with $\mathscr{V}(\mathfrak{a})$.

To prove the result, it suffices to show that the obvious map $A \to \mathrm{Map}(X, \mathcal{K})$ is injective, so that A can be identified with a subalgebra of $\mathrm{Map}(X, \mathcal{K})$. But this fact follows from Theorem A.2(2) since A reduced, so $\sqrt{\mathfrak{a}} = \mathfrak{a}$. □

Examples A.4. (1) Affine n-space $(\mathbb{A}^n, \mathcal{K}[x_1, \ldots, x_n])$ is an affine variety.

(2) Let (X, A) be an affine variety, and let $f \in A$. If $X_f = \{x \in X \mid f(x) \neq 0\}$ and if A_f is the localization of A at the multiplicative set $S = \{1, f, f^2, \ldots\}$, then (X_f, A_f) is an affine variety. For example, let $X = \mathbb{A}^{n^2}$ be affine n^2-space, identified with the set $\mathrm{M}_n(\mathcal{K})$ of $n \times n$ matrices over $\mathcal{K}$. Let $f = \det$ be the determinant polynomial. Then X_f identifies with the general linear group $\mathrm{GL}_n(\mathcal{K})$ of invertible $n \times n$ matrices.

(3) If B is a finite dimensional $\mathcal{K}$-algebra (not necessarily commutative), then the set of units $B^\times$ in B is an affine variety.

To finish the definition of $\mathsf{Affvar}_{\mathcal{K}}$, the category of affine varieties over $\mathcal{K}$, we define morphisms as follows.

Definition A.5. Let (X, A) and (Y, B) be affine varieties. A *morphism* from X to Y is a map $\varphi\colon X \to Y$ such that $\varphi^*(B) \subseteq A$.

A morphism $\varphi\colon X \to Y$ is completely determined by the comorphism $\varphi^*\colon B \to A$, once we identify X (resp., Y) with $\mathrm{Hom}_{\mathcal{K}\text{-alg}}(A, \mathcal{K})$ (resp., $\mathrm{Hom}_{\mathcal{K}\text{-alg}}(B, \mathcal{K})$). That is, given $x \in X$, $\varphi(x)(b) = x(\varphi^*(b))$. So, we obtain the following basic fact:

Theorem A.6. *The functor*

$$\mathsf{Affvar}_{\mathcal{K}} \longrightarrow \mathsf{Affalg}_{\mathcal{K}}, \quad (X, A) \longmapsto A$$

is a contravariant equivalence of categories.

Given an affine variety X, there is an important topology — the *Zariski topology* — defined on it. Let $A = \mathcal{O}(X)$. If $\mathfrak{a}$ is an ideal in A, let $\mathscr{V}(\mathfrak{a}) \subseteq X$ be the set of all $x \in X$ such that $f(x) = 0$, for all $f \in \mathfrak{a}$. Then the set $\mathsf{Zar}(X)$ of closed sets in the topology on X are those subsets $\mathscr{V}(\mathfrak{a})$ for ideals $\mathfrak{a} \subseteq A$. Since

(1) $\mathscr{V}(\mathfrak{a}) \cup \mathscr{V}(\mathfrak{b}) = \mathscr{V}(\mathfrak{a} \cap \mathfrak{b})$, for ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$,

(2) $\bigcap_i \mathscr{V}(\mathfrak{a}_i) = \mathscr{V}\left(\sum_i \mathfrak{a}_i\right)$, for any set of ideals $\mathfrak{a}_i \subseteq A$, i running over an index set I,

(3) $\emptyset = \mathscr{V}(A)$, and

(4) $X = \mathscr{V}(0)$,

$\mathsf{Zar}(X)$ is, in fact, the set of closed sets for a topology. As we have already seen, any closed subset $Z = \mathscr{V}(\mathfrak{a})$ is itself an affine variety with coordinate algebra $A/\sqrt{\mathfrak{a}}$. Generally, we speak of these as *closed subvarieties* of X. By the discussion above, the closed subsets of X are in one-to-one correspondence with ideals $\mathfrak{a}$ of A satisfying $\sqrt{\mathfrak{a}} = \mathfrak{a}$; also, for two such ideals $\mathfrak{a}, \mathfrak{b}$, $\mathscr{V}(\mathfrak{a}) \subseteq \mathscr{V}(\mathfrak{b}) \Longleftrightarrow \mathfrak{a} \supseteq \mathfrak{b}$.

The following theorem, whose proof is an easy exercise (see Exercise A.2), summarizes some basic properties of the Zariski topology.

Theorem A.7. *Let X be an affine variety with coordinate algebra A.*

(1) *X is irreducible if and only if A is an integral domain. More generally, a closed subset $\mathscr{V}(\mathfrak{a})$ is irreducible if and only if $\sqrt{\mathfrak{a}}$ is a prime ideal in A.*

(2) *X can be written as a finite union*

$$X = X_1 \cup X_2 \cup \cdots \cup X_m$$

of irreducible, closed subvarieties which is irredundant in the sense that there are no containments $X_i \subseteq X_j$, for $i \neq j$. Furthermore, this decomposition is unique up to order. (The X_i are called the irreducible components of X.)

(3) *X is a noetherian topological space in the sense that any decreasing chain $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ of closed subspaces eventually stabilizes, i.e., there exists $n > 0$ with $X_n = X_{n+1} = \ldots$.*

(4) *For any nonunit $f \in A$, the subset X_f is a nonempty open subset of X. The collection of all such X_f, $f \in A$, forms a base for the Zariski topology on X.*

Given a commutative ring A, its *Krull dimension*, $\operatorname{Kdim} A$, is the maximal length (possibly ∞) t of a proper chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_t$ of prime ideals $\mathfrak{p}_i$ in A.

Before stating the next theorem, we need to introduce the notion of the *tangent space* $T_x(X)$ of X at a point $x \in X$. Let $A = \mathcal{O}(X)$ and let $\mathfrak{m}_x$ be the kernel of the evaluation map $A \to \mathcal{K}$, $f \mapsto f(x)$. Let

$$T_x(X) := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*,$$

the linear dual of the vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Theorem A.8. *Suppose X is an affine variety.*

(1) $\dim X = \operatorname{Kdim} \mathcal{O}(X) < \infty$. *Also,* $\dim \mathbb{A}^n = n$.

(2) *If $X_1, \ldots, X_t$ are the irreducible components of X, then*

$$\dim X = \max_i \dim X_i.$$

(3) *Suppose that X is irreducible, so that $\mathcal{O}(X)$ is an integral domain. For $\mathcal{K}(X)$ the fraction field of $\mathcal{O}(X)$,*

$$\dim X = \text{tr.}\deg_{\mathcal{K}} \mathcal{K}(X),$$

the transcendency degree of $\mathcal{K}(X)$ over $\mathcal{K}$.

(4) *If Y is an irreducible closed subvariety of X, then* $\dim Y = \dim X$ *if and only if Y is an irreducible component of X of maximal dimension. In particular, if X is irreducible,* $\dim Y = \dim X$ *if and only if $Y = X$.*

(5) *For any $x \in X$,*

$$\dim X \leqslant \dim T_x(X) < \infty.$$

Furthermore, there exists a nonempty open subset X_{reg} of X such that for $x \in X_{\mathrm{reg}}$, $\dim X = \dim T_x(X)$. (Points $x \in X$ for which the equality $\dim X = \dim T_x(X)$ holds are called *smooth* or *regular* points of X.)

(6) *For $\varphi\colon X \to Y$ a morphism of affine varieties, and for $x \in X$ and $y = \varphi(x)$, φ induces a natural linear map $d\varphi\colon T_x(X) \to T_y(Y)$, called the differential of φ at x.*

Given $x \in X$, a *point derivation* (of $\mathcal{O}(X)$) at x is a linear map $\delta\colon \mathcal{O}(X) \to \mathcal{K}$ such that $\delta(fg) = f(x)\delta(g) + \delta(f)g(x)$. Then $T_x(X)$ can be naturally (and easily) identified with the space of point derivations at x. In this way, the differential in (6) is easy to construct, since a point derivation δ at x (of $\mathcal{O}(X)$) will clearly induce a point derivation $\delta \circ \varphi^*$ at $y = \varphi(x)$ (of $\mathcal{O}(Y)$).

Next observe that the category $\mathsf{Affvar}_{\mathcal{K}}$ of affine varieties over $\mathcal{K}$ has products. See Exercise A.3 for part of the proof.

Theorem A.9. *Let X, Y be affine varieties.*

(1) $\mathcal{O}(X) \otimes_{\mathcal{K}} \mathcal{O}(Y) \in \mathsf{Affalg}_{\mathcal{K}}$.

(2) *If Z is the (unique up to isomorphism) affine variety with $\mathcal{O}(Z) = \mathcal{O}(X) \otimes_{\mathcal{K}} \mathcal{O}(Y)$, then as a set Z identifies with the set-theoretic product $X \times Y$.*

(3) *If* $\pi_1\colon Z \to X$ *(resp.,* $\pi_2\colon Z \to Y$*) is the morphism of varieties induced by the algebra homomorphism* $\mathcal{O}(X) \to \mathcal{O}(X)\otimes_{\mathcal{K}}\mathcal{O}(Y)$ *(resp.,* $\mathcal{O}(Y) \to \mathcal{O}(X)\otimes_{\mathcal{K}}\mathcal{O}(Y)$*) given by* $a \mapsto a\otimes 1$ *(resp.,* $b \mapsto 1\otimes b$*), then, identifying* Z *with* $X \times Y$ *as sets,* $\pi_1((x,y)) = x$ *and* $\pi_2((x,y)) = y$*, for all* $x \in X, y \in Y$.

(4) Z *is the product of* X *and* Y *in the category* $\mathsf{Affvar}_{\mathcal{K}}$ *in the following sense: given any affine variety* W *and morphisms* $F\colon W \to X$, $G\colon W \to Y$*, there exists a unique morphism* $H\colon W \to Z$ *such that* $\pi_1 \circ H = F$ *and* $\pi_2 \circ H = G$*. For this reason, we denote* Z *simply by* $X \times Y$.

(5) $\dim X \times Y = \dim X + \dim Y$.

Of course, property (4) in the theorem follows immediately from the contravariant equivalence given by Theorem A.6.

Let X be an affine variety with coordinate algebra $A = \mathcal{O}(X)$. It remains to equate (X, A) with an object $(X, \mathcal{O}_X)$ in $\mathsf{Shv}_{\mathcal{K}}$. We have remarked above that the open subsets X_f, $f \in A$, form a base for the Zariski topology on X. Suppose that U is a nonempty open subset of X. We assign a $\mathcal{K}$-algebra $\mathcal{O}_X(U)$ of $\mathcal{K}$-valued functions on U as follows. By definition, $g\colon U \to \mathcal{K}$ belongs to $\mathcal{O}_X(U)$ provided, for any $x \in U$, there is an $f \in A$ such that $x \in X_f \subseteq U$ and $g|_{X_f} \in A_f$. It can be verified that $(X, \mathcal{O}_X) \in \mathsf{Shv}_{\mathcal{K}}$ and $\mathcal{O}_X(X) = A$. In this way, the pair (X, A) determines the pair $(X, \mathcal{O}_X)$, and conversely. (However, it is not true that every object $(X, \mathcal{O}_X)$ in $\mathsf{Shv}_{\mathcal{K}}$ is a variety.) This identification of (X, A) with $(X, \mathcal{O}_X)$ explains the notation $A = \mathcal{O}(X)$ we introduced before — $\mathcal{O}(X)$ is an abbreviation for $\mathcal{O}_X(X)$.

We define a *quasi-affine* variety to be an object $(U, \mathcal{O}_U)$ in $\mathsf{Shv}_{\mathcal{K}}$ in which U is an open subset of an affine variety X and the sheaf of $\mathcal{K}$-valued functions $\mathcal{O}_U$ is that induced on U by $\mathcal{O}_X$. In general, a quasi-affine variety U need not be affine. However, it will be a union of a finite number of affine open subvarieties U_i. If $x \in U$, its tangent space is calculated (in a well-defined way) by regarding x as a point in an open affine subvariety of U. The dimension of U is the topological dimension of U defined above. As in Theorem A.8(5), $\dim U = \dim T_x$, for all x lying in a nonempty open subset of U.

We conclude this section by the following remark on how some of the setup can be made to work over a not necessarily algebraically closed field.

Remark A.10. Let $\mathcal{k}$ be a subfield of $\mathcal{K}$. We say that the affine variety X with coordinate algebra A is *defined over* $\mathcal{k}$ (or that X is a $\mathcal{k}$-variety) provided there is a $\mathcal{k}$-subalgebra A_0 of A such that $A_0 \otimes_{\mathcal{k}} \mathcal{K} \xrightarrow{\sim} A$. In this case, any algebra homomorphism $f\colon A_0 \to \mathcal{k}$ extends uniquely to an algebra homomorphism $f \otimes \mathcal{K}\colon A \to \mathcal{K}$. Identifying X with $\mathrm{Hom}_{\mathcal{K}\text{-alg}}(A, \mathcal{K})$, we denote by $X(\mathcal{k})$ the subset of X consisting of all algebra homomorphisms

$f \otimes \mathcal{K}$, for $f \in \mathrm{Hom}_{k\text{-alg}}(A_0, k)$. We call $X(k)$ the set of *k-rational* points of X. The following are some examples:

(1) If $X = \mathbb{A}^n$ is an affine n-space, then

$$A = \mathcal{K}[x_1, \ldots, x_n] \cong k[x_1, \ldots, x_n] \otimes_k \mathcal{K}.$$

In this way, $\mathbb{A}^n$ is defined over k. Of course, $X(k)$ identifies with k^n.

(2) The general linear group $\mathrm{GL}_n(\mathcal{K})$ viewed as an affine variety has coordinate algebra $\mathcal{O}(\mathrm{GL}_n(\mathcal{K})) = \mathcal{K}[x_{1,1}, \ldots, x_{n,n}][\det^{-1}]$. Since the determinant polynomial has coefficients in the prime field, $\det \in k[x_{1,1}, \ldots, x_{n,n}]$, so the isomorphism

$$\mathcal{K}[x_{1,1}, \ldots, x_{n,n}][\det^{-1}] \cong k[x_{1,1}, \ldots, x_{n,n}][\det^{-1}] \otimes_k \mathcal{K}$$

gives $\mathrm{GL}_n(\mathcal{K})$ a structure as a k-variety. (In general, $\mathrm{GL}_n(\mathcal{K})$ may have many nonisomorphic structures as a k-variety — we have just provided one example.) We also have that the set of k-rational points $\mathrm{GL}_n(k)$ is the subgroup of $\mathrm{GL}_n(\mathcal{K})$ consisting of $n \times n$ invertible matrices with coefficients in k.

A.2. Varieties

In the previous section, we defined the notion of an affine variety. Now we introduce varieties that need not be affine. Such geometric spaces are obtained by patching together a finite collection of affine varieties in a consistent way. More precisely, we make the following definition.

Definition A.11. An object $(X, \mathcal{O}_X)$ in $\mathsf{Shv}_{\mathcal{K}}$ is a *variety* provided that $X = U_1 \cup \cdots \cup U_m$, where each U_i is a nonempty open subset such that the following conditions hold:

(1) For $1 \leqslant i \leqslant n$, $(U_i, \mathcal{O}_{U_i})$ is an affine variety.

(2) Given i, j, $U_i \cap U_j = (U_i)_{f_{i,j}}$, for some $f_{i,j} \in \mathcal{O}_X(U_i)$.

(3) For $1 \leqslant i, j \leqslant n$, $\{(x, x) \mid x \in U_i \cap U_j\}$ is a closed subset of $U_i \times U_j$.

For some explanation of the usefulness of condition (3), see Exercise A.5.

Examples A.12. (1) An important class of (nonaffine) varieties are the projective varieties $\mathbb{P}^n$, where n is a nonnegative integer. Thus, $\mathbb{P}^n$ is the set of lines in $\mathbb{A}^{n+1}$ through the origin; specifically,

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \backslash \{\mathbf{0}\}) / \sim,$$

where $\sim$ is the equivalence relation on $\mathbb{A}^{n+1} \backslash \{\mathbf{0}\}$ defined by putting

$$\boldsymbol{x} = (x_1, \ldots, x_{n+1}) \sim \boldsymbol{y} = (y_1, \ldots, y_{n+1}),$$

provided there exists $\xi \in \mathcal{K}$ such that $x_i = \xi y_i$, for all i. Denote the equivalence class of $(x_1, \ldots, x_{n+1})$ by $[x_1, \ldots, x_{n+1}]$. (More generally, if V

is a nonzero, finite dimensional vector space, write $\mathbb{P}(V)$ for the set of lines in V. So, if $\dim V = n+1$, then $\mathbb{P}(V)$ identifies with $\mathbb{P}^n$, once an ordered basis of V has been fixed.

The polynomial algebra $S = \mathcal{K}[x_1, \ldots, x_{n+1}]$ is an $\mathbb{N}$-graded algebra $S = \bigoplus_r S_r$, where S_r is the subspace of S spanned by the monomials in $x_1, \ldots, x_{n+1}$ of total degree r. For $1 \leqslant i \leqslant n+1$, let U_i be the subset of $\mathbb{P}^n$ consisting of all "points" with nonzero i-coordinate. Identify U_i with the affine space $\mathbb{A}^n$ having coordinate algebra $\mathcal{K}[x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_{n+1}/x_i]$. Then $\mathbb{P}^n = U_1 \cup \ldots \cup U_{n+1}$. The topology on $\mathbb{P}^n$ is defined by declaring a subset U of $\mathbb{P}^n$ to be open provided that $U \cap U_i$ is open in U_i, for all i. When U is open, $\mathcal{O}_{\mathbb{P}^n}(U)$ consists of those $f \in \mathrm{Map}(U, \mathcal{K})$ such that, for each i, $f|_{U \cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$. Then $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is a variety.

(2) An ideal $\mathfrak{a}$ in S is homogeneous provided that $\mathfrak{a} = \bigoplus \mathfrak{a}_i$, where $\mathfrak{a}_i = S_i \cap \mathfrak{a}$. Given a homogeneous ideal $\mathfrak{a}$, let $\mathcal{V}(\mathfrak{a})$ be the set of common zeros in $\mathbb{P}^n$ of homogeneous elements in $\mathfrak{a}$. Then $\mathcal{V}(\mathfrak{a})$ has a natural structure as a variety. We call such varieties *projective varieties.* Observe that $\mathcal{V}(\mathfrak{a}) = \emptyset$ (the "empty" variety) if and only if either $\mathfrak{a}$ is the augmentation ideal $S_+ := \bigoplus_{i>0} S_i$ or is S itself.

A morphism from a variety X to a variety Y is just a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the category $\mathsf{Shv}_{\mathcal{K}}$.

Theorem A.13. (1) *If U is an open subset of a variety X, then $(U, \mathcal{O}_U)$ is a variety.*

(2) *Products exist in the category of varieties. More precisely, if X, Y are varieties, there exists a variety Z together with morphisms $\pi_1 \colon Z \to X$ and $\pi_2 \colon Z \to Y$ satisfying the following universal property: given any variety W and morphisms $\varphi \colon W \to X$, $\psi \colon W \to Y$, there exists a unique morphism $\theta \colon W \to Z$ such that $\varphi = \pi_1 \circ \theta$ and $\psi = \pi_2 \circ \theta$. As a set, Z identifies with the cartesian product $X \times Y$ with $\pi_1((x,y)) = x, \pi_2((x,y)) = y$. For this reason, we denote Z by $X \times Y$.*

Examples A.14. (1) The map $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ given by

$$\big([x_1, \ldots, x_{m+1}], [y_1, \ldots, y_{n+1}]\big) \longmapsto [\ldots, x_i y_j, \ldots]$$

(ordered in some fixed way) defines $\mathbb{P}^m \times \mathbb{P}^n$ as a projective subvariety of $\mathbb{P}^{mn+m+n}$.

(2) Let M be an $(n+1)$-dimensional vector space, and let $1 \leqslant d \leqslant n+1$. Any d-dimensional subspace N of M determines a line in the dth exterior power $\Lambda^d M$, and hence a point in $\mathbb{P}(\Lambda^d M)$. The set $\mathfrak{G}_d(M)$ of such points in $\mathbb{P}(\Lambda^d M)$ is a closed (hence projective) subvariety, called the *d-Grassmannian variety.* For example, $\mathfrak{G}_1(M) = \mathbb{P}(M) \cong \mathbb{P}^n$.

(3) Let M be an $(n+1)$-dimensional vector space. Let $1 \leqslant d_1 < d_2 < \cdots < d_t \leqslant n+1$ be a sequence of integers. Using (2) above, any flag $M_1 \subset M_2 \subset \cdots \subset M_t$ of subspaces with $\dim M_i = d_i$, $1 \leqslant i \leqslant t$, determines a point in $\mathfrak{G}_{d_1}(M) \times \cdots \times \mathfrak{G}_{d_t}(M)$. By (1) and (2), the set of such points defines a projective variety $\mathfrak{F}(\boldsymbol{d}, M)$, where $\boldsymbol{d} = (d_1, \ldots, d_t)$. When $t = n+1$ (and hence $d_i = i$, for all i), this variety is denoted simply $\mathfrak{F}_{\mathrm{cpl}}(M)$ and called the *flag variety* of M. Its elements are called *complete flags*.

We will need the following result.

Theorem A.15. *Let $\varphi\colon X \to Y$ be a morphism of varieties.*

(1) (Chevalley) *The image $\varphi(X)$ contains a nonempty open subset of the closure $\overline{\varphi(X)}$.*

(2) *Assume that X is irreducible with dense image in Y (i.e., $\overline{\varphi(X)} = Y$). Let $r = \dim X - \dim Y$. For $y \in \varphi(X)$, $\dim \varphi^{-1}(y) \geqslant r$. In addition, there exists a nonempty open subvariety U of Y such that $U \subseteq \varphi(X)$, and, for $y \in U$, $\dim \varphi^{-1}(y) = r$.*

Remark A.16. Let $\mathcal{k}$ be a subfield of $\mathcal{K}$. The theory of varieties over $\mathcal{K}$ can be expanded to include a theory of varieties defined over the subfield $\mathcal{k}$. In this case, the structure sheaf $\mathcal{O}_X$ is obtained by "base change" from a sheaf of $\mathcal{k}$-valued functions, suitably defined. The space X will have a finite cover by open affine subvarieties U, each defined over $\mathcal{k}$ and with compatible $\mathcal{k}$-structures. At the level of projective varieties, we might be tempted to define a $\mathcal{k}$-projective variety to be one defined by homogeneous polynomials in $\mathcal{k}[x_1, \ldots, x_{n+1}]$. This definition leads to some difficulties when $\mathcal{k}$ is not perfect, but suffices for the applications in this book. The reader can check that when M is a finite dimensional vector space over $\mathcal{K}$ with a $\mathcal{k}$-structure, then the Grassmannian varieties $\mathfrak{G}_d(M)$ and the flag varieties $\mathfrak{F}(\boldsymbol{d}, M)$ are all defined over $\mathcal{k}$. We refer the reader to the references in the Notes for a more detailed discussion.

Finally, we say that a variety X is *complete* provided that, given any variety Y, the projection map $\pi_2\colon X \times Y \to Y$ is a closed mapping.

Theorem A.17. *Any projective variety X over $\mathcal{K}$ is complete.*

Lemma A.18. (1) *If $f\colon X \to Y$ is a morphism of varieties with X complete, then the image $f(X)$ is a closed subset of Y, and it is complete when viewed as a subvariety of Y.*

(2) *If X is an irreducible complete variety, then $\mathcal{O}_X(X) \cong \mathcal{K}$.*

(3) *Any irreducible, complete affine variety X is a single point.*

Proof. (1) By Exercise A.5, the graph Γ_f is closed in $X \times Y$. Thus, $f(X)$, which is the image of Γ_f under the projection $X \times Y \to Y$, is closed in Y. The completeness of X now easily implies that of $f(X)$.

(2) Any $f \in \mathcal{O}_X(X)$ defines a morphism $f\colon X \to \mathbb{A}^1$. Since $\mathbb{A}^1$ is an open subvariety of the complete variety $\mathbb{P}^1$, $\mathbb{A}^1$ is not complete. On the other hand, the only other closed, irreducible subsets of $\mathbb{A}^1$ are points, so f must be constant.

(3) Let X be irreducible, complete, and affine. Then $\mathcal{O}(X) = \mathcal{O}_X(X) \cong \mathcal{K}$ by (2). Thus, $X = \mathrm{Hom}_{\mathcal{K}\text{-}\mathsf{alg}}(\mathcal{O}(X), \mathcal{K})$ is a point. □

A.3. Affine algebraic groups

We outline some basic material on affine algebraic groups.

Definition A.19. An *affine algebraic group* over $\mathcal{K}$ is an affine variety G which is also a group with the property that the multiplication map $G \times G \to G$, $(x, y) \mapsto xy$, and the inverse map $G \to G$, $x \mapsto x^{-1}$, are both morphisms of affine varieties.

Let G be an affine algebraic group and let $A = \mathcal{O}(G)$. By Theorem A.6, multiplication $G \times G \to G$ is defined by a unique algebra homomorphism $\Delta\colon A \to A \otimes A$. Thus, if $\Delta(f) = \sum f_{(1)} \otimes f_{(2)}$, then $f(xy) = \sum f_{(1)}(x) f_{(2)}(y)$, for all $f \in A$, and $x, y \in G$. Also, the inverse map $G \to G$ is defined by an algebra homomorphism $\gamma\colon A \to A$. Thus, $\gamma(f)(x) = f(x^{-1})$, for all $f \in A$, and $x \in G$. Finally, let $\varepsilon\colon A \to \mathcal{K}$ be evaluation at the identity $e \in G$: $\varepsilon(f) = f(e)$. The group axioms readily imply the first assertion in the following result; see §5.1 for a discussion of Hopf algebras. The second assertion is readily obtained using the contravariant equivalence given in Theorem A.6.

Proposition A.20. *The 4-tuple $(A, \Delta, \gamma, \varepsilon)$ is a commutative Hopf algebra. Conversely, if A is a commutative Hopf algebra, which as an algebra belongs to $\mathsf{Affalg}_{\mathcal{K}}$, then there is an algebraic group G with coordinate algebra A.*

Given the commutative Hopf algebra A in the proposition, the corresponding algebraic group G is provided, as a variety, by Theorem A.6. The multiplication (resp., inverse map, identity element) are provided by Δ (resp., γ, ε).

Given two affine algebraic groups G and H, a *morphism* $\varphi\colon H \to G$ is defined to be a homomorphism of groups which is also a morphism of affine varieties. As a morphism of varieties, φ is determined by its comorphism $\varphi^*\colon \mathcal{O}(G) \to \mathcal{O}(H)$, and φ is a morphism of affine algebraic groups if and only if φ^* is a morphism of Hopf algebras.

In the following examples, we will often describe the group G in terms of its coordinate algebra A.

Examples A.21. (1) *The multiplicative group* $\mathbb{G}_m$ has coordinate algebra $\mathcal{O}(\mathbb{G}_m) = \mathcal{K}[x, x^{-1}]$, the algebra of Laurent polynomials in a variable x. The comultiplication Δ is defined by $\Delta(x^{\pm 1}) = x^{\pm 1} \otimes x^{\pm 1}$, while $\gamma(x^{\pm 1}) = x^{\mp 1}$, and $\varepsilon(x^{\pm 1}) = 1$. As an abstract group, $\mathbb{G}_m$ is isomorphic to the multiplicative group $\mathcal{K}^\times$ of nonzero elements in $\mathcal{K}$.

For any affine algebraic group G, the set $X(G) = \mathrm{Hom}(G, \mathbb{G}_m)$ is a group under the multiplication of $\mathcal{O}(G)$, called the *character group* of G. Each $\chi \in X(G)$ is called a *character* of G. Note that $X(G)$ is, in fact, the set of group-like elements[1] in $\mathcal{O}(G)$.

(2) *The n-dimensional torus* $T = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ (n copies) has coordinate algebra $\mathcal{O}(T) = \mathcal{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, the algebra of Laurent polynomials in n variables. Also, $\Delta(x_i^{\pm 1}) = x_i^{\pm 1} \otimes x_i^{\pm 1}$, while $\gamma(x_i^{\pm 1}) = x_i^{\mp 1}$, and $\varepsilon(x_i^{\pm 1}) = 1$. As an abstract group, $T \cong \mathcal{K}^\times \times \cdots \times \mathcal{K}^\times$ (n copies).

The character group $X(T)$ of a torus T is of great importance. Let $\varepsilon_i \colon T \to \mathbb{G}_m$ be the projection of T onto its i-component. It is easy to check that $X(T)$ is the free abelian group with basis $\varepsilon_1, \ldots, \varepsilon_n$. (In general, it is common to change notation and regard $X(T)$ as an additive group.)

(3) *The additive group* $\mathbb{G}_a$ has coordinate algebra $\mathcal{O}(\mathbb{G}_a) = \mathcal{K}[x]$, the polynomial algebra in a single variable x. Thus, as an affine variety, $\mathbb{G}_a \cong \mathbb{A}^1$. We have $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\gamma(x) = -x$, and $\varepsilon(x) = 0$. As an abstract group, $\mathbb{G}_a$ is isomorphic to the additive group of $\mathcal{K}$.

(4) *The general linear group* $\mathrm{GL}_n(\mathcal{K})$ has coordinate algebra $\mathcal{O}(\mathrm{GL}_n(\mathcal{K})) = \mathcal{K}[x_{1,1}, \ldots, x_{n,n}][\det^{-1}]$, which is the localization of the polynomial algebra in n^2 variables at the determinant polynomial det. Then $\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j}$, and $\Delta(\det^{-1}) = \det^{-1} \otimes \det^{-1}$ define an algebra map $\Delta \colon A \to A \otimes A$. The counit ε satisfies $\varepsilon(x_{i,j}) = \delta_{i,j}$ and $\varepsilon(\det^{-1}) = 1$. Finally, the antipode γ is defined by Cramer's rule: $\gamma(x_{i,j}) = (-1)^{i+j} A_{i,j} / \det$, where $A_{i,j}$ is the (i,j)-minor[2] of the $n \times n$ matrix $(x_{k,l})$, the determinant of its submatrix by deleting the ith column and the jth row. One can check that $\gamma(\det^{-1}) = \det$.

---

[1]An element x in a bialgebra A (over an arbitrary ring $\mathcal{R}$) is called group-like if $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$. This is equivalent to saying that there is an A-comodule V which is free and of rank 1 over $\mathcal{R}$ with $\tau(v) = v \otimes x$, for any $v \in V$, where $\tau \colon V \to V \otimes A$ is the comodule structure map.

[2]In some linear algebra textbooks, the (i,j)-minor $A_{i,j}$ of matrix $A = (x_{k,l})$ is defined as the determinant of its submatrix obtained by deleting its ith row and jth column. Our definition here makes the exposition of its generalization to quantum linear groups more natural; see Corollary B.9 and the definition before it.

Of course, $\mathbb{G}_m \cong \mathrm{GL}_1(\mathcal{K})$. Note also that $\mathrm{GL}(M)$, for any finite dimensional $\mathcal{K}$-vector space M, can be identified with $\mathrm{GL}_n(\mathcal{K})$, for $n = \dim M$, via a basis of M. So $\mathrm{GL}(M)$ is also an affine algebraic group.

(5) *The special linear group* $\mathrm{SL}_n(\mathcal{K})$ with coordinate algebra $\mathcal{O}(\mathrm{SL}_n(\mathcal{K})) = \mathcal{K}[x_{1,1}, \dots, x_{n,n}]/(\det -1)$, where $\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j}$. Also, $\gamma(x_{i,j}) = (-1)^{i+j} A_{i,j}$ (cf. Example (4) above), and $\varepsilon(x_{i,j}) = \delta_{i,j}$. As an abstract group, $\mathrm{SL}_n(\mathcal{K})$ is the group of all $n \times n$ matrices in $\mathcal{K}$ having determinant 1.

(6) Any finite group Γ is an affine algebraic group, in which the underlying variety is just a disjoint union of zero-dimensional varieties (points). The coordinate algebra $\mathcal{O}(\Gamma) = \bigoplus_{x\in\Gamma} \mathcal{K} e_x$, where $e_x e_y = \delta_{x,y} e_x$, for all $x, y \in \Gamma$. The coalgebra structure is defined by $\Delta(e_x) = \sum_{yz=x} e_y \otimes e_z$, $\varepsilon(e_x) = 1$ and $\gamma(e_x) = e_{x^{-1}}$.

The following result summarizes some elementary properties of affine algebraic groups.

Proposition A.22. *Let G, H be affine algebraic groups.*

(1) *The product variety $G \times H$ is also an affine algebraic group. As a group, it is isomorphic to the direct product of G and H.*

(2) *If K is a subgroup of G, then its closure $\overline{K}$ is an affine algebraic group, given its structure as a closed subvariety of G.*

(3) *If $\varphi\colon H \to G$ is a morphism of affine groups, the image* $\operatorname{Im}\varphi$ *of φ is a closed subgroup of G and the kernel* $\operatorname{Ker}\varphi$ *of φ is a closed subgroup of H.*

(4) *As a topological space, G has finitely many connected components. These components are disjoint and are the irreducible components of the variety G. The unique component G° containing the identity element $e \in G$ is a closed normal subgroup of G (and hence an affine algebraic group in its own right — called the connected component of the identity of G).*

(5) *The commutator subgroup $\mathscr{C}^1(G) := (G, G)$ is a closed subgroup of G.*

See Exercise A.8 for a sketch of part of the proof.

Example A.23. The groups in Examples A.21(1)–(5) are all connected. If G is finite, $G^\circ = \{e\}$. If $\mathcal{K}$ does not have characteristic 2, the *orthogonal group* $\mathrm{O}_n(\mathcal{K}) = \{g \in \mathrm{GL}_n(\mathcal{K}) \mid g^{-1} = g^{\mathsf{T}}\}$ has two connected components. In this case, $\mathrm{O}_n(\mathcal{K})^\circ = \mathrm{SO}_n(\mathcal{K})$, the subgroup of orthogonal matrices of determinant 1.

An affine algebraic group G is called *solvable* (resp., nilpotent, abelian) provided it is solvable (resp., nilpotent, abelian) as an abstract group. We say that G is *unipotent* provided that, for any realization of G as a closed subgroup of some $\mathrm{GL}_n(\mathcal{K})$, all the elements of G when viewed as matrices are unipotent (i.e., have only eigenvalues 1). For example, the additive group $\mathbb{G}_a$ is unipotent.

If G is an affine algebraic group, it has a unique maximal connected solvable normal closed subgroup denoted $R(G)$ and called the *solvable radical* of G. Similarly, G has a unique maximal connected unipotent normal closed subgroup denoted $R_u(G)$ and called the *unipotent radical.* We will call a connected affine algebraic group G *reductive* (resp., *semisimple*) provided $R_u(G) = \{e\}$ (resp., $R(G) = \{e\}$).

Example A.24. $\mathrm{GL}_n(\mathcal{K})$ is reductive, but not semisimple. The subgroup $R(G)$ consists of the nonzero scalar matrices (and is isomorphic to $\mathbb{G}_m$). The group $\mathrm{SL}_n(\mathcal{K})$ is semisimple.

An important tool in studying algebraic groups is the Lie algebra of the group. Let G be an affine algebraic group. Recall that the tangent space $T_e(G)$ of G at e can be identified with the space of point derivations $\mathcal{O}(G) \to \mathcal{K}$ at e. Thus, it is a subspace of $\mathcal{O}(G)^* = \mathrm{Hom}_{\mathcal{K}}(\mathcal{O}(G), \mathcal{K})$. On the other hand, since $\mathcal{O}(G)$ is a Hopf algebra, the space $\mathcal{O}(G)^*$ has an associative $\mathcal{K}$-algebra structure dual to the coalgebra structure on $\mathcal{O}(G)$; see Proposition 5.4. One can check that if $\mathsf{x}, \mathsf{y} \in T_e(G)$, then $[\mathsf{x}, \mathsf{y}] := \mathsf{xy} - \mathsf{yx}$ (in terms of the multiplication of $\mathcal{O}(G)^*$) is also a point derivation of $\mathcal{O}(G)$ at e. Therefore, we define a Lie algebra structure on the tangent space $T_e(G)$.

Definition A.25. The Lie algebra $\mathfrak{g} := T_e(G)$ endowed with the above-defined operation $[\,\cdot\,,\,\cdot\,]$ is called the *Lie algebra of* G.

For example, it is easy to see directly that the Lie algebra of $\mathrm{GL}_n(\mathcal{K})$ is isomorphic to the general linear Lie algebra $\mathfrak{gl}_n(\mathcal{K}) = \mathrm{M}_n(\mathcal{K})$ of $n \times n$ matrices over $\mathcal{K}$. Also, the Lie algebra of $\mathrm{SL}_n(\mathcal{K})$ is the special linear Lie algebra $\mathfrak{sl}_n(\mathcal{K})$ of $n \times n$ matrices of trace 0.

Theorem A.26. *Let G, H be affine algebraic groups over $\mathcal{K}$ with Lie algebras $\mathfrak{g}, \mathfrak{h}$, respectively.*

(1) $\dim \mathfrak{g} = \dim G$. *In particular, $\mathfrak{g} = 0$ if and only if G is finite.*

(2) *Any morphism $\varphi\colon G \to H$ of affine algebraic groups induces a Lie algebra homomorphism $d\varphi\colon \mathfrak{g} \to \mathfrak{h}$ called the differential of φ.*

Proof. By Theorem A.8, there exist points $x \in G$ which are regular, i.e., at which $\dim T_x(G) = \dim G$. Because G acts transitively on itself by right (or left) translation, *every* point in G is regular. In particular, $\dim G = \dim T_e(G) = \dim \mathfrak{g}$. This proves (1). For (2), the homomorphism $d\varphi$ is the restriction of $\varphi^{**}\colon \mathcal{O}(G)^* \to \mathcal{O}(H)^*$. $\square$

Some calculations of differentials are given in Exercise A.9.

Finite groups are often studied by means of their actions on sets. The same principle holds true for algebraic groups.

Definition A.27. If G is an affine algebraic group and X is a variety, then G acts regularly on X (on the left) provided there is a morphism $G \times X \to X$, $(g, x) \mapsto g{\cdot}x$, of varieties satisfying the following two conditions: (1) $e{\cdot}x = x$, for all $x \in G$; and (2) $(gh){\cdot}x = g{\cdot}(h{\cdot}x)$, for all $x \in X$, $g, h \in G$.

A regular right action of G on X is symmetrically defined by a morphism $X \times G \to X$.

If G acts regularly on X (either on the left or the right), we usually just say that X is a G-variety.

Examples A.28. (1) There are three natural ways to make $X = G$ into a G-variety: (i) $G \times X \to X$, $(g, x) \mapsto gx$; (ii) $G \times X \to X$, $(g, x) \mapsto xg^{-1}$; and (iii) $G \times X \to X$, $(g, x) \mapsto \operatorname{Int} g(x) := gxg^{-1}$. The third example is often called the *adjoint action* of G on itself.

(2) Let $X = \mathfrak{g}$. For $g \in G$, $\operatorname{Int} g\colon G \to G$ defined in (1) is a morphism, thus we have $\operatorname{Ad} g := d(\operatorname{Int} g)\colon \mathfrak{g} \to \mathfrak{g}$, by Theorem A.26(2). This procedure defines an action (or representation) of G on $\mathfrak{g}$: $(g, \mathsf{x}) \mapsto \operatorname{Ad} g(\mathsf{x})$, for $g \in G$ and $\mathsf{x} \in \mathfrak{g}$, also called the *adjoint action* (or *adjoint representation* of G on $\mathfrak{g}$). Regarding both G and $\mathfrak{g}$ as subsets of $\mathcal{O}(G)^*$, we easily check (see Exercise A.9) that $\operatorname{Ad} g(\mathsf{x}) = g\mathsf{x}g^{-1}$ in terms of the multiplication of $\mathcal{O}(G)^*$.

(3) Let H be a closed subgroup of G. There is a unique variety structure on the set G/H of left cosets gH of H in G with the following universal property: given any G-variety X such that, for some $x \in X$, $h{\cdot}x = x$, for all $h \in H$, there exists a unique morphism $\varphi\colon G/H \to X$ such that $\varphi(gH) = g{\cdot}x$, for all $g \in G$. The variety G/H is called the *quotient variety* of G by H. If $X = G/H$, the map $G \times G/H \to G/H$, $(g, xH) \mapsto gxH$, makes G/H into a G-variety. If, in addition, H is a normal subgroup, then the quotient variety G/H has the structure of an affine algebraic group, as well. Furthermore, the quotient morphism $\pi\colon G \to G/H$ is a morphism of algebraic groups.

We will need the following important result.

Theorem A.29. *Suppose G is an affine algebraic group which acts regularly on a variety X. For $x \in X$, let $\mathfrak{O}_x = G{\cdot}x \subseteq X$ be the corresponding G-orbit, and let $H = \{g \in G \mid g{\cdot}x = x\}$ be the isotropy subgroup of x.*

(1) *The G-orbit $\mathfrak{O}_x$ is an open, smooth subvariety of its closure $\overline{\mathfrak{O}_x}$ with*

$$\dim \mathfrak{O}_x = \dim G/H = \dim G - \dim H.$$

(2) *The boundary* $\overline{\mathfrak{O}}_x \backslash \mathfrak{O}_x$ *is a union of G-orbits having dimensions strictly smaller than* $\dim \mathfrak{O}_x$*. In particular, a G-orbit with minimal dimension is closed in X.*

Proof. (1) Because G acts regularly on X, the map $\varphi\colon G \to X$, $g \mapsto g{\cdot}x$, is morphism of varieties with $\varphi(G) = \mathfrak{O}_x$. By Theorem A.15(1), $\mathfrak{O}_x$ is open in its closure $\overline{\mathfrak{O}}_x$. Because the regular points of $\overline{\mathfrak{O}}_x$ contain an open dense set, $\mathfrak{O}_x$ itself must contain regular points. And, because G acts transitively on $\mathfrak{O}_x$, every point of $\mathfrak{O}_x$ is regular. Thus, $\mathfrak{O}_x$ is smooth. The dimension equality follows easily from Theorem A.15(2), since $\varphi^{-1}(g{\cdot}x) = gHg^{-1}$, for any $g \in G$.

(2) Clearly, $\mathfrak{O}_x$ is stable under the action of G, so its closure $\overline{\mathfrak{O}}_x$ (and therefore, its boundary $\overline{\mathfrak{O}}_x \backslash \mathfrak{O}_x$) is a union of G-orbits. Finally, since $\overline{\mathfrak{O}}_x \backslash \mathfrak{O}_x$ meets every irreducible component of $\overline{\mathfrak{O}}_x$ in a proper closed subset, it has strictly smaller dimension. □

A key theorem concerning actions of algebraic groups is the following famous *Borel fixed point theorem.*

Theorem A.30. *Suppose that B is a connected solvable affine algebraic group acting regularly on a complete variety X. Then B fixes a point $x \in X$.*

By a torus of an affine algebraic group G, we simply mean a closed subgroup T of G which, as an affine algebraic group, is isomorphic to a torus $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$. (Possibly, $T \cong \{e\}$, the trivial group.) A torus T in G is a *maximal torus* provided it is not properly contained in another torus in G. For dimension reasons, maximal tori exist. If G is an affine algebraic group, put $\mathscr{C}^1(G) = (G, G)$ (commutator subgroup), and, for $n > 1$, set $\mathscr{C}^n(G) = (\mathscr{C}^{n-1}(G), G)$, the subgroup of G generated by commutators $(x, y) = xyx^{-1}y^{-1}$, $x \in \mathscr{C}^{n-1}(G), y \in G$. The groups $\mathscr{C}^n(G)$ are closed subgroups of G.

Theorem A.31. *Let B be a connected solvable affine algebraic group.*

(1) *If $U = R_u(B)$ is the unipotent radical of B, then U is a closed, normal subgroup of B, and the quotient group B/U is a torus.*

(2) *If T is any maximal torus of B, then $B \cong U \rtimes T$ is the semidirect product of T and U.*

(3) *If T and T' are two maximal tori of B, then there exists*

$$x \in \mathscr{C}^\infty(B) := \bigcap_{n=1}^{\infty} \mathscr{C}^n(B)$$

such that $xTx^{-1} = T'$.

A *Borel subgroup* B of an affine algebraic group G is a subgroup which is maximal among the connected, closed, solvable subgroups of G.

Theorem A.32. *Let G be an affine algebraic group.*

(1) *Any two maximal tori T and T' are conjugate in G.*

(2) *Any two Borel subgroups B and B' are conjugate in G.*

(3) *Any maximal torus T is contained in a Borel subgroup B.*

(4) *A closed connected subgroup H of G is a Borel subgroup if and only if H is solvable and the quotient variety G/H is complete.*

Statements (1), (2), and (3) follow directly from Theorems A.30 and A.31, once statement (4) has been shown. The proof of (4) reduces easily to showing that if B is a Borel subgroup of G, then G/B is complete. A proof of this result requires some elementary facts about representations of affine algebraic groups which are indicated in §A.4. See Exercise A.21.

Now let G be a reductive group, let T be a maximal torus, and $X(T)$ be the character group of T (see Example A.21(1)). Let $N = N_G(T)$ be the normalizer of T in G. The quotient group $W := N/T$ is called the Weyl group of G (with respect to T).

We consider the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G. Since T acts as a (commutative) group of semisimple linear transformations, $\mathfrak{g}$ decomposes into eigenspaces for this action of T. Given $\alpha \in X(T)$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, t{\cdot}x = \alpha(t)x, \text{ for all } t \in T\}$. Moreover, T acts trivially on its own Lie algebra $\mathfrak{t} \subseteq \mathfrak{g}$, thus, $\mathfrak{t} \subseteq \mathfrak{g}_0$. In fact, it can be shown that $\mathfrak{t} = \mathfrak{g}_0$. We call $\alpha \in X(T)$ a *root* of T in G provided $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. Let $\Phi \subseteq X(T)$ be the set of roots, so that

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

For $\alpha \in \Phi$, the eigenspace $\mathfrak{g}_\alpha$ is called the *α-root space.* On the other hand, the action of N on $\mathfrak{g}$ induces an action of W on Φ. This action can also be obtained by restricting the action of W on $X(T)$.

Theorem A.33. *Let G be a reductive group, and let T be a maximal torus of G.*

(1) *The group W is a finite Coxeter group. Thus, the finite dimensional real space $\mathbb{E} := X(T) \otimes \mathbb{R}$ can be given the structure of a Euclidean space with inner product $(-,-)$ so that W consists of orthogonal linear transformations. Moreover, for any Borel subgroup B of G containing T, $B \cap N = T$.*

(2) *Φ is a finite abstract root system in $\mathbb{E} := X(T) \otimes \mathbb{R}$ in the sense of Theorem 0.35, and W becomes the Weyl group of the root system Φ. In*

particular, each $\alpha \in \Phi$ *defines a reflection* $s_\alpha \in W$ *by*

$$s_\alpha(x) = x - (x, \check{\alpha})\alpha, \quad \text{for } x \in \mathbb{E},$$

where $\check{\alpha} = 2\alpha/(\alpha, \alpha)$.

(3) *Each Borel subgroup* $B \supseteq T$ *gives a choice of the set* Φ^+ *of positive roots in such a way that the Lie algebra* $\mathfrak{b}$ *of* B *decomposes as*

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

Moreover, any choice of the set of positive roots arises in this way.

(4) *If* $\alpha \in \Phi$, $\dim \mathfrak{g}_\alpha = 1$. *Moreover, for* $\alpha \in \Phi$, G *contains a unique* 1*-dimensional subgroup* $U_\alpha \cong \mathbb{G}_a$ *normalized by* T. *Fix a choice of* Φ^+, *and let* Γ *be a closed subset of* Φ^+ *in the sense that* $\alpha, \beta \in \Gamma$ *with* $\alpha + \beta \in \Phi^+$ *implies* $\alpha + \beta \in \Gamma$. *Then the multiplication of* G *defines a variety isomorphism*

$$\text{mult}\colon \prod_{\alpha \in \Gamma} U_\alpha \longrightarrow U_\Gamma, \tag{A.3.1}$$

where the product is taken with respect to a fixed, but arbitrary, listing of elements in Γ, *and* U_Γ *is the closed subgroup of* G *generated by* U_α, *for* $\alpha \in \Gamma$. *In particular, if* B *is the Borel subgroup containing* T *corresponding to the choice of* Φ^+, *then*

$$\text{mult}\colon \prod_{\alpha \in \Phi^+} U_\alpha \longrightarrow U := R_u(B)$$

is an isomorphism of varieties.

(5) *For each root* $\alpha \in \Phi$, *there is a morphism* $h_\alpha\colon \mathbb{G}_m \to T$ *such that*

$$\xi \circ h_\alpha\colon \mathbb{G}_m \longrightarrow \mathbb{G}_m, \quad t \longmapsto t^{(\xi, \check{\alpha})}, \quad \text{for } \xi \in X(T).$$

The key to a proof of Theorem A.33(4) is to show that, given $x \in U_\alpha$ and $y \in U_\beta$, for $\alpha, \beta \in \Gamma$, the commutator $xyx^{-1}y^{-1}$ is a product of elements in U_γ, for various $\gamma \in \Gamma$ with $\mathrm{ht}(\gamma) > \max\{\mathrm{ht}(\alpha), \mathrm{ht}(\beta)\}$, where $\mathrm{ht}(\gamma)$ is the height of γ. In Chapter 4, a similar result is proved in Lemma 4.36 for $\mathrm{GL}_n(\Bbbk)$ ($\Bbbk$ an arbitrary field) using only elementary linear algebra; this result can serve as an example of the result presented here.

An important example of closed subset of Φ^+ is $\Phi_w := \{\alpha \in \Phi^+ \mid -w(\alpha) \in \Phi^+\}$, for a fixed $w \in W$. We will write U_w for U_{Φ_w}. The complement of Φ_w in Φ^+, i.e., $\Phi_w' := \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^+\}$, is also a closed subset of Φ^+.

The double coset decomposition of a reductive group with respect to a Borel subgroup plays an important role in the structure theory of the group. The following theorem gives the fundamental properties of such a decomposition.

Theorem A.34. *Let G be a reductive group, let T be a maximal torus of G, and let B be the Borel subgroup of G containing T corresponding to a choice of positive roots Φ^+. Let $N = N_G(T)$, and let $W = N/T$ be the Weyl group of G (with respect to T) with S the set of simple reflections with respect to the choice of Φ^+. For $w \in W$, denote by wB (resp., Bw) the coset n_wB (resp., Bn_w), where n_w is a representative of w in N. Then the following statements hold:*

(1) *For $w \in W$, $wBw^{-1} = B$ implies $w = 1$.*

(2) *For $w \in W$ and $s \in S$,*

$$BwB{\cdot}BsB = \begin{cases} BwsB, & \text{if } \ell(ws) > \ell(w); \\ BwsB \cup BwB, & \text{if } \ell(ws) < \ell(w). \end{cases}$$

(3) (Bruhat decomposition) *The group G decomposes into a disjoint union*

$$G = \bigcup_{w \in W} BwB.$$

(4) *Each double coset BwB ($w \in W$) is a subvariety of G, open in its closure, and the multiplication*

$$\mathrm{mult}\colon B \times \{n_w\} \times U_w \xrightarrow{\sim} BwB$$

is an isomorphism of varieties, where n_w is a fixed representative of w in N.

By Theorem A.34(4), if $w_0 \in W$ is the longest element, Bw_0B is open dense in G (called the *big cell* of G).

Remark A.35. A group G (not necessarily algebraic) has a BN-pair structure provided that G is generated by subgroups B and N satisfying the following properties:

(a) $T := B \cap N \lhd N$;

(b) $W := N/T$ is generated by a subset S consisting of elements of order 2;

(c) $sBs \neq B$, for all $s \in S$; and

(d) $wBs \subseteq BwsB \cup BwB$, for all $s \in S$, $w \in W$.

(In the above, if $n_w \in N$ represents $w \in W$, define $wB = n_wB$ and $Bw = Bn_w$.) Theorem A.34(1)–(2), together with Theorem A.33(1), shows that a reductive algebraic group G has a natural BN-pair structure. For further discussion, see [**LAI**, Ch. 4]. The Bruhat decomposition is a formal consequence of the axioms for a BN-pair. Most of the discussion of parabolic subgroups in the next section can also be generalized to the context of a group with BN-pair.

Example A.36. Let $G = \mathrm{GL}_n(\mathcal{K})$. We can describe some of the above results in concrete terms. A maximal torus T consists of the $n \times n$ invertible diagonal matrices. Then T has dimension n, and $X(T)$ has basis $\varepsilon_1, \ldots, \varepsilon_n$, where

$$\varepsilon_i(t) = t_i \quad \text{if} \quad t = \mathrm{diag}(t_1, \ldots, t_n) \in T.$$

Recall that, by convention, we write $X(T)$ as an additive group. Let B^+ be the subgroup of G consisting of all invertible upper triangular matrices, and let U^+ be the subgroup of upper unitriangular matrices. Then B^+ is a Borel subgroup with unipotent radical U^+.

For $1 \leqslant i, j \leqslant n$, let $E_{i,j}$ be the (i,j)th matrix unit in the space $\mathrm{M}_n(\mathcal{K})$; thus, the entries in $E_{i,j}$ are all equal to 0, except for a 1 in position (i,j). The adjoint action of G (and T) on its Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathcal{K})$ is by matrix conjugation. Then $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$. For $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$, $\mathfrak{g}_\alpha = \mathcal{K} E_{i,j}$. Also, $U_\alpha = \{I_n + tE_{i,j} \mid t \in \mathcal{K}\}$, where I_n is the identity matrix.

Next, $N(T)$ is the subgroup of $\mathrm{GL}_n(\mathcal{K})$ consisting of all invertible monomial matrices. So, $W = N(T)/T \cong \mathfrak{S}_n$. The root system Φ has type A_{n-1}.

Finally, $G = \mathrm{GL}_n(\mathcal{K})$ acts naturally on the variety $\mathfrak{F}_{\mathrm{cpl}}(\mathcal{K}^n)$ of complete flags in $\mathcal{K}^n$. If $v_1, \ldots, v_n$ is the standard basis of $\mathcal{K}^n$ and if $V_i = \mathrm{span}(v_1, \ldots, v_i)$, then B is the stabilizer in G of the flag $V_1 \subset V_2 \subset \cdots \subset V_n = V$, and $G/B \cong \mathfrak{F}_{\mathrm{cpl}}(\mathcal{K}^n)$.

In Chapter 4, we established many of the same results for the general linear group $\mathrm{GL}_n(k)$ over an arbitrary (in particular, finite) field k. See Theorem 4.37.

Let G be a semisimple group with fixed maximal torus T, Borel subgroup B, etc., as in the above theorem. Enumerate the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. For $1 \leqslant i \leqslant n$, there exists a unique $\varpi_i \in \mathbb{E}$ such that $\langle \varpi_i, \check{\alpha}_j \rangle = \delta_{i,j}$, whenever $1 \leqslant j \leqslant n$. The elements ϖ_i are called *fundamental dominant weights* for the root system Φ. Let X be the lattice (free abelian group) generated by the ϖ_i in $\mathbb{E}$. Then $\mathbb{E} \cong \mathbb{R} \otimes_{\mathbb{Z}} X$. Let $X^+ = \mathbb{N}\varpi_1 \oplus \cdots \oplus \mathbb{N}\varpi_n$, which is the cone of *dominant weights* of T. If $\mathbb{Z}\Phi$ is the set of $\mathbb{Z}$-linear combinations of roots, $\mathbb{Z}\Phi \subseteq X(T) \subseteq X$. By definition, G is *simply connected* provided $X = X(T)$, and it is *adjoint* if $X(T) = \mathbb{Z}\Phi$. In addition, given any lattice L of $\mathbb{E}$ satisfying $\mathbb{Z}\Phi \subseteq L \subseteq X$, there exists a semisimple group G over $\mathcal{K}$, unique up to isomorphism, with root system Φ such that $X(T) \cong L$. See the Notes for further references.

Example A.37. The semisimple algebraic group $\mathrm{SL}_n(\mathcal{K})$ is simply connected. Let T be the maximal torus of diagonal matrices. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leqslant i < n$. Thus, $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is a set of simple roots defined by the Borel subgroup of upper triangular unimodular matrices. Then the fundamental dominant weights $\varpi_1, \ldots, \varpi_{n-1}$ are given by $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$.

We will use a partial ordering on X defined as follows. For $\xi, \nu \in X$, write $\xi \leqslant \nu$ provided $\nu - \xi \in \mathbb{N}\Phi^+$, a nonnegative integer combination of positive roots. This partial ordering induces a partial ordering on the set X^+ of dominant weights.

A.4. Parabolic subgroups and the Chevalley–Bruhat ordering

This section relates the Chevalley–Bruhat ordering (see §4.1) on the Weyl group of a reductive group to the geometry of the group. We will make use of the results on parabolic subgroups of Coxeter groups presented in §4.3. We will also quote some results from [**LAI**], which are related to Theorem A.34, and which amount to the assertion that a reductive group G is a group with a BN-pair.

Let G be a connected affine algebraic group over $\mathcal{K}$. A subgroup P of G is called a *parabolic subgroup* provided that the quotient variety G/P is a complete variety. The following result summarizes some basic facts about parabolic subgroups.

Theorem A.38. *Let G be a reductive group.*

(1) *A closed subgroup P of G is parabolic if and only if P contains a Borel subgroup of G.*

(2) *If P is a parabolic subgroup of G, then P is connected and equal to its own normalizer in G.*

(3) *Suppose that Z' is a closed subvariety of a G-variety Z such that $P{\cdot}Z' = Z'$, for some parabolic subgroup P of G. Then $G{\cdot}Z'$ is closed in Z. In particular, if P and Q are parabolic subgroups of G both containing the same Borel subgroup B, then the product PQ is closed in G.*

Proof (sketch of (3)). The subset of $G/P \times Z$ consisting of all points (xP, z) such that $x^{-1}{\cdot}z \in Z'$ is closed. Because the projection $G/P \times Z \to Z$ is a closed map since G/P is complete, $G{\cdot}Z'$ is closed in Z, as required. □

From now on, assume that G is a reductive group. Let T be a fixed maximal torus, and let Φ be the root system of T in $\mathfrak{g}$. Fix a Borel subgroup $B \supset T$ corresponding to a set Φ^+ of positive roots. Let Π be the simple roots in Φ^+ and let $W = N(T)/T$ be the Weyl group of G. Any root $\alpha \in \Phi$ determines a reflection $s_\alpha \in W$. If $S = \{s_\alpha \mid \alpha \in \Pi\}$, then (W, S) is a Coxeter system by Theorem A.33(1).

For notational convenience, we will usually (without mention) identify the sets Π and S: $\alpha \in \Pi \leftrightarrow s_\alpha \in S$. For $I \subseteq \Pi$, let Φ_I be the closed subroot system of Φ generated by I: $\Phi_I = \Phi \cap \mathbb{Z}I$.

Theorem A.39. *Let G be a reductive group.*

(1) *The parabolic subgroups of G containing B are in one-to-one correspondence with subsets $I \subseteq \Pi$. If $I \subseteq \Pi$, the corresponding parabolic subgroup P_I is generated (as an abstract group) by B and the root subgroups $U_{-\alpha}$, $\alpha \in I$.*

(2) *Any parabolic subgroup P of G is conjugate to a unique $P_I \supseteq B$.*

(3) *For $I \subseteq \Pi$, if L_I is the subgroup of G generated by T and the $U_{\pm\alpha}$, for $\alpha \in I$, then L_I is a closed, reductive subgroup of G containing T and having root system Φ_I (called a Levi subgroup of P_I). Also, $P_I = L_I \rtimes R_u(P_I)$.*

Example A.40. In the notation of Example A.14(3), the parabolic subgroups of $\mathrm{GL}_n(\mathcal{K})$ are just the isotropy subgroups for the action of $\mathrm{GL}_n(\mathcal{K})$ on some $\mathfrak{F}(\boldsymbol{d}, V)$, for $V = \mathcal{K}^n$.

If H and K are subgroups of G at least one of which contains T, then, given $w \in W$, write HwK for Hn_wK, where $n_w \in N(T)$ is any coset representative of w. The definition of HwK does not depend on the choice of n_w. Recall from Theorem A.34(3) that the double cosets BwB, $w \in W$, are precisely the distinct (B, B)-cosets in G. For $I \subseteq \Pi$, let W_I be the parabolic subgroup of W defined by the set of reflections s_α, $\alpha \in I$. Let ${}^IW$ (resp., W^I) be the set of shortest right (resp., left) coset representatives of W_I in W. Thus, $W^I = {}^IW^{-1}$ and multiplication defines bijections $W_I \times {}^IW \xrightarrow{\sim} W$ and $W^I \times W_I \xrightarrow{\sim} W$. For $I, J \subseteq \Pi$, ${}^IW^J = {}^IW \cap W^J$ is a set of double coset representatives for $W_I \backslash W / W_J$. See §4.3 for more details.

Let $\leqslant$ be the Chevalley–Bruhat ordering on W; see §4.1. We can regard ${}^IW$, W^I, and ${}^IW^J$ as subposets.

Theorem A.41. *Let $I, J \subseteq \Pi$.*

(1) *${}^IW^J$ is a set of double coset representatives for $P_I \backslash P / P_J$ in G. In other words,*

$$G = \bigcup_{w \in {}^IW^J} P_I w P_J \quad \textit{(disjoint union).}$$

(2) *For $w \in {}^IW^J$, $P_I w P_J = BW_I w W_J B$.*

(3) *For $x, y \in {}^IW^J$, $P_I x P_J$ is contained in the Zariski closure $\overline{P_I y P_J}$ of the double coset $P_I y P_J$ if and only if $x \leqslant y$.*

(4) *In particular, for $x, y \in W$, $BxB \subseteq \overline{ByB}$ if and only if $x \leqslant y$.*

Proof. Statements (1) and (2) follow formally because the groups B and $N = N(T)$ define a BN-pair structure on G. See [**LAI**, Prop. 2, § 2.5, Ch. IV].

Next, we show why (4) holds. For $s \in S$, write $P_s = P_{\{s\}}$. Then $P_s = B \cup BsB$. If $y \in W$ has reduced expression $y = s_1 \cdots s_d$, then $P_{s_1} \cdots P_{s_d}$

is a closed subvariety of G (by Theorem A.38(3)), which is a union of the double cosets BxB, for $x \leqslant y$; see [**LAI**, §2.1, Ch. IV]. Since Bs_iB is open in P_s, it follows that

$$ByB = Bs_1B \cdots Bs_dB$$

is open (and dense) in $P_{s_1} \cdots P_{s_d}$. Therefore, $\overline{ByB} = P_{s_1} \cdots P_{s_d}$, completing the proof of (4).

To see (3), first suppose that $P_IxP_J \subseteq \overline{P_IyP_J}$, for $x, y \in {}^IW^J$. Because $BxB \subseteq P_IxP_J$, (2) and (4) imply that $x \leqslant ayb$, for some $a \in W_I$ and $b \in W_J$. Because x, y are shortest double coset representatives, this fact immediately implies that $x \leqslant y$, as required. Conversely, if $x \leqslant y$ both belong to ${}^IW^J$, then $BxB \subseteq \overline{P_IyP_J}$ by (4), which implies that $P_IxP_J \subseteq \overline{P_IyP_J}$. □

Let $P = P_I$ and $Q = P_J$ be two parabolic subgroups of G containing B. We can form the G-variety $X := G/P \times G/Q$. It has points (xP, yQ), and, for $g \in G$, $g{\cdot}(xP, yQ) := (gxP, gyQ)$. We form the following poset $(\Psi, \leqslant)$: the points of Ψ are just the orbits $\mathfrak{O} = G{\cdot}(xP, yQ)$ of G in X. If $\mathfrak{O}, \mathfrak{O}' \in \Psi$, put $\mathfrak{O} \leqslant \mathfrak{O}'$ if and only if $\mathfrak{O} \subseteq \overline{\mathfrak{O}'}$.

We can also consider the $P \times Q$-variety G in which $(x, y){\cdot}g := xgy^{-1}$, for $x \in P, y \in Q$, and $g \in G$. Let Ω be the set of orbits of $P \times Q$ on G, viewed as a poset by means of Zariski closure. Thus, given orbits $\mathfrak{O}, \mathfrak{O}' \in \Omega$, $\mathfrak{O} \leqslant \mathfrak{O}'$ if and only if $\mathfrak{O} \subseteq \overline{\mathfrak{O}'}$. By Theorem A.41, the poset $(\Omega, \leqslant)$ is isomorphic to the poset $({}^IW^J, \leqslant)$.

There is an evident bijection $\sigma\colon \Psi \xrightarrow{\sim} \Omega$ defined as follows. Given an orbit $\mathfrak{O} \in \Psi$, first form $Z := \mathfrak{O} \cap (P/P \times G/Q)$ and let $\sigma(\mathfrak{O})$ be the inverse image of Z under the map $G \to P/P \times G/Q$, $g \mapsto (P, gQ)$.

Proposition A.42. *The map σ is a poset isomorphism.*

Proof. By Theorem A.38(3), for $w \in {}^IW^J$, $G{\cdot}(P/P \times \overline{PwQ/Q})$ is closed in $G/P \times G/Q$. (Because the quotient map $G \to G/Q$ is an open map, $\overline{PwQ}/Q$ identifies with $\overline{PwQ/Q}$.) Thus,

$$\overline{G{\cdot}(P, wQ)} = G{\cdot}(P/P \times \overline{PwQ/Q}).$$

This easily implies that σ is a poset isomorphism. □

A.5. Representation theory: a first view

Let G be an affine algebraic group over an algebraically closed field $\mathcal{K}$. In this section, we briefly outline some elementary features of the representation theory of G, especially when G is semisimple.

Let M be a finite dimensional $\mathcal{K}G$-module with $\phi\colon G \to \mathrm{GL}(M)$, the corresponding group homomorphism. We say that M is a *rational G-module* if ϕ is a morphism of algebraic groups. More generally, if M is an arbitrary

$\mathcal{K}G$-module, i.e., we do not assume that M is finite dimensional, then M is a rational G-module provided it can be written as a union of finite dimensional submodules which are rational G-modules in the above sense. The representation $G \to \mathrm{GL}(V)$ associated with a rational G-module V is called a rational representation of G. Given two rational G-modules M and M', a morphism $f\colon M \to M'$ is simply a linear transformation from M to M' commuting with the action of G. In this way, we obtain the category G-mod of all rational G-modules.

As noted in Proposition A.20, $\mathcal{O}(G)$ carries the structure of a (commutative) Hopf algebra. Consider the category of right $\mathcal{O}(G)$-comodules for the underlying coalgebra $\mathcal{O}(G)$; see Definition 5.3. A right comodule M with structure map $\tau_M\colon M \to M \otimes \mathcal{O}(G)$ defines a left G-module structure on M as follows. Given $g \in G$ and $v \in M$, write $\tau_M(v) = \sum v_i \otimes a_i$ (finite sum), where $v_i \in M$ and $a_i \in \mathcal{O}(G)$. Then set

$$g{\cdot}v := \sum a_i(g)v_i.$$

Moreover, the local finiteness for $\mathcal{O}(G)$-comodules (see Exercise 5.2) implies the local finiteness of the resulting G-modules, and, clearly, any finite dimensional $\mathcal{O}(G)$-comodule gives a finite dimensional rational G-module. In this way, we obtain a functor comod-$\mathcal{O}(G) \to G$-mod.

Lemma A.43. *The functor* comod-$\mathcal{O}(G) \to G$-mod *defined above is an equivalence of categories.*

Proof. The functor in the other direction is obtained in the following way: If M is a left rational G-module with basis $\{v_i\}$, then the action of G can be expressed as $gv_j = \sum_i a_{i,j}(g)v_i$ (finite sum). The rationality of M implies $a_{i,j} \in \mathcal{O}(G)$. It is then easy to see that $\tau\colon M \to M \otimes \mathcal{O}(G)$ given by $\tau(v_j) = \sum_i v_i \otimes a_{i,j}$ defines a right $\mathcal{O}(G)$-comodule structure on M. $\square$

Examples A.44. (1) The coordinate algebra $\mathcal{O}(G)$ of G is a right $\mathcal{O}(G)$-comodule with the comultiplication as the structure map. Thus, $\mathcal{O}(G)$ is naturally a (left) rational G-module. In fact, the module structure can be defined as follows: given $f \in \mathcal{O}(G)$ and $g, x \in G$, $(g{\cdot}f)(x) = f(xg)$.

(2) If M and M' are rational G-modules, so is the tensor product $M \otimes M'$ (with the diagonal action of G). Also, the tensor powers $M^{\otimes r}$, the symmetric powers $\mathsf{S}^r(M)$, and the exterior powers $\Lambda^r M$ are rational G-modules. Moreover, if M is finite dimensional, the linear dual M^* and, more generally, the space $\mathrm{Hom}_{\mathcal{K}}(M, M')$ carry natural structures as rational G-modules.

(3) The 1-dimensional trivial G-module (any $g \in G$ acts as the identity operator) is a rational module. This module is usually denoted by $\mathcal{K}$.

The notion of a rational representation is required to prove the following basic result.

Theorem A.45. *If G is an affine algebraic group, then G is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathcal{K})$, for some positive integer n.*

A proof of this result, which uses Example A.44(1), is indicated in Exercise A.16. Of course, any closed subgroup of $\mathrm{GL}(M)$, for a finite dimensional vector space M over $\mathcal{K}$, is also an affine algebraic group over $\mathcal{K}$. Therefore, affine algebraic groups are often referred to as *linear algebraic groups.*

Now fix a maximal torus T of G, and let $X(T)$ be the character group of T. The integral group algebra $\mathbb{Z}X(T)$ of $X(T)$ has basis e^ξ, for $\xi \in X(T)$, and product rule $e^\xi e^\zeta = e^{\xi+\zeta}$ (recall that the abelian group $X(T)$ is usually written additively). For a rational T-module M and $\xi \in X(T)$, the ξ-weight space of T in M is the subspace $M_\xi := \{v \in M \mid t \cdot v = \xi(t)v,\ \forall t \in T\}$. If $M_\xi \neq 0$, then ξ is called a *weight* of T in M. We have

$$M = \bigoplus_{\xi \in X(T)} M_\xi. \tag{A.5.1}$$

(See Exercise A.15.) If T is a maximal torus in G and if M is a finite dimensional rational G-module, we can regard M as a rational T-module, and consider the decomposition (A.5.1). The *character* of M is then

$$\operatorname{ch} M := \sum_{\xi \in X(T)} \dim M_\xi e^\xi \in \mathbb{Z}X(T). \tag{A.5.2}$$

Now assume that G is a connected semisimple group over $\mathcal{K}$. Fix a maximal torus T and a Borel subgroup $B \supset T$ determining the positive roots Φ^+ and simple roots Π, as discussed in Theorem A.34. For simplicity, assume that G is simply connected; see the discussion before Example A.37. This means that $X(T) = X$, which is the free $\mathbb{Z}$-module with fundamental dominant weights $\varpi_1, \ldots, \varpi_n$ as a basis. The set $X(T)$ is partially ordered as follows. For $\xi, \nu \in X(T)$, write $\xi \leqslant \nu$ if and only if $\nu - \xi \in \mathbb{N}\Phi^+$.

The Weyl group W of G acts naturally on $X(T)$, putting, for $w \in W$ and $\xi \in X(T)$, $w(\xi)(t) := \xi(n_w^{-1} t n_w)$, where $n_w \in N_G(T)$ represents w. Thus, if M is a rational G-module, W permutes the weights of T in M. In particular, if M is finite dimensional,

$$\operatorname{ch} M \in (\mathbb{Z}X(T))^W, \tag{A.5.3}$$

the subspace of W-fixed points in $\mathbb{Z}X(T)$. If P is any parabolic subgroup containing T, a similar result holds for a rational P-module M, namely, $\operatorname{ch} M \in (\mathbb{Z}X(T))^{W_P}$, where W_P is the Weyl group of a Levi factor of P containing T. This observation will be used sometimes in the next section.

Any $\xi \in X(T)$ can be "inflated" to a character on B by means of the quotient morphism $B \to B/U \cong T$, where $U = R_u(B)$. By abuse of notation, given $\xi \in X(T)$, let $\mathcal{K}_\xi$ also denote the 1-dimensional rational B-module defined by ξ. It has basis vector v satisfying $ut \cdot v = \xi(t)v$, for $u \in U$ and $t \in T$. Sometimes it will be useful to denote $\mathcal{K}_\xi$ simply by ξ.

Theorem A.46. *Let G be a simply connected, semisimple algebraic group.*

(1) *If L is a simple rational G-module, then B stabilizes a unique one-dimensional subspace of L. Necessarily, this B-stable subspace has the form $\mathcal{K}_\xi$, for $\xi \in X^+$. We call ξ the highest weight of L; it can be characterized as the maximal weight in L with respect to the partial ordering $\leqslant$. Also, $\dim L_\xi = 1$.*

(2) *If L and L' are simple rational G-modules having the same highest weight ξ, then $L \cong L'$ as rational G-modules.*

(3) *Conversely, given $\xi \in X^+$, there exists a simple rational G-module of highest weight ξ.*

Proof. (1) Generally, for any weight ν in a rational G-module M, if $v \in M_\nu$ and $x \in U_\alpha$, for $\alpha \in -\Phi^+$,

$$x.v = v + \sum_{\zeta<\nu} v_\zeta, \quad \zeta \in X(T),\ v_\zeta \in M_\zeta. \tag{A.5.4}$$

See Exercise A.19. Now let $\xi \in X(T)$ be a weight of T in L such that there exists $0 \neq v^+ \in L_\xi$ for which the line $\mathcal{K}v^+$ is B-stable. If $B' = U' \rtimes T$ is the Borel subgroup opposite to B (generated by T and $U_{-\alpha}$, $\alpha \in \Phi^+$),[3] $B'B = U'B$ is dense in G, so $L = \mathcal{K}B'B.v^+ = \mathcal{K}U'.v^+$. By (A.5.4), ξ is the maximal weight in L and $\dim L_\xi = 1$. Finally, the weights of T in L are stable under the Weyl group W, so that if $\alpha \in \Pi$, then $s_\alpha(\xi) \leqslant \xi$, i.e., $(\xi, \check{\alpha}) \in \mathbb{N}$, so that $\xi \in X^+$. This proves (1).

(2) If $v^+ \in L$ and $u^+ \in L'$ are nonzero vectors of weight ξ, and if $\widetilde{L} \subseteq L \oplus L'$ is the submodule generated by (v^+, u^+), then the projection maps $\widetilde{L} \to L$ and $\widetilde{L} \to L'$ are isomorphisms.

(3) See Exercise A.22. □

In view of this theorem, we can label the (isomorphism classes of) rational simple G-modules by the set X^+ of dominant weights. For $\xi \in X^+$, let $L(\xi)$ be a fixed rational simple G-module of highest weight ξ.

Remarks A.47. (1) Our assumption that G is simply connected is largely a convenience. Otherwise, the rational simple modules are indexed by those dominant weights ξ belonging to $X(T)$.

[3] In the notation of Theorem A.34, $B' = w_0^{-1}Bw_0$.

(2) The classification of the simple rational representations of a reductive group G can easily be reduced to the case of semisimple groups, making use of the semisimple derived group G'. For example, the dominant weights for the simply connected group $\mathrm{SL}_n(\mathcal{K})$ correspond to partitions $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ with at most $n-1$ nonzero parts. In the notation of Example A.36, the fundamental dominant weights on the maximal torus T of $\mathrm{SL}_n(\mathcal{K})$ consisting of diagonal matrices are given in Example A.37. Thus, the correspondence from the set of partitions to X^+ is given by $\lambda \mapsto \sum_{i=1}^{n-1}(\lambda_i - \lambda_{i+1})\varpi_i$, taking $\lambda_n = 0$. Now the reader can check that the rational simple representations of $\mathrm{GL}_n(\mathcal{K})$ are indexed by (weakly) decreasing sequences $\mu = (\mu_1, \dots, \mu_n)$ of integers (not necessarily nonnegative). The restriction of $L(\mu)$ to $\mathrm{SL}_n(\mathcal{K})$ is the simple $\mathrm{SL}_n(\mathcal{K})$-module of highest weight $\xi := \sum_{i=1}^{n-1}(\mu_i - \mu_{i+1})\varpi_i$.

Let $\mathscr{K}_0(G)$ be the Grothendieck group of the full subcategory of G-mod whose objects are the finite dimensional rational G-modules. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of finite dimensional rational G-modules, then $\mathrm{ch}\, M = \mathrm{ch}\, M' + \mathrm{ch}\, M''$. Hence, there is a group homomorphism

$$\mathrm{ch}\colon \mathscr{K}_0(G) \longrightarrow (\mathbb{Z}X(T))^W.$$

Theorem A.46 implies the following fundamental fact.

Corollary A.48. *There is an isomorphism*

$$\mathrm{ch}\colon \mathscr{K}_0(G) \xrightarrow{\sim} (\mathbb{Z}X(T))^W.$$

Actually, it is clear that ch is a ring isomorphism as well. A similar result holds if G is replaced by a parabolic subgroup $P \supseteq T$ and W by W_P.

A.6. Representations in positive characteristic; Frobenius morphisms

In some sense, the most interesting aspect of the representation theory of reductive algebraic groups occurs when the ground field has positive characteristic p. In fact, rational representations of a connected reductive group over an algebraically closed field $\mathcal{K}$ of characteristic 0 are all semisimple, and the representation theory of such a group is not much different from the representation theory of its Lie algebra, which is a finite dimensional reductive Lie algebra over $\mathcal{K}$.

However, when $\mathcal{K}$ has positive characteristic, a rational module for a connected reductive group need not be semisimple (so the usage of the name "reductive group" is an abuse of terminology from the viewpoint of representations). Non-semisimplicity makes the representation theory of algebraic groups in this case essentially different from the "classical" representation theory of finite dimensional reductive Lie algebras in characteristic 0, and enriches the representation theory of algebraic groups in many ways.

In this section, we will introduce a basic tool — Frobenius morphisms — in the representation theory of algebraic groups in positive characteristic. Thus, we assume that the ground field $\mathcal{K}$ has characteristic $p > 0$. If $q = p^r$ is a power of p with $r > 0$, then $\mathcal{K}$ contains a subfield $\mathbb{F}_q$ of q elements.

For a positive integer n, the map $F\colon \mathrm{GL}_n(\mathcal{K}) \to \mathrm{GL}_n(\mathcal{K})$ sending $x = (x_{i,j}) \in \mathrm{GL}_n(\mathcal{K})$ to $F(x) = (x_{i,j}^q)$ is clearly a group isomorphism and a variety morphism. Thus, F is an endomorphism of $\mathrm{GL}_n(\mathcal{K})$ (in the category of algebraic groups). However, F is not an automorphism of $\mathrm{GL}_n(\mathcal{K})$ (in the category of algebraic groups), since the inverse of F is no longer a variety morphism. The homomorphism F is called a (standard) *Frobenius morphism* of $\mathrm{GL}_n(\mathcal{K})$.

More generally, let G be an affine algebraic group and let $F\colon G \to G$ be a morphism. By Theorem A.45, G can be regarded as a closed subgroup of $\mathrm{GL}_n(\mathcal{K})$, for some positive integer n. We say that F is a *Frobenius morphism* on G if, for some r, F^r is *standard* in the sense that it agrees with the restriction of a standard Frobenius morphism on $\mathrm{GL}_n(\mathcal{K})$. For example, suppose that G is a closed subgroup of $\mathrm{GL}_n(\mathcal{K})$ defined on $\mathbb{F}_q$. That is, G, as a closed subvariety of $\mathrm{GL}_n(\mathcal{K})$, is defined by a set of polynomials in the $x_{i,j}$ with coefficients in $\mathbb{F}_q$. Then the Frobenius morphism $(x_{i,j}) \mapsto (x_{i,j}^q)$ of $\mathrm{GL}_n(\mathcal{K})$ restricts to a Frobenius morphism of G.

Remark A.49. Although the notion of a Frobenius morphism discussed here has its origins in the theory of algebraic groups, it has been adopted in Chapter 2 to the study of finite dimensional algebras. Conversely, the notion of Frobenius morphisms of algebraic groups can be recovered from that of Frobenius maps on vector spaces (§2.1) or Frobenius morphisms of algebras (§2.2); see Remark 2.5(2).

The following theorem has nothing directly to do with representation theory, but it is used in the construction of a Frobenius folding of an almost split sequence in §2.8.

Theorem A.50 (Lang–Steinberg)**.** *If G is a connected affine algebraic group and if F is a Frobenius morphism of G, then the map $\mathcal{L}\colon G \to G$, $g \mapsto g^{-1}F(g)$, is surjective.*

We only give a proof in the case where $\mathcal{K} = \overline{\mathbb{F}}_q$, the algebraic closure of the finite field $\mathbb{F}_q$. This special case is enough for the application in §2.8. A typical feature under the assumption $\mathcal{K} = \overline{\mathbb{F}}_q$ is that every element in G is of finite order (see Exercise A.25). A general proof using tangent spaces is sketched in Exercise A.27.

Proof. Consider the right action "$\triangleleft$" of G on itself given by the morphism

$$\varphi\colon G \times G \longrightarrow G, \quad (x,g) \longmapsto x \triangleleft g := g^{-1}xF(g).$$

Let $\mathfrak{O}$ be a closed orbit (of minimal dimension); see Theorem A.29. If $\mathfrak{O} = G$, then $e \in \mathfrak{O}$. Therefore, $G = \mathfrak{O}_e := \{g^{-1}F(g) \mid g \in G\} = \mathcal{L}(G)$, giving the theorem. So it remains to prove that $\mathfrak{O} = G$ or, equivalently, to prove that $\dim \mathfrak{O} = \dim G$.

Fix an $x \in \mathfrak{O}$ and consider the (surjective) orbit morphism $\psi\colon G \to \mathfrak{O}$, $g \mapsto x \triangleleft g$. By Theorem A.15(2), to prove the equality $\dim \mathfrak{O} = \dim G$, it suffices to prove that the stabilizer of x is a finite set, i.e., that the equation $x = g^{-1}xF(g)$ has only finitely many solutions $g \in G$. To see this, let $f(g) = xF(g)x^{-1}$. Then,

$$f^i(g) = xF(x)F^2(x)\cdots F^{i-1}(x)F^i(g)F^{i-1}(x^{-1})\cdots F^2(x^{-1})F(x)^{-1}x^{-1},$$

for all $i > 0$. But there exists an $m > 0$ such that $F^m(x) = x$ (see Exercise A.25). Thus, for $i = sm$,

$$f^{sm}(g) = \big(xF(x)F^2(x)\cdots F^{m-1}(x)\big)^s F^{sm}(g)\big(xF(x)F^2(x)\cdots F^{m-1}(x)\big)^{-s}.$$

If r is the order of the element $xF(x)F^2(x)\cdots F^{m-1}(x)$, then $f^{rm}(g) = F^{rm}(g)$. Since the equation $F^{rm}(g) = g$ has only finitely many solutions $g \in G$, the same is true for $f^{rm}(g) = g$. Consequently, $f(g) = g$ has only finitely many solutions. $\square$

A Frobenius morphism $F\colon G \to G$ relates several important objects. First, we have the finite group G^F of F-fixed points. When G is a reductive group, G^F is known as a finite group of Lie type. In particular, if G is almost simple (that is, semisimple with indecomposable root system) and simply connected, then, with few exceptions, the quotient of G^F by its center is a finite simple group. If we form $\overline{G} = G/C(G)$, $C(G)$ being the center of G, then $\overline{G}$ is a simple group, F induces a Frobenius morphism, denoted by F again, on $\overline{G}$, and the corresponding finite simple group can also be obtained as the normal subgroup of $\overline{G}^F$ generated by its unipotent elements (i.e., elements whose orders are powers of $p = \operatorname{char} \mathcal{K}$). See [**284**, §12.8] for more details. In this way, we obtain the various infinite series of finite simple groups, except for the cyclic groups of prime orders and the alternating groups $\mathfrak{A}_n := (\mathfrak{S}_n, \mathfrak{S}_n)$, for $n \geqslant 5$.

Examples A.51. (1) Let $G = \mathrm{GL}_n(\mathcal{K})$ (resp., $\mathrm{SL}_n(\mathcal{K})$), and let $F\colon G \to G$ be the standard Frobenius morphism $(x_{i,j}) \mapsto (x_{i,j}^q)$. Then $G^F = \mathrm{GL}_n(\mathbb{F}_q)$ (resp., $\mathrm{SL}_n(\mathbb{F}_q)$). For any field $\mathcal{k}$, let $\mathrm{PSL}_n(\mathcal{k})$ denote the quotient of $\mathrm{SL}_n(\mathcal{k})$ by its center. Then we obtain $\mathrm{PSL}_n(\mathbb{F}_q)$ (which is simple except for $n = 2$ and, in the meantime, $q = 2$ or 3) as the quotient of $\mathrm{SL}_n(\mathcal{K})^F$ by its center. The group $\mathrm{PSL}_n(\mathbb{F}_q)$ can also be viewed as the normal subgroup of $\mathrm{PSL}_n(\mathcal{K})^F$ generated by its unipotent elements. See [**284**, Cor. 12.6].

(2) Let $G = \mathrm{GL}_n(\mathcal{K})$ (resp., $\mathrm{SL}_n(\mathcal{K})$), $n \geqslant 3$, and let $F_0\colon G \to G$ be the standard Frobenius morphism $(x_{i,j}) \mapsto (x_{i,j}^q)$. Let $F = * \circ F_0$, where $*\colon G \to G$ is the evolution of G sending $g \in G$ to $(g^\intercal)^{-1}$ (with $g^\intercal$ denoting the transpose of g). Clearly, $*$ commutes with F_0, making $F^2 = F_0^2$ is a standard Frobenius morphism. Thus, F is a Frobenius morphism. One can check (Exercise A.26) that G^F is the unitary group $\mathrm{U}_n(\mathbb{F}_{q^2})$ (resp., the special unitary group $\mathrm{SU}_n(\mathbb{F}_{q^2})$). If we consider $G = \mathrm{PSL}_n(\mathcal{K})$ instead, then G^F contains the finite projective unitary subgroup $\mathrm{PSU}_n(\mathbb{F}_{q^2})$ (which is simple, except when $n = 3$ and $q = 2$) as the normal subgroup generated by its unipotent elements.

Again let G be a reductive group. By restriction, a rational representation of G gives rise to a representation of the finite group G^F over the field $\mathcal{K}$ (and so to a representation in the "defining characteristic"). Fortunately, there exists a set of simple representations of G which, upon restriction, is the complete set of simple representations of G^F. We will briefly indicate how this works at the end of this section. Thus, the representation theory of a finite group of Lie type in its defining characteristic heavily relies on the representation theory of the ambient algebraic group.

Another important object in the representation theory of algebraic groups with Frobenius morphisms is the notion of the Frobenius kernel. Although $F\colon G \to G$, for G an arbitrary affine algebraic group, is a group automorphism (see Exercise A.24), it is only an endomorphism of G as an algebraic group. Its comorphism $F^*\colon \mathcal{O}(G) \to \mathcal{O}(G)$ has a cokernel in the category $\mathsf{CommHopf}$ of commutative Hopf algebras, yielding a kernel in the dual category $\mathsf{Affgrp} := \mathsf{CommHopf}^{\mathrm{op}}$ (called the category of affine group schemes[4]). This kernel, denoted $\operatorname{Ker} F$, is called *the kernel* of F. We have an exact sequence in the category Affgrp:

$$E \longrightarrow \operatorname{Ker} F \longrightarrow G \xrightarrow{F} G \longrightarrow E,$$

where $E = \{e\}$ is the trivial group with only one element e.

One can check that $\mathcal{O}(\operatorname{Ker} F)$, the coordinate algebra of $\operatorname{Ker} F$ (i.e., the corresponding object of $\operatorname{Ker} F$ in $\mathsf{CommHopf}$), is obtained by forming the quotient algebra of $\mathcal{O}(G)$ modulo its ideal generated by $F^*(\operatorname{Ker} \varepsilon_{\mathcal{O}(G)})$ (which can be seen to be a Hopf ideal). It turns out that $\mathcal{O}(\operatorname{Ker} F)$ is a finite dimensional Hopf algebra (see Exercise A.28).

[4]Besides the formal definition as an object in the category $\mathsf{CommHopf}^{\mathrm{op}}$, an affine group scheme can also be defined more concretely as a representable functor from the category of commutative $\mathcal{K}$-algebras to the category of groups.

The (rational) $\operatorname{Ker} F$-modules are defined as $\mathcal{O}(\operatorname{Ker} F)$-comodules, which, in turn, are modules over the finite dimensional algebra $\mathcal{O}(\operatorname{Ker} F)^*$ (cf. Propositions 5.4 and 5.5). Therefore, the representation theory of the Frobenius kernel is the representation theory of the finite dimensional algebra $\mathcal{O}(\operatorname{Ker} F)^*$. Note that $\mathcal{O}(\operatorname{Ker} F)^*$ is a subalgebra of $\mathcal{O}(G)^*$, and it is easily checked that the conjugation of $x \in G$ (in terms of multiplication of $\mathcal{O}(G)^*$) leaves $\mathcal{O}(\operatorname{Ker} F)^*$ stable.

Upon restriction, a rational G-module gives a (rational) $\operatorname{Ker} F$-module in the above sense. If a simple G-module remains simple when restricted to $\operatorname{Ker} F$, we will say that the module is *infinitesimally simple.*

On the other hand, from a rational G-module M we can obtain another rational G-module $M^{(F)}$, called the *Frobenius twist*,[5] or, more precisely, the *F-twist*, of M, by letting G act through the Frobenius morphism F. That is, define a new action $\star$ of G on M by putting $x \star v = F(x)v$, for $x \in G$ and $v \in M$. Or, in terms of $\mathcal{O}(G)$-comodules, $M^{(F)}$ is defined by the structure map $F \circ \tau$ if M has structure map τ.

This setup is particularly useful in studying the representation theory of a reductive group G (in positive characteristic). In fact, in this case, infinitesimally simple modules constitute an "essential" part of the set of simple G-modules. In other words, all the simple rational G-modules can be expressed in terms of the infinitesimally simple modules. This important result, known as the Steinberg tensor product theorem, is stated without proof below.

To state the theorem, we remark that any semisimple affine algebraic group G over an algebraically closed field $\mathcal{K}$ of characteristic $p > 0$ can be linearized over the prime field $\mathbb{F}_p$ with a maximal torus T diagonalized. Then the standard Frobenius morphism $(x_{i,j}) \mapsto (x_{i,j}^p)$ of the ambient general linear group defines a Frobenius morphism $F\colon G \to G$. For this Frobenius morphism, we will write $M^{(p^i)}$ for the Frobenius twist $M^{(F^i)}$ of a G-module M.

A weight $\xi \in X(T)$ is called p-restricted (with respect to a prefixed set of positive roots) if $0 \leqslant \langle \xi, \check{\alpha} \rangle < p$, for all simple roots α. Suppose, in addition, that G is simply connected. Then we can express a dominant weight $\xi \in X(T)$ in *p-adic form* as

$$\xi = \xi_0 + p\xi_1 + p^2\xi_2 + \cdots + p^r\xi_r,$$

with $\xi_i \in X(T)$ p-restricted and $\xi_r \neq 0$.

Now we have the following theorem.

[5]Although the definition of Frobenius morphism presented in this section is compatible with that in Chapter 2, the Frobenius twist defined here is different from that defined in §2.2.

Theorem A.52 (Steinberg tensor product theorem). *Under the above assumptions and notation, if $\xi \in X(T)^+$, then there is a G-module isomorphism*

$$L(\xi) \cong L(\xi_0) \otimes L(\xi_1)^{(p)} \otimes L(\xi_2)^{(p^2)} \otimes \cdots \otimes L(\xi_r)^{(p^r)}.$$

Moreover, if $\xi \in X(T)$ is p-restricted, $L(\xi)$ is infinitesimally simple, and the set $\{L(\xi)|_{\mathrm{Ker}\,F} \mid \xi \text{ } p\text{-restricted}\}$ is a complete set of representatives of isoclasses of simple Ker F*-modules.*

For the Frobenius morphism F in the above theorem, the finite dimensional algebra $\mathcal{O}(\mathrm{Ker}\,F^r)^*$ is often denoted by $\mathbf{u}_r$ in the literature. The algebra $\mathbf{u}_1 = \mathcal{O}(\mathrm{Ker}\,F)^*$ is nothing but the restricted enveloping algebra of the Lie algebra $\mathfrak{g}$ of the group G.[6]

Another important result states that, with the above assumptions, a simple rational G-module $L(\xi)$ remains simple when restricted to G^F if and only if ξ is p-restricted. And, the set $\{L(\xi)|_{G^F} \mid \xi \text{ } p\text{-restricted}\}$ is a complete set of representatives of isoclasses of simple G^F-modules over $\mathcal{K}$. The same result holds if F is replaced by F^i, for some positive integer i, and if p-restricted weights are replaced by p^i-restricted weights, defined similarly.

A.7. Induced representations and the Weyl character formula

Although the theory developed in this and the next sections applies equally to groups in characteristic 0 and those in prime characteristic, we are mainly interested in the latter case. In fact, many characteristic p results have uninteresting formulations in characteristic 0, or correspond to considerably stronger results. As an example of the latter, the Kempf theorem A.56 can be strengthened to the so-called Borel–Weil–Bott theorem, which claims that, if $\xi + \rho \in X^+$, where ρ is half the sum of positive roots, then $R^i \operatorname{Ind}_B^G(w(\xi + \rho) - \rho) \cong L(\xi)$ if $\xi \in X^+$ and $i = \ell(w)$, and equals 0 otherwise.

In the previous paragraph, Ind_B^G denotes the induction functor from the category of rational B-modules to the category of rational G-modules. The functor Ind_B^G is left exact. For $i \geqslant 0$, $R^i \operatorname{Ind}_B^G$ is the ith right derived functor of Ind_B^G. As in the representation theory of finite groups, the theory of induced representations plays a central role in the representation theory of affine algebraic groups.

[6] A *restricted Lie algebra* $\mathfrak{g}$ over a field $\mathcal{k}$ of characteristic p is a $\mathcal{k}$-Lie algebra with an additional operation $\mathsf{x} \mapsto \mathsf{x}^{[p]}$, called the *p-operation* of $\mathfrak{g}$, with the property that $[\mathsf{x}^{[p]}, \mathsf{y}^{[p]}] = [\mathsf{x}, \mathsf{y}]^{[p]}$. The restricted enveloping algebra $\mathbf{u}(\mathfrak{g})$ of a restricted Lie algebra $\mathfrak{g}$ is the quotient algebra of $U(\mathfrak{g})$ modulo its ideal generated by $\mathsf{x}^p - \mathsf{x}^{[p]}$, for all $\mathsf{x} \in \mathfrak{g}$. In our case, the p-operation of the Lie algebra $\mathfrak{g}$ of the group G is given by the pth power in $\mathcal{O}(G)^*$ — if x is a point derivation of $\mathcal{O}(G)$ at e, then x^p, where product is formed in $\mathcal{O}(G)^*$, is also a point derivation of $\mathcal{O}(G)$ at e.

More generally, consider a closed subgroup H of an affine algebraic group G. Any rational G-module M is, by restriction to H, also a rational H-module, denoted $\mathrm{Res}_H^G M$. Clearly, $\mathrm{Res}_H^G\colon G\text{-}\mathsf{mod} \to H\text{-}\mathsf{mod}$ is an exact, additive functor. With this, we define the induction functor Ind_H^G in the first statement of the following theorem.

Theorem A.53. *Let H be a closed subgroup of an affine algebraic group G.*

(1) *The restriction functor Res_H^G admits a left exact, right adjoint — the induction functor $\mathrm{Ind}_H^G\colon H\text{-}\mathsf{mod} \to G\text{-}\mathsf{mod}$ from H-mod to G-mod.*

(2) *For $M \in G\text{-}\mathsf{mod}$ and $N \in H\text{-}\mathsf{mod}$,*

$$M \otimes \mathrm{Ind}_H^G N \cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G M \otimes N).$$

(3) *If K is a closed subgroup of H, then $\mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H = \mathrm{Ind}_K^G$.*

(4) *The category G-mod has enough injective objects.*

We sketch a proof of (1)–(3) in Exercise A.17. Then (4) follows once it is observed, by an elementary argument, that, if M is an injective rational H-module, then $\mathrm{Ind}_H^G M$ is an injective rational G-module. In particular, we can take $H = \{e\}$, where *every* rational H-module is injective, to conclude that $\mathrm{Ind}_{\{e\}}^G \mathrm{Res}_{\{e\}}^G M$ is an injective rational G-module for any rational G-module M. Clearly, $M \subseteq \mathrm{Ind}_{\{e\}}^G \mathrm{Res}_{\{e\}}^G M$. Now a general argument on essential extensions (cf. [**HAI**, Ch. 1, Th. 9.2]) shows that M has an injective envelope, usually denoted $I(M)$. It is an injective module, containing M as an essential submodule, and it is a direct summand of every injective module containing M.

Example A.54. Suppose that H is a parabolic subgroup of a connected affine group G. Then the rational G-module $\mathrm{Ind}_H^G \mathcal{K}$ obtained by inducing the trivial H-module $\mathcal{K}$ to G is the trivial G-module $\mathcal{K}$. In fact, as shown in Exercise A.17, for any rational H-module N, $\mathrm{Ind}_H^G N$ identifies with a $(\mathcal{O}(G) \otimes N)^H$ of H-fixed points. In particular, $N = \mathcal{K}$ and $\mathrm{Ind}_H^G \mathcal{K} = \mathcal{O}(G)^H = \mathcal{O}(G/H)$, the space of everywhere defined functions on the quotient variety G/H of left cosets. If H is parabolic, then G/H is complete, so that Lemma A.18(2) implies that $\mathrm{Ind}_H^G \mathcal{K} \cong \mathcal{K}$, the trivial module consisting of constant functions.

In particular, we can consider the right derived functors

$$R^n \mathrm{Ind}_H^G\colon H\text{-}\mathsf{mod} \longrightarrow G\text{-}\mathsf{mod}, \quad n \geqslant 0.$$

Given a rational H-module N, let $0 \to N \to I^0 \to I^1 \to \cdots$ be a resolution of N by injective H-modules, and set $R^n \mathrm{Ind}_H^G N$ equal to the nth cohomology of the complex $\mathrm{Ind}_H^G I^\bullet$. Each $R^n \mathrm{Ind}_H^G N$ is a rational G-module.

Theorem A.55. *Let H be a closed subgroup of G and let N be a rational H-module.*

(1) $R^n \operatorname{Ind}_H^G N = 0$, *for* $n > \dim G/H$.

(2) *If H is a parabolic subgroup and* $\dim N < \infty$, *then* $\dim R^n \operatorname{Ind}_H^G N < \infty$, *for all n.*

In fact, $R^n \operatorname{Ind}_H^G N$ can be interpreted in another way, namely, as a sheaf cohomology group $H^n(G/H, \mathscr{L}_N)$ of a (G-equivariant) quasi-coherent sheaf $\mathscr{L}_N$ on the quotient variety G/H. Then (1) follows from the Grothendieck vanishing theorem, while (2) follows since $\mathscr{L}_N$ is coherent if N is finite dimensional. (See [**147**, Ch. III, Th. 5.2].)

For the rest of this section and the next section, let G be a fixed simply connected, semisimple algebraic group over $\mathcal{K}$. We use the same notation as above. In particular, $B' = U' \rtimes T$ is the Borel subgroup containing T and opposite to B. For $\xi \in X(T)$, let ξ also denote the 1-dimensional B'-module defined by ξ.

Theorem A.56 (Kempf). *For $\xi \in X^+$, $R^n \operatorname{Ind}_{B'}^G(\xi) = 0$, for $n > 0$.*

When $\xi \in X^+$, the rational G-module

$$\nabla(\xi) := \operatorname{Ind}_{B'}^G \xi \tag{A.7.1}$$

is called the *costandard module* of highest weight ξ. We use Theorem A.56 to calculate the character and dimension of $\nabla(\xi)$.

Let $B' \subseteq P \subseteq Q$ be parabolic subgroups of G. Given a finite dimensional rational P-module N, form the "Euler characteristic"

$$\mathcal{E}_{P,Q}(N) := \sum_{n=0}^{\infty} (-1)^n \operatorname{ch} R^n \operatorname{Ind}_P^Q N. \tag{A.7.2}$$

By Theorem A.55, this definition makes sense! If $P = B'$ and $Q = G$, put simply $\mathcal{E} = \mathcal{E}_{B',G}$.

Because $\mathcal{E}_{P,Q}$ is additive on short exact sequences of rational P-modules, it is defined on the Grothendieck group $\mathscr{K}_0(P)$ of finite dimensional rational P-modules. Making use of the identifications in Corollary A.48, we view $\mathcal{E}_{P,Q}$ as a homomorphism

$$\mathcal{E}_{P,Q} \colon \mathbb{Z}X(T)^{W_P} \longrightarrow \mathbb{Z}X(T)^{W_Q} \tag{A.7.3}$$

of abelian groups.

Lemma A.57. (1) $\mathcal{E}(1) = 1$. *(Here $1 = e^0$.)*

(2) *If $B' \subseteq P \subseteq Q \subseteq R$, then $\mathcal{E}_{P,R} = \mathcal{E}_{Q,R} \circ \mathcal{E}_{P,Q}$.*

(3) *For $\eta \in \mathbb{Z}X(T)^{W_P}$ and $\xi \in \mathbb{Z}X(T)^{W_Q}$,*

$$\xi\mathcal{E}_{P,Q}(\eta) = \mathcal{E}_{P,Q}(\xi\eta).$$

Proof (sketch). (1) follows from Example A.54.

(2) The proof is actually quite formal and can be outlined as follows. By Theorem A.53(3), $\operatorname{Ind}_P^R = \operatorname{Ind}_Q^R \circ \operatorname{Ind}_P^Q$. On the other hand, Ind_P^Q takes injective objects in P-mod to injective (hence, acyclic) objects in Q-mod, so that there is, given a rational P-module M, a Grothendieck spectral sequence (see [**HAI**, Ch. VIII, Th 9.3])

$$E_2^{s,t} = R^s \operatorname{Ind}_Q^R(R^t \operatorname{Ind}_P^Q M) \Longrightarrow R^{s+t} \operatorname{Ind}_P^Q M.$$

Finally, for formal reasons, the Euler characteristic commutes with the differentials on the spectral sequence, implying the desired formula.

Statement (3) follows easily from Theorem A.53(2) and the identification of Grothendieck groups with character groups. □

Theorem A.58. *For $\xi \in X(T)$,*

$$\mathcal{E}(\xi) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\xi+\rho)}}{\sum_{w\in W}(-1)^{\ell(w)}e^{w\rho}}. \tag{A.7.4}$$

Proof (sketch). We break the proof down into several steps.

(1) Suppose that $\alpha \in \Pi$ is a simple root, and P is the "minimal" parabolic subgroup $P_\alpha = \langle B', U_\alpha\rangle$. Then a direct calculation shows that

$$\mathcal{E}_{B',P}(\xi) = e^{\xi} + e^{\xi-\alpha} + \ldots + e^{\xi-n\alpha}, \tag{A.7.5}$$

if $n = (\xi, \check{\alpha}) \geqslant 0$.

(2) If there exists a simple root α such that $(\xi,\check{\alpha}) = -1$, then $\mathcal{E}(\xi) = 0$. A direct check yields that $\mathcal{E}_{B',P_\alpha}(\xi) = 0$, so the claim follows from Lemma A.57(1).

(3) For $w \in W$ and $\xi \in X(T)$, put $w\cdot\xi = w(\xi+\rho)-\rho$ (the "dot" action). Then $\mathcal{E}(w\cdot\xi) = (-1)^{\ell(w)}\mathcal{E}(\xi)$. This formula holds, for $w = s_\alpha$, where $\alpha \in \Pi$, for $\mathcal{E}_{B',P_\alpha}$, and then for $\mathcal{E}$ by Lemma A.57.

(4) For $w \in W$, let $\varepsilon_w = (-1)^{\ell(w)}$, and, for $\xi \in X(T)$, let $A(\xi) := \sum_{w\in W}\varepsilon_w e^{w\xi}$. Then $A(\rho)^2 \in (\mathbb{Z}X(T))^W$, so by Lemma A.57(2),

$$\begin{aligned}
A(\rho)^2\mathcal{E}(\xi) &= \sum_{x,y\in W}\mathcal{E}(\xi+x^{-1}\rho+y\rho) = \sum_{u=x,\, v=xy}\varepsilon_v\mathcal{E}(\xi+u^{-1}\rho+u^{-1}v\rho)\\
&= \sum_{u,v\in W}\varepsilon_u\varepsilon_v\mathcal{E}(u\cdot(\xi+u^{-1}\rho+u^{-1}v\rho)) = \sum_{u,v\in W}\mathcal{E}(u\xi+v\rho+u\rho)\\
&= A(\xi+\rho)A(\rho)\mathcal{E}(1) = A(\xi+\rho)A(\rho).
\end{aligned}$$

Since $0 \neq A(\rho) = e^\rho\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})$ (Weyl's denominator formula), and since $\mathbb{Z}X(T)$ is an integral domain, we have (A.7.4), as required. □

Combining Theorems A.56 and (A.7.4), we obtain part (1) of the following important theorem. Part (2) then follows directly, except for the assertion that $\nabla(\xi)$ has a simple socle (see Exercise A.23). In the character formula given in (1), it would be possible to cancel an $e^{-\rho}$ from the numerator and denominator, but we prefer the given form since $w(\xi+\rho) = w\cdot\xi$ and $w\rho - \rho = w\cdot\rho$, in the notation of the dot action of W.

Theorem A.59. *Let $\xi \in X^+$.*

(1) (Weyl character and dimension formulas) *We have*

$$\operatorname{ch}\nabla(\xi) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\xi+\rho)-\rho}}{\sum_{w\in W}(-1)^{\ell(w)}e^{w\rho-\rho}}.$$

Therefore,

$$\dim\nabla(\xi) = \prod_{\alpha\in\Phi^+}\frac{(\xi+\rho,\check{\alpha})}{(\rho,\check{\alpha})}.$$

(2) *As a rational G-module, $\nabla(\xi)$ has simple socle $L(\xi)$, while other composition factors $L(\nu)$ satisfy $\nu < \xi$. (Thus, $L(\xi)$ occurs with multiplicity one in $\nabla(\xi)$.)*

(3) *When $\mathcal{K}$ has characteristic 0, any rational G-module is semisimple. Thus, in particular, $\nabla(\xi) \cong L(\xi)$.*

It is also the case that $\nabla(\xi)$ is finite dimensional. Let $w_0 \in W$ be the longest element, and, for $\xi \in X$, define $\xi^\star = -w_0(\xi)$. If $\xi \in X^+$, then $\xi^\star \in X^+$. For $\xi \in X^+$, put

$$\Delta(\xi) := \nabla(\xi^\star)^*. \tag{A.7.6}$$

The rational G-module $\Delta(\xi)$ is called the *standard module* of highest weight ξ. Theorem A.60 easily implies the following result.

Corollary A.60. *For $\xi \in X^+$, $\Delta(\xi)$ has top isomorphic to $L(\xi)$. The other composition factors $L(\nu)$ satisfy $\nu < \xi$. Finally,* $\operatorname{ch}\Delta(\xi) = \operatorname{ch}\nabla(\xi)$.

A.8. Higher Ext functors; Δ- and ∇-filtrations

We give a very brief account of some basic results on the higher Ext functors of rational modules for reductive groups. These results relate to the existence of certain filtrations, and will be useful in constructing interesting highest weight categories in Appendix C.

We have already used the fact that the category G-mod has enough injective objects. As a consequence, the usual cohomology machinery is available for the category G-mod. Thus, given $M, N \in G$-mod and a nonnegative integer n,

$$\operatorname{Ext}^n_G(M,N) := R^n\operatorname{Hom}_G(M,N)$$

is the nth right derived functor of the left exact functor $\mathrm{Hom}_G(M, -)\colon G\text{-}\mathsf{mod} \to \mathsf{Ab}$ (the category of abelian groups) evaluated at N. If $M = \mathcal{K}$, the extension group $\mathrm{Ext}^n_G(\mathcal{K}, N)$ is usually denoted simply $H^n(G, N)$; if, in addition, M is finite dimensional, then $\mathrm{Ext}^n_G(M, N) \cong H^n(G, M^* \otimes N)$.

We say a rational G-module M has a Δ-*filtration* provided there exists a decreasing filtration $M = M_0 \supseteq M_1 \supseteq \cdots$ such that $\bigcap_i M_i = 0$ and each nonzero $M_i/M_{i+1} \cong \Delta(\xi_i)$, for some $\xi_i \in X^+$. Dually, M has a ∇-filtration provided there is an increasing filtration $0 = M_0 \subseteq M_1 \subseteq \cdots$ such that $\bigcup M_i = M$ and each nonzero section $M_i/M_{i-1} \cong \nabla(\xi_i)$, for some $\xi_i \in X^+$.

Theorem A.61. *Let $\xi, \nu \in X^+$, and let M be a rational G-module.*

(1) $\dim \mathrm{Ext}^n_G(\Delta(\xi), \nabla(\nu)) = \delta_{\xi,\nu}\delta_{n,0}$.

(2) *For any integer $n \geqslant 0$,*

$$\mathrm{Ext}^n_G(\Delta(\xi), L(\nu)) \neq 0 \Longrightarrow \xi < \nu$$
$$\mathrm{Ext}^n_G(L(\nu), \nabla(\xi)) \neq 0 \Longrightarrow \xi < \nu.$$

(3) (Donkin, Scott) *Assume that, for any $\zeta \in X^+$,* $\dim \mathrm{Hom}_G(\Delta(\zeta), M) < \infty$. *Then M has a Δ-filtration if and only if* $\mathrm{Ext}^1_G(M, \nabla(\nu)) = 0$, *for all $\nu \in X^+$. Similarly, M has a ∇-filtration if and only if* $\mathrm{Ext}^1_G(\Delta(\nu), M) = 0$, *for all $\nu \in X^+$. In both these criteria, it suffices to confine ν to those dominant weights satisfying $\nu < \xi$ for those $\xi \in X^+$ such that $L(\xi)$ is a composition factor of M.*

Proof. We will sketch the arguments.

(1) Theorem A.60 and Corollary A.59 immediately imply that

$$\dim \mathrm{Ext}^0_G(\Delta(\xi), \nabla(\nu)) = \dim \mathrm{Hom}_G(\Delta(\xi), \nabla(\nu)) = \delta_{\xi,\nu}.$$

Thus, it suffices to show that

$$\mathrm{Ext}^n_G(\Delta(\xi), \nabla(\nu)) = \mathrm{Ext}^n_G(\Delta(\nu^\star), \nabla(\xi^\star)) = 0,$$

for all positive integers n. We can assume that ξ is not strictly larger than ν in the partial ordering $<$ on $X(T)$ (replacing ξ by $\nu^\star$ and ν by $\xi^\star$ if necessary). There is an injective resolution $\nu \to I_\bullet$ in B'-mod in which I^n is a summand of $\nu \otimes \mathcal{O}(U')^{\otimes n}$ with weights $> \nu$ if $n > 0$. By Theorem A.56, $R^n \mathrm{Ind}^G_{B'} \nu = 0$, so the resolution induces an injective resolution $\nabla(\nu) \to \mathrm{Ind}^G_{B'} I^\bullet$ of $\nabla(\nu)$. Thus, $\mathrm{Ext}^\bullet_G(\Delta(\xi), \nabla(\nu))$ is the cohomology of the complex

$$\mathrm{Hom}_G(\Delta(\xi), \mathrm{Ind}^G_{B'} I^\bullet) \cong \mathrm{Hom}_{B'}(\Delta(\xi), I^\bullet).$$

Since $\Delta(\xi)$ is a cyclic B'-module with generator of weight ξ ($\not> \nu$), this complex has zero cohomology in positive degree. This proves (1).

(2) follows by the same argument.

(3) Assume that M satisfies the Hom and Ext^1 conditions in (3). Choose minimal ν for which $L(\nu)$ is a composition factor of the socle of M. If $\tau \in X^+$ satisfies $\tau < \nu$, then form the short exact sequence $0 \to Q(\tau) \to \Delta(\tau) \to L(\tau) \to 0$, so that the composition factors $L(\sigma)$ of $Q(\tau)$ satisfy $\sigma < \tau$. By the long exact sequence of cohomology, $\mathrm{Ext}^1_G(L(\tau), M) = 0$, since $\mathrm{Hom}_G(Q(\tau), M) = 0$ and $\mathrm{Ext}^1_G(\Delta(\tau), M) = 0$. Thus, the inclusion $L(\nu) \subseteq M$ extends to an inclusion $\nabla(\nu) \subseteq M$. The hypothesis of (3) still applies to $M/\nabla(\nu)$ and

$$\dim \mathrm{Hom}_G(\Delta(\tau), M/\nabla(\nu)) = \dim \mathrm{Hom}_G(\Delta(\tau), M) - \delta_{\tau,\nu}.$$

Since X^+ is countable, we can construct a submodule M' of M with a ∇-filtration such that $\mathrm{Hom}_G(\Delta(\sigma), M/M') = 0$, for all $\sigma \in X^+$. Thus, $M = M'$. The converse follows from (1). $\square$

The injective envelope $I(\xi) = I(L(\xi))$, $\xi \in X^+$, has the property that $\mathrm{Ext}^1_G(\Delta(\zeta), I(\xi)) = 0$, for all $\zeta \in X^+$. Also, $I(\xi)$ is a direct summand of $\mathcal{O}(G) = \mathrm{Ind}_{\{e\}}^G \mathcal{K}$, so

$$\dim \mathrm{Hom}_G(\Delta(\zeta), I(\xi)) \leqslant \dim \mathrm{Hom}_G(\Delta(\zeta), \mathcal{O}(G)) = \dim \Delta(\zeta) < \infty.$$

Thus, $I(\xi)$ has a ∇-filtration.

Exercises and notes

Exercises

§A.1

A.1. Complete the details for the proofs of Examples A.4(2), (3).

A.2. Give the details of the proof of Theorem A.7.

A.3. Let X, Y be affine varieties.

(1) Show that the algebra $\mathcal{O}(X) \otimes_{\mathcal{K}} \mathcal{O}(Y)$ identifies with an algebra of functions on the product set $X \times Y$. (In particular, $\mathcal{O}(X) \otimes_{\mathcal{K}} \mathcal{O}(Y)$ is reduced, i.e., it has no nonzero nilpotent elements). Conclude that (1) of Theorem A.9 holds.

(2) Prove that there is an identification

$$\mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathcal{O}(X) \otimes_{\mathcal{K}} \mathcal{O}(Y), \mathcal{K}) \leftrightarrow \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathcal{O}(X), \mathcal{K}) \times \mathrm{Hom}_{\mathcal{K}\text{-alg}}(\mathcal{O}(Y), \mathcal{K})$$

of sets. Conclude that (2) and (3) of Theorem A.9 hold.

(3) Using the universal mapping properties of tensor products, deduce part (4) of Theorem A.9.

(4) Assume that X, Y are both irreducible, and show that $X \times Y$ is irreducible.

Hint: First, assume that $X \subseteq \mathbb{A}^m$ (resp., $Y \subseteq \mathbb{A}^n$) is the zero set of a prime ideal $\mathfrak{p} \subset \mathcal{K}[x_1, \ldots, x_m]$ (resp., $\mathfrak{q} \subset \mathcal{K}[y_1, \ldots, y_n]$). If I is the ideal generated by $\mathfrak{p}, \mathfrak{q}$ in $\mathcal{K}[x, y] := \mathcal{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$, check that $X \times Y$ is the zero set of I in $\mathbb{A}^{m+n}$. Thus, $X \times Y$ is a closed subvariety of $\mathbb{A}^{m+n}$. To see that it is irreducible, suppose that $f(x, y), g(x, y) \in \mathcal{K}[x, y]$ are such that $f(x, y)g(x, y)$ vanishes on $X \times Y$, but that $f(x, y)$ and $g(x, y)$ do not vanish on $X \times Y$. Then there exist $a, b \in X$ such that $f(a, y)$, $g(b, y)$ do not vanish on Y. Since Y is irreducible, $f(a, y)g(b, y)$ does not vanish on Y, and there exists a point $c \in Y$ for which $f(a, c)g(b, c) \neq 0$. However, $f(x, c)g(x, c)$ does vanish on X. Now use the irreducibility of X to obtain a contradiction.

§A.2

A.4. Verify Example A.14(1).

A.5. Let X be a variety over $\mathcal{K}$. Deduce that $\mathcal{D} := \{(x, x) \mid x \in X\}$ is a closed subspace of $X \times X$. Show that if $f \colon X \to Y$ is a morphism of varieties, then the graph $\Gamma_f := \{(x, f(x)) \mid x \in X\}$ is a closed subvariety of $X \times Y$.

A.6. Assume the notation of Example A.14(2). Show that the map $\iota_d \colon \mathfrak{G}_d(M) \to \mathbb{P}(\Lambda^d M)$ which sends a d-dimensional subspace N to the line $\Lambda^d N$ in $\Lambda^d M$ is injective. Let $v_1, \ldots, v_n$ be an ordered basis for M. If $\boldsymbol{i} = (i_1, i_2, \ldots, i_d)$ is an increasing sequence of integers i_j, $1 \leqslant i_j \leqslant n$, put $v_{\boldsymbol{i}} = v_{i_1} \wedge \cdots \wedge v_{i_d}$. The $v_{\boldsymbol{i}}$ thus form a basis of $\Lambda^d M$. Let $U_{\boldsymbol{i}}$ be the open subspace of $\mathbb{P}(\Lambda^d M)$ consisting of lines with nonzero $v_{\boldsymbol{i}}$-component. Prove that $\operatorname{Im} \iota_d \cap U_{\boldsymbol{i}}$ is a closed subvariety of $U_{\boldsymbol{i}}$. In this way, identify $\mathfrak{G}_d(M)$ as a closed subvariety of $\mathbb{P}(\Lambda^d M)$.

A.7. Verify Example A.14(3).

§A.3

A.8. (1) Prove assertions (1), (2), (4) of Proposition A.22 directly. Use Theorem A.15(1) to prove assertion (3) of Proposition A.22.

(2) Suppose that G is an affine algebraic group, and that $\mathcal{X} = \{X_i\}_{i \in I}$ is a family of irreducible (but not necessarily) closed subspaces of G, each containing the identity element e. Show that the subgroup H of G generated by the members of $\mathcal{X}$ is closed and connected.

Hint: We can assume, enlarging the family if necessary, that if $X \in \mathcal{X}$, then $X^{-1} \in \mathcal{X}$. Use Theorem A.15(1) to show that there is a sequence $i_1, \ldots, i_d$ of elements in I (with possible repetitions) such that $H = X_{i_1} \cdots X_{i_d}$. Conclude also that H is closed.

(3) Use (2) above to prove Proposition A.22(5).

Hint: First show that the normal subgroup $K := (G^\circ, G)$ is closed and connected by using (2) above with $\mathcal{X}$ consisting of the irreducible subsets $X_g := \{gxg^{-1}x^{-1} \mid x \in G^\circ\}$. Finally, show that K has finite index in (G, G), so that (G, G) is also closed. For this, show the image of (G, G) in the quotient group G/K is finite. (This requires a theorem of Baer: if N is a subgroup of M such that the set of commutators $(x, y) = xyx^{-1}y^{-1}$, $x \in M, y \in N$, is finite, then the subgroup (M, N) they generate is finite. See [**24**, p. 60].)

A.9. (1) Show that the multiplication $m\colon G \times G \to G$ has differential $dm\colon \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ sending (x, y) to $\mathsf{x} + \mathsf{y}$, for any $\mathsf{x}, \mathsf{y} \in \mathfrak{g}$.

(2) Let $\iota\colon G \to G$ be the morphism taking $x \in G$ to $x^{-1} \in G$. Show that $d\iota\colon \mathfrak{g} \to \mathfrak{g}$ takes $\mathsf{x} \in \mathfrak{g}$ to $-\mathsf{x}$.

(3) Fix $g \in G$, consider the inner automorphism $\operatorname{Int} g\colon G \to G$, $x \mapsto gxg^{-1}$. Show that $d(\operatorname{Int} g)\colon \mathfrak{g} \to \mathfrak{g}$ is the restriction of the automorphism of $\mathcal{O}(G)^*$ given by $u \mapsto gug^{-1}$, for $u \in \mathcal{O}(G)^*$ (and $g \in G$ is canonically regarded as an element of $\mathcal{O}(G)^*$). The automorphism $d(\operatorname{Int} g)$ of $\mathfrak{g}$ is usually denoted by $\operatorname{Ad} g$. In this way, we obtain a morphism $\operatorname{Ad}\colon G \to \operatorname{Aut}(\mathfrak{g})$. Prove that $d(\operatorname{Ad}) = \operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$, $\operatorname{ad} \mathsf{x}(\mathsf{y}) = [\mathsf{x}, \mathsf{y}]$.

A.10. Let A be an (associative) algebra over $\mathcal{k}$ and let V be a finite dimensional A-module. Prove that $\operatorname{Aut}_A(V)$ is an affine algebraic group. Show that $\operatorname{Aut}_A(V)$ is a dense subset of the affine variety $\operatorname{End}_A(V)$. Hence, $\dim \operatorname{Aut}_A(V) = \dim \operatorname{End}_A(V)$.

A.11. Form the $2m \times 2m$ matrix $J = \left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}\right)$, where I_m is the $m \times m$ identity matrix. Let G be the set of all $2m \times 2m$ invertible matrices g such that $gJg^{\mathsf{T}} = J$. Then G is a semisimple, simply connected group (the symplectic group). Illustrate the various parts of Theorems A.33 and A.34; see §4.6 and Example A.36 for related results for general linear groups.

A.12. Let G be an affine algebraic group over $\mathcal{K}$. Prove that if $x \in G$, then the variety G is smooth at x. If G is connected, conclude that the integral domain $\mathcal{O}(G)$ is algebraically closed in its fraction field $\mathcal{K}(G)$.

§A.4

A.13. Explicitly describe the parabolic subgroups containing a fixed Borel subgroup for the symplectic groups introduced in Exercise A.11.

A.14. Prove that, for $I, J \subseteq \Pi$, we have $P_I \cap P_J = P_{I \cap J}$.

§A.5

A.15. Let M be a rational module for a torus T. Prove the decomposition (A.5.1).

Hint: First show that $X(T)$ forms a basis for $\mathcal{O}(T)$, and observe that each $\chi \in X(T)$ is a group-like element in the Hopf algebra $\mathcal{O}(T)$. Next, let $\tau_M\colon M \to M \otimes \mathcal{O}(T)$ be the structure map on M. For $v \in M$, write $\tau_M(v) = \sum v_i \otimes \chi_i$, where $\chi_i \in X(T)$ are distinct. Use the identity $(\tau_M \otimes 1)\tau_M = (1 \otimes \Delta)\tau_M$ to conclude that $v_i \in M_{\chi_i}$.

A.16. Prove Theorem A.45.

Hint: Let M be a finite dimensional submodule of $\mathcal{O}(G)$ containing a set of algebra generators. Show that the resulting representation $\rho\colon G \to \mathrm{GL}(M)$ maps G isomorphically onto its image H, which is a closed subgroup of $\mathrm{GL}(M)$.

A.17. Let H be a closed subgroup of an affine algebraic group G. Given a rational H-module N, we regard $\mathcal{O}(G) \otimes N$ as a right H-module, by putting

$$(f \otimes w) \cdot h = f \cdot h \otimes h^{-1} \cdot w, \quad f \in \mathcal{O}(G), w \in N, h \in H,$$

where $f \cdot h \in \mathcal{O}(G)$ is defined by $(f \cdot h)(x) = f(hx)$, $x \in G$. This right action of H commutes with the left action of G, given by $g \cdot (f \otimes w) = (g \cdot f) \otimes w$. Thus, the space $(\mathcal{O}(G) \otimes N)^H$ of H-fixed points is a rational G-module.

(1) Define $\mathrm{Ev}\colon (\mathcal{O}(G) \otimes N)^H \to N$ by $\mathrm{Ev}(\sum f_i \otimes w_i) = \sum f_i(e)w_i$. Show that Ev defines an H-module morphism $\mathrm{Ev}\colon \mathrm{Res}^G_H(\mathcal{O}(G) \otimes N)^H \to N$.

(2) Let M be a rational G-module and let $a\colon \mathrm{Res}^G_H M \to N$ be an H-module morphism. Show there is a unique G-module morphism $A\colon M \to (\mathcal{O}(G) \otimes N)^H$ such that $\mathrm{Ev} \circ A = a$.

(3) Conclude that the right adjoint $\mathrm{Ind}^G_H\colon H\text{-}\mathsf{mod} \to G\text{-}\mathsf{mod}$ to Ind^G_H exists, and that Ev is the corresponding adjunction morphism $\mathrm{Res}^G_H \circ \mathrm{Ind}^G_H \to \mathrm{id}_{H\text{-}\mathsf{mod}}$.

(4) Using standard properties of adjunction maps, prove Theorem A.53(2), (3).

A.18. Let G be a closed, connected, solvable subgroup of $\mathrm{GL}(M)$, for some finite dimensional vector space M. Prove the Lie–Kolchen theorem: *G stabilizes a complete flag in M*. Conversely, show that if G is a subgroup of $\mathrm{GL}(M)$ which stablizes a complete flag in M, then G is solvable.

Hint: It suffices to show that G has a common eigenvector in M. Argue by induction on the $\dim G + \dim M$. We can assume that M is simple. By induction, the commutator subgroup G' has a common eigenvector in M. Since G' is a normal subgroup of G, if $g \in G$ and if $v \in M$ is a G'-eigenvector, then gv is also an eigenvector for G'. Because M is finite dimensional, there are only a finite number of such eigenvalues (viewed as characters $G' \to \mathbb{G}_m$). Now an elementary argument, using exterior powers, step (1) of Exercise A.20(1), and a continuity argument involving the connectedness of G shows that G, in fact, stabilizes an eigenspace for G'. By induction, we can thus assume that M is itself an eigenspace for G'. Every element in G' has determinant 1, so, since the elements of G' are scalar operators, G' is finite, whence $G' = \{e\}$ is trivial. Therefore, G consists of a commutating family of linear transformations, which thus has a common eigenvector. An alternative and much more sophisticated proof can be based on Theorem A.30 and Example A.14(3).

A.19. Suppose that M is a rational G-module and that $v \in M_\xi$ is a root vector, for some $\xi \in X(T)$. Let $\alpha \in \Phi$. Show that, for $x \in U_\alpha$, $x \cdot v = v + \sum_\zeta v_\zeta$, where $v_\zeta \in M_\zeta$ and $\zeta = \xi + r\alpha$, for some positive integer r.

Hint: The conjugation action of T on U_α induces an action of T as a group of automorphisms of the coordinate algebra $\mathcal{O}(U_\alpha)$. Then $\mathcal{O}(U_\alpha)$ is a polynomial algebra in a variable u satisfying $t{\cdot}u = \alpha(t)^{-1}u$.

A.20. A key step in proving the existence of quotient groups (see Example A.28(3)) is the following fundamental theorem: *If H is a closed subgroup of an affine algebraic group G, then there exists a (finite dimensional) rational representation $\rho\colon G \to \mathrm{GL}(M)$ such that H (and its Lie algebra $\mathfrak{h}$) is the stabilizer of a line $L \subseteq M$.* We sketch a proof of this result for H in the following steps.

(1) Let M be a finite dimensional vector space over $\mathcal{K}$, and let $\psi\colon M \to M$ be an invertible linear transformation, extending to an invertible $\mathcal{K}$-algebra homomorphism $\psi\colon \Lambda(M) \to \Lambda(M)$. Suppose that N is a subspace of dimension n of M. Prove that $\psi(N) = N$ if and only if $\psi(\Lambda^n N) = \Lambda^n N$.

(2) Now let I be the ideal in $\mathcal{O}(G)$ consisting of all functions which vanish on H. Because $\mathcal{O}(G)$ is noetherian, there exist $f_1, \dots, f_m \in I$ which generate the ideal I. Let M be the finite dimensional rational G-submodule of $\mathcal{O}(G)$ generated by $f_1, \dots, f_m$ and set $N = I \cap M$. Prove that $g \in G$ belongs to H if and only if $g \cdot N = N$. Now use (1) to complete the proof.

Remarks: (a) Although we have sketched the argument above for H, it can be extended to include the action of the Lie algebra. Of course, if $\phi\colon M \to M$ is a linear transformation, the appropriate action of ϕ on $\Lambda^n M$ in the Lie algebra situation is given by $\phi(v_1 \wedge \cdots \wedge v_n) = \sum_i v_1 \wedge \cdots \wedge v_{i-1} \wedge \phi(v_i) \wedge v_{i+1} \wedge \cdots \wedge v_n$, for $v_1, \dots, v_n \in M$.

(b) Suppose that $G \subseteq \mathrm{GL}(M)$, and let $\mathbb{P}(M) = \mathfrak{G}_1(M)$ be the corresponding projective space. (See Example A.14(2).) It can be proved that the quotient variety G/H is isomorphic to the orbit $\mathfrak{O}$ of $L \in \mathbb{P}(M)$.

A.21. Let G be an affine algebraic group and let B be a Borel subgroup of G. Prove that the quotient variety G/B is complete.

Hint: It can be assumed that B has maximal dimension among all Borel subgroups of G. By Theorem A.45 (see Exercise A.16) and Exercise A.20, it can be assumed that (i) $G \subseteq \mathrm{GL}(M)$, for some finite dimensional vector space M, and (ii) B (resp., its Lie algebra) is the stablizer in G (resp., the Lie algebra of G) of a line $M_1 \subseteq M$. Then B stablizes a complete flag $M_1 \subset M_2 \subset \cdots \subset M_n = M$ of M; see Exercise A.18. Let x be the point defined by this flag in the complete flag variety $X := \mathfrak{F}_{\mathrm{cpl}}(M)$. Let $\mathfrak{O}_x := G{\cdot}x$ be the G-orbit of x for the regular action of G on X. Using the Remark (b) to Exercise A.20, it can be shown that $G/B \cong \mathfrak{O}_x$. By Theorem A.29(2) and Exercise A.18, $\mathfrak{O}_x$ is a minimal orbit, and hence $\mathfrak{O}_x$ is closed in X. Therefore, $\mathfrak{O}_x$ is complete by Lemma A.18(1).

A.22. We sketch a proof of Theorem A.46.

(1) For $\alpha \in \Pi$, set $Q_\alpha = P_{\Pi \setminus \{\alpha\}}$ (a maximal parabolic subgroup). By Exercise A.20, there exists a rational G-module having a line $\mathcal{K}v$ stabilized by Q_α. Assume that v has weight ξ. Show that ξ is fixed by all s_β, $\beta \in \Pi\setminus\{\alpha\}$, so that $\xi = m\varpi_\alpha$, for some positive integer m, where ϖ_α is the fundamental weight corresponding to α. Let M' be the submodule

of M generated by v. Prove that M' has a simple quotient $L(m\varpi_\alpha)$ of highest weight $m\varpi_\alpha$.

(2) Let M be as in (1), and choose a basis $v_1, \ldots, v_n$ of M with $v = v_1$. For $g \in G$, write $g \cdot v_j = \sum_i a_{i,j}(g)v_j$. If $f := a_{1,1}$, then $b \cdot f = \xi(b)f$, for $b \in B$.

(3) Define $f \in \mathcal{O}(U'B)$ by $f(ub) = \varpi_\alpha(b)$. Regarding $f \in \mathcal{K}(G)$, we have $f^d \in \mathcal{O}(G)$ by (4). Hence, $f \in \mathcal{O}(G)$ using Exercise A.12. Conclude that G has a simple representation of highest weight ϖ_α.

(4) Let $\xi = \sum_{\alpha\in\Pi} n_\alpha \varpi_\alpha \in X^+$. Let

$$M(\xi) := \bigotimes L(\varpi_\alpha)^{\otimes n_\alpha}.$$

Show that M has a B-fixed line of weight ξ, and conclude that G has a simple rational module of highest weight ξ.

A.23. Using the definition $\nabla(\xi) = \mathrm{Ind}_B^G \xi$, show that $\nabla(\xi)$ has simple socle isomorphic to $L(\xi)$, and hence $\Delta(\xi)$ has simple top isomorphic to $L(\xi)$.

§A.6

A.24. Let $F\colon G \to G$ be a Frobenius morphism for the affine algebraic group G. Prove the following assertions:

(1) If H is an F-stable closed subgroup of G, then $F|_H\colon H \to H$ is also a Frobenius morphism. If, in addition, H is normal in G, then the homomorphism $\overline{F}\colon G/H \to G/H$ induced by F is also a Frobenius morphism.

(2) The morphism F is bijective. In other words, F is an isomorphism of G as a group.

Hint: : The injectivity is proved by reducing to the case of a standard Frobenius morphism. Then the surjectivity is trivial if G is finite, and it follows from dimension considerations if G is connected. The general case is reduced to these two extreme cases by using (1).

A.25. Prove that, for a Frobenius morphism $F\colon G \to G$, the set G^F of F-invariants of G is a finite subgroup of G. Moreover, if $\mathcal{K} = \overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$, then $G = \bigcup_{i\geqslant 1} G^{F^i}$. In particular, in this case, every element in G is of finite order.

A.26. Prove the assertions of Example A.51(2). Here the unitary group is defined with respect to the Hermitian form on the n-dimensional $\mathbb{F}_{q^2}$-vector space V defined by $\left(\sum_i a_i v_i, \sum_i b_i v_i\right) = \sum_i a_i b_i^q$, where $(v_1, \ldots, v_n)$ is the basis of V through which the group $\mathrm{GL}_n(\mathbb{F}_{q^2})$ acts on V.

A.27. Prove the Lang–Steinberg theorem (Theorem A.50) in the following steps:

(1) Suppose that $F\colon G \to G$ is a Frobenius morphism. Show that if F is standard, then its differential $dF\colon \mathfrak{g} \to \mathfrak{g}$ is the zero map. Prove that, in general, $dF\colon \mathfrak{g} \to \mathfrak{g}$ is a nilpotent map in the sense that $(dF)^r := \underbrace{dF \circ dF \circ \cdots \circ dF}_{r} = 0$, for large r.

(2) Fix $x \in G$, consider the morphism (of varieties) $\theta_x \colon G \to G$, $g \mapsto g^{-1}xF(g)x^{-1}$. Prove this morphism is dominant by showing that its differential $d\theta_x \colon \mathfrak{g} \to \mathfrak{g}$ is an isomorphism. Deduce that the G-orbit $\mathfrak{O}_x$ of x under the right G-action $x \triangleleft g = g^{-1}xF(g)$ is dense in G.

Hint: We have $d\theta_x = -\mathrm{id}_{\mathfrak{g}} + \mathrm{Ad}\, x \circ dF$, by Exercise A.9. Verify that $(\mathrm{Ad}\, x \circ dF)^r = \mathrm{Ad}\, y \circ (dF)^r$, for $r \geqslant 1$ with $y = xF(x) \cdots F^{r-1}(x)$. Thus, $(d\theta_x)^{p^t} = (-1)^p \mathrm{id}_{\mathfrak{g}}$, for large t, by the nilpontency of dF.

(3) Show that, for any $x \in G$, the orbit $\mathfrak{O}_x$ is closed in G, and hence equals G. In particular, this is the case if $x = e$, giving the theorem.

Hint: All orbits $\mathfrak{O}_x$ have the same dimension ($= \dim G$) and so must be closed in G, by Theorem A.29(2).

A.28. Let $F \colon G \to G$ be a Frobenius morphism of an affine algebraic group G. Prove that $\mathcal{O}(\mathrm{Ker}\, F)$ (hence $\mathcal{O}(\mathrm{Ker}\, F)^*$) is finite dimensional.

Hint: Clearly, we have $\mathcal{O}(\mathrm{Ker}\, F^{i+1}) \twoheadrightarrow \mathcal{O}(\mathrm{Ker}\, F^i)$, for any positive integer i. Hence, we can assume that F is standard.

§A.8

A.29. (1) Prove that, given dominant weights ξ and ν, if $\mathrm{Ext}^n_G(\Delta(\xi), \Delta(\nu)) \neq 0$, for some $n > 0$, then $\nu > \xi$. Similarly, show that if $\mathrm{Ext}^n_G(\nabla(\nu), \nabla(\xi)) \neq 0$, for some $n > 0$, then $\nu > \xi$.

(2) Let $\Lambda \subset X^+$ be a finite ideal, so that $\zeta \leqslant \xi \in \Lambda \Longrightarrow \zeta \in \Lambda$. Let $\xi \in X^+$, and let $I(\xi)_\Lambda$ be the largest submodule of the injective envelope $I(\xi)$ of $L(\xi)$ such that all the composition factors $L(\zeta)$ of $I(\xi)_\Lambda$ satisfy $\zeta \in \Lambda$. Prove that $I(\xi)_\Lambda$ has a ∇-filtration.

Notes

§§A.1–A.2: We have adopted a rather elementary approach. All of the results can be found in standard textbooks, e.g., Hartshorne [**147**, Ch. 1]. Atiyah–MacDonald [**CA**] collects together in a concise way the relevant commutative algebra, often in the form of exercises.

A proper discussion of algebraic geometry would include the theory of schemes. See [**147**] for this as well.

§§A.3–A.4: There are many good references to the theory of affine algebraic groups. Borel [**24**] and Springer [**280**] are classics. The treatise [**56**] by Demazure–Gabriel takes a functorial points of view, which is very useful for studying groups over fields of positive characteristic. These books present most of the needed algebraic geometry from scratch, paying attention to varieties defined over non-algebraically closed fields, a topic we have largely ignored above. Borel's book stops short of presenting the full classification of reductive groups over $\mathcal{K}$. This topic is completely covered by Springer [**280**, Ch. 9–10] in terms of the root datum. At the end of §A.3, we have stated the main result, but in the special case of semisimple groups.

See also Steinberg [**285**] and Geck [**124**] for other readable accounts. We have followed [**285**, §2.9] for Definition A.11. An approach via the theory of Chevalley groups appears in [**283**]. Many of these results are also nicely described in [**132**].

§A.5: The elementary theory of rational representations for an affine algebraic group is contained in [**280**, Ch. 2]. We have followed [**158**, §31.4] for the proof of the existence of simple modules of a given highest weight sketched in Exercise A.22.

§§A.6–A.8: A standard reference for almost all of the material in these sections, together with historical remarks, is the book by Jantzen [**165**].

The easy proof of the Lang–Steinberg theorem presented here is due to Müller [**226**], while the "standard" proof using tangent spaces contained in Exercise A.27 can be found in, for example, Borel [**24**] and Springer [**280**] (a "classical" proof, modified so that it applies to the generalized definition of Frobenius morphism, is sketched in Exercise A.27). For another discussion of Frobenius morphisms (from a slightly different point of view), see [**284**, §11].

The tensor product theorem was first proved by Steinberg in [**282**]. He also obtained there the connection between the rational simple G-modules and the simple G^F-modules over $\mathcal{K}$. The tensor product theorem is now usually proved using "infinitesimal methods" involving $\operatorname{Ker} F$ by means of an argument first given by Cline–Parshall–Scott [**44**].

For the discussion of the Euler characteristic operator, we have followed the elegant treatment given by Donkin [**82**].

The modules $\Delta(\xi)$ are often called Weyl modules in the literature. These can also be obtained by "reduction mod" p from the complex simple modules for the associated complex Lie algebra, using a minimal lattice, as explained in Steinberg [**283**]. In turn, $\nabla(\xi)$ arises from a maximal lattice. In fact, these assertions follow easily once the elementary properties of $\nabla(\xi)$ in §A.7 have been proved.

More generally, the study of the standard modules $\Delta(\xi)$ and costandard modules $\nabla(\xi)$ can be placed in the context of "Borel–Weil–Bott theory" for reductive groups over fields of arbitrary characteristic. From this point of view, the isomorphism (A.7.6) is just a special case of Serre duality for the cohomology of coherent sheaves on G/B.

The problem of determining the characters of the simple modules $L(\xi)$ in positive characteristics is a big unsolved problem. For a survey of progress up to 2004, together with many references, see Tanisaki [**291**]. When $G = \mathrm{SL}_n(\mathcal{K})$, the problem of determining the characters of the simple modules can be reduced (when $p \geqslant n$) to a concrete problem (in the spirit of almost split sequences studied in Chapter 2) involving Specht modules for symmetric groups [**235**, Th. 4.2].

Appendix B

Quantum linear groups through coordinate algebras

One can approach quantum groups by means of their coordinate algebras, which are (usually noncommutative and noncocommutative) Hopf algebras obtained by deforming the coordinate algebras of affine algebraic groups as studied in Appendix A. To obtain *quantum linear groups* — deformations of classical general or special linear groups — we must begin with the notion of a *quantum matrix space.* A particularly interesting property of a quantum matrix space in our context is that it has a coordinate algebra which, just like the coordinate algebra of a classical matrix space, is a graded algebra whose homogeneous components are coalgebras of finite rank over the ground ring. Moreover, the quantum Schur algebras discussed in Chapter 9 can be realized as the dual algebras of these homogeneous components. This appendix gives a brief account of this theory of quantum linear groups, with applications of this point of view to quantum Schur algebras.

Throughout this appendix, let $\mathcal{R}$ be a commutative ring over which all the structures we will consider are defined. We motivate the discussion by reviewing the classical case from Appendix A.

Fix a positive integer n. The set M_n of $n \times n$ matrices over $\mathcal{R}$ can be viewed as an affine n^2-space defined over $\mathcal{R}$ with coordinate algebra $A_n := \mathcal{O}(\mathrm{M}_n) = \mathcal{R}[x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{n,n}]$, the (commutative) polynomial algebra in n^2 generators $x_{i,j}$, $i, j = 1, \ldots, n$. As an $\mathcal{R}$-module, A_n is free with monomials in the $x_{i,j}$ as a basis. Matrix multiplication makes M_n

into a monoid, and A_n carries a bialgebra structure with comultiplication Δ and counit ε induced by the matrix multiplication and the identity matrix in M_n. In fact, the comultiplication and counit can be expressed in terms of the generators $x_{i,j}$ as follows:

$$\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j} \quad \text{and} \quad \varepsilon(x_{i,j}) = \delta_{i,j}.$$

Let Ω be a free $\mathcal{R}$-module of rank n with basis $\{\omega_1, \ldots, \omega_n\}$, by means of which the multiplicative monoid M_n acts from the left (resp., right) on Ω. Thus, Ω is a left (resp., right) M_n-module. This module structure can be interpreted in terms of a right (resp., left) A_n-comodule structure via the structure map $\tau\colon \Omega \to \Omega \otimes A_n$ (resp., $\rho\colon \Omega \to A_n \otimes \Omega$) given by

$$\tau(\omega_j) = \sum_i \omega_i \otimes x_{i,j} \qquad \left(\text{resp., } \rho(\omega_i) = \sum_j x_{i,j} \otimes \omega_j\right).$$

The actions of the monoid M_n on Ω extend to actions of M_n on the tensor algebra $\mathsf{T}(\Omega)$, symmetric algebra $\mathsf{S}(\Omega)$, and exterior algebra $\Lambda(\Omega)$. All of these actions can be interpreted in terms of A_n-comodules with structure maps obtained by extending τ (resp., ρ) to algebra homomorphisms.

Also, A_n is naturally graded by the total degrees of the monomials, and Δ maps a monomial to a sum of tensor products of monomials with the same total grade. Let $A_n(r)$ be the homogeneous component of A_n of grade r; that is, let $A_n(r)$ be the span of all monomials of total degree r. Thus, $A_n(r)$ is a free $\mathcal{R}$-module of rank $\binom{n^2+r-1}{r}$, and the above discussion implies that $A_n(r)$ is a subcoalgebra of A_n. The dual algebra $A_n(r)^*$ (see Proposition 5.4) of $A_n(r)$ is the (classical) Schur algebra $S(n,r)$.

B.1. Quantum linear algebra

As in §5.1, let $\mathsf{Bialg}_{\mathcal{R}}$ be the category of bialgebras over $\mathcal{R}$. We will call an object M in the opposite category $(\mathsf{Bialg}_{\mathcal{R}})^{\mathrm{op}}$ a *quantum matrix space* if the corresponding object in $\mathsf{Bialg}_{\mathcal{R}}$, usually called the *coordinate algebra* of M and denoted $\mathcal{O}(M)$, is generated, as an algebra, by n^2 generators $x_{i,j}$ $(i,j = 1, \ldots, n)$, for some $n \in \mathbb{N}$, with coalgebra structure given by

$$\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j} \quad \text{and} \quad \varepsilon(x_{i,j}) = \delta_{i,j}.$$

The most interesting quantum matrix spaces arise from the deformations of the classical matrix spaces M_n of all $n \times n$-matrices with entries in $\mathcal{R}$. We now construct the quantum analogue $\mathrm{M}_{n,v}$ of M_n with a single parameter

$v \in \mathcal{R}^{\times}$, the multiplicative group of invertible elements in $\mathcal{R}$. We also let $q = v^2$, whose role will be clear in §B.3.[1]

To obtain $\mathrm{M}_{n,v}$, we define, as the first step, an algebra $A_{n,v}$ as the candidate of the coordinate algebra $\mathcal{O}(\mathrm{M}_{n,v})$ of $\mathrm{M}_{n,v}$. Let $A_{n,v}$ be the associative algebra over $\mathcal{R}$ with n^2 generators $\chi_{i,j}$, $i, j = 1, \ldots, n$, and relations

$$\begin{aligned} \chi_{k,i}\chi_{k,j} &= v\chi_{k,j}\chi_{k,i}, && \text{for all } i > j; \\ \chi_{k,i}\chi_{l,i} &= v\chi_{l,i}\chi_{k,i}, && \text{for all } k > l; \\ \chi_{k,i}\chi_{l,j} &= \chi_{l,j}\chi_{k,i}, && \text{if } k > l \text{ and } i < j; \\ \chi_{k,i}\chi_{l,j} - \chi_{l,j}\chi_{k,i} &= (v - v^{-1})\chi_{l,i}\chi_{k,j}, && \text{if } k > l \text{ and } i > j. \end{aligned} \tag{B.1.1}$$

The relations (B.1.1) of $A_{n,v}$ can be interpreted in terms of a Yang–Baxter matrix (i.e., the matrix of a Yang–Baxter operator, which is a solution to the Yang–Baxter equation (B.1.3) below). Again let Ω be a free $\mathcal{R}$-module of rank n with basis $\{\omega_1, \ldots, \omega_n\}$. Consider the $\mathcal{R}$-module homomorphism $\mathscr{R}_v \colon \Omega \otimes \Omega \to \Omega \otimes \Omega$ defined by

$$\mathscr{R}_v(\omega_i \otimes \omega_j) = \begin{cases} \omega_i \otimes \omega_j, & \text{for } i < j; \\ v\omega_i \otimes \omega_i, & \text{for } i = j; \\ \omega_i \otimes \omega_j + (v - v^{-1})\omega_j \otimes \omega_i, & \text{for } i > j. \end{cases} \tag{B.1.2}$$

It is easy to check that the transformation $\mathscr{R} = \mathscr{R}_v$ is a solution of the Yang–Baxter equation

$$\mathscr{R}_{1,2} \circ \mathscr{R}_{1,3} \circ \mathscr{R}_{2,3} = \mathscr{R}_{2,3} \circ \mathscr{R}_{1,3} \circ \mathscr{R}_{1,2}, \tag{B.1.3}$$

where $\mathscr{R}_{s,t} \colon \Omega \otimes \Omega \otimes \Omega \to \Omega \otimes \Omega \otimes \Omega$ is the application of an $\mathcal{R}$-module homomorphism $\mathscr{R} \colon \Omega \otimes \Omega \to \Omega \otimes \Omega$ to the sth and tth factors, for $1 \leqslant s < t \leqslant 3$, viewed as $\Omega \otimes \Omega$.

Let R_v be the matrix of the $\mathcal{R}$-module homomorphism $\mathscr{R}_v$ with respect to the basis $\{\omega_i \otimes \omega_j \mid i, j = 1, \ldots, n\}$ ordered lexicographically. Also, let X be the $n \times n$ matrix $(\chi_{i,j})$, and let I_n be the $n \times n$ identity matrix. Form the $n^2 \times n^2$ matrices[2] $\mathsf{X}_1 = \mathsf{X} \otimes \mathsf{I}_n$ and $\mathsf{X}_2 = \mathsf{I}_n \otimes \mathsf{X}$.

Proposition B.1. *With the above conventions, the relations* (B.1.1) *can be stated in matrix form as*

$$\mathsf{R}_v\mathsf{X}_1\mathsf{X}_2 = \mathsf{X}_2\mathsf{X}_1\mathsf{R}_v. \tag{B.1.4}$$

The easy proof of Proposition B.1 is left for Exercise B.1.

The matrix form (B.1.4) sometimes is useful in deducing various identities in $A_{n,v}$. See, for example, the proof of Lemma B.11.

[1]In Appendix B, we do not distinguish q from q and v from v.

[2]The tensor product of two matrices $A = (a_{i,j})_{m \times n}$ and $B = (b_{k,l})_{r \times s}$ (with coefficients in the same ring) is the matrix $A \otimes B := (a_{i,j}b_{k,l})_{mr \times ns}$, whose row indices (i,k) and column indices (j,l) are ordered lexicographically, respectively.

We summarize certain basic properties of the algebra $A_{n,v}$ as follows.

Theorem B.2. (1) *$A_{n,v}$ is a free $\mathcal{R}$-module with basis*

$$\mathcal{B} = \left\{ \prod_{i,j} \chi_{i,j}^{t_{i,j}} \;\middle|\; t_{i,j} \in \mathbb{N} \right\}, \tag{B.1.5}$$

where the products are formed with respect to any fixed ordering of the $\chi_{i,j}$. If $\mathcal{R}$ is an integral domain, then $A_{n,v}$ is also an integral domain.

(2) *$A_{n,v}$ is a $\mathbb{Z}$-graded algebra with the homogeneous component $A_{n,v}(r)$ of grade r spanned by all monomials $\prod_{i,j} \chi_{i,j}^{t_{i,j}}$ with $\sum_{i,j} t_{i,j} = r$. Thus, $A_{n,v}(r)$ is a free $\mathcal{R}$-module of rank $\binom{n^2+r-1}{r}$.*

(3) *$A_{n,v}$ is a bialgebra with comultiplication Δ and counit ε defined on generators by*

$$\Delta(\chi_{i,j}) = \sum_k \chi_{i,k} \otimes \chi_{k,j} \quad \textit{and} \quad \varepsilon(\chi_{i,j}) = \delta_{i,j}. \tag{B.1.6}$$

(4) *The homogeneous components $A_{n,v}(r)$ are subcoalgebras of $A_{n,v}$.*

Proof (sketch)**.** The proof of (1) involves an application of the Bergman basis theorem (see [**20**]). Assertion (2) is clear from the defining relations — all of the relations are homogeneous (and of grade 2). The proof of (3) is routine, by checking that the $\Delta(\chi_{i,j})$ satisfy the relations satisfied by the $\chi_{i,j}$ (Exercise B.2). Now (4) is obvious from the definition of Δ. □

We can standardize the choice of the basis $\mathcal{B}$ by using the lexicographic ordering on the generators $\chi_{i,j}$. Thus, $\chi_{i,k} \leqslant \chi_{j,l}$ if $i < j$, or if $i = j$ and $k \leqslant l$. A monomial in the $\chi_{i,j}$ is called *standard* if its factors are arranged (perhaps weakly) increasingly from left to right with respect to this ordering. By Theorem B.2, the standard monomials form a basis for $A_{n,v}$.

Definition B.3. The basis of $A_{n,v}$ (resp., $A_{n,v}(r)$) consisting of standard monomials is called the *standard basis* of $A_{n,v}$ (resp., $A_{n,v}(r)$), and it is denoted $\boldsymbol{B}_{n,v}$ (resp., $\boldsymbol{B}_{n,v}(r)$).

The standard basis can be indexed by various sets occurring in Chapter 9 (and earlier). To do this, we recall some notational conventions. For any positive integer r, let $I(n,r)$ be the set of sequences of integers $\boldsymbol{i} = (i_1, \ldots, i_r)$ with $1 \leqslant i_k \leqslant n$, for all k. For $\boldsymbol{i} = (i_1, \ldots, i_r)$ and $\boldsymbol{j} = (j_1, \ldots, j_n)$ in $I(n,r)$, we let

$$\chi_{\boldsymbol{i},\boldsymbol{j}} = \chi_{i_1,j_1} \cdots \chi_{i_r,j_r} \in A_{n,v}(r).$$

We may also define a (right) action of the symmetric group $\mathfrak{S} = \mathfrak{S}_r$ on $I(n,r)$ by letting

$$\boldsymbol{i}w = (i_{w(1)}, \ldots, i_{w(r)}), \quad \text{for } \boldsymbol{i} \in I(n,r),\ w \in \mathfrak{S}_r.$$

Let $\Lambda(n,r)$ be the set of compositions of r into n parts, and, for $\lambda = (\lambda_1,\ldots,\lambda_n) \in \Lambda(n,r)$ (thus, $\lambda_i \geqslant 0$ and $\sum \lambda_i = r$), define

$$\boldsymbol{i}_\lambda = (\underbrace{1,\ldots,1}_{\lambda_1},\underbrace{2,\ldots,2}_{\lambda_2},\ldots,\underbrace{n,\ldots,n}_{\lambda_n}) \in I(n,r).$$

Let $\mathfrak{S}_\lambda$ be the stablizer of $\boldsymbol{i}_\lambda$ in $\mathfrak{S}$ (which coincides with the parabolic subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}$ defined in §9.1), and let ${}^\lambda\mathfrak{S}$ be the set of shortest right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}$. Clearly, every element in $I(n,r)$ can be uniquely expressed as $\boldsymbol{i}_\lambda w$, for $\lambda \in \Lambda(n,r)$ and $w \in {}^\lambda\mathfrak{S}$.

For $\lambda,\mu \in \Lambda(n,r)$, let ${}^\lambda\mathfrak{S}^\mu$ denote the set of shortest $(\mathfrak{S}_\lambda,\mathfrak{S}_\mu)$-double coset representatives. An element in $\boldsymbol{B}_{n,v}(r)$ can be uniquely expressed as $\chi_{\boldsymbol{i}_\lambda,\boldsymbol{i}_\mu w^{-1}}$, for $\lambda,\mu \in \Lambda(n,r)$ and $w \in {}^\lambda\mathfrak{S}^\mu$, and conversely, any element of the form $\chi_{\boldsymbol{i}_\lambda,\boldsymbol{i}_\mu w^{-1}}$, for $\lambda,\mu \in \Lambda(n,r)$ and $w \in {}^\lambda\mathfrak{S}^\mu$, belongs to $\boldsymbol{B}_{n,v}(r)$. This index system for $\boldsymbol{B}_{n,v}(r)$ is essential in the proof of the fact that the dual algebra of $A_{n,v}$ is isomorphic to the q-Schur algebra $S_q(n,r)$; see Lemma B.34 and Theorem B.37.

We may also index $\boldsymbol{B}_{n,v}(r)$ by $\Xi(n,r)$, the set of $n\times n$ matrices with entries in $\mathbb{N}$ summing to r. Recall that we defined in (8.2.3) a two-line array π_A associated with $A \in \Xi(n,r)$. For $A \in \Xi(n,r)$, the corresponding element in $\boldsymbol{B}_{n,v}(r)$ is $\chi_A = \chi_{\boldsymbol{i},\boldsymbol{j}}$, where $\pi_A = \binom{\boldsymbol{i}}{\boldsymbol{j}}$. We refer the reader to the discussion in §9.1 for the equivalence of this index system and the above system using $\lambda,\mu \in \Lambda(n,r)$ and $w \in {}^\lambda\mathfrak{S}^\mu$.

Remark B.4. Using $I(n,r)$, the comultiplication Δ and counit ε of $A_{n,v}(r)$ can be expressed in the following formulas: for $\boldsymbol{i},\boldsymbol{j} \in I(n,r)$,

$$\Delta(\chi_{\boldsymbol{i},\boldsymbol{j}}) = \sum_{\boldsymbol{k}\in I(n,r)} \chi_{\boldsymbol{i},\boldsymbol{k}} \otimes \chi_{\boldsymbol{k},\boldsymbol{j}} \quad\text{and}\quad \varepsilon(\chi_{\boldsymbol{i},\boldsymbol{j}}) = \delta_{\boldsymbol{i},\boldsymbol{j}}.$$

Definition B.5. The quantum matrix space $\mathrm{M}_{n,v}$ with coordinate algebra $\mathcal{O}(\mathrm{M}_{n,v}) = A_{n,v}$ is called the *standard quantum matrix space* of order n.

Note that if $v=1$, (B.1.1) is nothing but the commutative law. Thus, the classical $A_n = \mathcal{O}(\mathrm{M}_n)$ is recovered as a limiting case.

A left (resp., right) $\mathrm{M}_{n,v}$-module is interpreted (by definition!) as a right (resp., left) $A_{n,v}$-comodule. As an example, consider the "natural" left (resp., right) $\mathrm{M}_{n,v}$-module Ω, where Ω is a free $\mathcal{R}$-module of rank n. The $A_{n,v}$-comodule structure map $\tau\colon \Omega \to \Omega\otimes A_{n,v}$ (resp., $\rho\colon \Omega \to A_{n,v}\otimes\Omega$) is defined in terms of a prefixed basis $\{\omega_1,\ldots,\omega_n\}$ by

$$\tau(\omega_j) = \sum_i \omega_i \otimes \chi_{i,j} \quad \Big(\text{resp., } \rho(\omega_i) = \sum_j \chi_{i,j}\otimes\omega_j\Big), \tag{B.1.7}$$

for all j (resp., for all i).

Because $A_{n,v}$ is a bialgebra, there are $\mathrm{M}_{n,v}$-module structures on the tensor algebra $\mathsf{T}(\Omega) := \bigoplus_{r\in\mathbb{N}} \Omega^{\otimes r}$ such that the extended comodule structure maps $\tau\colon \mathsf{T}(\Omega) \to \mathsf{T}(\Omega) \otimes A_{n,v}$ and $\rho\colon \mathsf{T}(\Omega) \to A_{n,v} \otimes \mathsf{T}(\Omega)$ are algebra homomorphisms. In fact, using the notation $\omega_{\boldsymbol{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r}$ for $\boldsymbol{i} \in I(n,r)$, the structure maps τ and ρ can be expressed in terms of the basis elements of $\mathsf{T}(\Omega)$ as

$$\tau(\omega_{\boldsymbol{j}}) = \sum_{\boldsymbol{i}\in I(n,r)} \omega_{\boldsymbol{i}} \otimes \chi_{\boldsymbol{i},\boldsymbol{j}} \quad\text{and}\quad \rho(\omega_{\boldsymbol{i}}) = \sum_{\boldsymbol{j}\in I(n,r)} \chi_{\boldsymbol{i},\boldsymbol{j}} \otimes \omega_{\boldsymbol{j}}. \tag{B.1.8}$$

Recall that we defined the v-symmetric algebra $\mathsf{S}_v(\Omega)$ and the v-exterior algebra $\Lambda_v(\Omega)$ in Examples 0.17 and 0.18, respectively. We let $\mathsf{S}_v^r(\Omega)$ and $\Lambda_v^r(\Omega)$ be the homogeneous component of grade r (for $r \in \mathbb{N}$) of $\mathsf{S}_v(\Omega)$ and $\Lambda_v(\Omega)$, respectively. Of course, $\Lambda_v^r(\Omega) = 0$ if $r > n$.

Proposition B.6. *The algebras $\mathsf{S}_v(\Omega)$ and $\Lambda_v(\Omega)$, as well as their homogeneous components $\mathsf{S}_v^r(\Omega)$ and $\Lambda_v^r(\Omega)$, for $r \in \mathbb{N}$, inherit left and right $\mathrm{M}_{n,v}$-module structures from the $\mathrm{M}_{n,v}$-module structure on Ω defined by* (B.1.7).

Proof. The result will follow if we can show that the relation spaces R_v and R'_v defined in Examples 0.17 and 0.18, respectively, are $\mathrm{M}_{n,v}$-submodules of $\mathsf{T}(\Omega)$ with respect to both τ and ρ. It is a routine verification. Here is an example:

We show that, for $i < j$, $\tau(\omega_i \otimes \omega_j + v\omega_j \otimes \omega_i) \in R'_v \otimes A_{n,v}$. In fact,

$$\begin{aligned}
&\tau(\omega_i \otimes \omega_j + v\omega_j \otimes \omega_i)\\
&= \sum_{k,l} \omega_k \otimes \omega_l \otimes \chi_{k,i}\chi_{l,j} + v \sum_{k,l} \omega_k \otimes \omega_l \otimes \chi_{k,j}\chi_{l,i}\\
&= \sum_{k<l} \omega_k \otimes \omega_l \otimes (\chi_{k,i}\chi_{l,j} + v\chi_{k,j}\chi_{l,i}) + \sum_{k} \omega_k \otimes \omega_k \otimes (\chi_{k,i}\chi_{k,j} + v\chi_{k,j}\chi_{k,i})\\
&\qquad + \sum_{k<l} \omega_l \otimes \omega_k \otimes (\chi_{l,i}\chi_{k,j} + v\chi_{l,j}\chi_{k,i})\\
&= \sum_{k<l} \omega_k \otimes \omega_l \otimes (\chi_{l,j}\chi_{k,i} + v^{-1}\chi_{k,j}\chi_{l,i}) + \sum_{k} \omega_k \otimes \omega_k \otimes (\chi_{k,i}\chi_{k,j} + v\chi_{k,j}\chi_{k,i})\\
&\qquad + \sum_{k<l} v\omega_l \otimes \omega_k \otimes (v^{-1}\chi_{l,i}\chi_{k,j} + \chi_{l,j}\chi_{k,i})\\
&= \sum_{k<l} (\omega_k \otimes \omega_l + v\omega_l \otimes \omega_k) \otimes (\chi_{l,j}\chi_{k,i} + v^{-1}\chi_{k,j}\chi_{l,i})\\
&\qquad + \sum_{k} \omega_k \otimes \omega_k \otimes (\chi_{k,i}\chi_{k,j} + v\chi_{k,j}\chi_{k,i}),
\end{aligned}$$

which belongs to $R'_v \otimes A_{n,v}$, as required. The other verifications are left to the reader in Exercise B.3. $\square$

The $\mathrm{M}_{n,v}$-module structure on the v-exterior algebra $\Lambda_v(\Omega)$ is particularly useful in establishing a theory of "quantum linear algebra."

For $\boldsymbol{i} \in I(n,r)$, let

$$\hat{\omega}_{\boldsymbol{i}} = \omega_{i_1} \cdots \omega_{i_r} \in \Lambda_v^r(\Omega),$$

where the product is formed inside $\Lambda_v(\Omega)$.[3] We obtain from (B.1.8), for $\boldsymbol{j}$ (or $\boldsymbol{i}$) in $I(n,r)$, that

$$\tau(\hat{\omega}_{\boldsymbol{j}}) = \sum_{\boldsymbol{i}\in I(n,r)} \hat{\omega}_{\boldsymbol{i}} \otimes \chi_{\boldsymbol{i},\boldsymbol{j}} \text{ and } \rho(\hat{\omega}_{\boldsymbol{i}}) = \sum_{\boldsymbol{j}\in I(n,r)} \chi_{\boldsymbol{i},\boldsymbol{j}} \otimes \hat{\omega}_{\boldsymbol{j}}. \tag{B.1.9}$$

Now let $I_0(n,r)$ be the subset of $I(n,r)$ consisting of sequences $\boldsymbol{i} = (i_1, \dots, i_r)$ with $1 \leqslant i_1 < \cdots < i_r \leqslant n$. Clearly, $I_0(n,r) \neq \emptyset$ if and only if $r \leqslant n$.

Since $\hat{\omega}_{\boldsymbol{i}} = 0$ if there are two identical ingredients in $\boldsymbol{i}$, for $r \leqslant n$ and $\boldsymbol{j} \in I(n,r)$, $\hat{\omega}_{\boldsymbol{j}} \neq 0$ if and only if $\boldsymbol{j} = \boldsymbol{i}w$, for some $\boldsymbol{i} \in I_0(n,r)$ and $w \in \mathfrak{S}_r$. In particular, for $r \leqslant n$, the set $\{\hat{\omega}_{\boldsymbol{i}} \mid \boldsymbol{i} \in I_0(n,r)\}$ is a basis for $\Lambda_v^r(\Omega)$. Moreover,

$$\hat{\omega}_{\boldsymbol{i}w} = (-v)^{-\ell(w)}\hat{\omega}_{\boldsymbol{i}}, \quad \text{for } \boldsymbol{i} \in I_0(n,r),\ w \in \mathfrak{S}_r.$$

With these observations, we rewrite, for $\boldsymbol{j}$ (or $\boldsymbol{i}$) in $I_0(n,r)$, the formulas in (B.1.9) as

$$\tau(\hat{\omega}_{\boldsymbol{j}}) = \sum_{\boldsymbol{i}\in I_0(n,r)} \sum_{w\in\mathfrak{S}_r} \hat{\omega}_{\boldsymbol{i}w} \otimes \chi_{\boldsymbol{i}w,\boldsymbol{j}} = \sum_{\boldsymbol{i}\in I_0(n,r)} \hat{\omega}_{\boldsymbol{i}} \otimes \left(\sum_{w\in\mathfrak{S}_r} (-v)^{-\ell(w)} \chi_{\boldsymbol{i}w,\boldsymbol{j}} \right)$$

$$\rho(\hat{\omega}_{\boldsymbol{i}}) = \sum_{\boldsymbol{j}\in I_0(n,r)} \sum_{w\in\mathfrak{S}_r} \chi_{\boldsymbol{i},\boldsymbol{j}w} \otimes \hat{\omega}_{\boldsymbol{j}w} = \sum_{\boldsymbol{j}\in I_0(n,r)} \left(\sum_{w\in\mathfrak{S}_r} (-v)^{-\ell(w)} \chi_{\boldsymbol{i},\boldsymbol{j}w} \right) \otimes \hat{\omega}_{\boldsymbol{j}}.$$

It is easy to verify (see Exercise B.4) that

$$\sum_{w\in\mathfrak{S}_r} (-v)^{-\ell(w)} \chi_{\boldsymbol{i}w,\boldsymbol{j}} = \sum_{w\in\mathfrak{S}_r} (-v)^{-\ell(w)} \chi_{\boldsymbol{i},\boldsymbol{j}w},$$

for all $\boldsymbol{i}, \boldsymbol{j} \in I_0(n,r)$. We denote this element of $A_{n,v}$ by $\mathcal{D}_{\boldsymbol{i},\boldsymbol{j}}$. We finally arrive at the following formulas. For $\boldsymbol{j}$ (or $\boldsymbol{i}$) in $I_0(n,r)$,

$$\tau(\hat{\omega}_{\boldsymbol{j}}) = \sum_{\boldsymbol{i}\in I_0(n,r)} \hat{\omega}_{\boldsymbol{i}} \otimes \mathcal{D}_{\boldsymbol{i},\boldsymbol{j}}, \qquad \rho(\hat{\omega}_{\boldsymbol{i}}) = \sum_{\boldsymbol{j}\in I_0(n,r)} \mathcal{D}_{\boldsymbol{i},\boldsymbol{j}} \otimes \hat{\omega}_{\boldsymbol{j}}. \tag{B.1.10}$$

The case $n = r$ has particular importance. The set $I_0(n,n)$ has a unique element $\boldsymbol{i}_{\max} = (1, \dots, n)$. We denote by $\det_q := \mathcal{D}_{\boldsymbol{i}_{\max},\boldsymbol{i}_{\max}}$ (the reason for using the notation $\det_q$ instead of $\det_v$ will become clear in §B.3), and call

[3] Compare $\hat{\omega}_{\boldsymbol{i}}$ with ω_I defined in Example 0.18. Here we use *sequences* with elements in $\{1, \dots, n\}$ as indices; while in Example 0.18, the indices are *subsets* of $\{1, \dots, n\}$.

it the *quantum determinant* for $\mathrm{M}_{n,v}$. We have

$$\begin{aligned}\det_q &= \sum_{w\in\mathfrak{S}_n} (-v)^{-\ell(w)} \chi_{w(1),1}\cdots\chi_{w(n),n} \\ &= \sum_{w\in\mathfrak{S}_n} (-v)^{-\ell(w)} \chi_{1,w(1)}\cdots\chi_{n,w(n)}.\end{aligned} \tag{B.1.11}$$

When $v=1$, the classical determinant for M_n is recovered.

Theorem B.7. *The quantum determinant* $\det_q$ *is a central group-like element in* $A_{n,v}$.

Only the centrality of $\det_q$ is nontrivial, which will be proved at the end of this section.

In general, for $\boldsymbol{i}$ and $\boldsymbol{j}\in I_0(n,r)$ with $r\leqslant n$, the element $\mathcal{D}_{\boldsymbol{i},\boldsymbol{j}}$ will be called a *quantum minor* of $\det_q$. This notion is a generalization of the classical concept of a minor in a determinant.

For $\boldsymbol{i}\in I_0(n,r)$ with $r\leqslant n$, denote by $\boldsymbol{i}'\in I_0(n,n-r)$ the complement of $\boldsymbol{i}$. That is, $\boldsymbol{i}'$ is the unique element in $I_0(n,n-r)$ such that $\boldsymbol{i}\cup\boldsymbol{i}'=\boldsymbol{i}_{\max}$, as sets. For convenience, let $\mathcal{A}_{\boldsymbol{i},\boldsymbol{j}}:=\mathcal{D}_{\boldsymbol{j}',\boldsymbol{i}'}$ for $\boldsymbol{i}$ and $\boldsymbol{j}\in I_0(n,r)$. Let $|\boldsymbol{i}|$ be the sum of ingredients in the sequence $\boldsymbol{i}$, for $\boldsymbol{i}\in I_0(n,r)$.

Now we are ready to prove the following result.

Theorem B.8 (Laplace expansions). *Let* $\boldsymbol{i},\boldsymbol{j}\in I_0(n,r)$ *with* $r\leqslant n$. *Then*

$$\begin{aligned}\delta_{\boldsymbol{i},\boldsymbol{j}}\det_q &= \sum_{\boldsymbol{k}\in I_0(n,r)} (-v)^{|\boldsymbol{j}|-|\boldsymbol{k}|}\mathcal{D}_{\boldsymbol{i},\boldsymbol{k}}\mathcal{A}_{\boldsymbol{k},\boldsymbol{j}} = \sum_{\boldsymbol{k}\in I_0(n,r)} (-v)^{|\boldsymbol{k}|-|\boldsymbol{i}|}\mathcal{A}_{\boldsymbol{i},\boldsymbol{k}}\mathcal{D}_{\boldsymbol{k},\boldsymbol{j}} \\ &= \sum_{\boldsymbol{k}\in I_0(n,r)} (-v)^{|\boldsymbol{j}|-|\boldsymbol{k}|}\mathcal{D}_{\boldsymbol{k},\boldsymbol{i}}\mathcal{A}_{\boldsymbol{j},\boldsymbol{k}} = \sum_{\boldsymbol{k}\in I_0(n,r)} (-v)^{|\boldsymbol{k}|-|\boldsymbol{i}|}\mathcal{A}_{\boldsymbol{k},\boldsymbol{i}}\mathcal{D}_{\boldsymbol{j},\boldsymbol{k}}.\end{aligned}$$

Proof. Using (B.1.10) and Exercise B.5 (and noting that $\hat\omega_{\boldsymbol{k}'}\hat\omega_{\boldsymbol{l}}=0$ if $\boldsymbol{k}\neq\boldsymbol{l}$, for $\boldsymbol{k},\boldsymbol{l}\in I_0(n,r)$), we can make the following calculations:

$$\begin{aligned}\rho(\hat\omega_{\boldsymbol{i}}\hat\omega_{\boldsymbol{j}'}) &= \Bigg(\sum_{\boldsymbol{k}\in I_0(n,r)}\mathcal{D}_{\boldsymbol{i},\boldsymbol{k}}\otimes\hat\omega_{\boldsymbol{k}}\Bigg)\Bigg(\sum_{\boldsymbol{l}\in I_0(n,r)}\mathcal{D}_{\boldsymbol{j}',\boldsymbol{l}'}\otimes\hat\omega_{\boldsymbol{l}'}\Bigg) \\ &= \sum_{\boldsymbol{k}\in I_0(n,r)}\mathcal{D}_{\boldsymbol{i},\boldsymbol{k}}\mathcal{D}_{\boldsymbol{j}',\boldsymbol{k}'}\otimes\hat\omega_{\boldsymbol{k}}\hat\omega_{\boldsymbol{k}'} \\ &= \Bigg(\sum_{\boldsymbol{k}\in I_0(n,r)}(-v)^{r(r+1)/2-|\boldsymbol{k}|}\mathcal{D}_{\boldsymbol{i},\boldsymbol{k}}\mathcal{A}_{\boldsymbol{k},\boldsymbol{j}}\Bigg)\otimes\hat\omega_{\boldsymbol{i}_{\max}}.\end{aligned}$$

On the other hand, since $\hat\omega_{\boldsymbol{i}}\hat\omega_{\boldsymbol{i}'}=(-v)^{r(r+1)/2-|\boldsymbol{i}|}\hat\omega_{\boldsymbol{i}_{\max}}$, by Exercise B.5 again, we have

$$\rho(\hat\omega_{\boldsymbol{i}}\hat\omega_{\boldsymbol{j}'})=\delta_{\boldsymbol{i},\boldsymbol{j}}(-v)^{r(r+1)/2-|\boldsymbol{j}|}\det_q\otimes\hat\omega_{\boldsymbol{i}_{\max}}.$$

Therefore,

$$\delta_{\boldsymbol{i},\boldsymbol{j}} \det_q = \sum_{\boldsymbol{k}\in I_0(n,r)} (-v)^{|\boldsymbol{j}|-|\boldsymbol{k}|} \mathcal{D}_{\boldsymbol{i},\boldsymbol{k}} \mathcal{A}_{\boldsymbol{k},\boldsymbol{j}}.$$

Since $|\boldsymbol{k}| + |\boldsymbol{k}'| = n(n+1)/2$, for all $\boldsymbol{k}$, we also have

$$\delta_{\boldsymbol{i},\boldsymbol{j}} \det_q = \sum_{\boldsymbol{k}\in I_0(n,r)} (-v)^{|\boldsymbol{j}'|-|\boldsymbol{k}'|} \mathcal{D}_{\boldsymbol{i}',\boldsymbol{k}'} \mathcal{A}_{\boldsymbol{k}',\boldsymbol{j}'} = \sum_{\boldsymbol{k}\in I_0(n,r)} (-v)^{|\boldsymbol{k}|-|\boldsymbol{j}|} \mathcal{A}_{\boldsymbol{k},\boldsymbol{i}} \mathcal{D}_{\boldsymbol{j},\boldsymbol{k}}.$$

The other two formulas are proved similarly. □

In the special case $r = 1$, we write $\mathcal{A}_{i,j} = \mathcal{A}_{\boldsymbol{i},\boldsymbol{j}} = \mathcal{D}_{\boldsymbol{j}',\boldsymbol{i}'}$ if $\boldsymbol{i} = \{i\}$ and $\boldsymbol{j} = \{j\}$, which can be called the (i,j)-*minor*. The following corollary generalizes the classical expansions of a determinant along its rows or columns.

Corollary B.9. *Let* $1 \leqslant i, j \leqslant n$. *Then*

$$\begin{aligned}\delta_{i,j} \det_q &= \sum_{k=1}^{n} (-v)^{j-k} \chi_{i,k} \mathcal{A}_{k,j} = \sum_{k=1}^{n} (-v)^{k-i} \mathcal{A}_{i,k} \chi_{k,j} \\ &= \sum_{k=1}^{n} (-v)^{j-k} \chi_{k,i} \mathcal{A}_{j,k} = \sum_{k=1}^{n} (-v)^{k-i} \mathcal{A}_{k,i} \chi_{j,k}.\end{aligned} \tag{B.1.12}$$

Let X be the $n \times n$ matrix $(\chi_{i,j})$, and let A be the matrix $((-v)^{j-i}\mathcal{A}_{i,j})$. The first two expressions in (B.1.12) can be rewritten in matrix form as

$$\mathsf{XA} = \det_q \cdot \mathsf{I}_n = \mathsf{AX}, \tag{B.1.13}$$

where I_n is the $n \times n$ identity matrix. The centrality of $\det_q$ is an easy consequence, since

$$\mathsf{X}\cdot \det_q = \mathsf{X}\cdot(\det_q \cdot \mathsf{I}_n) = \mathsf{XAX} = (\det_q \cdot \mathsf{I}_n)\cdot\mathsf{X} = \det_q \cdot\mathsf{X}.$$

Thus, $\det_q$ commutes with all $\chi_{i,j}$, and so is central.

B.2. Quantum linear groups

As in §5.1, let $\mathsf{Hopf}_{\mathcal{R}}$ be the category of Hopf algebras over $\mathcal{R}$. An object G in the opposite category $(\mathsf{Hopf}_{\mathcal{R}})^{\mathrm{op}}$ is called a quantum group if the corresponding object $\mathcal{O}(G)$, called the *coordinate algebra* of G, in $\mathsf{Hopf}_{\mathcal{R}}$ is a deformation of the coordinate algebra of an affine group scheme over $\mathcal{R}$. Rather than making precise here what we mean by a "deformation," we illustrate the idea by means of important examples.

In this section, we define the quantum linear groups $\mathrm{GL}_{n,v}$ and $\mathrm{SL}_{n,v}$, which are deformations of the classical general and special linear groups GL_n and SL_n. Imitating the classical case, we define two algebras as candidates for "coordinate algebras."

Since the quantum determinant $\det_q$ is central in $A_{n,v}$, we can form the localization of $A_{n,v}$ at the multiplicative set $\{\det_q^r \mid r \in \mathbb{N}\}$ to obtain an algebra $\tilde{A}_{n,v} := A_{n,v}[\det_q^{-1}]$. For the details of localization theory of noncommutative rings, see [**222**]. On the other hand, form the quotient algebra $\bar{A}_{n,v} := A_{n,v}/(\det_q -1)$ of $A_{n,v}$ modulo its ideal generated by the central element $\det_q -1$. The algebras $\tilde{A}_{n,v}$ and $\bar{A}_{n,v}$ are candidates for the coordinate algebras of $\mathrm{GL}_{n,v}$ and $\mathrm{SL}_{n,v}$, respectively. To support the roles of these algebras, we need to define Hopf algebra structures on them. We begin with the following proposition, which summarizes certain fundamental properties of the algebras $\tilde{A}_{n,v}$ and $\bar{A}_{n,v}$.

Proposition B.10. (1) *The algebras $\tilde{A}_{n,v}$ and $\bar{A}_{n,v}$ are integral domains if $\mathcal{R}$ is an integral domain.*

(2) *$A_{n,v}$ is canonically a subalgebra of $\tilde{A}_{n,v}$, while $\bar{A}_{n,v}$ is canonically the quotient algebra of $\tilde{A}_{n,v}$ modulo its ideal generated by* $\det_q -1$.

(3) *Both $\tilde{A}_{n,v}$ and $\bar{A}_{n,v}$ inherit bialgebra structures from $A_{n,v}$.*

Proof. The assertions both that $\tilde{A}_{n,v}$ is an integral domain (under the assumption that $\mathcal{R}$ is an integral domain) and that $A_{n,v}$ is canonically a subalgebra of $\tilde{A}_{n,v}$ follow from the general theory of localization, since $A_{n,v}$ is an integral domain, by Theorem B.2(1), and since $\det_q$ is not a zero divisor. The integral property of $\bar{A}_{n,v}$ (under the assumption that $\mathcal{R}$ is an integral domain) thus follows from an $\mathcal{R}$-algebra isomorphism $\bar{A}_{n,v} \otimes \mathcal{R}[u, u^{-1}] \cong \tilde{A}_{n,v}$, for u an indeterminate over $\mathcal{R}$; see Exercise B.6. Moreover, since the quotient map $\pi\colon A_{n,v} \twoheadrightarrow \bar{A}_{n,v}$ sends $\det_q$ to 1, an invertible element, the universal property of a localization ensures that this quotient map factors through $\tilde{A}_{n,v}$, giving a surjective homomorphism $\tilde{\pi}\colon \tilde{A}_{n,v} \twoheadrightarrow \bar{A}_{n,v}$. Thus, $\bar{A}_{n,v}$ is canonically a quotient of $\tilde{A}_{n,v}$. Any element of $\tilde{A}_{n,v}$ has the form $f \cdot \det_q^{-m}$, for $f \in A_{n,v}$ and $m \in \mathbb{N}$. Since $\tilde{\pi}(f \cdot \det_q^{-m}) = \pi(f)$, $\operatorname{Ker} \tilde{\pi}$ is the ideal generated by $\det_q -1$. This completes the proof of (1) and (2).

For (3), there is an algebra homomorphism $A_{n,v} \xrightarrow{\Delta} A_{n,v} \otimes A_{n,v} \to \tilde{A}_{n,v} \otimes \tilde{A}_{n,v}$, sending $\det_q$ to $\det_q \otimes \det_q$, which is invertible in $\tilde{A}_{n,v} \otimes \tilde{A}_{n,v}$. Again, the universal property of the localization ensures the unique extension of Δ to an algebra homomorphism $\tilde{A}_{n,v} \to \tilde{A}_{n,v} \otimes \tilde{A}_{n,v}$ which is clearly a comultiplication. Similarly, the counit extends. Now consider $\bar{A}_{n,v}$. In $A_{n,v}$, $\Delta(\det_q -1) = \det_q \otimes (\det_q -1) + (\det_q -1) \otimes 1$ and $\varepsilon(\det_q -1) = 0$, so the ideal of $A_{n,v}$ generated by $\det_q -1$ is a biideal. Thus, the bialgebra structure on $A_{n,v}$ induces a bialgebra structure on the quotient algebra $\bar{A}_{n,v}$. □

To define antipodes for the algebras $\tilde{A}_{n,v}$ and $\bar{A}_{n,v}$, we need the following lemma.

Lemma B.11. *The following commutation formulas hold in $A_{n,v}$:*

$$\begin{aligned} \mathcal{A}_{k,i}\mathcal{A}_{k,j} &= v^{-1}\mathcal{A}_{k,j}\mathcal{A}_{k,i}, && \text{for all } i > j; \\ \mathcal{A}_{k,i}\mathcal{A}_{l,i} &= v^{-1}\mathcal{A}_{l,i}\mathcal{A}_{k,i}, && \text{for all } k > l; \\ \mathcal{A}_{k,i}\mathcal{A}_{l,j} &= \mathcal{A}_{l,j}\mathcal{A}_{k,i}, && \text{if } k > l \text{ and } i < j; \\ \mathcal{A}_{k,i}\mathcal{A}_{l,j} - \mathcal{A}_{l,j}\mathcal{A}_{k,i} &= (v^{-1} - v)\mathcal{A}_{l,i}\mathcal{A}_{k,j}, && \text{if } k > l \text{ and } i > j. \end{aligned} \tag{B.2.1}$$

Proof. Again, let $\mathsf{A} = ((-v)^{j-i}\mathcal{A}_{i,j})$. From the definition of the linear transformation $\mathscr{R}_v$ in (B.1.2), the matrix R_v given in Proposition B.1 is invertible with inverse $\mathsf{R}_{v^{-1}}$. Also, as in Proposition B.1, if we form $n^2 \times n^2$ matrices $\mathsf{P}_1 := \mathsf{P} \otimes \mathsf{I}_n$ and $\mathsf{P}_2 := \mathsf{I}_n \otimes \mathsf{P}$, for any $n \times n$ matrix P, then $(\mathsf{PQ})_1 = \mathsf{P}_1\mathsf{Q}_1$ and $(\mathsf{PQ})_2 = \mathsf{P}_2\mathsf{Q}_2$, for any $n \times n$ matrices P and Q. With these observations, and using (B.1.4) and (B.1.13), we have

$$\begin{aligned} \mathsf{R}_{v^{-1}}\mathsf{A}_1\mathsf{A}_2 &= \mathsf{R}_{v^{-1}}\mathsf{A}_1\mathsf{A}_2(\mathsf{R}_v\mathsf{X}_1\mathsf{X}_2)\mathsf{A}_2\mathsf{A}_1\mathsf{R}_{v^{-1}}\det_q^{-2} \\ &= \mathsf{R}_{v^{-1}}\mathsf{A}_1\mathsf{A}_2(\mathsf{X}_2\mathsf{X}_1\mathsf{R}_v)\mathsf{A}_2\mathsf{A}_1\mathsf{R}_{v^{-1}}\det_q^{-2} = \mathsf{A}_2\mathsf{A}_1\mathsf{R}_{v^{-1}}. \end{aligned}$$

This means that the set of elements $\{(-v)^{j-i}\mathcal{A}_{i,j}\}$ obeys the same commutation formulas as the set $\{\chi_{i,j}\}$, with v replacing by v^{-1}. This proves the lemma. □

Now we are ready to define the antipodes for $\tilde{A}_{n,v}$ and $\bar{A}_{n,v}$.

Theorem B.12. (1) *Define*

$$\gamma(\chi_{i,j}) = (-v)^{j-i}\mathcal{A}_{i,j}\det_q^{-1}. \tag{B.2.2}$$

Then γ extends to an algebra anti-endomorphism of $\tilde{A}_{n,v}$, providing an antipode that makes $\tilde{A}_{n,v}$ into a Hopf algebra.

(2) *The antipode γ maps the ideal of $\tilde{A}_{n,v}$ generated by* $\det_q - 1$ *into itself. Thus, $\bar{A}_{n,v}$ is a quotient Hopf algebra of $\tilde{A}_{n,v}$ with antipode γ sending $\chi_{i,j}$ to $(-v)^{j-i}\mathcal{A}_{i,j}$.*

Proof. (1) By Lemma B.11, the assignment (B.2.2) clearly extends to an algebra anti-homomorphism $\gamma\colon A_{n,v} \to \tilde{A}_{n,v}$. The antipode law (5.1.8) (as maps $A_{n,v} \to \tilde{A}_{n,v}$ at this moment) follows directly from an easy application of (B.1.12). In particular, $\det_q \gamma(\det_q) = 1 = \gamma(\det_q)\det_q$. That is, $\gamma(\det_q) = \det_q^{-1}$, which is an invertible element in $\tilde{A}_{n,v}$. Hence γ extends to an algebra anti-endomorphism $\gamma\colon \tilde{A}_{n,v} \to \tilde{A}_{n,v}$ satisfying the antipode law. This proves (1).

(2) Since $\gamma(\det_q - 1) = \det_q^{-1} - 1 = -\det_q^{-1}(\det_q - 1)$, the ideal of $\tilde{A}_{n,v}$ generated by $\det_q - 1$ is stable under γ. □

The quantum groups with coordinate algebras $\tilde{A}_{n,v}$ and $\bar{A}_{n,v}$ are denoted by $\mathrm{GL}_{n,v}$ and $\mathrm{SL}_{n,v}$ and they are called the *quantum general linear group* (of degree n) and the *quantum special linear group* (of degree n), respectively. Thus, we have $\mathcal{O}(\mathrm{GL}_{n,v}) = \tilde{A}_{n,v}$ and $\mathcal{O}(\mathrm{SL}_{n,v}) = \bar{A}_{n,v}$. When $v = 1$, the classical general linear group GL_n and special linear group SL_n are recovered.

As in the classical case, any quotient Hopf algebra of $\mathcal{O}(\mathrm{GL}_{n,v})$ (resp., $\mathcal{O}(\mathrm{SL}_{n,v})$) defines a *closed* subgroup of $\mathrm{GL}_{n,v}$ (resp., $\mathrm{SL}_{n,v}$). In particular, $\mathrm{SL}_{n,v}$ is a closed subgroup of $\mathrm{GL}_{n,v}$. We now briefly introduce certain interesting closed subgroups of $\mathrm{GL}_{n,v}$ and $\mathrm{SL}_{n,v}$.

For convenience, let G_v be $\mathrm{GL}_{n,v}$ or $\mathrm{SL}_{n,v}$ (and thus $\mathcal{O}(G_v) = \tilde{A}_{n,v}$ or $\bar{A}_{n,v}$). First, we mention three important closed subgroups of G_v.

Proposition B.13. (1) *The ideal of $\mathcal{O}(G_v)$ generated by all $\chi_{i,j}$ with $i > j$ (resp., with $i < j$) is a Hopf ideal. Thus, the corresponding quotient Hopf algebra defines a closed subgroup of G_v called the upper (resp., lower) triangular Borel subgroup and denoted B_v^+ (resp., B_v^-).*

(2) *The ideal of $\mathcal{O}(G_v)$ generated by all $\chi_{i,j}$ with $i \neq j$ is a Hopf ideal. Thus, the corresponding quotient Hopf algebra defines a closed subgroup of G_v, called the diagonal maximal torus and denoted T_v.*

The proof of the proposition is easy, and is left as Exercise B.7.

Remark B.14. Constructions of certain parabolic subgroups and their Levi decompositions are also available. We will not go into the details.

Remarks B.15. (1) Based on these closed subgroups, it is possible to talk about the root system and weights for G_v. In particular, the weight lattice of G_v is $X(T_v)$, the character group of the (ordinary) torus T_v, which is generated as a multiplicative group by $\chi_{i,i}$, for $1 \leqslant i \leqslant n$. If, for psychological reasons, we write $X(T_v)$ additively, then we write ε_i for $\chi_{i,i}$ if it is regarded as a character of T_v; cf. Example A.36. This gives the root system $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, the positive roots $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$, and the simple roots $\Pi = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \ldots, n-1\}$. This is a root system of type A_{n-1}. The fundamental weights in the lattice $X(T_v)$ are $\varpi_1 := \varepsilon_1$, $\varpi_2 := \varepsilon_1 + \varepsilon_2$, ..., $\varpi_{n-1} = \varepsilon_1 + \cdots + \varepsilon_{n-1}$; while a weight $\sum_{i=1}^n \mu_i \varepsilon_i$, $\mu_i \in \mathbb{Z}$, is dominant if and only if $\mu_1 \geqslant \cdots \geqslant \mu_{n-1} \geqslant \mu_n$.

(2) At least when $\mathcal{R} = \Bbbk$ is a field, the main results in the representation theory of the reductive algebraic groups (sketched in §A.5) can be carried over to the quantum linear groups, developed in terms of comodules over the coordinate algebras. For example, the simple modules for Borel subgroups are 1-dimensional, determined by a character of T_v, while a G_v-module induced from a 1-dimensional B_v^--module, if nonzero, has a simple socle, and all simple G_v-modules can be obtained in this way. Through this construction, we arrive at a classification of simple G_v-modules by highest T_v-weights. The simple G_v-modules are indexed by $X(T_v)^+$, the set of dominant weights in $X(T_v)$. The theory of cohomology of vector bundles on flag

varieties can be generalized to the quantum case in terms of the derived functors of the induction from B_v^--modules to G_v-modules. Costandard modules $\nabla(\xi)$, $\xi \in X(T_v)^+$, can be obtained using induction from the Borel subgroup B_v^-, as in the classical case (see §A.7). Based on these results, we obtain highest weight categories (in the sense of Definition C.8) of finite dimensional G_v-modules; see Example C.9 for the analogous situation for a semisimple group.

In the remainder of this section, we assume that v is a primitive ℓth root of 1, for an *odd* positive integer ℓ. We will exhibit an *infinitesimal theory* for $G_v = \mathrm{GL}_{n,v}$ or $\mathrm{SL}_{n,v}$.

We have the following result.

Lemma B.16. *Let v be a primitive ℓth root of 1 with ℓ odd.*

(1) *For any i, j, the element $\chi_{i,j}^\ell$ is in the center of $\mathcal{O}(\mathrm{M}_{n,v})$, $\mathcal{O}(\mathrm{GL}_{n,v})$, and $\mathcal{O}(\mathrm{SL}_{n,v})$.*

(2) *In $\mathcal{O}(\mathrm{M}_{n,v})$, $\mathcal{O}(\mathrm{GL}_{n,v})$, and $\mathcal{O}(\mathrm{SL}_{n,v})$ we have*

$$\Delta(\chi_{i,j}^\ell) = \sum_{k=1}^{n} \chi_{i,k}^\ell \otimes \chi_{k,j}^\ell, \quad \varepsilon(\chi_{i,j}^\ell) = \delta_{i,j}.$$

Proof. The proof requires only routine verification, involving the Gaussian polynomials $\left[\!\left[{m \atop n} \right]\!\right]$ defined in §0.4. We leave it as Exercise B.8(1). □

We now discuss the quantum and the classical linear groups simultaneously. Denote the coordinate functions of the classical M_n, GL_n, and SL_n by $x_{i,j}$, which differs from the notation $\chi_{i,j}$ for the coordinate functions of quantum $\mathrm{M}_{n,v}$, $\mathrm{GL}_{n,v}$, and $\mathrm{SL}_{n,v}$. We have the following result.

Proposition B.17. (1) *There is a bialgebra embedding*

$$\phi\colon \mathcal{O}(\mathrm{M}_n) \longrightarrow \mathcal{O}(\mathrm{M}_{n,v}), \quad x_{i,j} \longmapsto \chi_{i,j}^\ell.$$

(2) *The bialgebra embedding ϕ extends to Hopf algebra embeddings*

$$\phi\colon \mathcal{O}(\mathrm{GL}_n) \longrightarrow \mathcal{O}(\mathrm{GL}_{n,v}) \;\; \textit{and} \;\; \phi\colon \mathcal{O}(\mathrm{SL}_n) \longrightarrow \mathcal{O}(\mathrm{SL}_{n,v}).$$

Proof (sketch). (1) follows directly from Lemma B.16. The extensions in (2) can be obtained from the facts that $\phi(\det) = \det_q^\ell$ and $\gamma \circ \phi(x_{i,j}) = \phi \circ \gamma(x_{i,j})$. The detailed verifications are also left to the reader; see Exercise B.8(2)–(3). □

The homomorphisms ϕ defined in Proposition B.17 are called the *(quantum) co-Frobenius morphisms.* Dually, we have an epimorphism of "quantum semigroups" $F\colon \mathrm{M}_{n,v} \to \mathrm{M}_n$, and epimorphisms of quantum groups

$$F\colon \mathrm{GL}_{n,v} \longrightarrow \mathrm{GL}_n \;\; \text{and} \;\; F\colon \mathrm{SL}_{n,v} \longrightarrow \mathrm{SL}_n.$$

All these morphisms are called *(quantum) Frobenius morphisms.*

Again, let $G_v = \mathrm{GL}_{n,v}$ or $\mathrm{SL}_{n,v}$, and correspondingly, $G = \mathrm{GL}_n$ or SL_n. We have the Frobenius morphism $F\colon G_v \to G$ or, equivalently, the co-Frobenius morphism $\phi\colon \mathcal{O}(G) \to \mathcal{O}(G_v)$. The Frobenius morphism has a "kernel" (in the categorical sense) $G_v^{\mathrm{inf}} := \operatorname{Ker} F$, which is a closed subgroup of G_v defined by the ideal generated by all $x_{i,j}^{\ell} - \delta_{i,j}$. (A direct verification shows that the ideal generated by all $x_{i,j}^{\ell} - \delta_{i,j}$ is a Hopf ideal. Or, instead, this ideal can be regarded as the extension of the augmentation ideal $\operatorname{Ker}\varepsilon$ of $\mathcal{O}(G)$, while the latter is clearly generated by $x_{i,j} - \delta_{i,j}$.) The algebra $\mathcal{O}(G_v^{\mathrm{inf}})$ is free and of finite rank over $\mathcal{R}$, so G_v^{inf} is usually called the *infinitesimal quantum linear group.*

The exact sequence

$$1 \longrightarrow G_v^{\mathrm{inf}} \longrightarrow G_v \longrightarrow G \longrightarrow 1$$

(where 1 stands for the group with only the identity) gives a fundamental framework in which to discuss representations of G_v through representations of the infinitesimal quantum group G_v^{inf} (which are just representations of an algebra of finite rank) and representations of the algebraic group G. An ample theory has resulted from this setting. As an example, we conclude with the tensor product theorem.

A G-module ($= \mathcal{O}(G)$-comodule) V can be regarded as a G_v-module ($= \mathcal{O}(G_v)$-comodule) by pulling back along the Frobenius morphism F (that is, by pushing out along the co-Frobenius morphism ϕ). This G_v-module is called the *(quantum) Frobenius twist*, and is denoted by $V^{[F]}$.[4]

A dominant weight $\zeta \in X(T_v)^+$ is called ℓ-restricted if ζ is a linear combination of fundamental weights with all coefficients $< \ell$. A dominant weight ξ can be written in the (one-step) ℓ-adic expression as $\xi = \xi_{-1} + \ell\xi'$ with ξ' dominant and ξ_{-1} ℓ-restricted.

Theorem B.18 (tensor product theorem). *Suppose $\mathcal{R} = \Bbbk$ is a field. If $\xi \in X(T_v)^+$ with the (one step) ℓ-adic expression $\xi = \xi_{-1} + \ell\xi'$, then we have the following G_v-module isomorphism*

$$L_v(\xi) \cong L_v(\xi_{-1}) \otimes L(\xi')^{[F]},$$

where $L_v(\zeta)$ is the simple G_v-module with highest weight $\zeta \in X(T_v)^+$, while $L(\zeta)$ is the simple G-module with highest weight $\zeta \in X(T)^+$.

Remark B.19. If the ground field $\Bbbk$ has prime characteristic p, then, by Theorem A.52, the G-module $L(\xi')$ can be decomposed as $L(\xi_0) \otimes L(\xi_1)^{(p)} \otimes$

[4] Here, we use the same term and notation as in the similar situation in the representation theory of algebraic groups in prime characteristic; see §A.6. However, there is a key difference between the Frobenius twist in the two contexts. The twist here depends on the parameter q rather than the characteristic of the ground field.

$\cdots \otimes L(\xi_r)^{(p^r)}$, where $\xi' = \xi_0 + p\xi_1 + \cdots + p^r\xi_r$ is the p-adic expression of ξ' (with $\xi_0, \ldots, \xi_r$ p-restricted and $\xi_r \neq 0$). Therefore, in this case, we have an ℓ-p-mixed tensor product decomposition for $L_v(\xi)$:

$$L_v(\xi) \cong L_v(\xi_{-1}) \otimes L(\xi_0)^{[F]} \otimes L(\xi_1)^{(p)[F]} \otimes \cdots \otimes L(\xi_r)^{(p^r)[F]}.$$

B.3. Multiparameter quantum matrix spaces

In §B.1 we investigated the quantum matrix space $\mathrm{M}_{n,v}$ with a single parameter $v \in \mathcal{R}$. In fact, $\mathrm{M}_{n,v}$ is a special case of more general structures, namely, multiparameter quantum matrix spaces $\mathrm{M}^q_{n,\boldsymbol{v}}$. The most general setting involves $n(n-1)/2+1$ parameters. For convenience, we consider an $n \times n$ parameter matrix $\boldsymbol{v} = (v_{i,j})$ in which $v_{i,j} \in \mathcal{R}$ with $v_{i,i} = 1$, for all i, and $v_{i,j}v_{j,i} = 1$, for all i, j. Thus, $\boldsymbol{v}$ determines $n(n-1)/2$ free parameters. Let $0 \neq q \in \mathcal{R}$ be the extra parameter, called the *dominant parameter.*

Now let $A^q_{n,\boldsymbol{v}}$ be the associative algebra over $\mathcal{R}$ with generators $\chi_{i,j}$ $(i, j = 1, \ldots, n)$ and relations

$$\begin{aligned}
\chi_{k,i}\chi_{k,j} &= v_{i,j}^{-1}\chi_{k,j}\chi_{k,i}, && \text{for all } i>j;\\
\chi_{k,i}\chi_{l,i} &= qv_{k,l}\chi_{l,i}\chi_{k,i}, && \text{for all } k>l;\\
\chi_{k,i}\chi_{l,j} &= qv_{i,j}^{-1}v_{k,l}\chi_{l,j}\chi_{k,i}, && \text{if } k>l \text{ and } i<j;\\
v_{k,l}^{-1}\chi_{k,i}\chi_{l,j} - v_{i,j}^{-1}\chi_{l,j}\chi_{k,i} &= (q-1)\chi_{l,i}\chi_{k,j}, && \text{if } k>l \text{ and } i>j.
\end{aligned} \tag{B.3.1}$$

Definition B.20. The "space" $\mathrm{M}^q_{n,\boldsymbol{v}}$ with coordinate algebra $\mathcal{O}(\mathrm{M}^q_{n,\boldsymbol{v}}) = A^q_{n,\boldsymbol{v}}$ is called the *multiparameter quantum matrix space* of order n with parameter matrix $\boldsymbol{v}$ and dominant parameter q.

We now define a bialgebra structure on $A^q_{n,\boldsymbol{v}}$ in Theorem B.28 to support the above definition. If we put $v_{i,j} = v$, for all $i < j$ and $q = v^2$, then the standard quantum matrix space $\mathrm{M}_{n,v}$ defined in (B.1.1) is obtained.

Although we have defined a variety of quantum matrix spaces, we have, in fact, not wandered too far. We will show that, once the dominant parameter q is fixed, we can, given two parameter matrices $\boldsymbol{v}$ and $\boldsymbol{u}$, "twist" (in a precise sense) the algebra $A^q_{n,\boldsymbol{v}}$ to the algebra $A^q_{n,\boldsymbol{u}}$. In addition, the coalgebra structures on the $A^q_{n,\boldsymbol{v}}$, for a fixed q, which we have not defined yet, are the same! Thus, one can "twist" a standard quantum matrix space to obtain all multiparameter quantum matrix spaces.

Example B.21 (Two-parameter quantum matrix spaces). A two-parameter special subfamily of the multiparameter quantum matrix spaces can be obtained by considering all $A^q_{n,\boldsymbol{a}}$, where the parameter matrix $\boldsymbol{a}$ has the form $\boldsymbol{a} = (a_{i,j})$ with $a_{i,j} = a$, where $a \in \mathcal{R}$, for all $i < j$, is a fixed invertible element. In the literature, this system is often parametrized by parameters a and $b = q/a$. (Thus, the product ab of the two parameters a and b gives the dominant parameter q.) We denote the $A^q_{n,\boldsymbol{a}}$ as above by $A^{a,b}_n$ in terms of the parameters a and b. From (B.3.1), the defining relations of $A^{a,b}_n$ are

$$\begin{aligned} \chi_{k,i}\chi_{k,j} &= a\chi_{k,j}\chi_{k,i}, && \text{for all } i>j;\\ \chi_{k,i}\chi_{l,i} &= b\chi_{l,i}\chi_{k,i}, && \text{for all } k>l;\\ \chi_{k,i}\chi_{l,j} &= a^{-1}b\chi_{l,j}\chi_{k,i}, && \text{if } k>l \text{ and } i<j;\\ \chi_{k,i}\chi_{l,j}-\chi_{l,j}\chi_{k,i} &= (b-a^{-1})\chi_{l,i}\chi_{k,j}, && \text{if } k>l \text{ and } i>j. \end{aligned} \tag{B.3.2}$$

One can check directly (Exercise B.9(1)) that $A^{a,b}_n$ is a bialgebra with comultiplication Δ and counit ε defined by $\Delta(\chi_{i,j}) = \sum_k \chi_{i,k} \otimes \chi_{k,j}$ and $\varepsilon(\chi_{i,j}) = \delta_{i,j}$. If $a = v$ is a square root of q, then (B.1.1) is recovered, and we obtain $A^{v,v}_n = A_{n,v}$, as defined in §B.1.

The invariance of the coalgebra structures on $A^{a,b}_n$ with $ab = q$ fixed, called *hyperbolic invariance*, can be proved directly (i.e., without using the concept of 2-cocycles introduced below). We indicate a proof in Exercise B.9(2) so that the reader may gain an intuitive feeling about the invariance of the coalgebras.

We now return to general multiparameter quantum matrix spaces. The twisting of multiparameter quantum matrix spaces depends on the 2nd cohomology group of a suitable abelian group with coefficients in $\mathcal{R}^\times$, the multiplication group of invertible elements in $\mathcal{R}$. We begin by reviewing the relevant definitions.

Definition B.22. Let Λ be an abelian group written additively. A mapping $\zeta\colon \Lambda \times \Lambda \to \mathcal{R}^\times$ is called a 2-*cocycle* of Λ if it satisfies

$$\zeta(\lambda, \mu + \iota)\zeta(\mu, \iota) = \zeta(\lambda, \mu)\zeta(\lambda + \mu, \iota), \quad \text{for all } \lambda, \mu, \iota \in \Lambda. \tag{B.3.3}$$

A mapping $\zeta\colon \Lambda\times\Lambda \to \mathcal{R}^\times$ is called a 2-*coboundary* if there exists a mapping $\eta\colon \Lambda \to \mathcal{R}^\times$ such that

$$\zeta(\lambda, \mu) = \eta(\lambda)\eta(\mu)\eta(\lambda + \mu)^{-1}, \quad \text{for all } \lambda, \mu \in \Lambda. \tag{B.3.4}$$

The 2-cocycles of Λ with values in $\mathcal{R}^\times$ form an abelian group under function multiplication, and the set of 2-coboundaries is a subgroup. The quotient group, denoted $H^2(\Lambda, \mathcal{R}^\times)$, is called the *2nd cohomology group* of Λ with coefficients in $\mathcal{R}^\times$, and the coset of a 2-cocycle ζ with respect to the subgroup of 2-coboundaries is called the *cohomology class* of ζ. If 2-cocycles ζ and ξ belong to the same cohomology class, then they are termed cohomologous.

We call a 2-cocycle (a 2-coboundary) ζ *unitary* if, in addition,

$$\zeta(0,0) = 1.$$

Clearly, every 2-cocycle is cohomologous to a unitary 2-cocycle. In fact, if ξ is a 2-cocycle, then $\zeta\colon \Lambda \times \Lambda \to \mathcal{R}^\times$ defined by $\zeta(\lambda, \mu) = \xi(0,0)^{-1}\xi(\lambda, \mu)$ is a unitary 2-cocycle cohomologous to ξ (since a constant map $\Lambda \times \Lambda \to \mathcal{R}^\times$ is clearly a 2-coboundary). Moreover, it is clear that if two unitary 2-cocycles ξ and ζ are cohomologous, then $\xi\zeta^{-1}$ is a unitary 2-coboundary. It follows that $H^2(\Lambda, \mathcal{R}^\times)$ can be obtained by forming the quotient group of the group of unitary 2-cocyles modulo its subgroup of unitary 2-coboundaries.

We now explain how a 2-cocycle of Λ twists an algebra graded by Λ.

Suppose we are given an $\mathcal{R}$-algebra A which is graded by an abelian group Λ: $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ with $A_\lambda A_\mu \subseteq A_{\lambda+\mu}$. Let $\zeta \colon \Lambda \times \Lambda \to \mathcal{R}^\times$ be a unitary 2-cocycle. Define a new binary product $*$ on A by

$$x*y = \zeta(\lambda, \mu)xy, \quad \text{for all } x \in A_\lambda,\ y \in A_\mu. \tag{B.3.5}$$

Clearly, $*$ is $\mathcal{R}$-bilinear on $A_\lambda \times A_\mu$, for any $\lambda, \mu \in \Lambda$, and hence it defines a bilinear mapping $*\colon A \times A \to A$.

Proposition B.23. *Let $A = (A, \cdot\,)$ be an $\mathcal{R}$-algebra graded by an abelian group Λ as above. Then:*

(1) *For any unitary 2-cocycle ζ, A becomes an associative algebra graded by Λ under the operation $*$ defined by* (B.3.5). *We also have $x*y = xy$ if $x \in A_0$ or $y \in A_0$. In particular, the identity 1 of the algebra $A = (A, \cdot\,)$ is also the identity of the algebra $A^\zeta := (A, *)$.*

(2) *For unitary 2-cocycles ζ and ξ, $(A^\zeta)^\xi = A^{\zeta\xi}$. In particular, $(A^\zeta)^{\zeta^{-1}} = A$.*

(3) *If ζ is a unitary 2-coboundary as in* (B.3.4), *then there is an algebra isomorphism $\phi\colon A^\zeta \xrightarrow{\sim} A$ defined by $\phi(x) = \eta(\lambda)x$, for $x \in A_\lambda$. Therefore, given a unitary 2-cocycle ζ, the algebra A^ζ is determined, up to isomorphism, by the cohomology class of ζ.*

(4) *If $\psi\colon A \to B$ is a homomorphism of Λ-graded algebras, then $\psi\colon A^\zeta \to B^\zeta$, for any unitary 2-cocycle ζ of Λ, is also a graded algebra homomorphism.*

Proof. (1) Consider $x \in A_\lambda$, $y \in A_\mu$, and $z \in A_\iota$, for $\lambda, \mu, \iota \in \Lambda$. We have

$$(x*y)*z = \big(\zeta(\lambda,\mu)xy\big)*z = \zeta(\lambda,\mu)\zeta(\lambda+\mu,\iota)(xy)z, \text{ and}$$
$$x*(y*z) = x*\big(\zeta(\mu,\iota)yz\big) = \zeta(\lambda,\mu+\iota)\zeta(\mu,\iota)x(yz).$$

Using (B.3.3), we obtain that $(x*y)*z = x*(y*z)$.

By assumption, $\zeta(0,0) = 1$. We see from (B.3.3) that $\zeta(0,\iota) = \zeta(\lambda,0) = 1$, for $\iota, \lambda \in \Lambda$. This means that $x*y = xy$ if $x \in A_0$ or $y \in A_0$. In particular, the identity 1 of A serves as the identity in A^ζ.

(2) These statements are obvious from the definitions.

(3) If ζ satisfies (B.3.4), then, for $x \in A_\lambda$ and $y \in A_\mu$,

$$\phi(x*y) = \eta(\lambda+\mu)(x*y) = \eta(\lambda)\eta(\mu)xy = \phi(x)\phi(y).$$

Also, since ζ is unitary, $\eta(0) = 1$ from (B.3.4). Thus, $\phi(1) = 1$. Since ϕ is obviously bijective, it gives an isomorphism $A^\zeta \to A$.

(4) We have

$$\psi(x*y) = \zeta(\lambda,\mu)\psi(xy) = \zeta(\lambda,\mu)\psi(x)\psi(y) = \psi(x)*\psi(y).$$

Also, the identity element is preserved under twisting. Therefore, ψ is an algebra homomorphism $A^\zeta \to B^\zeta$. □

Now let

$$\mathcal{V} = \{\boldsymbol{v} = (v_{i,j})_{n\times n} \mid v_{i,j} \in \mathcal{R},\ v_{i,i} = 1 = v_{i,j}v_{j,i},\ \text{for all } i,j\}.$$

We define a *componentwise multiplication* on $\mathcal{V}$, that is, for $\boldsymbol{v} = (v_{i,j})$ and $\boldsymbol{u} = (u_{i,j})$, define $\boldsymbol{vu} = (v_{i,j}u_{i,j})$. Clearly, the set $\mathcal{V}$ is an abelian group under this multiplication, in which the matrix with all entries being 1 is the identity, and the inverse of $\boldsymbol{v} = (v_{i,j})$ is $\boldsymbol{v}^{-1} = (v_{i,j}^{-1})$.

Let $\Lambda := \mathbb{Z}^n$ be the free abelian group with basis $\{\theta_1,\dots,\theta_n\}$. The free algebra generated by all $\chi_{i,j}$ is $\Lambda\times\Lambda$-graded by giving the generator $\chi_{i,j}$ the grade (θ_i,θ_j). Since the relations (B.3.1) are homogeneous with respect to this grading, we obtain a $\Lambda\times\Lambda$-graded algebra structure on $A^q_{n,\boldsymbol{v}}$.

If $\zeta\colon \Lambda\times\Lambda \to \mathcal{R}^\times$ is a 2-cocycle of Λ, we put $\pi_{i,j}(\zeta) = \zeta(\theta_i,\theta_j)\zeta(\theta_j,\theta_i)^{-1}$. Clearly, the $n\times n$ matrix $\boldsymbol{\pi}(\zeta) = \big(\pi_{i,j}(\zeta)\big) \in \mathcal{V}$, and $\boldsymbol{\pi}$ defines a group homomorphism from the group of 2-cocycles of Λ to the group $\mathcal{V}$. Moreover, since any 2-coboundary ζ of Λ is symmetric in the sense that $\zeta(\lambda,\mu) = \zeta(\mu,\lambda)$ (by (B.3.4)), the homomorphism factors through the cohomology group $H^2(\Lambda,\mathcal{R}^\times)$, giving a group homomorphism $\boldsymbol{\pi}\colon H^2(\Lambda,\mathcal{R}^\times) \to \mathcal{V}$.

Lemma B.24. *The homomorphism* $\boldsymbol{\pi}\colon H^2(\Lambda,\mathcal{R}^\times) \to \mathcal{V}$ *is surjective.*

Proof. Let $\boldsymbol{v} = (v_{i,j}) \in \mathcal{V}$. Define a function $\zeta_{\boldsymbol{v}}\colon \Lambda\times\Lambda \to \mathcal{R}^\times$ by

$$\zeta_{\boldsymbol{v}}\Big(\sum_i a_i\theta_i, \sum_j b_j\theta_j\Big) = \prod_{i>j} v_{i,j}^{a_ib_j}.$$

An easy calculation shows that $\zeta_{\boldsymbol{v}}$ is a unitary 2-cocycle, and $\boldsymbol{\pi}(\zeta_{\boldsymbol{v}}) = \boldsymbol{v}$. □

Remark B.25. The homomorphism $\boldsymbol{\pi}\colon H^2(\Lambda,\mathcal{R}^\times) \to \mathcal{V}$ is, in fact, an isomorphism; see Exercise B.14.

Clearly, if ζ and ξ are 2-cocycles of Λ, then the mapping $(\zeta,\xi)\colon (\Lambda\times\Lambda)\times(\Lambda\times\Lambda) \to \mathcal{R}^\times$ defined by

$$(\zeta,\xi)((\lambda_1,\mu_1),(\lambda_2,\mu_2)) = \zeta(\lambda_1,\lambda_2)\xi(\mu_1,\mu_2), \quad \text{for all } \lambda_i,\mu_i \in \Lambda,$$

is a 2-cocycle of $\Lambda\times\Lambda$. In particular, (ζ,ζ^{-1}) is a unitary 2-cocycle, since

$$(\zeta,\zeta^{-1})((0,0),(0,0)) = \zeta(0,0)\zeta(0,0)^{-1} = 1.$$

The algebra $A^q_{n,\boldsymbol{v}}$ is graded by $\Lambda \times \Lambda$, so we can twist it by (ζ, ζ^{-1}), for any 2-cocycle ζ of Λ, to obtain $(A^q_{n,\boldsymbol{v}})^{(\zeta,\zeta^{-1})}$.

Theorem B.26. *Let $\zeta\colon \Lambda \times \Lambda \to \mathcal{R}^\times$ be a 2-cocycle of Λ. For $\boldsymbol{v} \in \mathcal{V}$ and $q \in \mathcal{R}^\times$, $(A^q_{n,\boldsymbol{v}})^{(\zeta,\zeta^{-1})} \cong A^q_{n,\boldsymbol{\pi}(\zeta)\boldsymbol{v}}$.*

Proof. We must verify that the generators $\chi_{i,j}$ of $A^q_{n,\boldsymbol{v}}$ satisfy, under the $*$ multiplication, the relations (B.3.1), where $\boldsymbol{\pi}(\zeta)\boldsymbol{v}$ takes the place of $\boldsymbol{v}$. Here, we verify the fourth relation.

Suppose $k > l$ and $i > j$. For convenience, write $\pi_{i,j} = \pi_{i,j}(\zeta)$. We have

$$\begin{aligned}
&\pi_{k,l}^{-1}v_{k,l}^{-1}\chi_{k,i}*\chi_{l,j} - \pi_{i,j}^{-1}v_{i,j}^{-1}\chi_{l,j}*\chi_{k,i}\\
&\quad= \pi_{k,l}^{-1}v_{k,l}^{-1}\zeta(\theta_k,\theta_l)\zeta(\theta_i,\theta_j)^{-1}\chi_{k,i}\chi_{l,j} - \pi_{i,j}^{-1}v_{i,j}^{-1}\zeta(\theta_l,\theta_k)\zeta(\theta_j,\theta_i)^{-1}\chi_{l,j}\chi_{k,i}\\
&\quad= \zeta(\theta_l,\theta_k)\zeta(\theta_i,\theta_j)^{-1}v_{k,l}^{-1}\chi_{k,i}\chi_{l,j} - \zeta(\theta_l,\theta_k)\zeta(\theta_i,\theta_j)^{-1}v_{i,j}^{-1}\chi_{l,j}\chi_{k,i}\\
&\quad= \zeta(\theta_l,\theta_k)\zeta(\theta_i,\theta_j)^{-1}(v_{k,l}^{-1}\chi_{k,i}\chi_{l,j} - v_{i,j}^{-1}\chi_{l,j}\chi_{k,i})\\
&\quad= (q-1)\zeta(\theta_l,\theta_k)\zeta(\theta_i,\theta_j)^{-1}\chi_{l,i}\chi_{k,j}\\
&\quad= (q-1)\chi_{l,i}*\chi_{k,j},
\end{aligned}$$

as required. The other verifications are left to the reader in Exercise B.10.

Therefore, we now have defined a natural algebra homomorphism $\beta_{\boldsymbol{v},\zeta}$: $A^q_{n,\boldsymbol{\pi}(\zeta)\boldsymbol{v}} \longrightarrow \big(A^q_{n,\boldsymbol{v}}\big)^{(\zeta,\zeta^{-1})}$. By Proposition B.23(2) and (4),

$$\begin{aligned}
A^q_{n,\boldsymbol{v}} = A^q_{n,\boldsymbol{\pi}(\zeta^{-1})\boldsymbol{\pi}(\zeta)\boldsymbol{v}} &\xrightarrow{\beta_{\boldsymbol{\pi}(\zeta)\boldsymbol{v},\zeta^{-1}}} \big(A^q_{n,\boldsymbol{\pi}(\zeta)\boldsymbol{v}}\big)^{(\zeta^{-1},\zeta)}\\
&\xrightarrow{\beta_{\boldsymbol{v},\zeta}} \Big(\big(A^q_{n,\boldsymbol{v}}\big)^{(\zeta,\zeta^{-1})}\Big)^{(\zeta^{-1},\zeta)} = A^q_{n,\boldsymbol{v}}.
\end{aligned}$$

The composite is the identity map on $A^q_{n,\boldsymbol{v}}$, which ensures the injectivity of $\beta_{\boldsymbol{\pi}(\zeta)\boldsymbol{v},\zeta^{-1}}$ and the surjectivity of $\beta_{\boldsymbol{v},\zeta}$. Therefore, $\beta_{\boldsymbol{v},\zeta}$ is an isomorphism, since $\boldsymbol{v}$ and ζ are arbitrary. □

Corollary B.27. *For any $\boldsymbol{v} \in \mathcal{V}$, the algebra $A^q_{n,\boldsymbol{v}}$ is free as an $\mathcal{R}$-module with basis*

$$\mathcal{B} = \left\{ \prod_{i,j} \chi_{i,j}^{t_{i,j}} \;\middle|\; t_{i,j} \in \mathbb{N} \right\}, \tag{B.3.6}$$

where the products are formed with respect to any fixed ordering of the $\chi_{i,j}$. In particular, $A^q_{n,\boldsymbol{v}}$ has the standard basis $\boldsymbol{B}^q_{n,\boldsymbol{v}} = \bigcup_{r\in\mathbb{N}} \boldsymbol{B}^q_{n,\boldsymbol{v}}(r)$, where $\boldsymbol{B}^q_{n,\boldsymbol{v}}(r) = \{\chi_{\boldsymbol{i}_\lambda,\boldsymbol{i}_\mu w^{-1}} \mid \lambda,\mu \in \Lambda(n,r),\ w \in {}^\lambda(\mathfrak{S}_r)^\mu\}$ is the standard basis for the homogeneous component $A^q_{n,\boldsymbol{v}}(r)$ of grade r in $A^q_{n,\boldsymbol{v}}$. Moreover, $A^q_{n,\boldsymbol{v}}$ is an integral domain if $\mathcal{R}$ is an integral domain.

Proof. The defining relations (B.3.1) show that the algebra $A^q_{n,\boldsymbol{v}}$ is spanned by $\mathcal{B}$. To see the linear independence of $\mathcal{B}$ and the integral property (under the assumption that $\mathcal{R}$ is an integral domain), extend $\mathcal{R}$, if necessary, to contain a square root v of q. Then, by Theorem B.26, the corresponding properties of $A_{n,v}$ stated in Theorem B.2(1) ensure the results here. $\square$

Consider now the coalgebra structures on $A^q_{n,\boldsymbol{v}}$.

Theorem B.28. *If $q \in \mathcal{R}^\times$, then for any $\boldsymbol{v} \in \mathcal{V}$, the algebra $A^q_{n,\boldsymbol{v}}$ admits a bialgebra structure, whose comultiplication Δ and counit ε are defined on generators by*

$$\Delta(\chi_{i,j}) = \sum_k \chi_{i,k} \otimes \chi_{k,j} \quad \text{and} \quad \varepsilon(\chi_{i,j}) = \delta_{i,j}. \tag{B.3.7}$$

Moreover, all $A^q_{n,\boldsymbol{v}}$ with the same dominant parameter q are isomorphic as coalgebras.

Proof. First, *assume* that there is a bialgebra structure on $A^q_{n,\boldsymbol{v}}$, for some $\boldsymbol{v} \in \mathcal{V}$, with coalgebra as defined in (B.3.7). We make the following

Claim: *Identifying $A^q_{n,\boldsymbol{v}}$ and $(A^q_{n,\boldsymbol{v}})^{(\zeta,\zeta^{-1})}$ as $\mathcal{R}$-modules, the coalgebra structure on $A^q_{n,\boldsymbol{v}}$, together with the algebra structure of $(A^q_{n,\boldsymbol{v}})^{(\zeta,\zeta^{-1})}$, also defines a bialgebra structure on $(A^q_{n,\boldsymbol{v}})^{(\zeta,\zeta^{-1})}$, for any 2-cocycle ζ of Λ.*

Clearly, with this result, we can transfer the bialgebra structure of $A^q_{n,\boldsymbol{v}}$ to a bialgebra structure on the algebra $A^q_{n,\boldsymbol{u}}$, for any $\boldsymbol{u} \in \mathcal{V}$, keeping the coalgebra structures unchanged.

Denote the multiplication of $(A^q_{n,\boldsymbol{v}})^{(\zeta,\zeta^{-1})}$ by $*$ again. We must show that $\Delta(y*z) = \Delta(y)*\Delta(z)$ and $\varepsilon(y*z) = \varepsilon(y)*\varepsilon(z)$ with the assumptions that $\Delta(yz) = \Delta(y)\Delta(z)$ and $\varepsilon(yz) = \varepsilon(y)\varepsilon(z)$. It suffices to consider the case in which y and z are homogeneous with respect to the $\Lambda \times \Lambda$-grading with, say, grades (λ, μ) and (ϱ, η), respectively.

By (B.3.7), $\Delta(y)$ and $\Delta(z)$ can be written as finite sums

$$\Delta(y) = \sum_{\iota \in \Lambda} y'_{\lambda,\iota} \otimes y''_{\iota,\mu} \quad \text{and} \quad \Delta(z) = \sum_{\kappa \in \Lambda} z'_{\varrho,\kappa} \otimes z''_{\kappa,\eta},$$

where the subscripts of y', y'', z', and z'' indicate the $\Lambda \times \Lambda$-grades of the elements. Then

$$\begin{aligned}
&\Delta(y*z) \\
&\quad = \zeta(\lambda,\varrho)\zeta(\mu,\eta)^{-1}\Delta(yz) = \zeta(\lambda,\varrho)\zeta(\mu,\eta)^{-1}\sum_{\iota,\kappa} y'_{\lambda,\iota}z'_{\varrho,\kappa} \otimes y''_{\iota,\mu}z''_{\kappa,\eta} \\
&\quad = \zeta(\lambda,\varrho)\zeta(\mu,\eta)^{-1}\sum_{\iota,\kappa}\zeta(\lambda,\varrho)^{-1}\zeta(\iota,\kappa)y'_{\lambda,\iota}*z'_{\varrho,\kappa} \otimes \zeta(\iota,\kappa)^{-1}\zeta(\mu,\eta)y''_{\iota,\mu}*z''_{\kappa,\eta} \\
&\quad = \sum_{\iota,\kappa} y'_{\lambda\iota}*z'_{\varrho,\kappa} \otimes y''_{\iota,\mu}*z''_{\kappa,\eta} = \Delta(y)*\Delta(z).
\end{aligned}$$

Observe that $\varepsilon(y) \neq 0$, for a $\Lambda \times \Lambda$-homogeneous element y, only if y has grade (λ, λ), for some $\lambda \in \Lambda$. Thus, to prove $\varepsilon(y * z) = \varepsilon(y) * \varepsilon(z)$, we may assume y and z have grades (λ, λ) and (ϱ, ϱ), respectively. This gives $y * z = yz$, and the required equality is trivial, proving the claim.

To complete the proof of the theorem, we enlarge $\mathcal{R}$, if necessary, so that it contains a square root v of q. Then all the bialgebra structures can be obtained from the bialgebra structure of $A_{n,v}$ given by Theorem B.2(3). For $\boldsymbol{v} \in \mathcal{V}$, all commutations between the generators $\chi_{i,j}$ involve only coefficients in $\mathcal{R}$, so the comultiplication and counit defined by (B.3.7) are realized within $\mathcal{R}$, defining a bialgebra structure on $A^q_{n,\boldsymbol{v}}$, as required. □

Since all $A^q_{n,\boldsymbol{v}}$ have the same coalgebra structure, the quantum determinant $\det_q$ in $A_{n,v}$ defined by (B.1.11) is also a group-like element in $A^q_{n,\boldsymbol{v}}$. (The reason why we use $\det_q$, instead of $\det_v$, to denote the quantum determinant is now obvious — the quantum determinant depends only on the dominant parameter q.) As above, sometimes it is necessary, for technical reasons, to enlarge the ring $\mathcal{R}$. But, the element $\det_q$ exists in $A^q_{n,\boldsymbol{v}}$, for any $\boldsymbol{v} \in \mathcal{V}$; see the expression in (B.3.8). When we work in $A^q_{n,\boldsymbol{v}}$, the element $\det_q$ will be called the *quantum determinant* of $A^q_{n,\boldsymbol{v}}$.

To obtain an expression for $\det_q$ in terms of the generators $\chi_{i,j}$ and the multiplication on $A^q_{n,\boldsymbol{v}}$, we need an element v_w, for $w \in \mathfrak{S}_n$. The element is defined using the cocycle $\zeta_{\boldsymbol{v}}$ given in the proof of Lemma B.24:

$$v_w := \prod_{i=1}^{n-1} \zeta_{\boldsymbol{v}}\big(\theta_{w(1)} + \cdots + \theta_{w(i)}, \theta_{w(i+1)}\big) = \prod_{\substack{i<j \\ w(i)>w(j)}} v_{w(i),w(j)}.$$

Proposition B.29. *Let $\boldsymbol{v} = (v_{i,j}) \in \mathcal{V}$. Then in $A^q_{n,\boldsymbol{v}}$,*

$$\begin{aligned} \det_q &= \sum_{w \in \mathfrak{S}_n} (-q)^{-\ell(w)} v_w^{-1} \chi_{w(1),1} \cdots \chi_{w(n),n} \\ &= \sum_{w \in \mathfrak{S}_n} (-1)^{-\ell(w)} v_w \chi_{1,w(1)} \cdots \chi_{n,w(n)}. \end{aligned} \tag{B.3.8}$$

If $v_{i,j} = v$, for all $i < j$, then $v_w = v^{-\ell(w)}$. If, in addition, $q = v^2$, then (B.1.11) is recovered from (B.3.8). Thus, (B.3.8) holds in $A_{n,v}$.

Proof. As in the proof of Theorem B.28, we *assume* that (B.3.8) holds in $A^q_{n,\boldsymbol{v}}$, for some $\boldsymbol{v} \in \mathcal{V}$, and then *prove* it holds in $A^q_{n,\boldsymbol{uv}} = \big(A^q_{n,\boldsymbol{v}}\big)^{(\zeta_{\boldsymbol{u}},\zeta_{\boldsymbol{u}}^{-1})}$, for $\boldsymbol{u} \in \mathcal{V}$. We have, for $w \in \mathfrak{S}_n$, that

$$\begin{aligned} \chi_{w(1),1} * \cdots * \chi_{w(n),n} &= u_w \chi_{w(1),1} \cdots \chi_{w(n),n}; \text{ and} \\ \chi_{1,w(1)} * \cdots * \chi_{n,w(n)} &= u_w^{-1} \chi_{1,w(1)} \cdots \chi_{n,w(n)}. \end{aligned}$$

Here $*$ denotes the multiplication in $A^q_{n,\boldsymbol{uv}}$. Substitution to (B.3.8) yields the same formulas in $A^q_{n,\boldsymbol{uv}}$. □

Generally speaking, $\det_q$ is no longer a central element, but it still behaves reasonably — it commutes with $\chi_{i,j}$, up to the multiple of a product of parameters. More precisely, we have the following result.

Proposition B.30. *In the algebra $A^q_{n,\boldsymbol{v}}$, for $\boldsymbol{v} = (v_{i,j}) \in \mathcal{V}$,*

$$\chi_{i,j} \det_q = q^{i-j}\Big(\prod_k v_{i,k}v_{k,j}\Big) \det_q \chi_{i,j}.$$

Proof. Again, twist the formula from $A^q_{n,\boldsymbol{v}}$ to $A^q_{n,\boldsymbol{uv}}$ by means of the 2-cocycle $\zeta_{\boldsymbol{u}}$. The details are left to the reader as Exercise B.11. □

Because of Proposition B.30, the set $\{\det_q^r \mid r \in \mathbb{N}\}$ is a so-called "Ore set" in $A^q_{n,\boldsymbol{v}}$, so that one can form the localization of $A^q_{n,\boldsymbol{v}}$ at $\det_q$ to obtain an algebra $\tilde{A}^q_{n,\boldsymbol{v}} := A^q_{n,\boldsymbol{v}}[\det_q^{-1}]$, as in §B.2. For the details of the localization theory of noncommutative rings, the reader is again referred to the textbook [**222**]. It is easy to check that the bialgebra structure of $A^q_{n,\boldsymbol{v}}$ extends to $\tilde{A}^q_{n,\boldsymbol{v}}$ by making $\det_q^{-1}$ a group-like element; coalgebra structures of $\tilde{A}^q_{n,\boldsymbol{v}}$ are all isomorphic.

There is another way to obtain the algebras $\tilde{A}^q_{n,\boldsymbol{v}}$ by directly twisting the algebra $\tilde{A}_{n,v}$ defined in §B.2. In fact, since $\det_q$ is homogeneous with respect to the $\Lambda \times \Lambda$-grading we used above, the algebra $\tilde{A}_{n,v}$ is $\Lambda \times \Lambda$-graded. We can go through the procedure of twisting $\tilde{A}_{n,v}$ to obtain $\tilde{A}^q_{n,\boldsymbol{v}}$, for all $\boldsymbol{v} \in \mathcal{V}$, and, in the meantime, we can prove the invariance of the coalgebra structures, as we did in this section for $A^q_{n,\boldsymbol{v}}$.

Thus, we have assertion (1) of the following theorem.

Theorem B.31. (1) *The localization $\tilde{A}^q_{n,\boldsymbol{v}}$ of $A^q_{n,\boldsymbol{v}}$ at the quantum determinant* $\det_q$ *is a bialgebra, and the coalgebra structures on $\tilde{A}^q_{n,\boldsymbol{v}}$, for all $\boldsymbol{v} \in \mathcal{V}$, are all isomorphic.*

(2) *The bialgebra $\tilde{A}^q_{n,\boldsymbol{v}}$ is, in fact, a Hopf algebra, whose antipode $\gamma_{\boldsymbol{v}}$ satisfies*

$$\gamma_{\boldsymbol{vu}} = \eta_{\boldsymbol{v}}(\lambda)^{-1}\eta_{\boldsymbol{v}}(\mu)\gamma_{\boldsymbol{u}}$$

on the $\Lambda \times \Lambda$-homogeneous component of grade (λ, μ), for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$, where $\eta_{\boldsymbol{v}} \colon \Lambda \to \mathcal{R}^\times$ is the coboundary defined by $\eta_{\boldsymbol{v}}\big(\sum_i a_i\theta_i\big) := \prod_{i>j} v_{i,j}^{a_ia_j}$.

The proof of assertion (2) is left as Exercise B.12. A precise formula of $\gamma_{\boldsymbol{v}}$ is also given in the exercise.

So far, we have obtained a *multiparameter quantum general linear group* $\mathrm{GL}^q_{n,\boldsymbol{v}}$ by letting $\mathcal{O}(\mathrm{GL}^q_{n,\boldsymbol{v}}) := \tilde{A}^q_{n,\boldsymbol{v}}$, for any $\boldsymbol{v} \in \mathcal{V}$.

Remark B.32. It is now natural to ask if there is a multiparameter quantum special linear group associated with a given $\boldsymbol{v} \in \mathcal{V}$ and $q \in \mathcal{R}^{\times}$. Although the ideal generated by $\det_q -1$ of $\mathcal{O}(\mathrm{M}^q_{n,\boldsymbol{v}}) = A^q_{n,\boldsymbol{v}}$ (resp., of $\mathcal{O}(\mathrm{GL}^q_{n,\boldsymbol{v}}) = \tilde{A}^q_{n,\boldsymbol{v}}$) is always a biideal (resp., a Hopf ideal), the quotient algebra of $A^q_{n,\boldsymbol{v}}$ (resp., $\tilde{A}^q_{n,\boldsymbol{v}}$) modulo that ideal may degenerate if $\det_q$ is not central. For example, if $\det_q$ does not commute with a generator $\chi_{i,j}$, then $\chi_{i,j}$ will vanish in the quotient. Therefore, only the case in which $\det_q$ is central can give interesting multiparameter quantum special linear groups. From Proposition B.30, we see that the condition that $\det_q$ be central is equivalent to the condition that

$$q^i \prod_k v_{i,k} = q^j \prod_k v_{j,k}, \quad \text{for all } i, j.$$

When this condition is satisfied, we have a *multiparameter quantum special linear group* $\mathrm{SL}^q_{n,\boldsymbol{v}}$ with $\mathcal{O}(\mathrm{SL}^q_{n,\boldsymbol{v}}) := A^q_{n,\boldsymbol{v}}/(\det_q -1) = \tilde{A}^q_{n,\boldsymbol{v}}/(\det_q -1)$.

B.4. An application: quantum Schur algebras

As an application of the theory developed in this appendix, we show how the quantum Schur algebra[5] $S_q(n,r)$ can be realized as the dual algebra of the homogeneous component of $A_{n,v}(r)$ of grade r. This component, according to Theorem B.2, is a coalgebra, which is free as an $\mathcal{R}$-module of rank $\binom{n^2+r-1}{r}$. Moreover, for q fixed, Theorem B.28 assures that the coordinate algebras $A^q_{n,\boldsymbol{v}} = \mathcal{O}(\mathrm{M}^q_{n,\boldsymbol{v}})$ of the multiparameter quantum matrix spaces $\mathrm{M}^q_{n,\boldsymbol{v}}$ have the same coalgebra structure as v varies over $\mathcal{V}$. Thus, we may use any of the coalgebras $A^q_{n,\boldsymbol{v}}(r)$, the homogeneous components of $A^q_{n,\boldsymbol{v}}$ of grade r, to obtain the dual algebra $A_{n,v}(r)^* := \mathrm{Hom}_{\mathcal{R}}(A_{n,v}(r), \mathcal{R})$; see Proposition 5.4.

As in §B.1, let Ω be a free $\mathcal{R}$-module of rank n with basis $\{\omega_1, \omega_2, \dots, \omega_n\}$, and define a "natural" right $A^q_{n,\boldsymbol{v}}$-comodule on Ω by the structure map

$$\tau \colon \Omega \longrightarrow \Omega \otimes A^q_{n,\boldsymbol{v}}, \quad \omega_j \longmapsto \sum_i \omega_i \otimes \chi_{i,j}.$$

This comodule structure map extends to an $\mathcal{R}$-algebra homomorphism $\tau\colon \mathsf{T}(\Omega) \to \mathsf{T}(\Omega) \otimes A^q_{n,\boldsymbol{v}}$ making the tensor algebra $\mathsf{T}(\Omega)$ of Ω into an $A^q_{n,\boldsymbol{v}}$-comodule. Clearly, (B.1.8) still holds. It is also clear that the homogeneous component $\mathsf{T}^r(\Omega) = \Omega^{\otimes r}$ of grade r of $\mathsf{T}(\Omega)$ is an $A^q_{n,\boldsymbol{v}}$-subcomodule and that $\tau(\Omega^{\otimes r}) \subseteq \Omega^{\otimes r} \otimes A^q_{n,\boldsymbol{v}}(r)$. Hence, τ induces a right $A^q_{n,\boldsymbol{v}}(r)$-comodule structure on $\Omega^{\otimes r}$, whose comodule structure map, by abuse of notation, is

[5]Since we do not perform base change in the appendix, we write $S_q(n,r)$ instead of $S_q(n,r)_{\mathcal{R}}$. Also, we do not omit the subscript q to emphasize the role of the dominant parameter q.

still denoted by τ. Via this comodule structure, the dual algebra $A^q_{n,\boldsymbol{v}}(r)^*$ has a left action on $\Omega^{\otimes r}$ given by the formula:

$$f\cdot \omega = (\mathrm{id}_{\Omega^{\otimes r}} \otimes f)\circ \tau(\omega), \quad \text{for } f \in A^q_{n,\boldsymbol{v}}(r)^*,\ \omega \in \Omega^{\otimes r}. \tag{B.4.1}$$

Lemma B.33. *The action of $A^q_{n,\boldsymbol{v}}(r)^*$ on $\Omega^{\otimes r}$ is faithful.*

Proof. If $f \in A^q_{n,\boldsymbol{v}}(r)^*$ acts trivially on $\Omega^{\otimes r}$, then, for any $\boldsymbol{j} \in I(n,r)$,

$$0 = f\cdot \omega_{\boldsymbol{j}} = \sum_{\boldsymbol{i}} f(\chi_{\boldsymbol{i},\boldsymbol{j}})\omega_{\boldsymbol{i}}.$$

It follows that $f(\chi_{\boldsymbol{i},\boldsymbol{j}}) = 0$, for all $\boldsymbol{i}$ and $\boldsymbol{j} \in I(n,r)$. Thus $f = 0$. □

Since the structure map $\tau\colon \mathsf{T}(\Omega) \to \mathsf{T}(\Omega) \otimes A^q_{n,\boldsymbol{v}}$ involves the multiplication of $A^q_{n,\boldsymbol{v}}$, the $A^q_{n,\boldsymbol{v}}(r)$-comodule structure, hence the $A^q_{n,\boldsymbol{v}}(r)^*$-module structure, on $\Omega^{\otimes r}$ may depend on $\boldsymbol{v}$. To meet the preexisting definition of quantum Schur algebras in Chapter 9, we must choose a suitable $\boldsymbol{v}$, i.e., a suitable quantum matrix space.

Take the matrix $\boldsymbol{e} = (e_{i,j})$ with all $e_{i,j} = 1$ as the parameter matrix. Thus, from (B.3.1) (or (B.3.2)), the relations in $A^q_{n,\boldsymbol{e}}$ ($= A^{1,q}_n$ in the notation of two-parameter quantum matrix spaces introduced in Example B.21) are

$$\begin{aligned} \chi_{k,i}\chi_{k,j} &= \chi_{k,j}\chi_{k,i}, && \text{for all } i,\ j,\ k; \\ \chi_{k,i}\chi_{l,j} &= q\chi_{l,j}\chi_{k,i}, && \text{if } k > l \text{ and } i \leqslant j; \\ \chi_{k,i}\chi_{l,j} - \chi_{l,j}\chi_{k,i} &= (q-1)\chi_{l,i}\chi_{k,j}, && \text{if } k > l \text{ and } i > j. \end{aligned} \tag{B.4.2}$$

To get the action of $A^q_{n,\boldsymbol{e}}(r)^*$ on certain basis elements of $\Omega^{\otimes r}$, use the standard basis $\boldsymbol{B}^q_{n,\boldsymbol{e}}(r)$ of $A^q_{n,\boldsymbol{e}}(r)$ as given in Corollary B.27, denoting the basis elements by $\chi_{\boldsymbol{i}_\lambda,\boldsymbol{i}_\mu w^{-1}}$, for $\lambda,\mu \in \Lambda(n,r)$ and $w \in {}^\lambda\mathfrak{S}^\mu$. Denote the corresponding dual basis of $A^q_{n,\boldsymbol{e}}(r)^*$ by $f^w_{\lambda,\mu}$. That is,

$$f^w_{\lambda,\mu}(\chi_{\boldsymbol{i}_{\lambda'},\boldsymbol{i}_{\mu'}w'^{-1}}) := \delta_{\lambda,\lambda'}\delta_{\mu,\mu'}\delta_{w,w'}.$$

Lemma B.34. *Let $\lambda,\mu,\mu' \in \Lambda(n,r)$ and $w \in {}^\lambda\mathfrak{S}^\mu$. Then*

$$f^w_{\lambda,\mu}\cdot \omega_{\boldsymbol{i}_{\mu'}} = \delta_{\mu,\mu'} \sum_{\substack{d\in {}^\lambda\mathfrak{S} \\ \mathfrak{S}_\lambda d\mathfrak{S}_\mu = \mathfrak{S}_\lambda w \mathfrak{S}_\mu}} q^{\ell(d)}\omega_{\boldsymbol{i}_\lambda d}.$$

Proof. We have $\tau(\omega_{\boldsymbol{i}_{\mu'}}) = \sum_{\lambda'\in\Lambda(n,r),\, d\in {}^{\lambda'}\mathfrak{S}} \omega_{\boldsymbol{i}_{\lambda'}d} \otimes \chi_{\boldsymbol{i}_{\lambda'}d,\boldsymbol{i}_{\mu'}}$, and thus

$$f^w_{\lambda,\mu}\cdot \omega_{\boldsymbol{i}_{\mu'}} = \sum_{\lambda'\in\Lambda(n,r),\, d\in {}^{\lambda'}\mathfrak{S}} f^w_{\lambda,\mu}(\chi_{\boldsymbol{i}_{\lambda'}d,\boldsymbol{i}_{\mu'}})\omega_{\boldsymbol{i}_{\lambda'}d}.$$

If $\lambda' \neq \lambda$ or $\mu' \neq \mu$, then $f^w_{\lambda,\mu}(\chi_{\boldsymbol{i}_{\lambda'}d,\boldsymbol{i}_{\mu'}}) = 0$. The result for $\mu \neq \mu'$ follows, and the case $\mu' = \mu$ simplifies to

$$f^w_{\lambda,\mu} \cdot \omega_{\boldsymbol{i}_\mu} = \sum_{d \in {}^\lambda\mathfrak{S}} f^w_{\lambda,\mu}(\chi_{\boldsymbol{i}_\lambda d,\boldsymbol{i}_\mu})\omega_{\boldsymbol{i}_\lambda d}.$$

Using the defining relations (B.4.2), we see that $\chi_{\boldsymbol{i}_\lambda d,\boldsymbol{i}_\mu} = q^{\ell(d)}\chi_{\boldsymbol{i}_\lambda,\boldsymbol{i}_\mu d^{-1}y^{-1}}$, for any $y \in \mathfrak{S}_\lambda$. Therefore,

$$f^w_{\lambda,\mu}(\chi_{\boldsymbol{i}_\lambda d,\boldsymbol{i}_\mu}) = \begin{cases} q^{\ell(d)}, & \text{if } \mathfrak{S}_\lambda d\mathfrak{S}_\mu = \mathfrak{S}_\lambda w\mathfrak{S}_\mu; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the formula in the lemma holds in the $\mu = \mu'$ case. $\square$

Now we define a linear map $\mathscr{T} \colon \Omega \otimes \Omega \to \Omega \otimes \Omega$ by the rule

$$\mathscr{T}(\omega_i \otimes \omega_j) = \begin{cases} q\omega_j \otimes \omega_i, & \text{if } i \leqslant j; \\ (q-1)\omega_i \otimes \omega_j + \omega_j \otimes \omega_i, & \text{if } i > j. \end{cases} \tag{B.4.3}$$

We have the following results, which link the $A^q_{n,e}$-comodule structure on $\Omega^{\otimes r}$ with the action of the Hecke algebra $H_{\mathcal{R}} = H(\mathfrak{S}_r)_{\mathcal{R}}$ given in (9.1.1)

Proposition B.35. (1) *The above-defined map $\mathscr{T}$ is a right $A^q_{n,e}$-comodule homomorphism. Thus, for any $m < r$, the map $\mathscr{T}_{m,m+1} \colon \Omega^{\otimes r} \to \Omega^{\otimes r}$, which is the application of $\mathscr{T}$ to the mth and the $(m+1)$th factors of the tensor space, viewed as $\Omega \otimes \Omega$, is a right $A^q_{n,e}$-comodule homomorphism.*

(2) *The assignment $T_{s_m} \mapsto \mathscr{T}_{m,m+1}$ for $s_m = (m, m+1) \in \mathfrak{S}_r$, $m = 1, \ldots, r-1$, defines a (left and right) $H_{\mathcal{R}}$-module structure on $\Omega^{\otimes r}$.*

Proof. For (1), it suffices to check that $\tau \circ \mathscr{T}(\omega_i \otimes \omega_j) = (\mathscr{T} \otimes \mathrm{id}) \circ \tau(\omega_i \otimes \omega_j)$, for all i, j, where τ defines the comodule structure of $\Omega \otimes \Omega$, and id stands for the identity map on $A^q_{n,e}$. We give a verification for the case $i < j$, leaving the other cases as an exercise; see Exercise B.15.

$$\begin{aligned}
\tau \circ \mathscr{T}(\omega_i \otimes \omega_j) &= \tau(q\omega_j \otimes \omega_i) = q\sum_{k,l} \omega_l \otimes \omega_k \otimes \chi_{l,j}\chi_{k,i} \\
&= \sum_{k>l} \omega_l \otimes \omega_k \otimes \chi_{k,i}\chi_{l,j} + q\sum_{k} \omega_k \otimes \omega_k \otimes \chi_{k,i}\chi_{k,j} \\
&\qquad + \sum_{k<l} \omega_l \otimes \omega_k \otimes \big(q\chi_{k,i}\chi_{l,j} + (q-1)\chi_{l,i}\chi_{k,j}\big) \\
&= \sum_{k>l} \big(\omega_l \otimes \omega_k + (q-1)\omega_k \otimes \omega_l\big) \otimes \chi_{k,i}\chi_{l,j} + \sum_{k \leqslant l} q\omega_l \otimes \omega_k \otimes \chi_{k,i}\chi_{l,j} \\
&= \sum_{k,l} \mathscr{T}(\omega_k \otimes \omega_l) \otimes \chi_{k,i}\chi_{l,j} = (\mathscr{T} \otimes \mathrm{id}) \circ \tau(\omega_i \otimes \omega_j),
\end{aligned}$$

as required.

We now consider (2). From the defining relations (4.4.1) of a Hecke algebra, we see that, in the case of $W = \mathfrak{S}_r$, the only nontrivial relations requiring verification are $\mathscr{T} \circ \mathscr{T} = (q-1)\mathscr{T} + q$ (on $\Omega \otimes \Omega$) and $\mathscr{T}_{1,2} \circ \mathscr{T}_{2,3} \circ \mathscr{T}_{1,2} = \mathscr{T}_{2,3} \circ \mathscr{T}_{1,2} \circ \mathscr{T}_{2,3}$ (on $\Omega \otimes \Omega \otimes \Omega$). The verifications are routine, involving case-by-case exhibitions. For example, if $i < j$, then

$$\omega_i \otimes \omega_j \stackrel{\mathscr{T}}{\longmapsto} q\omega_j \otimes \omega_i \stackrel{\mathscr{T}}{\longmapsto} (q-1)q\omega_j \otimes \omega_i + q\omega_i \otimes \omega_j,$$

which equals the image of $\omega_i \otimes \omega_j$ under the map $(q-1)\mathscr{T} + q$. Also, if $i \leqslant j \leqslant k$, both $\mathscr{T}_{1,2} \circ \mathscr{T}_{2,3} \circ \mathscr{T}_{1,2}$ and $\mathscr{T}_{2,3} \circ \mathscr{T}_{1,2} \circ \mathscr{T}_{2,3}$ take $\omega_i \otimes \omega_j \otimes \omega_k$ to $q^3\omega_k \otimes \omega_j \otimes \omega_i$. For the other cases, see Exercise B.15 again. □

Remark B.36. The $H_{\mathcal{R}}$-module structure on $\Omega^{\otimes r}$ defined here looks different from that given in (9.1.1). However, (see Exercise B.16) these two (right) module structures are isomorphic via the map $\omega_{\boldsymbol{i}} \mapsto q^{-\operatorname{inv}(\boldsymbol{i})}\omega_{\boldsymbol{i}}$, where $\operatorname{inv}(\boldsymbol{i})$ stands for the number of inversions in the sequence $\boldsymbol{i}$, or, using the notation of Chapter 9, $\omega_{\boldsymbol{i}_\lambda d} \mapsto q^{-\ell(d)}\omega_{\boldsymbol{i}_\lambda d}$, for $\lambda \in \Lambda(n,r)$ and $d \in {}^\lambda\mathfrak{S}_r$. Therefore, the $H_{\mathcal{R}}$-module $\Omega^{\otimes r}$ defined here is, in fact, isomorphic to $\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{\mathcal{R}}$ via the map $\omega_{\boldsymbol{i}_\lambda w} \mapsto q^{-\ell(d)}x_\lambda T_d$.

Theorem B.37. *The $\mathcal{R}$-algebras $A^q_{n,\boldsymbol{v}}(r)^*$ and $S_q(n,r)$ are isomorphic. Moreover, if we identify the $H_{\mathcal{R}}$-module $\Omega^{\otimes r}$ with $\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{\mathcal{R}}$ via the map $\omega_{\boldsymbol{i}_\lambda d} \mapsto q^{-\ell(d)}x_\lambda T_d$, then $f^w_{\lambda,\mu} = \zeta^w_{\lambda,\mu}$ as defined in* (9.1.2).

Proof. Lemma B.33 shows that the algebra $A^q_{n,\boldsymbol{v}}(r)^*$ is realized as a subalgebra of $\operatorname{End}_{\mathcal{R}}(\Omega^{\otimes r})$, while Proposition B.35 ensures that the actions of the algebras $A^q_{n,\boldsymbol{v}}(r)^*$ and $H_{\mathcal{R}}$ on $\Omega^{\otimes r} \cong \bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{\mathcal{R}}$ (see Remark B.36) commute. This means, by the definition of quantum Schur algebras, that $A^q_{n,\boldsymbol{v}}(r)^*$ is realized as a subalgebra of $S_q(n,r)$. If $\mathcal{R} = \mathcal{k}$ is a field, the required isomorphism follows immediately from the dimension comparison, since both of these algebras have dimension $\binom{n^2+r-1}{r}$. To see the isomorphism in the general case, it suffices to show that $f^w_{\lambda,\mu} = \zeta^w_{\lambda,\mu}$. Identify $\Omega^{\otimes r}$ with $\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{\mathcal{R}}$ as in the theorem, then the formula in Lemma B.34 becomes

$$f^w_{\lambda,\mu}\cdot x_{\mu'} = \delta_{\mu,\mu'} \sum_{\substack{d\in{}^\lambda\mathfrak{S} \\ \mathfrak{S}_\lambda d\mathfrak{S}_\mu = \mathfrak{S}_\lambda w\mathfrak{S}_\mu}} x_\lambda T_d = \delta_{\mu,\mu'} \sum_{\substack{y\in\mathfrak{S}_\lambda,\, d\in{}^\lambda\mathfrak{S} \\ \mathfrak{S}_\lambda d\mathfrak{S}_\mu = \mathfrak{S}_\lambda w\mathfrak{S}_\mu}} T_{yd} = \delta_{\mu,\mu'} \sum_{x\in\mathfrak{S}_\lambda w\mathfrak{S}_\mu} T_x,$$

which is exactly the definition of $\zeta^w_{\lambda,\mu}$; see (9.1.2). □

Remark B.38. From the defining relations (B.4.2), it is clear that the invertibility of q is unnecessary in the definition of $A^q_{n,\boldsymbol{e}}$, and the standard basis $\boldsymbol{B}^q_{n,\boldsymbol{e}}$ can be realized without q^{-1}. Thus, we can define $A^q_{n,\boldsymbol{e}}(r)$, and hence $A^q_{n,\boldsymbol{e}}(r)^*$, over the ring $\mathcal{A} = \mathbb{Z}[q]$. Moreover, the isomorphism sending $f^w_{\lambda,\mu}$ to $\zeta^w_{\lambda,\mu}$ is also realized within $\mathcal{A}$. The only change in our argument is that the isomorphism $\Omega^{\otimes r} \to \bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{\mathcal{A}}$ must be revised as an isomorphism of the $\mathcal{A}$-submodule of $\Omega^{\otimes r}$ spanned by $q^{\ell(d)}\omega_{\boldsymbol{i}_\lambda d}$ ($\lambda \in \Lambda(n,r)$, $d \in {}^\lambda\mathfrak{S}$) with $\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{\mathcal{A}}$. This remark shows that the integral q-Schur algebra $S_q(n,r)$ over $\mathcal{A}$ (see Definition 9.2) can also be obtained by duality.

Exercises and notes

Exercises

§B.1

B.1. Verify that the matrix relation (B.1.4) in terms of the Yang–Baxter matrix is equivalent to the relations (B.1.1).

B.2. Check, for $A_{n,v}$, that the elements $\Delta(\chi_{i,j})$ satisfy the relations satisfied by the elements $\chi_{i,j}$.

B.3. Complete the verification for Proposition B.6.

B.4. Verify that

$$\sum_{w\in\mathfrak{S}_r} (-v)^{-\ell(w)}\chi_{\boldsymbol{i}w,\boldsymbol{j}} = \sum_{w\in\mathfrak{S}_r} (-v)^{-\ell(w)}\chi_{\boldsymbol{i},\boldsymbol{j}w},$$

for all $\boldsymbol{i},\boldsymbol{j} \in I_0(n,r)$.

Hint: Show that, for any $w \in \mathfrak{S}_r$, $\chi_{\boldsymbol{i}w,\boldsymbol{j}} = \chi_{\boldsymbol{i},\boldsymbol{j}w^{-1}}$.

B.5. Given $\boldsymbol{i} \in I_0(n,r)$, for $r \leqslant n$, let $\boldsymbol{i}' \in I_0(n, n-r)$ be the complement of $\boldsymbol{i}$. Prove that in $\Lambda_v(\Omega)$,

$$\omega_{\boldsymbol{i}}\omega_{\boldsymbol{i}'} = (-v)^{r(r+1)/2-|\boldsymbol{i}|}\omega_{\boldsymbol{i}_{\max}}.$$

§B.2

B.6. Prove that there is an $\mathcal{R}$-algebra isomorphism $\bar{A}_{n,v}\otimes\mathcal{R}[u,u^{-1}] \cong \tilde{A}_{n,v}$, where u is an indeterminate over $\mathcal{R}$.

Hint: First, we have $\bar{A}_{n,v} \otimes \mathcal{R}[u,u^{-1}] \cong (A_{n,v} \otimes \mathcal{R}[u,u^{-1}])/(\det_q -1)$. Now define an algebra isomorphism $(A_{n,v} \otimes \mathcal{R}[u,u^{-1}])/(\det_q -1) \to (A_{n,v} \otimes \mathcal{R}[u,u^{-1}])/(\det_q -u)$ by sending $\chi_{1,j}$ to $u^{-1}\chi_{1,j}$ and $\chi_{i,j}$, with $i > 1$, to $\chi_{i,j}$. Finally, $(A_{n,v} \otimes \mathcal{R}[u,u^{-1}])/(\det_q -u)$ is isomorphic to $\tilde{A}_{n,v}$.

B.7. Verify that the ideals defining the Borel subgroups $B_v^{\pm}$ and the maximal torus T_v are Hopf ideals, and thus complete the definitions of $B_v^{\pm}$ and T_v as closed subgroups of G_v.

B.8. (1) Give a proof for Lemma B.16.

(2) Prove that in $\mathcal{O}(\mathrm{M}_{n,v})$, $\det(\chi_{i,j}^{\ell}) = \det_q^{\ell}$, where $\det(\chi_{i,j}^{\ell})$ denotes the ordinary determinant of the matrix $(\chi_{i,j}^{\ell})$ with commutative entries.

(3) Prove that the embeddings $\phi\colon \mathcal{O}(\mathrm{GL}_n) \to \mathcal{O}(\mathrm{GL}_{n,v})$ and $\phi\colon \mathcal{O}(\mathrm{SL}_n) \to \mathcal{O}(\mathrm{SL}_{n,v})$ are compatible with antipodes.

Hint: Let $q = v^2$, which is also a primitive ℓth root of 1. The verification involves Gaussian polynomials $\left[\!\left[{m \atop r}\right]\!\right]$. Use Exercise 0.13.

§B.3

B.9. Consider the two-parameter family of quantum linear matrix spaces given in Example B.21.

(1) Give a direct verification (i.e., without reference to Theorem B.28) that the maps $\Delta(\chi_{i,j}) = \sum_k \chi_{i,k} \otimes \chi_{k,j}$ and $\varepsilon(\chi_{i,j}) = \delta_{i,j}$ extend to $\mathcal{R}$-algebra homomorphisms $\Delta\colon A_n^{a,b} \to A_n^{a,b} \otimes A_n^{a,b}$ and $\varepsilon\colon A_n^{a,b} \to \mathcal{R}$, making $A_n^{a,b}$ into an $\mathcal{R}$-bialgebra.

(2) (Hyperbolic invariance of two-parameter quantum matrix spaces) Let a, b and a', b' be invertible elements in $\mathcal{R}$ with $ab = a'b'$. Give a direct proof (i.e., without reference to Theorem B.28) that $A_n^{a,b} \cong A_n^{a',b'}$ as coalgebras.

Hint: Denote the generators of $A_n^{a,b}$ and $A_n^{a',b'}$ by $\chi_{i,j}$ and $\chi'_{i,j}$, respectively. Let $\kappa := a'/a = b/b'$. Define an $\mathcal{R}$-linear map $\phi_\kappa\colon A_n^{a,b} \to A_n^{a',b'}$ by sending $\chi_{\boldsymbol{i},\boldsymbol{j}}$ ($\boldsymbol{i}, \boldsymbol{j} \in I(n,r)$, for some $r \in \mathbb{N}$) to $\kappa^{\mathrm{inv}(\boldsymbol{i})-\mathrm{inv}(\boldsymbol{j})}\chi'_{\boldsymbol{i},\boldsymbol{j}}$, where $\mathrm{inv}(\boldsymbol{i})$ stands for the number of inversions in the sequence $\boldsymbol{i} = (i_1, \dots, i_r) \in I(n,r)$. Prove that ϕ_κ is a coalgebra isomorphism.

B.10. Complete the verifications in the proof of Theorem B.26.

B.11. Carry out the necessary verification in the proof of Proposition B.30.

B.12. Carry out the necessary verification in the proof of Theorem B.31(2). Moreover, prove the following expression for $\gamma_{\boldsymbol{v}}$, for $\boldsymbol{v} = (v_{i,j}) \in \mathcal{V}$, on the generators $\chi_{i,j}$:

$$\gamma_{\boldsymbol{v}}(\chi_{i,j}) = (-1)^{n+j} q^{j-i} \tilde{v}_{j,i} \det_q^{-1} \sum_{\substack{w \in \mathfrak{S}_n \\ w(n)=i}} (-1)^{\ell(w)} \tilde{v}_w \tilde{\chi}_w,$$

where $\tilde{v}_{j,i} := \prod_{k>j;\, l>i} v_{j,k} v_{l,i}$, $\tilde{v}_w := \prod_{k<l<n;\, w(k)>w(l)} v_{w(k),w(l)}$, while

$$\tilde{\chi}_w := \chi_{1,w(1)} \cdots \chi_{j-1,w(j-1)} \chi_{j+1,w(j)} \cdots \chi_{n,w(n-1)}.$$

Although the fact that the homomorphism $\boldsymbol{\pi}$ defined in Lemma B.24 is an isomorphism (see Remark B.25) is not necessary for theory of multiparameter quantum matrix spaces developed in §B.3, the following two exercises provide a proof of this fact, which we give for completeness. Exercise B.13 establishes the well-known fact that the second cohomology $H^2(G, A)$ of a group G with values in an abelian group A can be interpreted as the set of equivalence classes of the central extensions of G by A. This result is used in Exercise B.14 to establish that $\boldsymbol{\pi}$ is an isomorphism.

B.13. (Group extensions and the 2nd cohomology group) Let G be a group and let A be an abelian group (written multiplicatively). By a *central extension* of G by A we mean an exact sequence $1 \to A \xrightarrow{i} \widehat{G} \xrightarrow{p} G \to 1$ of groups such that $\operatorname{Im} i$ is central in $\widehat{G}$. Two central extensions $1 \to A \to \widehat{G} \to G \to 1$ and $1 \to A \to \widehat{G}' \to G \to 1$ of G by A are said to be *equivalent* provided there exists an isomorphism $\widehat{G} \xrightarrow{\sim} \widehat{G}'$ making the diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & \widehat{G} \arrow[r] \arrow[d, "\wr"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] & \widehat{G}' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

commutative. A central extension $1 \to A \xrightarrow{i} \widehat{G} \xrightarrow{p} G \to 1$ is *split* if there exists a homomorphism $c\colon G \to \widehat{G}$ such that $p \circ c = \mathrm{id}_G$ or, equivalently, if $\widehat{G} \cong A \times G$ (with i and p the corresponding inclusion and projection maps).

Now assume that $G = \Lambda$ is abelian (written additively). Given a central extension $1 \to A \xrightarrow{i} \widehat{\Lambda} \xrightarrow{p} \Lambda \to 1$, let $c\colon \Lambda \to \widehat{\Lambda}$ be a map of sets satisfying $p \circ c = \mathrm{id}_\Lambda$. Define $\zeta\colon \Lambda \times \Lambda \to \widehat{\Lambda}$ by

$$\zeta(\lambda, \mu) = c(\lambda + \mu)c(\lambda)^{-1}c(\mu)^{-1}, \quad \text{for all } \lambda, \mu \in \Lambda.$$

(1) Prove that ζ has its values in A, and that ζ is a 2-cocycle of Λ, which is unitary if and only if $c(1_\Lambda) = 1_{\widehat{\Lambda}}$.

(2) Show that if $c'\colon G \to \widehat{G}$ is a map also satisfying $p \circ c' = \mathrm{id}_G$, then the corresponding ζ' differs from ζ by a 2-coboundary, i.e., ζ and ζ' define the same cohomology class in $H^2(\Lambda, A)$.

(3) Conversely, given a 2-cocycle of Λ with value in A, show that it defines a central extension of Λ by A. Show that 2-cocycles in the same cohomology class define equivalent extensions. Conclude that the cohomology group $H^2(\Lambda, A)$ *classifies* the central extensions of Λ by A, up to equivalence.

B.14. Assume that $\Lambda \cong \mathbb{Z}^n$ with basis $\theta_1, \dots, \theta_n$. Let $\mathcal{V}_n(A)$ be the set of $n \times n$ matrices $\boldsymbol{v} = (v_{i,j})$ with entries in A satisfying $v_{i,i} = 1$ and $v_{i,j}v_{j,i} = 1$, for all i, j. Make $\mathcal{V}_n(A)$ into a group by introducing entry-wise multiplication. Prove that $H^2(\Lambda, A) \cong \mathcal{V}_n(A)$ in the following steps:

(1) Given a 2-cocycle ζ of Λ in A, define $\boldsymbol{\pi}(\zeta) = (\pi_{i,j}(\zeta)) \in \mathcal{V}_n(A)$ by putting $\pi_{i,j}(\zeta) = \zeta(\theta_i, \theta_j)\zeta(\theta_j, \theta_i)^{-1}$. Mimic the proof of Lemma B.24 to conclude that the map $\zeta \mapsto \boldsymbol{\pi}(\zeta)$ defines a surjective homomorphism $\boldsymbol{\pi}\colon H^2(\Lambda, A) \to \mathcal{V}_n(A)$.

(2) To see that $\boldsymbol{\pi}$ is an isomorphism, suppose that ζ is a 2-cocycle such that $\boldsymbol{\pi}(\zeta) = 1$ or, equivalently, that ζ is symmetric in the sense that $\zeta(\theta_i, \theta_j) = \zeta(\theta_j, \theta_i)$. Show that the extension $\widehat{\Lambda}$ of Λ by A defined by ζ is abelian and therefore splits, since Λ is *free* abelian. Conclude ζ is a coboundary.

§B.4

B.15. Complete the necessary verifications of Proposition B.35.

B.16. Establish the isomorphism between the $H_{\mathcal{R}}$-module $\Omega^{\otimes r}$ defined by Proposition B.35(2) and that defined by (9.1.1); see Remark B.36.

B.17. Suppose $\mathcal{R} = \mathcal{k}$ is a field. Let v be a primitive ℓth root of unity with ℓ odd. Let $q = v^2$. Show that there is a Frobenius morphism $F\colon \mathcal{S}_q(n, \ell r) \to \mathcal{S}(n, r)$, where $\mathcal{S}(n, r)$ is the classical Schur algebra (or, equivalently, the quantum Schur algebra with dominant parameter 1). Thus, an $\mathcal{S}(n, r)$-module V has a Frobenius twist V^F. Derive the tensor product theorem for Schur algebras from Theorem B.18.

Notes

The general point of view of this appendix follows the treatment in Parshall-Wang [**236**] and Donkin [**84**]. The reader can consult these works for a discussion of the representation theory of q-Schur algebras, especially at roots of unity, which is not really treated in this book.

§§B.1–B.2: The definitions of standard quantum matrix spaces and quantum linear groups by quantum coordinate algebras can be traced back to early works by Russian mathematicians. See, for example, Faddeev–Reshetkhin–Takhtadjian [**111**]. Manin [**215**] explained these structures in terms of so-called quantum vector spaces and certain products of these vector spaces. A systematic theory of standard quantum matrix spaces and quantum linear groups in terms of quantum coordinate algebras, including structures and representations, was developed by Parshall–Wang [**236**]. The material contained in these two sections is taken mainly from [**236**]. It is possible to go further in the direction of quantum linear algebra. For example, a quantum Cayley–Hamilton theory was developed by Zhang [**308**].

§B.3: The multiparameter matrix bialgebras were independently defined by Sudbery [**287**], Reshetikhin [**241**], and Artin–Schelter–Tate [**7**]. We mainly follow Artin–Schelter–Tate [**7**] to establish, in our notation and terminology, the invariance of the coalgebra structures on the multiparameter matrix bialgebras $A^q_{n,\mathbf{v}}$, for a fixed dominant parameter q. The bialgebra $A^q_{n,\mathbf{e}}$(that is, $A^{1,q}_n$, in the notation of two-parameter quantum matrix spaces) with parameter $(1, q)$ was discussed in detail by Dipper–Donkin [**71**] in relation to the theory of q-Schur algebras. The family of two-parameter matrix bialgebras introduced in Example B.21 was defined and investigated by Takeuchi [**289**]. Du–Parshall–Wang [**108**] proved the "hyperbolic invariance" (see Exercise B.9) of these two-parameter matrix bialgebras. This invariance is a special case of Artin–Schelter–Tate invariance.

§B.4: The fact that q-Schur algebras can be realized as dual algebras of homogeneous components of suitable quantum matrix bialgebras was independently proved by Dipper–Donkin [**71**] (using quantum matrix bialgebras $A^{1,q}_n$) and by Parshall–Wang [**236**] (using standard quantum matrix bialgebras $A_{n,v}$). Certain results (for example, the quasi-heredity of the q-Schur algebras over a field) on the structure and representation theory of q-Schur algebras were obtained by Parshall–Wang [**236**] from the investigation of the structure and representations of quantum linear groups.

Appendix C

Quasi-hereditary and cellular algebras

Quasi-hereditary algebras make up a class of finite dimensional algebras which possesses certain strong homological properties and which arises naturally in representation theory. In particular, if A is a quasi-hereditary algebra, the category A-mod of finite dimensional A-modules is a *highest weight category* in a sense which closely models module categories for algebraic groups, Lie algebras, quantum groups, ... Apart from this connection, the importance of quasi-hereditary algebras also lies in ring theory itself, since many interesting algebras turn out to be quasi-hereditary.

A cellular algebra is a finite dimensional algebra having a basis (the *cellular basis*) which reflects an important symmetry structure of the algebra. The basis leads to a class of important representations for the algebra, called *cell representations*. Many algebras, such as Hecke algebras, q-Schur algebras, Brauer algebras, and Temperley–Lieb algebras, are cellular.

This appendix develops, from an elementary point of view, the basic theory of quasi-hereditary/cellular algebras to a degree sufficient for this book. Further results and more details of some proofs are indicated in the exercises, as well as in the references mentioned in the Notes at the end.

§§C.1–C.2 introduce the basics of quasi-hereditary algebras over a field $\Bbbk$. Then §C.3 digresses to collect together some results from commutative algebra which are needed later. In addition, these results can be used to construct a "Brauer theory" for algebras over regular rings of Krull dimension $\leqslant 2$, a theory sketched in the exercises. In §C.4, we give a brief introduction of the theory of integral quasi-hereditary algebras, quasi-hereditary algebras over commutative, noetherian rings. This setup is applied in §C.5 to certain natural endomorphism algebras $\mathrm{End}_H(T)$, for algebras H having what we call a *Specht datum.* The main result, given in Theorem C.29, plays an important role in the theory of Schur algebras discussed in Chapter 9. Finally, §C.6 defines cellular algebras and develops some elementary results about them.

For a ring A, if M is an A-A-bimodule, denote by ${}_AM$ (resp., M_A) the corresponding left (resp., right) A-module obtained from M. For example, if M is an ideal $\mathfrak{J}$ in A, ${}_A\mathfrak{J}$ just means that $\mathfrak{J}$ is to be regarded as a left A-module. Unless otherwise mentioned, modules are always taken to be left modules, and they are assumed to be finitely generated. In this appendix, A-mod denotes the category of finitely generated left A-modules.

Let $\mathrm{rad}(A)$ be the Jacobson radical of A. When A is a finite dimensional algebra over a field, $\mathrm{rad}(A)$ is the largest two-sided nilpotent ideal of A.

C.1. Heredity ideals

Throughout §§C.1–C.2, $\mathscr{k}$ is a fixed field and A is a finite dimensional $\mathscr{k}$-algebra. We will often work with idempotent ideals of A, i.e., (two-sided) ideals $\mathfrak{J}$ satisfying $\mathfrak{J}^2 = \mathfrak{J}$. As the following elementary result shows, such ideals arise very naturally.

Lemma C.1. (1) *An ideal $\mathfrak{J}$ in the algebra A is idempotent if and only if $\mathfrak{J} = AeA$, for some idempotent element $e \in A$.*

(2) *Given an idempotent ideal $\mathfrak{J}$, the algebra eAe is semisimple if and only if $\mathfrak{J}\cdot \mathrm{rad}(A)\cdot\mathfrak{J} = 0$.*

Proof. We first prove (1). If $\mathfrak{J} = AeA$ for an idempotent $e \in A$, then obviously $\mathfrak{J}^2 = \mathfrak{J}$.

Conversely, let $\mathfrak{J}$ be an idempotent ideal in A. If A is a semisimple algebra, then A is a finite direct product of various algebras of the form $\mathrm{M}_n(\mathcal{D})$, where $\mathcal{D}$ is a division algebra over $\mathscr{k}$. In this case, $\mathfrak{J}$ (and, in fact, *any* ideal of A) is a direct product of some of the $\mathrm{M}_n(\mathcal{D})$, so $\mathfrak{J} = AeA$ for the idempotent e, which is just the sum of the identity elements of the factors $\mathrm{M}_n(\mathcal{D})$ contained in $\mathfrak{J}$. Thus, the lemma holds when A is semisimple. More generally, write $\mathfrak{N} = \mathrm{rad}(A)$ and let

$$\pi\colon A \longrightarrow \bar{A} := A/\mathfrak{N}$$

be the quotient map. The image $\bar{\mathfrak{J}} = \pi(\mathfrak{J})$ of $\mathfrak{J}$ in $\bar{A}$ is an idempotent ideal in the semisimple algebra $\bar{A}$. Thus, $\bar{\mathfrak{J}} = \bar{A}\bar{e}\bar{A}$, for some idempotent $\bar{e} \in \bar{A}$. It is well known that there exists an idempotent $e \in A$ such that $\pi(e) = \bar{e}$. (See Exercise C.1.) Finally, we have $AeA + \mathfrak{N} = \mathfrak{J} + \mathfrak{N}$ so that

$AeA+\mathfrak{N}^m = \mathfrak{J}+\mathfrak{N}^m$, for any positive integer m. Thus, $AeA = \mathfrak{J}$, as required for (1).

Assertion (2) follows from the elementary fact that the radical of eAe is $e\mathfrak{N}e$. □

The following notion will be basic for the development of quasi-hereditary algebras.

Definition C.2. An ideal $\mathfrak{J}$ of A is said to be a *heredity ideal* if the following three conditions hold:

(HI1) $\mathfrak{J}^2 = \mathfrak{J}$.

(HI2) $\mathfrak{J}$ is projective as a left A-module.

(HI3) The $\Bbbk$-algebra $E = \mathrm{End}_A({}_A\mathfrak{J})$ is semisimple.

We will develop several properties of heredity ideals. The two lemmas below provide slightly alternative characterizations of heredity ideals.

Lemma C.3. *The conditions* (HI1) *and* (HI2) *in Definition C.2 are equivalent to the conditions* (HI2) *and*

(HI1′) $\mathrm{Hom}_A({}_A\mathfrak{J}, {}_A(A/\mathfrak{J})) = 0$.

Proof. First, assume (HI1) and (HI2) hold. If $f \in \mathrm{Hom}_A({}_A\mathfrak{J}, {}_A(A/\mathfrak{J}))$, then $f(\mathfrak{J}) = f(\mathfrak{J}\mathfrak{J}) = \mathfrak{J}f(\mathfrak{J}) = 0$. Thus, (HI1′) and (HI2) hold.

Conversely, assume (HI1′) and (HI2) hold, but that $\mathfrak{J} \neq \mathfrak{J}^2$. Clearly, ${}_A(\mathfrak{J}/\mathfrak{J}^2)$, as an A-module, is a homomorphic image of a finite direct sum, say M, of copies of ${}_A(A/\mathfrak{J})$. On the other hand, by (HI2), ${}_A\mathfrak{J}$ is a projective A-module, so the quotient morphism ${}_A\mathfrak{J} \to {}_A(\mathfrak{J}/\mathfrak{J}^2)$ lifts to a nonzero morphism ${}_A\mathfrak{J} \to M$. Hence, there is a nonzero morphism ${}_A\mathfrak{J} \to {}_A(A/\mathfrak{J})$, contradicting condition (HI1′). Therefore, $\mathfrak{J}^2 = \mathfrak{J}$. □

Lemma C.4. *Let $\mathfrak{J}$ be an idempotent ideal in A. Then $\mathfrak{J}$ is a heredity ideal if and only if the following two conditions hold:*

(HI2′) *The multiplication map $\mu\colon Ae \otimes_{eAe} eA \to AeA$ is bijective, where $\mathfrak{J} = AeA$, for some idempotent $e \in A$.*

(HI3′) $\mathfrak{J}\cdot \mathrm{rad}(A)\cdot\mathfrak{J} = 0$.

Proof. To begin, assume that $\mathfrak{J}$ is an idempotent ideal in the algebra A (say, $\mathfrak{J} = AeA$, $e^2 = e$) such that ${}_A\mathfrak{J}$ is projective. We will show that condition (HI2′) holds. For any left A-module M, consider the map

$$\mu_M\colon Ae \otimes_{eAe} eA \otimes_A M \longrightarrow M, \quad ae \otimes eb \otimes m \longmapsto aebm.$$

Certainly, μ_M is an A-module map. If $M = Ae$, then the A-module map $\nu\colon Ae \to Ae\otimes_{eAe}eA\otimes_A Ae$ sending ae to $ae\otimes e\otimes e$ is clearly the inverse of μ_{Ae}. Therefore, μ_{Ae} is bijective. Consequently, μ_M is bijective when M is a direct summand of a finite direct sum of the module Ae. Since $\mathfrak{J} = AeA = \sum Aea$, a running over a finite subset of A, we see that $\mathfrak{J}$ is the homomorphic image of a finite direct sum of copies of Ae. Now the projectivity of ${}_A\mathfrak{J}$ ensures that $\mathfrak{J} = AeA$ is a direct summand of a finite direct sum of copies of Ae. Therefore, μ_{AeA} is bijective. Since $eA \otimes_A \mathfrak{J} \cong e\mathfrak{J} = eA$, condition (HI2$'$) holds.

Next, observe that the map

$$E := \mathrm{Hom}_A(Ae \otimes_{eAe} eA, Ae \otimes_{eAe} eA) \longrightarrow \mathrm{Hom}_{eAe}(eA, eA) =: F,$$

which sends $\varphi \in E$ to $[\varphi]\colon eA \to eA$ defined by $[\varphi](ea) = \varphi(e \otimes ea) = e\varphi(e \otimes ea) \in eAe \otimes_{eAe} eA \cong eA$ is an isomorphism of algebras. (In fact, this isomorphism is just the usual isomorphism arising from the adjoint associativity of the functor $-\otimes_{eAe} eA$ and the functor $\mathrm{Hom}_A(Ae, -)$ [**HAII**, Prop. 2.6.3], once it is observed that $\mathrm{Hom}_A(Ae, Ae \otimes_{eAe} eA) \cong eAe \otimes_{eAe} eA \cong eA$.) Thus, if eAe is semisimple (i.e., if $\mathfrak{J} \cdot \mathrm{rad}(A) \cdot \mathfrak{J} = 0$, by Lemma C.1(2)), then E is also semisimple. On the other hand, as an eAe-module, $eA = eAe \oplus eA(1-e)$. Let $f \in F$ be the corresponding projection of eA onto eAe. Then f is an idempotent, and

$$(eAe)^{\mathrm{op}} \cong fFf \cong \mathrm{Hom}_F(Ff, Ff)^{\mathrm{op}}.$$

Thus, if E is semisimple, then the isomorphic algebra F is semisimple, and hence eAe is also semisimple as the endomorphism algebra of a semisimple module.

Now we can prove the lemma. First, if $\mathfrak{J}$ is a heredity ideal, then ${}_A\mathfrak{J}$ is idempotent and projective so that (HI2$'$) holds by the above discussion. Then (HI2$'$) and the previous paragraph show that eAe is semisimple, so (HI3$'$) holds. Conversely, assume that conditions (HI2$'$) and (HI3$'$) hold. Then the previous paragraph again says that $E = \mathrm{End}_A({}_A\mathfrak{J})$ is semisimple. It remains to show that ${}_A\mathfrak{J}$ is projective. But eAe is semisimple, so eA is a projective left eAe-module. Thus, ${}_A\mathfrak{J} \cong Ae \otimes_{eAe} eA$ is isomorphic to a direct summand of a finite direct sum of copies of $Ae \otimes_{eAe} eAe \cong Ae$, and hence it is projective, as required. □

We can now establish several basic properties of heredity ideals. Recall that A is said to have finite global dimension provided there is an integer n_0 such that $\mathrm{Ext}^n_A(M, N) = 0$, for all finite dimensional A-modules M, N and for $n \geqslant n_0$.[1] If $\mathfrak{J}$ is an ideal in A, the quotient map $A \to A/\mathfrak{J}$ induces a functor $i^*\colon A/\mathfrak{J}\text{-}\mathsf{mod} \to A\text{-}\mathsf{mod}$ of module categories. Given an $A/\mathfrak{J}$-module M, it is convenient to denote the A-module i^*M simply by M again: the algebra A simply acts on M through the quotient map $A \to A/\mathfrak{J}$ and the

[1] In defining finite global dimension, one can work with either the category of finite dimensional A-modules or the category of all A-modules. This well-known fact follows easily from [**HAII**, Th. 4.1.2].

given action of $A/\mathfrak{J}$ on M. Moreover, if M, N are $A/\mathfrak{J}$-modules, then i^* induces a graded morphism

$$\mathrm{Ext}^\bullet_{A/\mathfrak{J}}(M,N) \longrightarrow \mathrm{Ext}^\bullet_A(M,N) \tag{C.1.1}$$

of Ext-groups. In fact, an element $\xi \in \mathrm{Ext}^n_{A/\mathfrak{J}}(M,N)$ can be represented by an equivalence class of n-extensions

$$0 \longrightarrow N \longrightarrow Q_1 \longrightarrow \cdots \longrightarrow Q_n \longrightarrow M \longrightarrow 0 \tag{C.1.2}$$

of $A/\mathfrak{J}$-modules; see [**HAII**, Vista 3.4.6]. Then applying the functor i^* to (C.1.2) gives an n-extension of A-modules and so defines the map in (C.1.1) in degree n.

Proposition C.5. *Let $\mathfrak{J}$ be a heredity ideal in a finite dimensional algebra A.*

(1) *$\mathfrak{J}^{\mathrm{op}}$ is a heredity ideal in the opposite algebra A^{op}.*

(2) *For $A/\mathfrak{J}$-modules M, N, the morphism given in* (C.1.1) *is an isomorphism.*

(3) *The algebra A has finite global dimension if and only if the quotient algebra $A/\mathfrak{J}$ has finite global dimension.*

Proof. Statement (1) is clear from Lemma C.4 since the conditions that $\mathfrak{J}$ be heredity given there are left-right symmetric.

Now we sketch the proofs of (2) and (3). Trivially, (C.1.1) is an isomorphism in homological degree 0 for all $A/\mathfrak{J}$-modules M, N. Next, observe that given any $A/\mathfrak{J}$-module N, the contravariant long exact sequence of $\mathrm{Ext}^\bullet_A$ for the short exact sequence $0 \to {}_A\mathfrak{J} \to A \to A/\mathfrak{J} \to 0$ yields, for $n \geqslant 1$, a surjection $\mathrm{Ext}^{n-1}_A(\mathfrak{J}, N) \to \mathrm{Ext}^n_A(A/\mathfrak{J}, N)$. Thus, if $n > 1$, $\mathrm{Ext}^n_A(A/\mathfrak{J}, N) = 0$ since ${}_A\mathfrak{J}$ is projective. But Lemma C.3 implies there is no nonzero morphism ${}_A\mathfrak{J} \to N$ (since ${}_A\mathfrak{J}$ is projective and N is a homomorphic image of a direct sum of copies of $A/\mathfrak{J}$). Hence, $\mathrm{Ext}^1_A(A/\mathfrak{J}, N) = 0$. Thus, given any $A/\mathfrak{J}$-module N, $A/\mathfrak{J}$ and hence all projective $A/\mathfrak{J}$-modules are acyclic for the functor $\mathrm{Hom}_A(-, N)$, where N is regarded as an A-module. Therefore, if M is an $A/\mathfrak{J}$-module, we can compute the groups $\mathrm{Ext}^n_A(M,N)$ using an $A/\mathfrak{J}$-projective resolution of M; see [**HAII**, Exer. 2.4.3]. Hence, the isomorphism (C.1.1) follows from the isomorphism in homological degree 0, proving (2).

Now if A has finite global dimension, then (2) implies that $A/\mathfrak{J}$ has finite global dimension as well. So to prove (3), assume that $A/\mathfrak{J}$ has finite global dimension. Thus, assume that there is an integer n_0 such that $\mathrm{Ext}^n_{A/\mathfrak{J}}(M,N) = 0$, for $n \geqslant n_0$ and for all finite dimensional $A/\mathfrak{J}$-modules M, N. We claim that $\mathrm{Ext}^n_A(M,N) = 0$, for all finite dimensional A-modules M, N and all $n > n_0+1$. We can assume that M and N are both simple modules (by repeated use of the long exact sequences of $\mathrm{Ext}^\bullet_A$). If $\mathfrak{J}M = 0 = \mathfrak{J}N$, our claim is clear since, then, M, N are $A/\mathfrak{J}$-modules. If $\mathfrak{J}M \neq 0$, define the A-module Q by the short exact sequence

$$0 \longrightarrow Q \longrightarrow Ae \otimes_{eAe} eM \xrightarrow{\text{mult}} M \longrightarrow 0. \tag{C.1.3}$$

Then $\mathfrak{J}Q = AeQ = 0$ so Q is an $A/\mathfrak{J}$-module. Also, because eAe is semisimple, $Ae \otimes_{eAe} eM$ is a projective A-module. In addition, if $\mathfrak{J}N \neq 0$, then $eN \neq 0$, and we can form a short exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{Hom}_{eAe}(eA, eN) \longrightarrow Q' \longrightarrow 0 \tag{C.1.4}$$

with $\mathfrak{J}Q' = 0$ and $\mathrm{Hom}_{eAe}(eA, eN)$ an injective A-module. (See Exercise C.2.) Now, dimension shifting (possibly twice using (C.1.3) and (C.1.4)) implies our claim. □

C.2. Quasi-hereditary algebras and highest weight categories

We now are ready to define the notion of a quasi-hereditary algebra. The idea is simply that a quasi-hereditary algebra should be one which is "stratified" by semisimple algebras. More precisely:

Definition C.6. A finite dimensional algebra A over the field $\Bbbk$ is called *quasi-hereditary* provided there is a sequence

$$0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_t = A$$

of ideals in A such that $\mathfrak{J}_i/\mathfrak{J}_{i-1}$ is a heredity ideal in $A/\mathfrak{J}_{i-1}$, for $i = 1, 2, \ldots, t$. Such a sequence of ideals is called a *heredity chain* in A.

Making use of our discussion of heredity ideals, we can immediately draw some conclusions about a quasi-hereditary algebra A.

Theorem C.7. *Let A be a quasi-hereditary algebra with heredity chain $\{\mathfrak{J}_i\}_{i=0}^t$.*

(1) *The opposite algebra A^{op} is a quasi-hereditary algebra with heredity chain $\{\mathfrak{J}_i^{\mathrm{op}}\}$ induced from that of A.*

(2) *For $0 \leqslant i < t$, the quotient map $A \to A/\mathfrak{J}_i$ induces a (graded) isomorphism*

$$\mathrm{Ext}^\bullet_{A/\mathfrak{J}_i}(M, N) \cong \mathrm{Ext}^\bullet_A(M, N),$$

for all $A/\mathfrak{J}_i$-modules M, N.

(3) *The algebra A has finite global dimension.*

Proof. (1) follows from Proposition C.5(1). The isomorphism in (2) is the composite of the isomorphisms

$$\mathrm{Ext}^\bullet_{A/\mathfrak{J}_i}(M, N) \xrightarrow{\sim} \mathrm{Ext}^\bullet_{A/\mathfrak{J}_{i-1}}(M, N) \xrightarrow{\sim} \cdots$$
$$\xrightarrow{\sim} \mathrm{Ext}^\bullet_{A/\mathfrak{J}_0}(M, N) = \mathrm{Ext}^\bullet_A(M, N)$$

which are guaranteed by Proposition C.5(2). Finally, (3) follows from Proposition C.5 by induction on the length of the heredity chain. □

It will be very convenient to recast the above discussion in purely module-theoretic terms. Call an abelian category $\mathcal{C}$ a finite $\Bbbk$-category provided $\mathcal{C}$ is equivalent to the category A-mod of finite dimensional modules, for a finite dimensional algebra A over $\Bbbk$. The algebra A is not uniquely determined by the category $\mathcal{C}$: Morita equivalent algebras have equivalent module categories; see [**BAII**, §3.12]. Given a finite $\Bbbk$-category $\mathcal{C}$, let $\Lambda = \Lambda(\mathcal{C})$ be a finite set which indexes the distinct isoclasses of simple objects in $\mathcal{C}$. Given $\lambda \in \Lambda$, let $L(\lambda) \in \mathcal{C}$ be a representative from the isomorphism class of simple objects corresponding to λ. Also, let $P(\lambda)$ (resp., $I(\lambda)$) be a projective (resp., injective) object in $\mathcal{C}$ with top (resp., socle) $L(\lambda)$. Both $P(\lambda)$ and $I(\lambda)$ are unique up to isomorphism.

Definition C.8. A *highest weight category* is a finite $\Bbbk$-category $\mathcal{C}$, together with a poset structure $\leqslant$ on the set $\Lambda = \Lambda(\mathcal{C})$, such that the following statements hold:

(HWC1) For $\lambda \in \Lambda$, there is given an object $\Delta(\lambda) \in \mathcal{C}$ which has simple top $L(\lambda)$ and, in addition, has the property that all composition factors $L(\mu)$ of the radical $\mathrm{rad}(\Delta(\lambda))$ satisfy $\mu < \lambda$.

(HWC2) For $\lambda \in \Lambda$, there exists a filtration

$$P(\lambda) = F_0^\lambda \supset F_1^\lambda \supset \cdots \supset F_{t_\lambda}^\lambda = 0$$

such that $F_0^\lambda/F_1^\lambda \cong \Delta(\lambda)$ and, for $0 < i < t_\lambda$, $F_i^\lambda/F_{i+1}^\lambda \cong \Delta(\mu_i)$, for some $\mu_i \in \Lambda$ (which depends on λ) satisfying $\mu_i > \lambda$.

We call Λ the *weight poset* of $\mathcal{C}$, and refer to the elements $\lambda \in \Lambda$ as *weights*.

Example C.9. Let G be a semisimple, simply connected algebraic group over an algebraically closed field $\mathcal{K}$. We use the notation of §A.5. Recall that there is a partial ordering defined on the set X^+ of dominant weights: $\xi, \zeta \in X^+$, $\xi \leqslant \zeta$ provided $\zeta - \xi \in \mathbb{N}\Phi^+$. Let Λ be a finite order ideal in the poset X^+. Let $\mathcal{C}[\Lambda]$ be the full subcategory of G-mod whose objects consist of finite dimensional rational G-modules having composition factors $L(\xi)$ satisfying $\xi \in \Lambda$. We will check that $\mathcal{C}[\Lambda]$ is a highest weight category with weight poset Λ.

First, the objects $\Delta(\xi)$, $\xi \in \Lambda$, are as defined in (A.7.6). Corollary A.60 establishes that these modules satisfy condition (HWC1).

Next, let $\Lambda^\star$ be the image of Λ under the poset isomorphism $X^+ \to X^+$, $\xi \mapsto \xi^\star := -w_0(\xi)$. Then $\Lambda^\star$ is also an order ideal in X^+. If $\xi \in \Lambda$, Exercise A.29(2) constructs the injective envelope $I(\xi^\star)_{\Lambda^\star}$ of $L(\xi^\star)$ in $\mathcal{C}[\Lambda^\star]$.

Furthermore, Exercise A.29(1) there implies that $I(\xi^\star)_{\Lambda^\star}$ has a ∇-filtration, with lowest "section" $\nabla(\xi^\star)$ and other sections $\nabla(\zeta^\star)$ with $\zeta > \xi$. Since $L(\xi^\star)^* \cong L(\xi)$, it follows that $P(\xi) := I(\xi^\star)^*$ is the projective cover of $L(\xi)$ in $\mathcal{C}[\Lambda]$, and it satisfies the required conditions in (HWC2). Thus, $\mathcal{C}[\Lambda]$ is a highest weight category.

Theorem C.10. *A finite dimensional $\Bbbk$-algebra A is quasi-hereditary if and only if the category A-mod of finite dimensional A-modules is a highest weight category with respect to a poset structure on $\Lambda = \Lambda(A\text{-mod})$.*

Proof. Suppose A is a quasi-hereditary algebra with defining sequence $0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_t = A$. For $0 < i \leqslant t$, consider the distinct indecomposable summands of the projective $A/\mathfrak{J}_{i-1}$-module $\mathfrak{J}_i/\mathfrak{J}_{i-1}$. Each such projective module is labelled by its simple top, so denote them by $\Delta_i(\lambda)$, for λ running over some subset Λ_i of Λ.

Claim: *If $i \neq j$, then $\Lambda_i \cap \Lambda_j = \emptyset$.*

Otherwise, suppose that $i < j$ and that $\lambda \in \Lambda_i \cap \Lambda_j$. Because $A/\mathfrak{J}_{j-1}$ is a homomorphic image of $A/\mathfrak{J}_{i-1}$, and $\Delta_j(\lambda)$ has top $L(\lambda)$, we find that $\Delta_j(\lambda)$ is a homomorphic image of the projective $A/\mathfrak{J}_{i-1}$-module $\Delta_i(\lambda)$. So, there exists a nonzero $A/\mathfrak{J}_{i-1}$-module homomorphism $\mathfrak{J}_i/\mathfrak{J}_{i-1} \to \mathfrak{J}_j/\mathfrak{J}_{j-1}$. However, since $\mathfrak{J}_i/\mathfrak{J}_{i-1}$ is an idempotent ideal in $A/\mathfrak{J}_{i-1}$ and $\mathfrak{J}_i \subseteq \mathfrak{J}_{j-1}$, the existence of such a homomorphism is absurd. This establishes the claim.

Moreover, any simple A-module $L(\lambda)$ is a homomorphic image of some $\mathfrak{J}_i/\mathfrak{J}_{i-1}$, hence of some $\Delta_i(\lambda)$. Thus, Λ is a disjoint union of the Λ_i. Given $\lambda \in \Lambda$, there is a unique i, for which $\lambda \in \Lambda_i$ and we put $\Delta(\lambda) = \Delta_i(\lambda)$, as defined above.

Define a poset structure $<$ on Λ by putting $\lambda < \mu$ if and only if $\lambda \in \Lambda_i$ and $\mu \in \Lambda_j$, for some $i > j$.

We verify the highest weight category axioms. First, any $\Delta(\lambda)$ has, by definition, simple top $L(\lambda)$, for $\lambda \in \Lambda_j$, say. Suppose that $L(\mu)$ is a composition factor of the radical $\mathrm{rad}(\Delta(\lambda))$ of $\Delta(\lambda)$. For some i, we have $\mu \in \Lambda_i$. If $i < j$, there exists a nonzero $A/\mathfrak{J}_{i-1}$-homomorphism $\mathfrak{J}_i/\mathfrak{J}_{i-1} \to \mathfrak{J}_j/\mathfrak{J}_{j-1}$ by the projectivity of the $A/\mathfrak{J}_{i-1}$-module $\mathfrak{J}_i/\mathfrak{J}_{i-1}$. As we showed above, the existence of such a homomorphism is impossible. Thus, $i \geqslant j$. If $i = j$, again there is a nonzero morphism $f\colon \bar{\mathfrak{J}} = \mathfrak{J}_i/\mathfrak{J}_{i-1} \to \mathrm{rad}(\bar{\mathfrak{J}})$. Thus, by Lemma C.4, letting $\bar{A} = A/\mathfrak{J}_{i-1}$,

$$f(\bar{\mathfrak{J}}) = f(\bar{\mathfrak{J}}^2) = \bar{\mathfrak{J}} f(\bar{\mathfrak{J}}) \subseteq \bar{\mathfrak{J}} \cdot \mathrm{rad}(\bar{\mathfrak{J}}) = \bar{\mathfrak{J}} \cdot \mathrm{rad}(A) \cdot \bar{\mathfrak{J}} = 0,$$

a contradiction. It follows that $i > j$. By definition of the poset structure on Λ, this means that $\mu < \lambda$. Thus, condition (HWC1) holds.

Next, we verify that the projective module $P(\lambda)$ satisfies condition (HWC2). Write $P(\lambda) \cong Ae$, for some primitive idempotent $e \in A$. For any i, $(\mathfrak{J}_i/\mathfrak{J}_{i-1})e \cong \mathfrak{J}_ie/\mathfrak{J}_{i-1}e$ is a direct summand of $\mathfrak{J}_i/\mathfrak{J}_{i-1}$ as a left A-module. Hence,

$$\mathfrak{J}_iP(\lambda)/\mathfrak{J}_{i-1}P(\lambda) \cong \mathfrak{J}_ie/\mathfrak{J}_{i-1}e$$

is a direct sum of various $\Delta(\nu)$, for $\nu \in \Lambda_i$. Now suppose that $\lambda \in \Lambda_i$. By definition, $\Delta(\lambda)$ is an $A/\mathfrak{J}_{i-1}$-direct summand of $\mathfrak{J}_i/\mathfrak{J}_{i-1}$, so there is a surjective A-module morphism $\mathfrak{J}_i \longrightarrow \Delta(\lambda)$. Since $\mathfrak{J}_i$ is an idempotent ideal, $\mathfrak{J}_i\Delta(\lambda) = \Delta(\lambda)$ and hence $\mathfrak{J}_iL(\lambda) = L(\lambda)$. Therefore, the covering morphism $P(\lambda) \longrightarrow L(\lambda)$ restricts to a surjective morphism $\mathfrak{J}_iP(\lambda) \longrightarrow L(\lambda)$. This means that $\mathfrak{J}_iP(\lambda) = P(\lambda)$. On the other hand, if $\mathfrak{J}_{i-1}P(\lambda) = P(\lambda)$, then $\mathfrak{J}_{i-1}L(\lambda) = L(\lambda)$, and $\mathrm{Hom}_A(\mathfrak{J}_{i-1}, L(\lambda)) \neq 0$. This fact implies the existence of an index $j < i$, for which $\mathrm{Hom}_A(\mathfrak{J}_j/\mathfrak{J}_{j-1}, L(\lambda)) \neq 0$, and $\lambda \in \Lambda_j \cap \Lambda_i$, a contradiction. Thus, $\mathfrak{J}_{i-1}P(\lambda)$ is a proper submodule of $P(\lambda)$, and $P(\lambda)/\mathfrak{J}_{i-1}P(\lambda)$ has simple top $L(\lambda)$. It follows that

$$P(\lambda)/\mathfrak{J}_{i-1}P(\lambda) \cong \mathfrak{J}_iP(\lambda)/\mathfrak{J}_{i-1}P(\lambda) \cong \Delta(\lambda).$$

Thus, omitting repetitions, $\mathfrak{J}_\bullet P(\lambda)$ defines a filtration of $P(\lambda)$ with top section $\Delta(\lambda)$ and lower sections of the form $\Delta(\mu)$, for $\mu \in \Lambda_1 \cup \cdots \cup \Lambda_{i-1}$ and hence, for $\mu > \lambda$. We have shown that A-mod is a highest weight category with poset $(\Lambda, \leqslant)$.

For the converse implication, see Exercise C.3. □

Corollary C.11. *Let A and B be finite dimensional $\Bbbk$-algebras such that A-mod and B-mod are equivalent as $\Bbbk$-categories (i. e., A and B are Morita equivalent algebras over $\Bbbk$). Then A is a quasi-hereditary algebra if and only if B is a quasi-hereditary algebra.*

In order to indicate the structure of the highest weight category A-mod, we often say that A is quasi-hereditary *with poset* Λ. The modules $\Delta(\lambda)$, $\lambda \in \Lambda$, are called the *standard* objects in the highest weight category A-mod. Now consider the opposite algebra A^{op}. We can identify A^{op}-mod with the category of finite dimensional right A-modules. For $\lambda \in \Lambda$, let $L'(\lambda)$ be the simple right A-module whose linear dual is isomorphic to $L(\lambda)$. Since A^{op} is quasi-hereditary, the standard objects $\Delta(\lambda)^{\mathrm{op}}$, viewed as right A-modules, exist. Let

$$\nabla(\lambda) := \mathrm{Hom}_{\Bbbk}(\Delta(\lambda)^{\mathrm{op}}, \Bbbk).$$

Then $\nabla(\lambda)$ is a left A-module. The modules $\nabla(\lambda)$, $\lambda \in \Lambda$, are called *costandard* objects in the highest weight category A-mod.

The following result follows easily from the discussion; see Exercise C.4.

Lemma C.12. *Let A-mod be a highest weight category with weight poset Λ. For $\lambda, \mu \in \Lambda$, if $\mathrm{Ext}^1_A(\Delta(\lambda), L(\mu)) \neq 0$, then $\mu > \lambda$. Similarly, if $\mathrm{Ext}^1_A(L(\lambda), \nabla(\mu)) \neq 0$, then $\mu > \lambda$.*

Making use of this result, we can prove the following strong homological property of the standard and costandard modules in a highest weight category.

If A-mod is a highest weight category, let A-$\mathsf{mod}(\Delta)$ (resp., A-$\mathsf{mod}(\nabla)$) be the full subcategory of A-mod consisting of all modules M which have a filtration $M = F_0 \supset F_1 \supset \cdots \supset F_m = 0$ with sections F_i/F_{i+1} isomorphic to modules of the form $\Delta(\lambda_i)$ (resp., $\nabla(\lambda_i)$), for some $\lambda_i \in \Lambda$.

Proposition C.13. *Let A-mod be a highest weight category with weight poset Λ.*

(1) *For $M \in A$-$\mathsf{mod}(\Delta)$ and $N \in A$-$\mathsf{mod}(\nabla)$, $\mathrm{Ext}^n_A(M,N) = 0$, for all $n > 0$.*

(2) *For $n > 0$ and $\lambda, \mu \in \Lambda$, if $\mathrm{Ext}^n_A(\Delta(\lambda), L(\mu)) \neq 0$, then $\mu > \lambda$. Hence, if $\mathrm{Ext}^n_A(\Delta(\lambda), \Delta(\mu)) \neq 0$, for some $n > 0$, then $\mu > \lambda$.*

(3) *For $n > 0$ and $\lambda, \mu \in \Lambda$, if $\mathrm{Ext}^n_A(L(\lambda), \nabla(\mu)) \neq 0$, then $\lambda > \mu$. Hence, if $\mathrm{Ext}^n_A(\nabla(\lambda), \nabla(\mu)) \neq 0$, for some $n > 0$, then $\lambda > \mu$.*

(4) *For $n \geqslant 0$ and $\lambda, \mu \in \Lambda$,*

$$\dim \mathrm{Ext}^n_A(\Delta(\lambda), \nabla(\mu)) = \delta_{n,0}\delta_{\lambda,\mu} \dim \mathrm{End}_A(L(\lambda)).$$

Proof. First, suppose that $\mathrm{Ext}^1_A(\Delta(\lambda), \nabla(\mu)) \neq 0$, for some $\lambda, \mu \in \Lambda$. Then for a composition factor $L(\tau)$ of $\Delta(\lambda)$, $\mathrm{Ext}^1_A(L(\tau), \nabla(\mu)) \neq 0$, so that $\lambda \geqslant \tau > \mu$. Similarly, for a composition factor $L(\xi)$ of $\nabla(\mu)$, $\mathrm{Ext}^1_A(\Delta(\lambda), L(\xi)) \neq 0$ so that $\lambda < \xi \leqslant \mu$. This contradiction proves that $\mathrm{Ext}^1_A(\Delta(\lambda), \nabla(\mu)) = 0$, for all λ, μ. Induction on the number of Δ-sections of M and ∇-sections of N and a standard argument using the long exact sequence of $\mathrm{Ext}^\bullet_A$ prove (1) in the special case $n = 1$. Now an easy induction argument on n, using dimension shifting, shows that $\mathrm{Ext}^n_A(M,N) = 0$, for any positive integer n, whenever $M \in A$-$\mathsf{mod}(\Delta)$ and $N \in A$-$\mathsf{mod}(\nabla)$. This proves (1).

We have already observed that (2) holds if $n = 1$. So assume $n > 1$. Dimension shifting on the short exact sequence

$$0 \longrightarrow L(\mu) \longrightarrow \nabla(\mu) \longrightarrow K(\mu) \longrightarrow 0$$

and using (1), we find by induction that if $\mathrm{Ext}^n_A(\Delta(\lambda), L(\mu)) \neq 0$, then $\lambda < \tau < \mu$, for some τ such that $L(\tau)$ is a composition factor of $K(\mu)$. This proves (2).

A dual argument shows that (3) holds. Now (4) is easy from (1). $\square$

Let A-mod be a highest weight category with weight poset Λ. Define a *tilting module* to be an A-module M which belongs to both A-$\mathsf{mod}(\Delta)$ and to A-$\mathsf{mod}(\nabla)$, i.e., $M \in A$-$\mathsf{mod}(\Delta\!\!\!\!\nabla) := A$-$\mathsf{mod}(\Delta) \cap A$-$\mathsf{mod}(\nabla)$.

Proposition C.14 (Ringel). *Let A-mod be a highest weight category with weight poset Λ.*

(1) *For any $\lambda \in \Lambda$, there exists an indecomposable tilting module $X(\lambda) \in A\text{-mod}(\text{✡})$ having composition factor $L(\lambda)$ of multiplicity 1, and satisfying the further property that all other composition factors $L(\mu)$ satisfy $\mu < \lambda$. The module $X(\lambda)$ is unique up to isomorphism with these properties.*

(2) *Any $M \in A\text{-mod}(\text{✡})$ is a direct sum of copies of the $X(\lambda)$.*

(3) *Let $X \in A\text{-mod}(\text{✡})$ contain at least one direct summand isomorphic to $X(\lambda)$, for each $\lambda \in \Lambda$. Then the algebra $E := \mathrm{End}_A(X)$ is also quasi-hereditary. In fact, E-mod is a highest weight category with weight poset Λ^{op}, the poset opposite to Λ.*

We will sketch a proof of this result in the exercises; see Exercise C.6. The algebra E is often called the *Ringel dual* of A.

C.3. Regular rings of Krull dimension at most 2

We pause for a digression into the commutative algebra of regular rings of Krull dimension $\leqslant 2$. As a main goal, we establish a result due to Auslander and Goldman (see Theorem C.17) which provides a criterion for the projectivity of certain modules over such rings. This result has several important consequences, which will be useful to us in the next two sections on quasi-hereditary algebras. Of course, the ring $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ of integral Laurent polynomials in a variable v is a regular ring of Krull dimension 2. Hence, the results of this and the following sections have applications in Chapter 9 to Schur algebras.

We assume the reader has some familiarity with the theory of (commutative) regular rings. Everything that is needed can be found in [**HAII**, Ch. 4]. Throughout, $\mathcal{R}$ denotes a commutative ring.

As indicated in Appendix A, the *Krull dimension* $\mathrm{Kdim}\,\mathcal{R}$ of $\mathcal{R}$ is the maximal length d of a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$ of prime ideals in $\mathcal{R}$. If no such integer d exists, we set $\mathrm{Kdim}\,\mathcal{R} = \infty$.

If $\mathcal{R}$ is a local ring, i.e., if $\mathcal{R}$ is noetherian and has a unique maximal ideal $\mathfrak{m}$, then $\mathrm{Kdim}\,\mathcal{R} \leqslant \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2 < \infty$ [**HAII**, p. 105], where $\mathscr{k} := \mathcal{R}/\mathfrak{m}$ is the residue field. When $\mathrm{Kdim}\,\mathcal{R} = \dim_{\mathscr{k}} \mathfrak{m}/\mathfrak{m}^2$, then $\mathcal{R}$ is called *regular*. A regular local ring must be a domain [**HAII**, Prop. 4.4.5].

A general commutative noetherian ring $\mathcal{R}$ is called *regular* if its localization $\mathcal{R}_\mathfrak{p}$ is a regular local ring, for every $\mathfrak{p} \in \mathrm{Spec}\,\mathcal{R}$.

Let $\mathcal{R}$ be a local ring and let M be a nonzero finitely generated $\mathcal{R}$-module. A *regular sequence* on M is a sequence $\{x_1, \ldots, x_n\}$ of elements in $\mathfrak{m}$ such that each x_i $(1 \leqslant i \leqslant n)$ is not a zero divisor on $M_{i-1} :=$

$M/(x_1, \dots, x_{i-1})M$ (i.e., if $0 \neq m \in M_{i-1}$, then $0 \neq x_i m$), where $M_0 = M$. The length of any two maximal sequences on M are the same, and the *depth* of M, $\operatorname{depth} M$, is the common length of the maximal regular sequences on M. Also, $\operatorname{depth} \mathcal{R} \leqslant \operatorname{Kdim} \mathcal{R}$.

Returning to a general $\mathcal{R}$, the *projective dimension* $\operatorname{pdim} M$ of an $\mathcal{R}$-module M is the minimum integer n such that there is a resolution of M by projective modules

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

If no such integer n exists, we set $\operatorname{pdim} M = \infty$. Define the *global dimension* $\operatorname{gldim} \mathcal{R} := \sup_M \operatorname{pdim} M$, taken over all $\mathcal{R}$-modules M. (When $\mathcal{R}$ is noetherian, the supremum can be taken over all finitely generated $\mathcal{R}$-modules.)

The formula in part (1) of the result below is known as the *Auslander–Buchsbaum equality.* A proof of this proposition can be found in [**HAII**, Ths. 4.4.15, 4.4.16, Prop. 4.4.1].

Proposition C.15. *Let $\mathcal{R}$ be a local ring.*

(1) *If M is a finitely generated, nonzero $\mathcal{R}$-module having finite projective dimension, then*

$$\operatorname{depth} M + \operatorname{pdim} M = \operatorname{depth} \mathcal{R}.$$

(2) *$\mathcal{R}$ is a regular local ring if and only if* $\operatorname{gldim} \mathcal{R} < \infty$. *In this case,* $\operatorname{gldim} \mathcal{R} = \operatorname{Kdim} \mathcal{R} = \operatorname{depth} \mathcal{R}$. *Also, for any finitely generated $\mathcal{R}$-module M,* $\operatorname{pdim} M < \infty$.

An important application of the Auslander–Buchsbaum formula is the Auslander–Goldman result for a regular ring of Krull dimension $\leqslant 2$. We need a lemma first.

Lemma C.16. *Let $\mathcal{R}$ be a local ring and assume M, N are finitely generated $\mathcal{R}$-modules such that* $\operatorname{Hom}_{\mathcal{R}}(M, N) \neq 0$. *Then, for $i = 1, 2$,*

$$\operatorname{depth} N \geqslant i \Longrightarrow \operatorname{depth} \operatorname{Hom}_{\mathcal{R}}(M, N) \geqslant i.$$

In particular, if $\operatorname{depth} N \leqslant 2$, *then* $\operatorname{depth} \operatorname{Hom}_{\mathcal{R}}(M, N) \geqslant \operatorname{depth} N$.

Proof. First, $\operatorname{Hom}_{\mathcal{R}}(M, N)$ is a finitely generated $\mathcal{R}$-module. If $x \in \mathfrak{m}$ is not a zero divisor of N, then the map $x \colon N \to N$ sending $n \in N$ to xn is injective. Thus, we obtain a short exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

and, hence, the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{R}}(M, N) \xrightarrow{x^*} \operatorname{Hom}_{\mathcal{R}}(M, N) \longrightarrow \operatorname{Hom}_{\mathcal{R}}(M, N/xN).$$

Since $x^*(f) = xf$, for $f \in \mathrm{Hom}_{\mathcal{R}}(M,N)$, we see that x is not a zero divisor on $\mathrm{Hom}_{\mathcal{R}}(M,N)$. Hence, depth $\mathrm{Hom}_{\mathcal{R}}(M,N) \geqslant 1$, establishing the $i = 1$ case. Also, by the Nakayama lemma, $\mathrm{Hom}_{\mathcal{R}}(M,N)/x\,\mathrm{Hom}_{\mathcal{R}}(M,N)$ is a nonzero submodule of $\mathrm{Hom}_{\mathcal{R}}(M,N/xN)$. In particular, $\mathrm{Hom}_{\mathcal{R}}(M,N/xN) \neq 0$.

If $y \in \mathfrak{m}$ is not a zero divisor on N/xN, then, by the previous paragraph, y is not a zero divisor on $\mathrm{Hom}_{\mathcal{R}}(M,N/xN)$ and on its nonzero submodule $\mathrm{Hom}_{\mathcal{R}}(M,N)/x\,\mathrm{Hom}_{\mathcal{R}}(M,N)$. The $i = 2$ case thus follows. □

Given an $\mathcal{R}$-module M, it is not necessarily true that its dual $M^* = \mathrm{Hom}_{\mathcal{R}}(M,\mathcal{R})$ is a projective $\mathcal{R}$-module. Projectivity of M^* certainly does hold if M is already projective over $\mathcal{R}$ (a fact frequently used later in this appendix, often without mention). However, we have the following remarkable result.

Theorem C.17 (Auslander–Goldman lemma). *If $\mathcal{R}$ is a regular ring of Krull dimension $\leqslant 2$ and M is a finitely generated $\mathcal{R}$-module, then $M^* := \mathrm{Hom}_{\mathcal{R}}(M,\mathcal{R})$ is projective. Thus, if $M \cong M^{**}$, then M is projective.*

Proof. Since M is projective if and only if $M_{\mathfrak{p}}$ is projective, for every prime ideal $\mathfrak{p}$ of $\mathcal{R}$ [**BAII**, §7.4], we may assume that $\mathcal{R}$ is a local regular ring.

We can assume that $M^* \neq 0$ (i.e., M is not a torsion module); otherwise, there is nothing to prove. For a regular local ring $\mathcal{R}$, depth $\mathcal{R} = \mathrm{Kdim}\,\mathcal{R}$ and pdim $M^* < \infty$ by Proposition C.15(2). Hence, the assumption $\mathrm{Kdim}\,\mathcal{R} \leqslant 2$ implies that depth $\mathcal{R} \leqslant 2$. By Lemma C.16 and the Auslander–Buchsbaum equality,

$$\text{depth } \mathcal{R} \leqslant \text{depth } M^* + \text{pdim } M^* = \text{depth } \mathcal{R},$$

and, thus, depth $M^* = \mathrm{Kdim}\,\mathcal{R}$ and pdim $M^* = 0$. It follows that M^* is projective. □

In the modular representation theory of a finite group G, classical Brauer theory relates the characteristic zero representation theory of G with the representation theory of G over a field of positive characteristic, making use of an intermediate discrete valuation ring. Using the above results, it is possible to develop a Brauer theory in which the discrete valuation ring is replaced by a regular local ring of Krull dimension $\leqslant 2$. The following fundamental result is key to this Brauer theory, which will be described in more detail in Exercises C.9 and C.10.

Corollary C.18. *Let $\mathcal{R}$ be a regular ring of Krull dimension $\leqslant 2$ with fraction field $\mathcal{K}$. Let A be an $\mathcal{R}$-algebra which is projective and finitely* *generated as an* $\mathcal{R}$*-module. Suppose that* $\boldsymbol{V}$ *is a finitely generated* $A_{\mathcal{K}}$*-module. Then* $\boldsymbol{V} = V_{\mathcal{K}}$*, for some* A*-lattice* V*. (In other words,* V *is an* A*-submodule of* $\boldsymbol{V}$ *which is projective and finitely generated as an* $\mathcal{R}$*-module.)*

Proof. Choose a $\mathcal{K}$-basis $\{x_1, \ldots, x_n\}$ for $\boldsymbol{V}$ and let $V_0 = \sum_{i=1}^{n} Ax_i$. Then V_0 is a finitely generated A-submodule of $\boldsymbol{V}$, and $\boldsymbol{V} = (V_0)_{\mathcal{K}}$. Now Theorem C.17 implies that the right A-module $V_0^* = \operatorname{Hom}_{\mathcal{R}}(V_0, \mathcal{R})$ is projective and finitely generated over $\mathcal{R}$. Hence, the A-module $V = V_0^{**}$ is finitely generated and projective as an $\mathcal{R}$-module. Since V_0 is $\mathcal{R}$-torsion free, the natural evaluation map

$$\operatorname{Ev}\colon V_0 \longrightarrow V_0^{**} = \operatorname{Hom}_{\mathcal{R}}(\operatorname{Hom}_{\mathcal{R}}(V_0, \mathcal{R}), \mathcal{R}), \quad v \longmapsto [f \mapsto f(v)]$$

defines an injection of A-modules which becomes an isomorphism upon applying (the exact localization functor) $(-)_{\mathcal{K}} = - \otimes_{\mathcal{R}} \mathcal{K}$. In other words, $\boldsymbol{V} \cong V_0^{**} \otimes_{\mathcal{R}} \mathcal{K}$. Identifying V with an A-submodule of $\boldsymbol{V}$ gives $V_{\mathcal{K}} \cong \boldsymbol{V}$, as required. □

Corollary C.19. *Let* $\mathcal{R}$ *be a regular ring of Krull dimension* $\leqslant 2$ *with fraction field* $\mathcal{K}$*. Suppose* F *is a finitely generated, projective* $\mathcal{R}$*-module and* $\boldsymbol{V}$ *is a subspace of the* $\mathcal{K}$*-space* $\boldsymbol{F} := F_{\mathcal{K}}$*. Then* $V := F \cap \boldsymbol{V}$ *is* $\mathcal{R}$*-projective.*

Proof. Because F is $\mathcal{R}$-projective, the canonical evaluation map

$$\operatorname{Ev}_F\colon F \xrightarrow{\sim} F^{**} = \operatorname{Hom}_{\mathcal{R}}(\operatorname{Hom}_{\mathcal{R}}(F, \mathcal{R}), \mathcal{R})$$

is an isomorphism which localizes to the canonical isomorphism

$$\operatorname{Ev}_{\boldsymbol{F}} = \operatorname{Ev} \otimes_{\mathcal{R}} \mathcal{K}\colon \boldsymbol{F} \xrightarrow{\sim} \boldsymbol{F}^{**} = \operatorname{Hom}_{\mathcal{K}}(\operatorname{Hom}_{\mathcal{K}}(\boldsymbol{F}, \mathcal{K}), \mathcal{K}).$$

Also, Ev_F restricts V to give the evaluation map $\operatorname{Ev}_V\colon V \to V^{**}$. (Observe that the cokernel of the natural map $F^* \to V^*$ is a torsion module, so that V^{**} identifies as an $\mathcal{R}$-submodule of F^{**}.) Identifying $\boldsymbol{V}$ with $\boldsymbol{V}^{**}$ via the restriction to $\boldsymbol{V}$ of $\operatorname{Ev}_{\boldsymbol{F}}$, it follows that Ev_V is an isomorphism of $\mathcal{R}$-modules. Therefore, Theorem C.17 implies that V is $\mathcal{R}$-projective. □

We will often use this corollary as follows. Suppose H is an algebra over $\mathcal{R}$ and V, W are two H-modules which are projective and finitely generated as $\mathcal{R}$-modules. Let $F = \operatorname{Hom}_{\mathcal{R}}(V, W)$, which is finitely generated and projective as an $\mathcal{R}$-module. Then $\boldsymbol{V} := \operatorname{Hom}_H(V, W)_{\mathcal{K}} \cong \operatorname{Hom}_{H_{\mathcal{K}}}(V_{\mathcal{K}}, W_{\mathcal{K}})$ is a subspace of $F_{\mathcal{K}}$, so that $\boldsymbol{V} \cap F = \operatorname{Hom}_H(V, W)$ is finitely generated and projective over $\mathcal{R}$.

If M is an $\mathcal{R}$-module, let $\operatorname{Supp}(M)$ be its support, i.e., the set of all $\mathfrak{p} \in \operatorname{Spec} \mathcal{R}$ such that $M_{\mathfrak{p}} \neq 0$.

Lemma C.20. *Let $\mathcal{R}$ be a regular ring of Krull dimension $\leqslant 2$, and let A be an $\mathcal{R}$-algebra, finitely generated and projective as an $\mathcal{R}$-module. Suppose X, Y are A-modules which are finitely generated and projective as $\mathcal{R}$-modules. Assume $\mathrm{Ext}^1_A(X,Y) \neq 0$. Then there exists $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}^1_A(X,Y))$ such that $\mathrm{Ext}^1_{A_{\mathcal{k}}}(X_{\mathcal{k}}, Y_{\mathcal{k}}) \neq 0$, where $\mathcal{k}$ is the residue field $\mathcal{k}(\mathfrak{p}) = \mathcal{R}_{\mathfrak{p}}/\mathfrak{p}\mathcal{R}_{\mathfrak{p}}$.*

Proof. Since $\mathrm{Ext}^1_A(X,Y) \neq 0$, by the local-global property ($M = 0 \iff M_{\mathfrak{p}} = 0$, $\forall \mathfrak{p} \in \mathrm{Spec}\,\mathcal{R}$, for any finitely generated $\mathcal{R}$-module M), there is a prime ideal $\mathfrak{p}$ such that $\mathrm{Ext}^1_A(X,Y)_{\mathfrak{p}} \neq 0$. If $\mathfrak{p} = 0$ is in $\mathrm{Supp}(\mathrm{Ext}^1_A(X,Y))$ the assertion is clear, since $\mathrm{Ext}^1_{A_{\mathcal{k}}}(X_{\mathcal{k}}, Y_{\mathcal{k}})$ identifies in this case with the localization $\mathrm{Ext}^1_A(X,Y)_{\mathfrak{p}}$. If $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}^1_A(X,Y))$ has height 1, then the localization $\mathcal{R}_{\mathfrak{p}}$ is a discrete valuation ring. In this case,

$$\mathrm{Ext}^1_{A_{\mathfrak{p}}}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}) \cong \mathrm{Ext}^1_A(X,Y)_{\mathfrak{p}} \neq 0 \Longrightarrow \mathrm{Ext}^1_{A_{\mathcal{k}(\mathfrak{p})}}(X_{\mathcal{k}(\mathfrak{p})}, Y_{\mathcal{k}(\mathfrak{p})}) \neq 0;$$

see Exercise C.8.

It remains to check the case when $\mathcal{R}$ has Krull dimension 2, and $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}^1_A(X,Y))$ is a maximal ideal of height 2 such that there is no smaller prime ideal in $\mathrm{Supp}(\mathrm{Ext}^1_A(X,Y))$. Localizing at $\mathfrak{p}$, we can assume that $\mathcal{R}$ is a regular local ring. Choose $p \in \mathfrak{p}$ so that $\mathcal{R}/p\mathcal{R}$ is regular of Krull dimension 1 [**HAII**, Exer. 4.4.1], and so is a discrete valuation ring. We claim that $\mathrm{Ext}^1_{A/pA}(X/pX, Y/pY) \cong \mathrm{Ext}^1_A(X, Y/pY) \neq 0$. Assuming this claim, the result follows from the discrete valuation ring case above. But the natural map $\mathrm{Ext}^1_A(X,Y) \to \mathrm{Ext}^1_A(X, Y/pY)$ is nonzero, since otherwise multiplication by p is surjective as an endomorphism of $\mathrm{Ext}^1_A(X,Y)$. But, using the Nakayama lemma, this would imply that $\mathrm{Ext}^1_A(X,Y) = 0$ (since it is a finitely generated $\mathcal{R}$-module), a contradiction. Thus, $\mathrm{Ext}^1_A(X, Y/pY) \neq 0$, as required. $\square$

Proposition C.21. *Let $\mathcal{R}$ be a regular ring of Krull dimension $\leqslant 2$, and let A be an $\mathcal{R}$-algebra, finitely generated and projective as an $\mathcal{R}$-module. Suppose that X, Y are A-modules which are finitely generated and projective over $\mathcal{R}$. If $\mathrm{Ext}^n_A(X,Y) \neq 0$, for some $n \geqslant 1$, then there exists $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}^n_A(X,Y))$ such that, if $\mathcal{k} = \mathcal{R}_{\mathfrak{p}}/\mathfrak{p}\mathcal{R}_{\mathfrak{p}}$, then $\mathrm{Ext}^n_{A_{\mathcal{k}}}(X_{\mathcal{k}}, Y_{\mathcal{k}}) \neq 0$. If $\mathcal{k}'$ is any field extension of $\mathcal{k}$, we have $\mathrm{Ext}^n_{A_{\mathcal{k}'}}(X_{\mathcal{k}'}, Y_{\mathcal{k}'}) \neq 0$.*

Proof. We argue by induction on $n \geqslant 1$. The case $n = 1$ is handled by Lemma C.20. So assume the $n \geqslant 2$ and the result is true for Ext^m_A, when $m < n$. Form a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow X \longrightarrow 0$$

of A-modules, where P is a projective and finitely generated A-module. Then

$$\mathrm{Ext}^{n-1}_A(K,Y) \cong \mathrm{Ext}^n_A(X,Y), \quad \text{for all } n \geqslant 2.$$

So, the hypothesis that $\mathrm{Ext}^n_A(X,Y) \neq 0$ implies that $\mathrm{Ext}^{n-1}_A(K,Y) \neq 0$. Since K is necessarily projective and finitely generated over $\mathcal{R}$,

$$\mathrm{Ext}^{n-1}_{A_{\mathcal{k}(\mathfrak{p})}}(K_{\mathcal{k}(\mathfrak{p})}, Y_{\mathcal{k}(\mathfrak{p})}) \neq 0, \quad \text{for some } \mathfrak{p},$$

by induction. However, X and necessarily P are projective $\mathcal{R}$-modules, so that the above short exact sequence splits as a sequence of $\mathcal{R}$-modules. Hence, we also have a short sequence

$$0 \longrightarrow K_{\mathcal{k}(\mathfrak{p})} \longrightarrow P_{\mathcal{k}(\mathfrak{p})} \longrightarrow X_{\mathcal{k}(\mathfrak{p})} \longrightarrow 0$$

of $A_{\mathcal{k}(\mathfrak{p})}$-modules in which $P_{\mathcal{k}(\mathfrak{p})}$ is a projective $A_{\mathcal{k}(\mathfrak{p})}$-module. Therefore, $\mathrm{Ext}^n_{A_{\mathcal{k}(\mathfrak{p})}}(X_{\mathcal{k}(\mathfrak{p})}, Y_{\mathcal{k}(\mathfrak{p})}) \neq 0$. □

We now apply the Auslander–Goldman lemma (Theorem C.17) to link base change with Ext vanishing. If X, Y are A-modules for an $\mathcal{R}$-algebra A, then, for any commutative $\mathcal{R}$-algebra $\mathcal{R}'$, there is a natural homomorphism

$$\mathrm{Hom}_A(X,Y)_{\mathcal{R}'} \longrightarrow \mathrm{Hom}_{A_{\mathcal{R}'}}(X_{\mathcal{R}'}, Y_{\mathcal{R}'}).$$

In the following result, we provide a condition which guarantees that this homomorphism is an isomorphism.

Theorem C.22. *Let $\mathcal{R}$ be a regular ring of Krull dimension $\leqslant 2$ and let A be an $\mathcal{R}$-algebra which is finitely generated and projective as an $\mathcal{R}$-module. Suppose M, N are A-modules, finitely generated and projective over $\mathcal{R}$. If $\mathrm{Ext}^i_A(M,N) = 0$, for $i = 1, 2$, then*

$$\mathrm{Hom}_A(M,N)_{\mathcal{R}'} \cong \mathrm{Hom}_{A_{\mathcal{R}'}}(M_{\mathcal{R}'}, N_{\mathcal{R}'}), \tag{C.3.1}$$

for any commutative $\mathcal{R}$-algebra $\mathcal{R}'$.

Proof. Let

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a resolution of M by projective, finitely generated A-modules P_i. The hypothesis implies that the complex

$$0 \longrightarrow \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(P_0,N) \longrightarrow \mathrm{Hom}_A(P_1,N) \longrightarrow \mathrm{Hom}_A(P_2,N) \longrightarrow \mathrm{Hom}_A(P_3,N)$$

is exact. Also, because the ring $\mathcal{R}$ has Krull dimension at most 2, the remarks after the proof of Corollary C.19 imply that the terms in the above complex are $\mathcal{R}$-projective. If X is the kernel of the map $\mathrm{Hom}_A(P_2,N) \to \mathrm{Hom}_A(P_3,N)$, the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & X & \longrightarrow & \mathrm{Hom}_A(P_2,N) & \longrightarrow & \mathrm{Hom}_A(P_3,N) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & X^{**} & \longrightarrow & \mathrm{Hom}_A(P_2,N)^{**} & \longrightarrow & \mathrm{Hom}_A(P_3,N)^{**}
\end{array}$$

By the 5-Lemma, we have $X \cong X^{**}$, and hence X is also $\mathcal{R}$-projective. It follows that the acyclic complex

$$0 \longrightarrow \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_A(P_0,N) \longrightarrow \mathrm{Hom}_A(P_1,N) \longrightarrow X \longrightarrow 0 \tag{C.3.2}$$

splits as a complex of $\mathcal{R}$-modules (in the sense that the kernels and cokernels of the various maps are $\mathcal{R}$-direct summands). Hence, (C.3.2) remains acyclic after applying the functor $- \otimes_{\mathcal{R}} \mathcal{R}'$. Since P_0 and P_1 are A-projective, $\mathrm{Hom}_A(Q,N)_{\mathcal{R}'} \cong \mathrm{Hom}_{A_{\mathcal{R}'}}(Q_{\mathcal{R}'}, N_{\mathcal{R}'})$, for $Q = P_0$, P_1. Thus,

$$\mathrm{Hom}_A(M,N)_{\mathcal{R}'} \cong \mathrm{Hom}_{A_{\mathcal{R}'}}(M_{\mathcal{R}'}, N_{\mathcal{R}'}),$$

again by the 5-Lemma. This completes the proof. □

C.4. Integral quasi-hereditary algebras

This section generalizes the notion of a quasi-hereditary algebra over a field to the notion of a quasi-hereditary algebra over a noetherian domain. Specializaton to the case in which the base ring is a regular local ring results in a vanishing property for tilting modules. We do not develop a complete theory for integral quasi-hereditary algebras, but instead focus on those results needed for Chapter 9.

To begin with, let $\mathcal{R}$ be an arbitrary commutative, noetherian domain with fraction field $\mathcal{K}$. Let A be an arbitrary $\mathcal{R}$-algebra, finitely generated and projective as an $\mathcal{R}$-module. An ideal $\mathfrak{J}$ of A is a *heredity ideal* provided that

(IHI1) $A/\mathfrak{J}$ is $\mathcal{R}$-projective;

(IHI2) $\mathfrak{J}^2 = \mathfrak{J}$;

(IHI3) $\mathfrak{J}$ is projective as a left A-module;

(IHI4) $E := \mathrm{End}_A(\mathfrak{J})$ is $\mathcal{R}$-semisimple.

Here an $\mathcal{R}$-algebra E is $\mathcal{R}$-semisimple provided, for every $\mathfrak{p} \in \operatorname{Spec} \mathcal{R}$, the residue algebra $E(\mathfrak{p}) = E \otimes_{\mathcal{R}} \mathcal{k}(\mathfrak{p})$ is a semisimple algebra over the residue field $\mathcal{k}(\mathfrak{p}) := \mathcal{R}_{\mathfrak{p}}/\mathfrak{p}\mathcal{R}_{\mathfrak{p}}$. In case each $E(\mathfrak{p})$ is a split semisimple algebra, e.g., if E is a split semisimple $\mathcal{R}$-algebra in (IHI4) — in particular, if E is a direct product of matrix algebras $\mathrm{M}_n(\mathcal{R})$ — we call $\mathfrak{J}$ a *split* heredity ideal.

When $\mathcal{R} = \mathcal{k}$ is a field, the above definition agrees identically with the definition of a heredity ideal given in §C.1.

The algebra A is $\mathcal{R}$-*quasi-hereditary* (or a *quasi-hereditary algebra* over $\mathcal{R}$) if there exists a sequence

$$0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_t = A \tag{C.4.1}$$

of ideals in A such that $\mathfrak{J}_i/\mathfrak{J}_{i-1}$ is a heredity ideal in $A/\mathfrak{J}_{i-1}$, for $0 < i \leqslant t$. Such a sequence (C.4.1) is called a *heredity chain* in A. If each $\mathrm{End}_{A/\mathfrak{J}_{i-1}}(\mathfrak{J}_i/\mathfrak{J}_{i-1})$ is a split $\mathcal{R}$-semisimple algebra, then A is called a *split* quasi-hereditary algebra.

We also have the following easy result.

Proposition C.23. *Suppose A is $\mathcal{R}$-quasi-hereditary. Then the algebra $A_{\mathcal{k}} = A \otimes_{\mathcal{R}} \mathcal{k}$ is quasi-hereditary in the sense of Definition C.6, for any field $\mathcal{k}$ which is an $\mathcal{R}$-algebra.*

Proof. Let A be $\mathcal{R}$-quasi-hereditary with heredity chain (C.4.1) and let $\bar{\mathfrak{J}}_{i,\mathcal{k}}$ be the image of $\mathfrak{J}_{i,\mathcal{k}} := \mathfrak{J}_i \otimes_{\mathcal{R}} \mathcal{k}$ in $A_{\mathcal{k}}$. Then the idempotent ideal $\bar{\mathfrak{J}}_{1,\mathcal{k}} \cong \mathfrak{J}_{1,\mathcal{k}}$ is a projective $A_{\mathcal{k}}$-module. Also, the projectivity of $\mathfrak{J}_1$ implies $\mathrm{End}_A(\mathfrak{J}_1)_{\mathcal{k}} \cong \mathrm{End}_{A_{\mathcal{k}}}(\mathfrak{J}_{1,\mathcal{k}})$ which is semisimple. Inductively, we see that $A_{\mathcal{k}}$ has a heredity chain

$$0 = \bar{\mathfrak{J}}_{0,\mathcal{k}} \subset \bar{\mathfrak{J}}_{1,\mathcal{k}} \subset \cdots \subset \bar{\mathfrak{J}}_{t,\mathcal{k}} = A_{\mathcal{k}},$$

proving the assertion. □

Definition C.24. Let A be an $\mathcal{R}$-quasi-hereditary algebra. Then A is $\mathcal{R}$-quasi-hereditary with poset Λ if, for any field $\mathcal{k}$ which is an $\mathcal{R}$-algebra, $A_{\mathcal{k}}$ is quasi-hereditary with poset Λ. Also, an A-module M, which is finitely generated and projective as an $\mathcal{R}$-module, is an *integral standard* (resp., *integral costandard*) module in A-mod corresponding to $\lambda \in \Lambda$ if, for any field $\mathcal{k}$ which is an $\mathcal{R}$-algebra, $M_{\mathcal{k}}$ is a standard (resp., costandard) module in $A_{\mathcal{k}}$-mod corresponding to λ.

In the next section, we will need the result below, which provides conditions under which the algebra A is quasi-hereditary. In the hypothesis, we require only that the noetherian domain $\mathcal{R}$ be normal (i.e., integrally closed in its fraction field $\mathcal{K}$.) This condition is automatic if $\mathcal{R}$ is regular.

Proposition C.25. *Assume that $\mathcal{R}$ is a normal noetherian domain with fraction field $\mathcal{K}$. Let A be an $\mathcal{R}$-algebra which is finitely generated and projective as an $\mathcal{R}$-module. Suppose that $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ is a family of A-modules such that the following conditions hold.*

(1) *For $\lambda \in \Lambda$, $\Delta(\lambda)$ is finitely generated and projective as an $\mathcal{R}$-module.*

(2) *For $\lambda \in \Lambda$, there exists a finitely generated projective A-module $P(\lambda)$ which has a decreasing filtration $F^\lambda_\bullet$: $P(\lambda) = F^\lambda_0 \supset F^\lambda_1 \supset \cdots \supset F^\lambda_{t_\lambda} = 0$ with sections $F^\lambda_i/F^\lambda_{i+1} \cong \Delta(\nu_{\lambda,i})$, for some $\nu_{\lambda,i} \in \Lambda$.*

(3) *For $\lambda \in \Lambda$, $\Delta(\lambda)_{\mathcal{K}}$ is an absolutely simple $A_{\mathcal{K}}$-module. For $\lambda \neq \mu \in \Lambda$, $\Delta(\lambda)_{\mathcal{K}}$ is not isomorphic to $\Delta(\mu)_{\mathcal{K}}$.*

(4) *In the Grothendieck group* $\mathscr{K}_0(A_{\mathcal{K}})$ *of finitely generated* $A_{\mathcal{K}}$*-modules,*

$$[P(\lambda)_{\mathcal{K}}] = [\Delta(\lambda)_{\mathcal{K}}] + \sum_{\zeta > \lambda} m_{\zeta,\lambda}[\Delta(\zeta)_{\mathcal{K}}], \quad \text{for all} \quad \lambda \in \Lambda.$$

(5) $P := \bigoplus_{\lambda\in\Lambda} P(\lambda)$ *is a progenerator for* A-mod. *In other words,* P *is projective and every finitely generated* A*-module is a homomorphic image of a finite direct sum of copies of* P.

Then A *is a split quasi-hereditary algebra over* $\mathcal{R}$ *with poset* Λ *and standard objects* $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$.

Proof. We prove that A is a split quasi-hereditary algebra over $\mathcal{R}$. The second assertion is left to the reader; see Exercise C.13.

Let $A' = \mathrm{End}_A(P)^{\mathrm{op}}$. Since P is a projective A-module, A' is finitely generated and projective as an $\mathcal{R}$-module. Also,

$$\mathscr{F} = \mathrm{Hom}_A(P, -)\colon A\text{-mod} \longrightarrow A'\text{-mod}$$

is an equivalence of module categories. The correspondence $\mathfrak{J} \mapsto \mathfrak{J}' := \mathscr{F}(\mathfrak{J}P) = \mathrm{Hom}_A(P, \mathfrak{J}P)$ defines a bijection between ideals in A and those in A'. In addition, a sequence of ideals $0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_t = A$ is a (split) heredity chain in A if and only if $0 = \mathfrak{J}'_0 \subset \mathfrak{J}'_1 \subset \cdots \subset \mathfrak{J}'_t = A'$ is a (split) heredity chain in A'; see Exercise C.12. Thus, it suffices to prove that A' is a split quasi-hereditary algebra over $\mathcal{R}$.

For $\lambda \in \Lambda$, let $P'(\lambda) = \mathscr{F}(P(\lambda))$ and $\Delta'(\lambda) = \mathscr{F}(\Delta(\lambda))$. It is easy to see that the hypotheses (1)–(5) hold, for the modules $P'(\lambda)$, $\Delta'(\lambda)$, $\lambda \in \Lambda$.

Let $\Lambda_1 \subseteq \Lambda$ be the set of maximal elements in the poset Λ. By condition (4), there exists an idempotent $e'_1 \in A'$ so that $A'e'_1 \cong \bigoplus_{\lambda\in\Lambda_1} P'(\lambda)$. Also, conditions (1)–(3) imply that $P'(\lambda) \cong \Delta'(\lambda)$, for all $\lambda \in \Lambda_1$. Therefore, given $\mu \in \Lambda$, in the filtration

$$F^{\mu}_{\bullet}\colon P'(\mu) = F^{\mu}_0 \supset F^{\mu}_1 \supset \cdots \supset F^{\mu}_{t_\mu} = 0$$

of $P'(\mu)$ with sections $F^{\mu}_i/F^{\mu}_{i+1} \cong \Delta(\nu_{\mu,i})$, for some $\nu_{\mu,i} \in \Lambda$, we can assume that the sections $\Delta'(\lambda)$, for $\lambda \in \Lambda_1$, are concentrated in the "tail" of the filtration, i.e., there exist an integer j_μ and nonnegative integers $m_{\zeta,\mu}$ such that $F^{\mu}_{j_\mu} = \bigoplus_{\zeta\in\Lambda_1} m_{\zeta,\mu}\Delta'(\zeta)$, but, for $i < j_\mu$, $\nu_{\mu,i} \notin \Lambda_1$. By (1) and (3), $\mathrm{Hom}_{A'}(\Delta'(\lambda), \Delta'(\zeta)) = 0$, for $\lambda \neq \zeta$. The ideal $\mathfrak{J}'_1 = A'e'_1A'$ in A' is the trace ideal of $A'e'_1$ in A', namely, $\mathfrak{J}'_1$ is the submodule of A' generated by the images of all homomorphisms $A'e'_1 \to A'$. Thus, $\mathfrak{J}'_1 \cong \bigoplus_{\mu\in\Lambda} F^{\mu}_{j_\mu}$. Because the $\Delta'(\lambda)$, $\lambda \in \Lambda_1$, are all projective $\mathcal{R}$-modules, it follows that $A'/\mathfrak{J}'_1$ is also a projective $\mathcal{R}$-module. By construction, $\mathfrak{J}'_1$ is projective as a left A'-module. Hence, conditions (IHI1)–(IHI3) hold.

For $\lambda \in \Lambda$, $\mathrm{End}_{A'}(\Delta'(\lambda))$ is a finitely generated $\mathcal{R}$-module such that $\mathrm{End}_{A'}(\Delta'(\lambda))_{\mathcal{K}} \cong \mathrm{End}_{A'_{\mathcal{K}}}(\Delta'(\lambda)_{\mathcal{K}}) \cong \mathcal{K}$. Since $\mathcal{R}$ is normal, $\mathrm{End}_{A'}(\Delta'(\lambda)) \cong \mathcal{R}$. Thus, $\mathrm{End}_{A'}(\mathfrak{J}'_1)$ is a direct product of matrix algebras $\mathrm{M}_n(\mathcal{K})$. So (IHI4) holds, and, in fact, $\mathfrak{J}'_1$ is a split hereditary ideal.

Let $A'_1 = A'/\mathfrak{J}'_1$. For $\lambda \in \Lambda\backslash\Lambda_1$, put

$$P'_1(\lambda) = P'(\lambda)/\mathfrak{J}'_1 P'(\lambda) = P'(\lambda)/F^{\lambda}_{j_\lambda}.$$

Then each $P'_1(\lambda)$ is a projective A'_1-module. Together with the family $\{\Delta'(\lambda)\}_{\lambda\in\Lambda\backslash\Lambda_1}$ of A'_1-modules, all the hypotheses (1)–(5) are satisfied. It follows by induction on the cardinality of Λ that A' is a split $\mathcal{R}$-quasi-hereditary algebra with poset Λ and standard objects $\Delta'(\lambda)$. □

In the rest of the section, we will consider quasi-hereditary algebras over a regular ring $\mathcal{R}$ of Krull dimension $\leqslant 2$ in order to develop the associated tilting module theory.

Tilting modules for quasi-hereditary algebras over a field were introduced in §C.2; see Proposition C.14. We now introduce their integral version.

Definition C.26. Let A be an $\mathcal{R}$-quasi-hereditary algebra with poset Λ. An A-module M, which is finitely generated and projective as an $\mathcal{R}$-module, is an *integral tilting module* if, for any field $\mathcal{k}$ which is an $\mathcal{R}$-algebra, $M_{\mathcal{k}}$ has both a Δ-filtration and a ∇-filtration.

In other words, M is an integral tilting module if and only if $M_{\mathcal{k}}$ is a tilting module, for every such field $\mathcal{k}$. If integral standard and costandard objects exist in A-mod in the sense of Definition C.24, then an A-module, which has both integral Δ- and ∇-filtrations, is an integral tilting module; see §9.5 for the example of quantum Schur algebras.

The following homological vanishing property holds for integral tilting modules.

Theorem C.27. *Assume that $\mathcal{R}$ is a regular ring of Krull dimension $\leqslant 2$. Let A be an $\mathcal{R}$-quasi-hereditary algebra with poset Λ. Let M, N be A-modules which are finitely generated and projective as $\mathcal{R}$-modules. If $M_{\mathcal{k}}$ (resp., $N_{\mathcal{k}}$) has a Δ-filtration (resp., ∇-filtration), for any field $\mathcal{k}$ which is an $\mathcal{R}$-algebra, then $\mathrm{Ext}^n_A(M,N) = 0$, for all $n > 0$. In particular, if M, N are integral tilting modules, then $\mathrm{Ext}^n_A(M,N) = 0$, for all $n > 0$.*

Proof. By Definition C.26, the second assertion is clearly a special case of the first assertion. The first assertion follows from Theorem C.13(1) by using Proposition C.21. □

C.5. Algebras with a Specht datum

The main result of this brief section, given in Theorem C.29, provides a way to obtain a split quasi-hereditary algebra as a certain endomorphism algebra. Throughout this section, we assume that $\mathcal{R}$ is a regular ring of Krull dimension $\leqslant 2$. In the applications, given in Chapter 9, $\mathcal{R}$ will be the ring $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in a variable v. Let $\mathcal{K}$ be the fraction field of $\mathcal{R}$.

We begin with the following definition which introduces the basic setup.

Definition C.28. Let H be an $\mathcal{R}$-algebra, free of finite rank as an $\mathcal{R}$-module. Assume that $H_{\mathcal{K}}$ is a split semisimple algebra. A triple (Λ, T, S) is called a *Specht datum* of H provided the following conditions hold:

(SD1) Λ is a poset, T and S are functions from Λ to the set of right H-modules which are finitely generated and projective over $\mathcal{R}$;

(SD2) For $\lambda \in \Lambda$, $S(\lambda)_{\mathcal{K}}$ is an absolutely simple $H_{\mathcal{K}}$-module. If λ, ζ are distinct elements in Λ, then $S(\lambda)_{\mathcal{K}}$ is not isomorphic to $S(\zeta)_{\mathcal{K}}$;

(SD3) For $\lambda \in \Lambda$, $T(\lambda)_{\mathcal{K}} \cong S(\lambda)_{\mathcal{K}} \oplus \big(\bigoplus_{\zeta > \lambda} d_{\zeta,\lambda} S(\zeta)_{\mathcal{K}}\big)$, for nonnegative integers $d_{\zeta,\lambda}$;

(SD4) For $\lambda \in \Lambda$, $T(\lambda)$ has an increasing filtration

$$0 = G_\lambda^0 \subset G_\lambda^1 \subset \cdots \subset G_\lambda^{t_\lambda} = T(\lambda)$$

with sections $G_\lambda^{i+1}/G_\lambda^i \cong S(\nu_{\lambda,i})$, where $\nu_{\lambda,i} \in \Lambda$, for $0 \leqslant i < t_\lambda$. Furthermore, for any $\mu \in \Lambda$,

$$\mathrm{Ext}^1_H(T(\lambda)/G_\lambda^i, T(\mu)) = 0, \quad 0 \leqslant i \leqslant t_\lambda.$$

Let (Λ, T, S) be a Specht datum. For an (arbitrary) collection $\{m_\lambda\}_{\lambda \in \Lambda}$ of positive integers, put

$$T = T(\Lambda) := \bigoplus_{\lambda \in \Lambda} m_\lambda T(\lambda) \quad \text{and} \quad A = A(\Lambda) := \mathrm{End}_H(T(\Lambda)). \qquad \text{(C.5.1)}$$

If the multiplicities m_λ are changed, then the algebra A is replaced by a Morita equivalent algebra. We have the following result.

Theorem C.29. *Let $\mathcal{R}$ be a regular ring of Krull dimension $\leqslant 2$. Let (Λ, T, S) be a Specht datum as given in Definition C.28. Let $T = T(\Lambda)$ and $A = A(\Lambda)$ be as in* (C.5.1). *For $\lambda \in \Lambda$, put $\Delta(\lambda) = \mathrm{Hom}_H(S(\lambda), T) \in A$-mod. Then A is a split quasi-hereditary algebra over $\mathcal{R}$ with poset Λ and standard modules $\{\Delta(\lambda)\}_{\lambda \in \Lambda}$.*

Proof. We verify the hypotheses (1)–(5) of Proposition C.25, using the fact that $\mathcal{R}$ is normal since it is regular. First, hypothesis (1) follows immediately from Corollary C.19. Next, given $\lambda \in \Lambda$, put $P(\lambda) := \mathrm{Hom}_H(T(\lambda), T)$. For a given i, let $F_i^\lambda := \mathrm{Hom}_H(T(\lambda)/G_\lambda^i, T)$. Then

$$P(\lambda) = F_0^\lambda \supset F_2^\lambda \supset \cdots \supset F_{t_\lambda}^\lambda = 0$$

is a filtration of $P(\lambda)$. Apply the functor $\mathrm{Hom}_H(-, T)$ to the short exact sequence $0 \to G_\lambda^{i+1}/G_\lambda^i \to T(\lambda)/G_\lambda^i \to T(\lambda)/G_\lambda^{i+1} \to 0$ and use the Ext_H^1-vanishing property contained in condition (SD4) in Definition C.28 to obtain that $F_i^\lambda/F_{i+1}^\lambda \cong \Delta(\nu_{\lambda,i})$. Thus, hypothesis (2) holds. Hypotheses (3) and (4) of Proposition C.25 are clear since $H_{\mathcal{K}}$ is semisimple.

Finally, by construction, $A = \bigoplus_{\lambda\in\Lambda} m_\lambda P(\lambda)$, so that $P := \bigoplus_{\lambda\in\Lambda} P(\lambda)$ is a progenerator for A-mod. Therefore, hypothesis (5) holds. □

C.6. Cellular algebras

In this final section, we introduce cellular algebras and their representations.

Definition C.30 (Graham–Lehrer). Let A be an algebra over a commutative ring $\mathcal{R}$. A 4-tuple (Λ, I, C, τ) is called a *cellular datum* of A if the following conditions hold:

(CA1) Λ is a poset and $\{I(\lambda)\}_{\lambda\in\Lambda}$ is a collection of disjoint (nonempty) finite sets;

(CA2) $C\colon \bigcup_{\lambda\in\Lambda} I(\lambda) \times I(\lambda) \to A$ is an injective map whose image

$$\{C_{i,j}^\lambda := C(i,j) \mid \lambda \in \Lambda,\ i, j \in I(\lambda)\}$$

is an $\mathcal{R}$-basis for A;

(CA3) τ is an $\mathcal{R}$-linear anti-involution of A such that $\tau(C_{i,j}^\lambda) = C_{j,i}^\lambda$, for all $\lambda \in \Lambda$ and $i, j \in I(\lambda)$; and

(CA4) if $\lambda \in \Lambda$ and $i, j \in I(\lambda)$, then, for any $a \in A$,

$$aC_{i,j}^\lambda \equiv \sum_{i'\in I(\lambda)} f_a^\lambda(i', i) C_{i',j}^\lambda \mod A^{>\lambda}, \tag{C.6.1}$$

where $f_a^\lambda(i', i) \in \mathcal{R}$ is independent of j, and $A^{>\lambda}$ is the $\mathcal{R}$-submodule spanned by all $C_{i,j}^\mu$ with $\mu > \lambda$, $i, j \in I(\mu)$.[2]

An algebra which possesses a cellular datum is called a *cellular algebra*, and the basis $\{C_{i,j}^\lambda \mid \lambda \in \Lambda, i, j \in I(\lambda)\}$ is called a *cellular basis*.

Applying τ to (C.6.1) yields

$$C_{j,i}^\lambda \tau(a) \equiv \sum_{i'\in I(\lambda)} f_a^\lambda(i', i) C_{j,i'}^\lambda \mod A^{>\lambda}. \tag{C.6.2}$$

[2]For convenient comparison with quasi-hereditary algebras, the order used in the definition here is opposite to the one used in Graham–Lehrer's original definition.

For simplicity, we often write $f_a(i', i)$ for $f_a^\lambda(i', i)$.

A cellular basis has remarkable properties, especially in the construction of representations of A.

Let A be a cellular algebra with cellular datum (Λ, I, C, τ). Let $A^{\geqslant\lambda}$ be the $\mathcal{R}$-submodule spanned by all $C_{i,j}^\mu$ with $\mu \geqslant \lambda$, $i, j \in I(\mu)$. Clearly, both $A^{\geqslant\lambda}$ and $A^{>\lambda}$ are two-sided ideals. Hence, $A^\lambda := A^{\geqslant\lambda}/A^{>\lambda}$ is an A-A-bimodule which is also an ideal of the quotient algebra $A/A^{>\lambda}$. This bimodule is a direct sum of the left (resp., right) A-modules $A_{\bullet,j}^\lambda$ (resp., $A_{j,\bullet}^\lambda$), $j \in I(\lambda)$, where $A_{\bullet,j}^\lambda$ (resp., $A_{j,\bullet}^\lambda$) is spanned by the images of $C_{i,j}^\lambda$ (resp., $C_{j,i}^\lambda$), for all $i \in I(\lambda)$.

Let $\mathsf{C}(\lambda)$ be a free $\mathcal{R}$-module with basis c_i^λ, $i \in I(\lambda)$. The A-action on $\mathsf{C}(\lambda)$ defined by

$$a \cdot \mathsf{c}_i^\lambda := \sum_{i' \in I(\lambda)} f_a(i', i)\mathsf{c}_{i'}^\lambda, \quad \text{for all } a \in A, \tag{C.6.3}$$

gives rise to a representation of A. We call $\mathsf{C}(\lambda)$ a *cell representation* or a *cell module*. Let $\mathsf{C}(\lambda)^\tau$ be the right A-module with the A-action twisted by τ. Thus, $\mathsf{C}(\lambda)^\tau = \mathsf{C}(\lambda)$ as an $\mathcal{R}$-module with the right A-action $\mathsf{c}_i^\lambda * a := \tau(a)\mathsf{c}_i^\lambda$, for all $a \in A$.

Lemma C.31. (1) *If $C_{i',j'}^\mu \mathsf{C}(\lambda) \neq 0$, then $\lambda \geqslant \mu$. Hence, $\mathsf{C}(\lambda)$ is naturally an $A/A^{>\lambda}$-module.*

(2) *For every $\lambda \in \Lambda$ and $j \in I(\lambda)$, the map sending c_i^λ to $C_{i,j}^\lambda + A^{>\lambda}$, for all $i \in I(\lambda)$, defines an A-module isomorphism $\mathsf{C}(\lambda) \cong A_{\bullet,j}^\lambda$. Similarly, $\mathsf{C}(\lambda)^\tau \cong A_{j,\bullet}^\lambda$, for all $j \in I(\lambda)$.*

(3) *There is a natural isomorphism of A-A-bimodules*

$$m_\lambda \colon \mathsf{C}(\lambda) \otimes_{\mathcal{R}} \mathsf{C}(\lambda)^\tau \longrightarrow A^\lambda \tag{C.6.4}$$

defined by $m_\lambda(\mathsf{c}_i^\lambda \otimes \mathsf{c}_j^\lambda) = C_{i,j}^\lambda + A^{>\lambda}$, for all $i, j \in I(\lambda)$.

Proof. If $C_{i',j'}^\mu \mathsf{C}(\lambda) \neq 0$, (C.6.3) implies that $C_{i',j'}^\mu C_{i,j}^\lambda \not\equiv 0 \bmod A^\lambda$, for some $i \in I(\lambda)$ (and all $j \in I(\lambda)$). Since $C_{i',j'}^\mu C_{i,j}^\lambda \in A^{\geqslant\mu}$, it follows that $\lambda \geqslant \mu$, proving (1). Now (2) and (3) are clear from the definition. $\square$

We also observe from (C.6.1) and (C.6.2) that, for any $\lambda \in \Lambda$ and $i, j, k, l \in I(\lambda)$,

$$\begin{aligned} C_{i,j}^\lambda C_{k,l}^\lambda &\equiv \sum_{k' \in I(\lambda)} f_{C_{i,j}^\lambda}(k', k) C_{k',l}^\lambda \bmod A^{>\lambda} \\ &\equiv \sum_{j' \in I(\lambda)} f_{C_{l,k}^\lambda}(j', j) C_{i,j'}^\lambda \bmod A^{>\lambda}. \end{aligned}$$

Thus, all $f_{C^\lambda_{i,j}}(k',k) = f_{C^\lambda_{l,k}}(j',j) = 0$ if $k' \neq i$ or $j' \neq l$, and $f_{C^\lambda_{i,j}}(i,k) = f_{C^\lambda_{l,k}}(l,j)$ is independent of i,l. Hence, for any $\lambda \in \Lambda$ and $j,k \in I(\lambda)$, there is a unique element $g(j,k) = g^\lambda(j,k) \in \mathcal{R}$ such that, for all $i,l \in I(\lambda)$,

$$C^\lambda_{i,j} C^\lambda_{k,l} \equiv g(j,k) C^\lambda_{i,l} \mod A^{>\lambda}. \tag{C.6.5}$$

Define the symmetric bilinear form

$$\beta_\lambda \colon \mathsf{C}(\lambda) \times \mathsf{C}(\lambda) \longrightarrow \mathcal{R} \tag{C.6.6}$$

by putting $\beta_\lambda(\mathsf{c}^\lambda_i, \mathsf{c}^\lambda_j) = g(i,j)$, for all $i,j \in I(\lambda)$. In the following result, we regard this bilinear form as a pairing $\beta_\lambda \colon \mathsf{C}(\lambda)^\tau \times \mathsf{C}(\lambda) \to \mathcal{R}$.

Lemma C.32. (1) *As a pairing, β_λ is associative in the sense that*

$$\beta_\lambda(\mathsf{c}^\lambda_j a, \mathsf{c}^\lambda_i) = \beta_\lambda(\mathsf{c}^\lambda_j, a\mathsf{c}^\lambda_i), \quad \text{for all } i,j \in I(\lambda) \text{ and } a \in A.$$

Hence, it induces an $\mathcal{R}$-linear map $\varphi_\lambda \colon \mathsf{C}(\lambda)^\tau \otimes_A \mathsf{C}(\lambda) \to \mathcal{R}$. In particular, the image $\mathrm{Im}(\varphi_\lambda)$ is an ideal of $\mathcal{R}$ generated by all the $g(j,i)$.

(2) *If $a, a' \in \mathsf{C}(\lambda)$ and $b, b' \in \mathsf{C}(\lambda)^\tau$, then the associative relations*

$$m_\lambda(a \otimes b) \cdot a' = a \cdot \beta_\lambda(b, a') = a \cdot \varphi_\lambda(b \otimes a'),$$
$$b \cdot m_\lambda(a' \otimes b') = \beta_\lambda(b, a') \cdot b' = \varphi_\lambda(b \otimes a') \cdot b'$$

hold. Here $\mathsf{C}(\lambda)$ is regarded as an $\mathcal{R}$-$\mathcal{R}$-bimodule via $a \cdot r = ra$, for all $a \in \mathsf{C}(\lambda)$, $r \in \mathcal{R}$.

(3) *If there exists $z \in \mathsf{C}(\lambda)$ such that $\mathcal{I}_z := \{\beta_\lambda(y,z) \mid y \in \mathsf{C}(\lambda)\} = \mathcal{R}$, then*

(a) $\mathrm{Hom}_A(\mathsf{C}(\lambda),\ \mathsf{C}(\mu)) = 0$, *unless $\lambda \leqslant \mu$;*

(b) $\mathrm{Hom}_A(\mathsf{C}(\lambda), \mathsf{C}(\lambda)) \cong \mathcal{R}$.

Proof. By (C.6.3) and the definitions,

$$\beta_\lambda(\mathsf{c}^\lambda_j a, \mathsf{c}^\lambda_i) = \sum_{j' \in I(\lambda)} f_{\tau(a)}(j',j) \beta_\lambda(\mathsf{c}^\lambda_{j'}, \mathsf{c}^\lambda_i) = \sum_{j' \in I(\lambda)} f_{\tau(a)}(j',j) g(j',i).$$

Similarly, $\beta_\lambda(\mathsf{c}^\lambda_j, a\mathsf{c}^\lambda_i) = \sum_{i' \in I(\lambda)} g(j,i') f_a(i',i)$. But, for fixed $i_0, j_0 \in I(\lambda)$,

$$\begin{aligned} \sum_{j' \in I(\lambda)} f_{\tau(a)}(j',j) g(j',i) C^\lambda_{i_0,j_0} &\equiv (C^\lambda_{i_0,j} a) C^\lambda_{i,j_0} = C^\lambda_{i_0,j}(a C^\lambda_{i,j_0}) \\ &\equiv \sum_{i' \in I(\lambda)} g(j,i') f_a(i',i) C^\lambda_{i_0,j_0} \mod A^{>\lambda}. \end{aligned}$$

Hence, $\beta_\lambda(\mathsf{c}^\lambda_j a, \mathsf{c}^\lambda_i) = \beta_\lambda(\mathsf{c}^\lambda_j, a\mathsf{c}^\lambda_i)$, proving (1).

To prove (2), we may assume $a = \mathsf{c}^\lambda_i, b = \mathsf{c}^\lambda_j$, and $a' = \mathsf{c}^\lambda_{i'}, b' = \mathsf{c}^\lambda_{j'}$. Then

$$m_\lambda(\mathsf{c}^\lambda_i \otimes \mathsf{c}^\lambda_j) \cdot \mathsf{c}^\lambda_{i'} = (C^\lambda_{i,j} + A^{>\lambda}) \mathsf{c}^\lambda_{i'} = g(j,i') \mathsf{c}^\lambda_i = \mathsf{c}^\lambda_i \cdot \beta_\lambda(\mathsf{c}^\lambda_j, \mathsf{c}^\lambda_{i'}).$$

The second equality can be proved similarly.

It remains to prove (3). By (2) and Lemma C.31(2),

$$\mathsf{C}(\lambda) \supseteq Az \supseteq A^{\lambda}z = \operatorname{span}\{x\beta_\lambda(y,z) \mid x, y \in \mathsf{C}(\lambda)\} = \mathcal{I}_z\mathsf{C}(\lambda).$$

Thus, $\mathcal{I}_z = \mathcal{R}$ implies $\mathsf{C}(\lambda) = Az = A^{\lambda}z$.

If $\operatorname{Hom}_A(\mathsf{C}(\lambda), \mathsf{C}(\mu)) \neq 0$, then there exist $x \in \mathsf{C}(\lambda)$, $a \in A^{\lambda}$, and $f \in \operatorname{Hom}_A(\mathsf{C}(\lambda), \mathsf{C}(\mu))$ such that $af(x) = f(ax) \neq 0$. Hence, $\lambda \leqslant \mu$ by Lemma C.31(1). Finally, choose $y \in \mathsf{C}(\lambda)$ such that $\beta_\lambda(y,z) = 1$, and define a map

$$\gamma\colon \operatorname{Hom}_A(\mathsf{C}(\lambda), \mathsf{C}(\lambda)) \longrightarrow \mathcal{R}, \quad f \longmapsto \beta_\lambda(y, f(z)).$$

Since, for $f \in \operatorname{Hom}_A(\mathsf{C}(\lambda), \mathsf{C}(\lambda))$,

$$f(z) = f(z\beta_\lambda(y,z)) = m_\lambda(z \otimes y)f(z) = \beta_\lambda(y, f(z))z,$$

$\mathsf{C}(\lambda) = Az$ implies that γ is an isomorphism of algebras. □

In the rest of the section, let $\mathcal{R} = \mathcal{k}$ be a field. We discuss the representation theory of finite dimensional cellular algebras over $\mathcal{k}$.

Theorem C.33. *Suppose A is a finite dimensional cellular $\mathcal{k}$-algebra having cellular datum (Λ, I, C, τ) and let $\Lambda_1 = \{\lambda \in \Lambda \mid \beta_\lambda \neq 0\}$.*

(1) *For any $\lambda \in \Lambda_1$,*

$$\operatorname{rad}(\mathsf{C}(\lambda)) = \{v \in \mathsf{C}(\lambda) \mid \beta_\lambda(y,v) = 0, \text{ for all } y \in \mathsf{C}(\lambda)^{\tau}\},$$

and $L(\lambda) := \mathsf{C}(\lambda)/\operatorname{rad}(\mathsf{C}(\lambda))$ is simple.

(2) *Let $\lambda \in \Lambda_1$. If $L(\lambda)$ is a composition factor of $\mathsf{C}(\mu)$, then $\lambda \leqslant \mu$. Also, $[\mathsf{C}(\lambda){:}L(\lambda)] = 1$.*

(3) *For any $\lambda \in \Lambda_1$, $L(\lambda)$ is absolutely simple.*

(4) *$\{L(\lambda) \mid \lambda \in \Lambda_1\}$ is a complete set of all nonisomorphic simple A-modules.*

(5) *If A is semisimple, then $L(\lambda) = \mathsf{C}(\lambda)$, for all $\lambda \in \Lambda = \Lambda_1$. Therefore, A is split semisimple.*

Proof. Let $\mathfrak{r} = \{v \in \mathsf{C}(\lambda) \mid \beta_\lambda(y,v) = 0, \text{ for all } y \in \mathsf{C}(\lambda)^{\tau}\}$. If $\lambda \in \Lambda_1$, then $\mathfrak{r} \neq \mathsf{C}(\lambda)$. For $0 \neq z \in \mathsf{C}(\lambda)/\mathfrak{r}$, write $z = z_1 + \mathfrak{r}$. Since $z_1 \notin \mathfrak{r}$, there exists an element $y \in \mathsf{C}(\lambda)^{\tau}$ such that $\beta_\lambda(y, z_1) = 1$. So, for any $x \in \mathsf{C}(\lambda)$, Lemma C.32(2) implies $x = x\beta_\lambda(y,z_1) = m_\lambda(x \otimes y)z_1 \in Az_1$. This shows that $\mathsf{C}(\lambda) = Az_1 = A^{\lambda}z_1$ and $\mathsf{C}(\lambda)/\mathfrak{r} = Az$. Hence, $\mathsf{C}(\lambda)/\mathfrak{r}$ is a simple left A-module, and, consequently, $\mathfrak{r} \supseteq \operatorname{rad}(\mathsf{C}(\lambda))$. If $\operatorname{rad}(\mathsf{C}(\lambda)) \neq \mathfrak{r}$, then there is a simple module L in $\operatorname{top}(\mathsf{C}(\lambda))$ such that the A-module epimorphism $\eta\colon \mathsf{C}(\lambda) \to L$ does not map $\mathfrak{r}$ to zero. Therefore, $\eta(\mathfrak{r}) = L$. Thus, $\eta(z_1) = \eta(u)$, for some $u \in \mathfrak{r}$, but $\beta_\lambda(y,u) = 0$ as $u \in \mathfrak{r}$. By Lemma C.32(2) again,

$$\begin{aligned}\eta(z_1) &= \eta(z_1\beta_\lambda(y,z_1)) = m_\lambda(z_1 \otimes y)\eta(z_1)\\ &= \eta(m_\lambda(z_1 \otimes y)u) = \eta(z_1\beta_\lambda(y,u)) = 0.\end{aligned}$$

So $\eta(\mathsf{C}(\lambda)) = 0$, a contradiction. Therefore, $\mathrm{rad}(\mathsf{C}(\lambda)) = \mathfrak{r}$, proving (1).

We now prove (2). If $L(\lambda)$ is a composition factor of $\mathsf{C}(\mu)$, then there is a nonzero A-module homomorphism $\varphi\colon \mathsf{C}(\lambda) \to \mathsf{C}(\mu)/N$, for some A-submodule $N \subset \mathsf{C}(\mu)$ such that $\mathrm{Im}(\varphi) \cong L(\lambda)$. Since $\mathsf{C}(\lambda) = A^\lambda z_1$, for some $z_1 \in \mathsf{C}(\lambda)$ as above, Lemma C.31(1) immediately implies that $\mu \geqslant \lambda$. Now assume that $\mu = \lambda$, and let $\bar{\varphi}\colon \mathsf{C}(\lambda)/\mathfrak{r} \to \mathsf{C}(\lambda)/N$ be the map induced by φ. Given $0 \neq z \in \mathsf{C}(\lambda)/\mathfrak{r}$, write $\bar{\varphi}(z) = z' + N$, for some $z' \in \mathsf{C}(\lambda)$. Then, for any $x \in \mathsf{C}(\lambda)$,

$$\begin{aligned}\varphi(x) &= \varphi(x\beta_\lambda(y, z_1)) = m_\lambda(x \otimes y)\varphi(z)\\ &= m_\lambda(x \otimes y)(z' + N) = \beta_\lambda(y, z')x + N.\end{aligned}$$

The fact that $\varphi \neq 0$ implies that $\beta_\lambda(y, z') \neq 0$. Thus, φ is surjective, and, hence, $\mathsf{C}(\lambda)/N = \mathrm{Im}(\varphi) \cong L(\lambda)$, forcing $N = \mathrm{rad}(\mathsf{C}(\lambda))$ by (1). Consequently, $[\mathsf{C}(\lambda) : L(\lambda)] = 1$.

As for (3), $\mathrm{Hom}_A(L(\lambda), L(\lambda)) \subseteq \mathrm{Hom}_A(\mathsf{C}(\lambda), L(\lambda))$. By the argument above for (2), $\mathrm{Hom}_A(\mathsf{C}(\lambda), L(\lambda)) \cong \mathscr{k}$. Hence, $\mathrm{End}_A(L(\lambda)) \cong \mathscr{k}$.

Given $\lambda, \mu \in \Lambda_1$, we have, by (2), $L(\lambda) \cong L(\mu)$ implies $\lambda \leqslant \mu \leqslant \lambda$. Take a linear ordering $\lambda_1, \lambda_2, \ldots$ on Λ such that $i \leqslant j$ whenever $\lambda_i \geqslant \lambda_j$ and let $\mathfrak{J}_s$ be spanned by all $C_{i,j}^{\lambda_l}$ with $l \leqslant s$ and $i, j \in I(\lambda_l)$. Then

$$0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_m = A \tag{C.6.7}$$

is a filtration of two-sided ideals such that $\mathfrak{J}_s/\mathfrak{J}_{s-1} \cong A^{\lambda_s}$. If L is a simple A-module, choose s minimal so that $\mathfrak{J}_s L \neq 0$. Then $A^{\lambda_s} L$ makes sense and is not zero. By Lemma C.31(2), there exists a $j \in I(\lambda_s)$ such that the A-submodule $A_{\bullet,j}^{\lambda_s} L$ of L is nonzero. Thus, $L = A_{\bullet,j}^{\lambda_s} L = A_{\bullet,j}^{\lambda_s} v$, for some $v \in L$, and the map $x \mapsto xv$ defines an A-module epimorphism $A_{\bullet,j}^{\lambda_s} \to L$. Now, if $C_{i,j}^{\lambda_s} v \neq 0$, for some $i \in I(\lambda)$, this epimorphism implies that there exists an $i' \in I(\lambda)$ such that $C_{i,j}^{\lambda_s} C_{i',j}^{\lambda_s} \not\equiv 0 \bmod A^{>\lambda_s}$. In other words, $\beta_{\lambda_s} \neq 0$. Hence, $\lambda_s \in \Lambda_1$ and $L \cong \mathrm{top}\,\mathsf{C}(\lambda_s) \cong L(\lambda_s)$, proving (4).

Finally, if A is semisimple then any left A-module is semisimple. Thus, $\mathrm{rad}(\mathsf{C}(\lambda)) = 0$, and hence, $L(\lambda) = \mathsf{C}(\lambda)$, for any $\lambda \in \Lambda = \Lambda_1$. Now $\mathrm{End}_A(L(\lambda)) = \mathscr{k}$ implies that A is split semisimple. $\square$

Remarks C.34. (1) If A is a cellular algebra, then the opposite algebra A^{op} is also cellular with the same cellular datum. We can identify A^{op}-mod with the category mod-A of finite dimensional right A-modules. Let $\mathsf{C}(\lambda)^{\mathrm{op}}$ be the cell module corresponding to $\lambda \in \Lambda$. Then, $\mathsf{C}(\lambda)^{\mathrm{op}}$ has an $\mathscr{R}$-basis $\{\mathsf{b}_i^\lambda\}_{i \in I(\lambda)}$ with the right A-action defined by

$$\mathsf{b}_i^\lambda \cdot a = \sum_{i' \in I(\lambda)} f_{\tau(a)}(i', i)\mathsf{b}_{i'}^\lambda$$

(cf. (C.6.2)). Thus, the map $\mathsf{b}_i^\lambda \mapsto \mathsf{c}_i^\lambda$ defines a right A-module isomorphism $\mathsf{C}(\lambda)^{\mathrm{op}} \cong \mathsf{C}(\lambda)^\tau$.

(2) Given a cellular algebra A over a field $\mathcal{k}$, the anti-involution τ on A defines naturally a contravariant equivalence

$$\mathfrak{d}\colon A\text{-}\mathsf{mod} \longrightarrow A\text{-}\mathsf{mod}$$

such that, for an A-module M, $\mathfrak{d}(M) = (M^*)^\tau = (M^\tau)^*$. We claim that $\mathfrak{d}(L) \cong L$, for every simple A-module L. Indeed, by Theorem C.33(4), we assume $L = L(\lambda)$, for some $\lambda \in \Lambda_1$. Then $L(\lambda) = \mathsf{C}(\lambda)/\operatorname{rad}(\mathsf{C}(\lambda))$, where the rad is defined by the bilinear form

$$\beta_\lambda\colon \mathsf{C}(\lambda)^{\mathrm{op}} \times \mathsf{C}(\lambda) \longrightarrow \mathcal{k}.$$

Let $L(\lambda)^{\mathrm{op}} = \mathsf{C}(\lambda)^{\mathrm{op}}/\operatorname{rad}(\mathsf{C}(\lambda)^{\mathrm{op}})$. Then β_λ induces a nondegenerate bilinear form

$$\bar\beta_\lambda\colon L(\lambda)^{\mathrm{op}} \times L(\lambda) \longrightarrow \mathcal{k}.$$

Hence, there is a linear isomorphism $L(\lambda) \xrightarrow{\sim} (L(\lambda)^{\mathrm{op}})^*, v \mapsto \bar\beta_\lambda(-,v)$. By Lemma C.32(1), this linear isomorphism is an A-modules isomorphism. Hence, $\mathfrak{d}L(\lambda) \cong L(\lambda)$. Thus, the module ${}^{\mathfrak{d}}\mathsf{C}(\lambda) := \mathfrak{d}(\mathsf{C}(\lambda))$ has socle $L(\lambda)$ and the same composition factors as $\mathsf{C}(\lambda)$.

The functor $\mathfrak{d}$ is called a *strong duality functor* in the sense that $\mathfrak{d}^2 \cong \mathrm{id}_{A\text{-}\mathsf{mod}}$ and $\mathfrak{d}(L) \cong L$, for every simple A-module L.

As usual, denote the projective cover of $L(\lambda)$ by $P(\lambda)$. A sequence of submodules of an A-module M

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is called a *cell filtration* of M if every section M_i/M_{i-1} is isomorphic to $\mathsf{C}(\lambda)$, for some $\lambda \in \Lambda$.

Theorem C.35. *Suppose A is a finite dimensional cellular $\mathcal{k}$-algebra having cellular datum (Λ, I, C, τ). For $\lambda \in \Lambda_1$, there exists a cell filtration of the projective cover $P(\lambda)$ such that if $[P(\lambda){:}\mathsf{C}(\mu)]$ denotes the number of sections of the filtration isomorphic to $\mathsf{C}(\mu)$, then $[P(\lambda) : \mathsf{C}(\mu)] \neq 0 \Longrightarrow \mu \geqslant \lambda$ and $[P(\lambda){:}\mathsf{C}(\lambda)] = 1$.*

Proof. By the proof of Theorem C.33(4), for $\lambda \in \Lambda_1$, $A^{\geqslant\lambda}L(\lambda)$ makes sense and equals $L(\lambda)$. Thus, $A^{\geqslant\lambda}P(\lambda) = P(\lambda)$, since it is a submodule of $P(\lambda)$ which covers the top $L(\lambda)$. Now, for any projective A-module P and any ideal $\mathfrak{J}$ of A, $\mathfrak{J} \otimes_A P \cong \mathfrak{J}P$. Thus, $A^{\geqslant\lambda} \otimes P(\lambda)$ is isomorphic under multiplication to $P(\lambda)$. If we order $\{\mu \in \Lambda \mid \mu \geqslant \lambda\} = \{\lambda_1, \ldots, \lambda_t = \lambda\}$ such that $i < j$ whenever $\lambda_i > \lambda_j$, then we have a filtration of $A^{\geqslant\lambda}$ as in (C.6.7) with $t = m$. Tensoring this filtration with $P(\lambda)$ gives a filtration of $P(\lambda)$ by the projectivity of $P(\lambda)$. Now the sections of this filtration are of the form $A^{\lambda_i} \otimes_A P(\lambda) \cong \mathsf{C}(\lambda_i) \otimes_{\Bbbk} \mathsf{C}(\lambda_i)^\tau \otimes_A P(\lambda)$ $(1 \leqslant i \leqslant t)$, which is isomorphic to a direct sum of copies of $\mathsf{C}(\lambda_i)$, by Lemma C.31. In particular, $A^\lambda \otimes_A P(\lambda) \cong \mathsf{C}(\lambda)$, since, as a $\Bbbk$-space,

$$\begin{aligned}\mathsf{C}(\lambda)^\tau \otimes_A P(\lambda) &\cong \mathrm{Hom}_{\Bbbk}(\mathsf{C}(\lambda)^\tau \otimes_A P(\lambda), \Bbbk)\\ &\cong \mathrm{Hom}_A(\mathsf{C}(\lambda)^\tau, P(\lambda)^*)\\ &\cong \mathrm{Hom}_A(P(\lambda), {}^{\eth}\mathsf{C}(\lambda)) \cong \Bbbk,.\end{aligned}$$

by Remark C.34(2). All the assertions now follow easily. □

Corollary C.36. *Let A be a finite dimensional cellular algebra over a field $\mathcal{K} = \Bbbk$ with a cellular datum (Λ, I, C, τ) and assume $\beta_\lambda \neq 0$, for all $\lambda \in \Lambda$. Then A is a (split) quasi-hereditary algebra.*

Proof. By Theorems C.33(2) and C.35, the category A-mod is a highest weight category. The assertion now follows from Theorem C.10. □

Exercises and notes

Exercises

§§C.1–C.2

C.1. Let A be a finite dimensional algebra over $\Bbbk$ with Jacobson radical $\mathfrak{N}$. Suppose that $\bar{e} \in \bar{A} = A/\mathfrak{N}$ is an idempotent. Show there exists an idempotent $e \in A$ lifting $\bar{e}$, i.e., satisfying $\pi(e) = \bar{e}$ if $\pi\colon A \longrightarrow \bar{A}$ is the quotient morphism.

Hint: By induction on the degree of nilpotency of $\mathfrak{N}$, we can assume that $\mathfrak{N}^2 = 0$. In this case, let $a \in A$ satisfy $\pi(a) = \bar{e}$. Show that $e := -3a^4 + 4a^3$ is an idempotent in A satisfying $\pi(e) = \bar{e}$.

C.2. (1) Complete the dimension shifting argument in the proof of Proposition C.5 (2).

(2) Under the hypothesis of Proposition C.5, let N be a simple A-module such that $\mathfrak{J}N \neq 0$. Define an inclusion $N \hookrightarrow \mathrm{Hom}_{eAe}(eA, eN)$ of left A-modules. Show that there is an isomorphism

$$\mathrm{Hom}_A(-, \mathrm{Hom}_{eAe}(eA, eN)) \cong \mathrm{Hom}_{eAe}(eA \otimes_A -, eN)$$

of functors on A-mod. Using the semisimplicity of eAe, conclude that the right-hand side of this expression is exact, and so $\mathrm{Hom}_{eAe}(eA, eN)$ is an injective A-module.

C.3. Complete the proof of Theorem C.10 by showing that if A-mod is a highest weight category, then A is a quasi-hereditary algebra.

Hint: Choose a maximal element $\lambda \in \Lambda$. Thus, $\Delta(\lambda)$ is projective; hence, $\Delta(\lambda) \cong Ae$, for some primitive idempotent $e \in A$. Let $\mathfrak{J}_1$ be the ideal in A which is generated by the images of all A-module morphisms $\Delta(\lambda)$

$\to {}_AA$. Show that $\mathfrak{J}_1 \cong n\Delta(\lambda)$, for some positive integer n. Thus, $\mathfrak{J}_1$ is the image of the evaluation map $\mathrm{Hom}_A(Ae, A) \otimes Ae \to A$, $f \otimes x \mapsto f(x)$. But $\mathrm{Hom}_A(Ae, A) \cong eA$, so $\mathfrak{J}_1 \cong AeA$. Also, $eAe \cong \mathrm{Hom}_A(Ae, Ae) \cong \mathrm{End}_A(\Delta(\lambda)) \cong \mathrm{End}_A(L(\lambda)$ is a division ring. Conclude that $\mathfrak{J}_1$ is a heredity ideal. Continue with A replaced by $A/\mathfrak{J}_1$.

C.4. Prove Lemma C.12.

C.5. Let $\mathcal{C}$ be a highest weight category with weight poset Λ. Let Γ be an order ideal in Λ, i.e., if $\lambda \in \Gamma$, and $\mu \in \Lambda$ satisfies $\mu \leqslant \lambda$, then $\mu \in \Gamma$. Let $\Omega = \Lambda \backslash \Gamma$ be the complementary order coideal.

(1) For $\omega \in \Omega$, let $e_\omega \in A$ be a primitive idempotent such that $P(\omega) \cong Ae_\omega$. Form the idempotent $e = e_\Omega = \sum_{\omega \in \Omega} e_\omega$, and let $\mathfrak{J} = AeA$. Show that $A/\mathfrak{J}$-mod identifies with the full subcategory $\mathcal{C}[\Gamma]$ of $\mathcal{C}$ consisting of all objects which have composition factors $L(\gamma)$, $\gamma \in \Gamma$. Show that $\mathcal{C}[\Gamma]$ is a highest weight category with weight poset Γ.

(2) Let $\mathcal{C}(\Omega) = eAe$-mod. Show that $\mathcal{C}(\Omega)$ is a highest weight category with weight poset Ω. Let $j^*: \mathcal{C} \to \mathcal{C}(\Omega)$ be the (exact) functor defined by $j^*(M) = eM$. Determine the effect of j^* on standard and costandard objects in $\mathcal{C}$.

C.6. Let $\mathcal{C} = A$-mod be a highest weight category with weight poset Λ.

(1) Let $M \in \mathcal{C}(\Delta)$. Show that if $F_\bullet: M = F_0 \supset F_1 \supset \cdots \supset F_t = 0$ is any Δ-filtration of M and $\lambda \in \Lambda$, prove that the number $[M : \Delta(\lambda)]_{F_\bullet}$ of occurrences of $\Delta(\lambda)$ as a section F_i/F_{i+1} in $F_\bullet$ equals

$$\dim \mathrm{Hom}_A(M, \nabla(\lambda))/\dim \mathrm{End}_A(L(\lambda)).$$

Hence, $[M : \Delta(\lambda)]_{F_\bullet}$ is independent of $F_\bullet$; denote this multiplicity by $[M : \Delta(\lambda)]$. Formulate a similar result for the multiplicity $[N : \nabla(\lambda)]$ of $\nabla(\lambda)$ as a section in a ∇-filtration of N.

(2) Prove *Brauer–Humphreys reciprocity*: For $\lambda, \mu \in \Lambda$,

$$[P(\lambda) : \Delta(\mu)] = [\nabla(\mu) : L(\lambda)] \text{ and } [I(\lambda) : \nabla(\mu)] = [\Delta(\mu) : L(\lambda)].$$

Suppose that $\mathcal{C}$ has a strong duality, i.e., a contravariant equivalence $\mathfrak{d}: \mathcal{C} \xrightarrow{\sim} \mathcal{C}^{\mathrm{op}}$ satisfying $\mathfrak{d}^2 \cong \mathrm{id}_{\mathcal{C}}$ and $\mathfrak{d}L(\lambda) \cong L(\lambda)$, for all $\lambda \in \Lambda$. Show that $\mathfrak{d}\nabla(\lambda) \cong \Delta(\lambda)$, so that $[\nabla(\mu) : L(\lambda)] = [\Delta(\mu) : L(\lambda)]$.

(3) Let $\mathcal{C}(\Delta)$ (resp., $\mathcal{C}(\nabla)$) be the full subcategory of $\mathcal{C}$ consisting of all objects M which have a filtration $M = F_0 \supset F_1 \supset \cdots \supset F_t = 0$ with sections $F_i/F_{i+1} \cong \Delta(\lambda)$ (resp., $\cong \nabla(\lambda)$), for some $\lambda \in \Lambda$. If $M \in \mathcal{C}(\Delta)$ (resp., $\mathcal{C}(\nabla)$), we say that M has a Δ- (resp., ∇-) filtration. Prove the Donkin-Scott criterion that $M \in \mathcal{C}$ has a Δ-filtration (resp., ∇-filtration) if and only if $\mathrm{Ext}^1_A(M, \nabla(\lambda)) = 0$ (resp., $\mathrm{Ext}^1_A(\Delta(\lambda), M) = 0$), for all $\lambda \in \Lambda$.

Hint: Suppose M satisfies $\mathrm{Ext}^1_A(M, \nabla(\lambda)) = 0$, for all λ. Choose $\lambda \in \Lambda$ minimal for which there exists a nonzero morphism $f: M \to \nabla(\lambda)$. Show $\mathrm{Im}(f) \cong L(\lambda)$. Apply $\mathrm{Hom}_A(M, -)$ to the exact sequence $0 \to Q \to \Delta(\lambda) \to L(\lambda) \to 0$, and conclude that there exists a surjective morphism

$F \colon M \to \Delta(\lambda)$ since $\mathrm{Hom}_A(M, Q) = 0$. By the long exact sequence of cohomology, conclude that $\mathrm{Ext}^1_A(F, \nabla(\mu)) = 0$, for all μ. Conclude by induction on $\dim M$ that M has a Δ-filtration.

(4) Let $\mathcal{C}(\text{✡}) = \mathcal{C}(\Delta) \cap \mathcal{C}(\nabla)$. Any $M \in \mathcal{C}(\text{✡})$ is called a *tilting module*. Show that if M, N are tilting modules, then $\mathrm{Ext}^n_A(M, N) = 0$, for all positive n.

(5) Show that, for $\lambda \in \Lambda$, there exists a unique (up to isomorphism) indecomposable tilting module $X(\lambda)$ such that λ is *the* maximal $\mu \in \Lambda$ with $[X(\lambda) : L(\mu)] \neq 0$. Necessarily, $[X(\lambda) : L(\lambda)] = 1$.

Hint: Fix λ and choose a listing $\lambda_1, \ldots, \lambda_t$ of the order ideal $(-\infty, \lambda]$ such that $\lambda_i > \lambda_j \Longrightarrow i < j$. Thus, $\lambda_1 = \lambda$. The group $\mathrm{Ext}^1_A(\Delta(\lambda_2), \Delta(\lambda_1))$ is a right module over the division algebra $\mathcal{D} := \mathrm{End}_A(\Delta(\lambda_2))$. Let $m_1 = \dim_{\mathcal{D}} \mathrm{Ext}^1_A(\Delta(\lambda_2), \Delta(\lambda_1))_{\mathcal{D}}$, and choose an extension

$$0 \longrightarrow \Delta(\lambda) \longrightarrow E_2 \longrightarrow m_1\Delta(\lambda_2) \longrightarrow 0,$$

so that the various pullbacks through the maps $\Delta(\lambda_2) \to m_1\Delta(\lambda_2)$ form a $\mathcal{D}$-basis for $\mathrm{Ext}^1_A(\Delta(\lambda_2), \Delta(\lambda_1))_{\mathcal{D}}$. Continue this process to obtain a module $E_t \in \mathcal{C}(\Delta)$ such that $\mathrm{Ext}^1_A(\Delta(\mu), E_t) = 0$, for all μ. By (3), E_t is a tilting module.

(6) Let X be a tilting module. Prove that $X \cong \bigoplus m_\lambda X(\lambda)$, for nonnegative integers m_λ. If each $m_\lambda \neq 0$, then X is called a *complete tilting module.* If X is a complete tilting module, prove that the endomorphism algebra $B = \mathrm{End}_A(X)$ is a quasi-hereditary algebra. In fact, B-mod is a highest weight category with weight poset the opposite poset Λ^{op}.

(7) Assume $\Bbbk$ is algebraically closed. Let $M \in \mathcal{C}(\Delta)$. Prove, for $\lambda \in \Lambda$, the multiplicity $[M : \Delta(\lambda)]$ equals the rank of the bilinear form

$$\mathrm{Hom}_A(M, X(\lambda)) \otimes_{\Bbbk} \mathrm{Hom}_A(P(\lambda), M) \longrightarrow \mathrm{Hom}_A(P(\lambda), X(\lambda)) \cong \Bbbk$$

defined by composition.

(8) Suppose that each tilting module $X(\lambda)$, $\lambda \in \Lambda$, is projective (resp., injective) in $\mathcal{C}$. Also, assume that, given any λ, $\Delta(\lambda)$ and $\nabla(\lambda)$ have the same image in the Grothendieck group $\mathscr{K}_0(\mathcal{C})$ — e.g., $\mathcal{C}$ has a duality in the sense of (2) above. Prove that A is a semisimple algebra.

Hint: Argue by induction on the "height" of λ to show that $\Delta(\lambda) = \nabla(\lambda) = I(\lambda)$. Give an example to show that if the Grothendieck hypothesis is dropped, the result fails.

(9) Let X be a complete tilting module, for $\mathcal{C}$ as above, and let $\mathsf{Add}\, X$ be the full additive subcategory of $\mathcal{C}$ having as objects direct summands of finite direct sums of copies of X. Prove that A has a finite resolution $0 \to A \to X^\bullet \to 0$ in which each $X^i \in \mathsf{Add}\, X$. (This property, together with the fact that X has finite projective dimension in $\mathcal{C}$ and the fact that $\mathrm{Ext}^n_A(X, X) = 0$, for positive n, are usually taken as the characterizing property of tilting modules in the theory of finite dimensional algebras.)

§C.3

C.7. Let $\mathcal{R}$ be a commutative, noetherian ring. Prove the following statements directly (or look them up in a textbook on commutative algebra).

(1) $\mathcal{R}$ is regular if and only if the polynomial ring $\mathcal{R}[X]$ is regular. Also, $\operatorname{Kdim}\mathcal{R}[X] = 1 + \operatorname{Kdim}\mathcal{R}$.

(2) $\mathcal{R}$ is regular if and only if $\mathcal{R}_{\mathfrak{m}}$ is regular, for all maximal ideals $\mathfrak{m}$ in $\mathcal{R}$.

(3) If $S \subset \mathcal{R}$ is a multiplicative set and if $\mathcal{R}$ is regular, then the localization $S^{-1}\mathcal{R}$ is also regular.

(4) The ring $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ of integral Laurent polynomials in an indeterminate v is regular of Krull dimension 2.

C.8. Assume that A is an algebra over a discrete valuation ring $\mathcal{R}$ having residue field $k = \mathcal{R}/(\pi)$. Assume that A is a finitely generated $\mathcal{R}$-module. Let X, Y be finitely generated A-modules such that $\operatorname{Ext}^1_A(X, Y) \neq 0$. Show that $\operatorname{Ext}^1_{A_k}(X_k, Y_k) \neq 0$.

Hint: First, check that $\operatorname{Ext}^1_{A_k}(X_k, Y_k) \cong \operatorname{Ext}^1_A(X, Y)_k$. Show that multiplication by π induces an exact sequence

$$\operatorname{Ext}^1_A(X, Y) \xrightarrow{\pi} \operatorname{Ext}^1_A(X, Y) \longrightarrow \operatorname{Ext}^1_A(X, Y/\pi Y),$$

so that if the desired conclusion fails, $\operatorname{Ext}^1_A(X, Y) = 0$ by the Nakayama lemma.

The goal of Exercises C.9–C.10 is to establish a Brauer theory over a regular local ring $\mathcal{R}$ of Krull dimension at most 2. Let $\mathcal{K}$ be the fraction field and $k = \mathcal{R}/\mathfrak{m}$ the residue field of $\mathcal{R}$. Let A be an $\mathcal{R}$-algebra which is $\mathcal{R}$-free of finite rank. Suppose that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are the distinct (up to isomorphism) simple $A_{\mathcal{K}}$-modules. By Corollary C.18, $\boldsymbol{X}_i \cong X_{i,\mathcal{K}}$, for A-lattices X_i, $1 \leqslant i \leqslant m$.

C.9. Prove that

(1) Any simple A_k-module L is a composition factor of some $X_{i,k}$, $1 \leqslant i \leqslant m$.

(2) If $\boldsymbol{X} \cong X_{\mathcal{K}} \cong X'_{\mathcal{K}}$, for A-lattices X and X', then X_k and X'_k have the same A_k-composition factors (with the same multiplicities).

Hint: First, replace $\mathcal{R}$ by $\hat{\mathcal{R}} := \varprojlim \mathcal{R}/\mathfrak{m}^i$, and $\mathcal{K}$ by the fraction field $\hat{\mathcal{K}}$ of $\hat{\mathcal{R}}$. Thus, $\mathcal{R}$ and $\hat{\mathcal{R}}$ have the same residue field k. Then consider a projective cover $P \twoheadrightarrow L$ of A_k-modules and, by [**112**, 12.9], lift P to a projective A-module $\tilde{P}$ such that $P = \tilde{P}_k$. Now, use the isomorphism $\operatorname{Hom}_{A_k}(P, X_{i,k}) \cong \operatorname{Hom}_A(\tilde{P}, X_i)_k$.

C.10. Let $L(1), \ldots, L(n)$ be the distinct simple A_k-modules. By (2) in the above exercise, the multiplicities $d_{i,j} = [X_{j,k} : L(i)]$ are independent of the choice of X_j. Let $D = (d_{i,j})$ be the corresponding $n \times m$ decomposition matrix. Also, let $C = (c_{i,j})$ be the $n \times n$ Cartan matrix of A_k, i.e., if $P(j)$ denotes the projective cover of $L(j)$ in A_k-mod, then $c_{i,j} = [P(j) : L(i)]$. Suppose that both $A_{\mathcal{K}}$ and $A_k/\operatorname{rad}(A_k)$ are split semisimple. Show that $C = D \cdot D^t$.

§§C.4–C.5

C.11. Let $\mathcal{R}$ be an arbitrary commutative, noetherian domain, and let E be an $\mathcal{R}$-algebra and let $e \in E$ be an idempotent. Suppose that E is a (split) semisimple algebra over $\mathcal{R}$. Show that eEe is a (split) semisimple algebra over $\mathcal{R}$.

Hint: Use the isomorphism $eEe \cong \mathrm{End}_E(Ee)^{\mathrm{op}}$ to show that eEe behaves well with respect to base change to a field $\mathcal{k}$.

C.12. (See [**BAII**, §3.12] for details dealing with parts (1)–(3).) Let A be an algebra over a commutative ring $\mathcal{R}$. Let P be a finitely generated progenerator for A-mod and put $A' = \mathrm{End}_A(P)^{\mathrm{op}}$. Let $\mathscr{F} = \mathrm{Hom}_A(P, -)\colon A\text{-mod} \to A'\text{-mod}$. Let $Q = \mathscr{F}(A)$.

(1) Prove directly that $A^{\mathrm{op}} \cong \mathrm{End}_{A'}(Q)$. Show directly that Q is a finitely generated progenerator of A'-mod.

(2) Let $\mathscr{G} := \mathrm{Hom}_{A'}(Q, -)\colon A'\text{-mod} \to A\text{-mod}$. Prove directly that $\mathscr{F}$ and $\mathscr{G}$ are inverse equivalences of categories.

(3) Show that $\mathfrak{J} \mapsto \mathscr{F}(\mathfrak{J}P) = \mathrm{Hom}_A(P, \mathfrak{J}P)$ defines a bijection between ideals of A and ideals of A'.

(4) Suppose that $\mathfrak{J}$ is an ideal in A and let $\mathfrak{J}' = \mathscr{F}(\mathfrak{J}P)$. Show $\mathrm{End}_A(\mathfrak{J})$ is (split) semisimple if and only if $\mathrm{End}_{A'}(\mathfrak{J}')$ is (split) semisimple. Show that $\mathfrak{J}$ is projective as a left A-module if and only if $\mathfrak{J}'$ is projective as a left A'-module. Finally, prove that $\mathfrak{J}$ is an idempotent ideal if and only if $\mathfrak{J}'$ is an idempotent ideal. Conclude that $\mathfrak{J}$ is a heredity ideal in A if and only if $\mathfrak{J}'$ is a heredity ideal in A'.

(5) If $\mathfrak{J}$ is an ideal in A, show that $P/\mathfrak{J}P$ is a progenerator for the $\mathcal{R}$-algebra $A/\mathfrak{J}$. Also, if $\mathfrak{J}' = \mathscr{F}(\mathfrak{J}P)$, show that $A'/\mathfrak{J}' \cong \mathrm{End}_{A/\mathfrak{J}}(P/\mathfrak{J}P)$.

(6) Show that $0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_t = A$ is a heredity chain in A if and only if $0 = \mathfrak{J}'_0 \subset \mathfrak{J}'_1 \subset \cdots \subset \mathfrak{J}'_t = A'$ is a heredity chain in A', where $\mathfrak{J}'_i = \mathscr{F}(\mathfrak{J}_i P)$, for $i = 1, \ldots, t$. Hence, A is (split) quasi-hereditary over $\mathcal{R}$ if and only if A' is also.

C.13. Complete the details of the proof of Proposition C.25.

§C.6

C.14. Suppose that A is a finite dimensional algebra over a field $\mathcal{k}$. Assume that A has an anti-involution τ. Prove that A is cellular in the sense of Definition C.30 if and only if the following conditions hold: A has a filtration $0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \cdots \subset \mathfrak{J}_t = A$ by a sequence of τ-stable ideals such that, for each $1 \leqslant i \leqslant t$, $\mathfrak{J}_i/\mathfrak{J}_{i-1}$ contains a left $A/\mathfrak{J}_{i-1}$-ideal W_i, giving rise to a commutative diagram

$$\begin{tikzcd}
\mathfrak{J}_i/\mathfrak{J}_{i-1} \arrow[r, "\alpha"] \arrow[d, "\tau"'] & W_i \otimes_{\mathcal{k}} \tau(W_i) \arrow[d, "(1,2)\circ(\tau\otimes\tau)"] \\
\mathfrak{J}_i/\mathfrak{J}_{i-1} \arrow[r, "\alpha"'] & W_i \otimes_{\mathcal{k}} \tau(W_i)
\end{tikzcd}$$

in which α is an isomorphism of A-A-bimodules, and $(1, 2)$ is the "switching map" on the tensor products. We also assume that, for each i, $\mathfrak{J}_{i-1}$ has a τ-stable complement in $\mathfrak{J}_i$.

Notes

§§C.1–C.2: The theory of quasi-hereditary algebras and highest weight categories was first developed in papers [**45, 234, 267**] by Cline–Parshall–Scott. The original idea was to model algebraically some of the geometric features of the theory of perverse sheaves used in the proof (by Brylinski–Kashiwara [**30**] and Beilinson–Bernstein [**15**], independently) of the Kazhdan–Lusztig conjecture [**177**] for the category $\mathcal{O}$ associated with a complex simple Lie algebra. For other early contributions to the theory of quasi-hereditary algebras, see [**77, 78, 80, 232**]. Surveys of quasi-hereditary algebras can be found, for example, in [**81, 233, 72**].

The theory of stratified algebras, as defined in [**47**] (and studied there at both the field and integral level) provides an important extension of the theory of quasi-hereditary algebras that is useful in representation theory.

The theory of tilting modules for quasi-hereditary algebras was first developed by Ringel [**250**]. For the fact that the Cartan matrix of a quasi-hereditary algebra has determinant 1 (discussed in Exercise C.6), see [**31**].

§C.3: Besides [**HAII**], the short book by Serre [**269**] is recommended for the theory of regular rings needed in this section. The Auslander–Buchsbaum formula (Proposition C.15) was first given in [**9**].

Exercises C.9 and C.10 are taken from Du–Parshall–Scott [**105**, 1.1.2–1.1.3]. The approach to the Brauer theory sketched in these Exercises was first given by Geck–Rouquier [**127**].

§C.4: The theory of integral quasi-hereditary algebras was introduced in [**46**]; see [**110**] for other applications of the theory. The converse of Proposition C.23 is also true; see [**46**, 3.3(a)].

§C.5: Although the formal notion of a Specht datum for an algebra is new, the ideas of this section are essentially contained in Du–Parshall–Scott [**106**].

§C.6: Cellular algebras were introduced by Graham–Lehrer [**134**]. The original idea came from the cellular property (Proposition 8.30) they observed from the Kazhdan–Lusztig basis of the Hecke algebra associated with a symmetric group. However, in contrast with the canonical bases, the definition of cellular bases directly reflects the structure of the algebra and is suitable for any ground ring. It turns out that many well-known algebras such as Brauer algebras and Ariki–Koike algebras are cellular algebras. Geck [**126**] recently proved that the Hecke algebras associated with finite Weyl groups are cellular.

Our treatment in the section largely follows from [**134**] with one exception where we replace the dual module $\mathsf{C}(\lambda)^*$ of a cell module $\mathsf{C}(\lambda)$, used in [**134**], by the module $\mathsf{C}(\lambda)^{\mathrm{op}}$, used in [**109**]. With this modification, we establish the associativity relations given in Lemma C.32. Moreover, $\mathsf{C}(\lambda)^{\mathrm{op}}$ is the dual module of the co-standard module $\nabla(\lambda)$ in the quasi-hereditary case.

For further investigations on the representation theory of general cellular algebras, see König–Xi [**182, 183**]. The idea of cellular bases has also been generalized to obtain a new formulation for quasi-hereditary algebras; see [**109, 99**]. See also [**47**, §1.2] for more on strong duality functors.

Exercise C.14 is taken from König–Xi [**182**].

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