Mathematical
Surveys
and
Monographs

Volume 162

Elliptic Equations in Polyhedral Domains

Vladimir Maz′ya
Jürgen Rossmann

American Mathematical Society

http://dx.doi.org/10.1090/surv/162

Elliptic Equations in Polyhedral Domains

Mathematical
Surveys
and
Monographs

Volume 162

Elliptic Equations in Polyhedral Domains

Vladimir Maz′ya
Jürgen Rossmann

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 35J57, 35J58, 35J25, 35J40, 35J08, 35J05, 35Q30.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-162

Library of Congress Cataloging-in-Publication Data

Maz′ia, V. G.
Elliptic equations in polyhedral domains / Vladimir Maz′ya, Jürgen Rossmann.
p. cm. — (Mathematical surveys and monographs ; v. 162)
Includes bibliographical references and index.
ISBN 978-0-8218-4983-5 (alk. paper)
1. Differential equations, Elliptic. 2. Polyhedra—Models. 3. Boundary value problems. I. Rossmann, J. (Jürgen), 1954– II. Title.

QA377.M296 2010
515′.3533—dc22

2009053203

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 15 14 13 12 11 10

Contents

Historical remarks

1. Bibliographical notes to chapters

Chapter 1 (smooth domains and isolated singularities). In addition to the references given in Sections 1.1 and 1.2, we note that a historical survey on elliptic boundary value problems in domains with smooth boundaries and in domains with isolated singularities on the boundary can be found in the book [**84**], which contains many references related to this topic. Therefore, in what follows, we will refer only to works dealing with boundary singularities of positive dimension.

Chapter 2 (Dirichlet problem, nonintersecting edges). The material in this chapter is an adaption to the Dirichlet problem of a more general framework in the papers [**118, 119, 120**] by Maz'ya and Plamenevskiĭ. The main difference is that, in contrast to these papers, we allow the right-hand side of the differential equation to belong to a weighted Sobolev or Hölder space of negative order. An earlier exposition of solvability and regularity results in Hilbert-Sobolev spaces of integer order was given by the same authors in [**114**]. A theory of the Dirichlet problem in Hilbert-Sobolev spaces of fractional order was developed by Dauge [**31**]. In [**118, 119, 120**], arbitrary elliptic equations supplied with different boundary conditions on the faces of a n-dimensional dihedron were considered.

In particular, as shown in [**118**], the boundary value problem is solvable in the weighted Sobolev space $V_\delta^{l,p}$ if the kernel and cokernel of the operator of the corresponding parameter-depending model problem in the plane cross-section angle (cf. Section 2.3) are trivial. This condition can be easily checked for the Dirichlet problem and for a broad class of strongly elliptic problems. In general, the algebraic verification of the triviality of the kernel and cokernel just mentioned is an open problem, but the answer is known in some special cases, see Maz'ya and Plamenevskiĭ [**112, 114, 115**], Maz'ya [**103**], Komech [**73**], Eskin [**51**]. It is proved by Kozlov [**77**] that, under some requirements on the elliptic operator, one can achieve the triviality of the kernel of the model problem in question by prescribing a finite number of complementary conditions on the edge.

Note that the results in [**118**] were derived using an operator multiplier theorem for the Fourier transform. The approach in the present book goes up to the paper [**119**], where estimates of solutions in weighted Hölder spaces were obtained by means of point estimates for Green's functions.

Various aspects of the elliptic theory for manifolds with edges (parametrices, Fredholm property, index) were studied in numerous works by Schulze and his collaborators by methods of the theory of pseudo-differential operators (see for example the monograph by Nazaikinskiĭ, Savin, Schulze and Sternin [**154**]).

Properties of the Dirichlet problem for the Laplacian stated in Subsections 2.6.6 and 2.8.6 are corollaries of the general Theorems 2.6.5 and 2.8.8. However, particular cases of these results were obtained previously by specific methods of the theory of second order elliptic equations with real coefficients. For instance, coercive estimates of solutions of the Dirichlet problem for second order elliptic equations in the weighted spaces $V_\delta^{2,2}$ were obtained by Kondrat'ev [**75**]. The paper [**12**] of Apushkinskaya and Nazarov is dedicated to Hölder estimates for solutions to the Dirichlet problem for quasilinear elliptic equations in domains with smooth closed edges of arbitrary dimension.

Chapters 3 and 4 (Dirichlet problem in domains of polyhedral type). Pointwise estimates for Green's matrix of the Dirichlet problem for strongly elliptic equations of higher order were obtained in our paper [**129**]. In the same paper, one can find estimates of solutions in weighted L_p-Sobolev spaces similar to those in Sections 3.5 and 4.1. The Hölder estimates in Sections 3.6 and 4.2 were not published before.

In the paper [**113**] Maz'ya and Plamenevskiĭ introduced a large class of multi-dimensional manifolds with edges of different dimensions intersecting under nonzero angles. This class of manifolds contains polyhedra in $\mathbb{R}^N$ as a very special case. A solvability theory for general elliptic boundary value problems on such manifolds in weighted L_2-Sobolev spaces was developed in [**116**] by an induction argument in dimensions of singular strata. It is assumed in this paper that kernels and cokernels of all model problems generated by edges of different dimensions are trivial, which is the case, in particular, for the Dirichlet problem. This material is reproduced in the book by Nazarov and Plamenevskiĭ [**160**].

A L_2-theory for the Dirichlet problem for general elliptic equations in three-dimensional polyhedral domains was also established in the papers by Lubuma, Nicaise [**92**] and Nicaise [**164**]. Some regularity results related to the Dirichlet problem for the Laplace equation in a polyhedral domain were obtained by Hanna and Smith [**65**], Grisvard [**58, 60**], Dauge [**31**], Ammann and Nistor [**11**]. Buffa, Costabel and Dauge [**18**] stated regularity assertions for the Laplace and Maxwell equations in isotropic and anisotropic weighted Sobolev spaces. The Dirichlet problem for the Lamé system (and for the Laplace equation as a particular case) in a broad class of piecewise smooth domains without cusps was investigated in detail by Maz'ya and Plamenevskiĭ [**124**].

Chapter 5 (Miranda-Agmon maximum principle). The main results of this chapter were obtained in our papers [**129**] and [**130**], the Hölder estimates for the derivatives of Green's matrix in convex polyhedral type domains presented in Subsection 5.1.5 were proved by Guzman, Leykekhman, Rossmann and Schatz [**64**].

The history starts with the estimate

$$\|u\|_{C^{m-1}(\overline{\mathcal{G}})} \le c\Big(\sum_{k=1}^{m}\Big\|\frac{\partial^{k-1}u}{\partial n^{k-1}}\Big\|_{C^{m-k}(\partial\mathcal{G})} + \|u\|_{L_1(\mathcal{G})}\Big), \tag{11.6.29}$$

for solutions of strongly elliptic equations $Lu = 0$ of order $2m > 2$ proved in the case of smooth boundaries by Miranda [**144, 145**] for two-dimensional and by Agmon

[**5**] for higher-dimensional domains. SCHULZE [**182, 183**] justified analogous C^k-estimates for solutions of strongly elliptic systems and for more general boundary conditions $D_\nu^{m_k} = g_k$ on $\partial\mathcal{G}$, where $m_k \le 2m-1$.

MAZ'YA and PLAMENEVSKIĬ [**122**] proved the estimate (11.6.29) for solutions of the biharmonic equation in a three-dimensional domain with conical vertices. As shown independently in MAZ'YA, ROSSMANN [**130**] and PIPHER, VERCHOTA [**169**], this estimate fails if the dimension is greater than 3. In [**169, 170**] PIPHER and VERCHOTA proved the estimate (11.6.29) for solutions of the biharmonic and polyharmonic equations in Lipschitz domains.

Chapter 6 (systems of second order, nonintersecting edges). The results of this chapter are borrowed from our paper [**133**]. Even when dealing only with the Dirichlet problem, we obtain new results in comparison with Chapter 2. Here the data and the solutions belong to a wider class of spaces with nonhomogeneous norms which include classical nonweighted Sobolev spaces. These spaces were earlier used in the paper [**128**] of MAZ'YA and ROSSMANN, where general elliptic boundary value problems were considered under the assumption of the unique solvability of model problems in a plane cross-section angle.

The first treatment of the Neumann problem for the equation $\Delta u = 0$ in the presence of a smooth edge on the boundary was given as early as 1916 by CARLEMAN [**19**], who used methods of potential theory. For the same problem see the works by MAZ'YA and PLAMENEVSKIĬ [**112, 115**] and SOLONNIKOV and ZAJACKOWSKI [**204**], where solutions in the spaces $W_\delta^{l,2}$ were considered. Analogous results in the weighted Sobolev spaces $W_\delta^{l,p}$ and weighted Hölder spaces $C_\delta^{l,\sigma}$ were obtained in the preprint [**190**] by Solonnikov. Furthermore, the Green's function for the Neumann problem was estimated in [**190**]. An L_2-theory for more general boundary value problems including the Neumann problem was developed in papers by NAZAROV [**155, 156**], ROSSMANN [**177**], NAZAROV, PLAMENEVSKIĬ [**158, 159**] (see also the book of NAZAROV and PLAMENEVSKIĬ [**160**]). The elliptic oblique derivative problem in domains with nonintersecting edges was treated by MAZ'YA and PLAMENEVSKIĬ [**112**].

NAZAROV and SWEERS [**161**] investigated the $W^{2,2}$-solvability of the biharmonic equation with prescribed boundary value of the solution and its Laplacian in a three-dimensional domain with variable opening at the edge, where some interesting effects arise for a critical opening.

If the domain is smooth and the role of an edge is played by a smooth surface of codimension 1 in the boundary separating different boundary conditions, another approach to mixed problems based on the Wiener-Hopf method was used starting in the 1960s (see the monograph by ESKIN [**50**]). A similar approach proved to be effective in the study of boundary value problems for domains with two-dimensional cracks and interior cuspidal edges (see DUDUCHAVA and WENDLAND [**39**], DUDUCHAVA and NATROSHVILI [**38**], CHKADUA [**20**], CHKADUA, DUDUCHAVA [**21**] *et al.*). In particular in [**39**], the Wiener-Hopf method was developed for systems of boundary pseudo-differential equations which allowed to manage without the factorization of corresponding matrix symbols and to investigate the asymptotics of the solution to the crack problem in an anisotropic medium.

Exterior cuspidal edges which require different methods were studied by DAUGE [**35**], SCHULZE, TARKHANOV [**187**], RABINOVICH, SCHULZE, TARKHANOV [**173,**

174], Maz'ya, Netrusov, Poborchi [**111**] and Maz'ya, Poborchi [**126**].

Chapters 7 and 8 (second order systems in domains of polyhedral type). These chapters contain a somewhat extended exposition of the results obtained by the authors in [**133, 134, 135**]. New features in comparison with Chapters 3 and 4 are the use of nonhomogeneous Sobolev and Hölder norms, and the inclusion of the Neumann problem.

The Neumann problem for the Laplace equation in a polyhedral cone was earlier studied in the preprint [**57**] of Grachev and Maz'ya, where the authors obtained estimates for the solutions in weighted Sobolev and Hölder spaces and pointwise estimates of the Green's matrix. Dauge [**34**] proved regularity assertions for solutions of the Neumann problem for second order elliptic equations with real coefficients in nonweighted L_p-Sobolev spaces. Regularity results in weighted L_2-Sobolev spaces for general self-adjoint systems were proved by Nazarov and Plamenevskiĭ [**157**]. The behavior of the solution of the Neumann problem for the Lamé system near the vertex of a polyhedron is studied in the book by Grisvard [**62**].

Mixed boundary value problems for the Laplace equation with Dirichlet and Neumann conditions are considered e.g. in the above mentioned works by Dauge [**34**] and Grisvard [**62**]. The same problems were studied by Ebmeyer [**44**], Ebmeyer and Frehse [**45**] for nonlinear second order equations in N-dimensional domains, $N \geq 3$, with piecewise smooth boundaries. Nicaise [**163**] obtained regularity results for solutions of mixed boundary value problems to the Lamé system in L_2-Sobolev spaces.

Maz'ya [**102, 103**] and Dauge [**32**] studied oblique derivative problems in domains of polyhedral type. Transmission problems in polyhedral domains were handled in papers by Costabel, Dauge, Nicaise [**29**], Chikouche, Mercier, Nicaise [**22, 23**], Knees [**72**], Elschner, Rehberg, Schmidt [**48**], Elschner, Kaiser, Rehberg, Schmidt [**47**].

The conditions ensuring the solvability and regularity of solutions near the vertices depend on information about eigenvalues of the operator pencils $A_k(\lambda)$ and $\mathfrak{A}_j(\lambda)$ introduced in Section 8.1. Information of this nature is collected in the book by Kozlov, Maz'ya and Rossmann [**85**]. The pencil generated by the Neumann problem for elliptic differential operators of arbitrary order was investigated by Kozlov and Maz'ya [**80**]. Assuming that the cone is convex, it was shown by Escobar [**49**] and in another way by Maz'ya [**107**] that the first positive eigenvalue of the pencil $\delta + \lambda(\lambda + N - 2)$ with zero Neumann conditions satisfies the sharp inequality $\lambda_1 \geq 1$. Earlier Dauge [**34**] found a rougher estimate $\lambda_1 > (\sqrt{5} - 1)/2$ in the three-dimensional case.

For special problems and special domains, eigenvalues of operator pencils generated by the Neumann problem were calculated numerically by Leguillon and Sanchez-Palencia [**90**], Dimitrov [**40**], Dimitrov, Andrä and Schnack [**41**] *et al.*

Chapter 9 (Stokes and Navier-Stokes systems, nonintersecting edges). This chapter is an extended version of our paper [**136**] concerning the mixed boundary value problem for the Stokes system in a dihedron. Some related results can be found in the earlier paper of Solonnikov [**189**] and Maz'ya, Plamenevskiĭ and Stupelis [**125**], where the Dirichlet problem and a particular mixed boundary

value problem were studied in connection with a nonlinear hydrodynamical problem with free boundary. A detailed exposition of the results obtained in [**125**] can be found in STUPELIS [**193**]. In contrast to [**136**], the paper [**125**] deals with solutions in weighted Sobolev and Hölder spaces with homogenous norms.

Chapters 10 and 11 (Stokes and Navier-Stokes systems, domains of polyhedral type). These chapters contain results obtained by the authors in [**136**]–[**140**] and [**179**]. The starting point for the development of this theory was the paper by MAZ'YA and PLAMENEVSKIĬ [**124**] dedicated to the Dirichlet problem.

The inequality (11.3.3) for the eigenvalues of the pencil generated by the Dirichlet problem for the Stokes system obtained by MAZ'YA and PLAMENEVSKIĬ in [**123**] was the first result of this nature. More estimates for the eigenvalues can be found in the paper by DAUGE [**33**]. A detailed analysis of these eigenvalues including a variational principle for real ones was developed by KOZLOV, MAZ'YA and SCHWAB [**86**] (see also the book [**85**]). The only paper, where the eigenvalues corresponding to the Neumann were touched upon, is that of KOZLOV and MAZ'YA [**79**]. Some results corresponding to various mixed type problems were obtained by KOZLOV, MAZ'YA and ROSSMANN [**83**] (see also the book [**85**]).

The results in Section 11.5 have a long history which begins with ODQUIST's inequality

$$\|u\|_{L_\infty(\mathcal{G})} \le c\,\|u\|_{L_\infty(\partial\mathcal{G})}$$

for the solutions of the Stokes system (11.6.1) (see [**166**]). A proof of this inequality for domains with smooth boundaries is given e.g. in the book by LADYZHENSKAYA [**89**]. We refer also to the papers of MAZ'YA and KRESIN [**108**], NAUMANN [**153**], KRATZ [**87**] and MAREMONTI [**93**]. Using point estimates of the Green's matrix, MAZ'YA and PLAMENEVSKIĬ [**123, 124**] proved this inequality for solutions of the Stokes system in three-dimensional domains with conical points and in domains of polyhedral type.

For the nonlinear problem (11.6.1), (11.6.2), SOLONNIKOV [**191**] showed that solutions satisfy the estimate (11.6.21) with a certain unspecified function F if the boundary $\partial\mathcal{G}$ is smooth. An estimate of this form can be also deduced from the results in a paper of MAREMONTI and RUSSO [**94**]. MAZ'YA and PLAMENEVSKIĬ [**124**] proved for domains of polyhedral type that the solution u of (11.6.1), (11.6.2) with finite Dirichlet integral is continuous in $\overline{\mathcal{G}}$ if h is continuous on $\partial\mathcal{G}$. However, the paper [**124**] contains no estimates for the maximum modulus of u. In our paper [**140**], we proved the inequality (11.6.21) for domains of polyhedral type and obtained the representation $F(t) = c_0 t\, e^{c_1 t/\nu}$ for the function c.

2. Bibliographical notes to other related material

The whole theme of elliptic boundary value problems in nonregular domains is so rich that obviously we could touch upon only a small part of it. In order to illustrate the variety of results in this area, we give here some references related to topics outside of this book without aiming at complete satisfaction to a certain extent.

Asymptotics of solutions near edges and vertices. The asymptotic expansions of solutions near boundary singularities are not treated in this book, but

this theme was thoroughly studied simultaneously with solvability properties and became a broad area of research. The asymptotics of solutions of the Dirichlet problem for elliptic equations of second order in a neighborhood of an edge was described by Kondrat'ev [**76**] and Nikishkin [**165**] and for the Laplace equation by Grisvard [**61**]. Asymptotic formulas for solutions to general elliptic boundary value problems were proved by Maz'ya and Plamenevskiĭ [**114**], Maz'ya and Rossmann [**127, 128**], Dauge [**30**], Nazarov and Plamenevskiĭ [**160**]. It was assumed in the last works that the edges do not contain "critical" points, i.e. that there is no bifurcation in singularities. The case of critical edge points was discussed in the papers by Rempel and Schulze [**175**], and Schulze [**184, 185**]. Explicit asymptotic formulas for such cases were derived by Costabel and Dauge [**25**], Maz'ya and Rossmann [**132**]. The asymptotics of solutions near polyhedral vertices was studied by von Petersdorff and Stephan [**202**] and Dauge [**36**] for second order equations. The last paper is a masterful survey of the area. We also mention a comprehensive study of singularities of solutions to the Maxwell equation by Costabel and Dauge [**26, 27, 28**].

Lipschitz graph and other domains. Needless to say, there are other areas in the theory of elliptic boundary value problems differing both by classes of domains and the methods of research. First of all, there exists a rich theory dealing with Lipschitz graph boundaries and based on refined methods of harmonic analysis. We refer only to the survey monograph by Kenig [**71**] and more recent works by Adolfsson, Pipher [**4**], Brown, Perry, Shen [**17**], Brown [**15**], Brown, Shen [**16**], Deuring, von Wahl [**37**], Dindoš, Mitrea [**42**], Ebmeyer [**44**], Ebmeyer, Frehse [**45, 46**], Jakab, Mitrea, Mitrea [**66**], Jerison, Kenig [**67**], Mayboroda, Mitrea [**95, 96**], Mitrea [**146**], Mitrea, Monniaux [**147**], Mitrea, Taylor [**148**]–[**151**], Pipher, Verchota [**168, 171**], Shen [**180, 181**] and Verchota [**198, 199**].

Successful attempts to apply these methods, which are based on the so-called Rellich's identity, to non-Lipschitz graph polyhedral domains in $\mathbb{R}^3$ and $\mathbb{R}^4$ were undertaken by Verchota [**197**], Verchota and Vogel [**200, 201**], Venouziou and Verchota [**196**].

Asymptotic formulas for solutions of the Dirichlet problem for strongly elliptic equations of arbitrary order near the Lipschitz graph boundary were found by Kozlov and Maz'ya [**82**]. The same boundary value problem with data in Besov spaces was treated in Maz'ya, Mitrea and Shaposhnikova [**109**] under an assumption on the boundary formulated in terms of the space BMO. Sharp conditions of the $W^{2,2}$-solvability of the Dirichlet problem for the Laplace equation in a domain in C^1 but not in C^2 were derived in Maz'ya [**101**].

Additional information was derived for boundary value problems in arbitrary convex domains (Kadlec [**68**], Adolfsson [**1, 2**], Adolfsson, Jerison [**3**], Fromm [**52, 53**], Fromm, Jerison [**54**], Kozlov, Maz'ya [**82**], Maz'ya [**107**], Mayboroda, Maz'ya [**97**]).

Introducing classes of Lipschitz graph domains characterized in terms of Sobolev multipliers, Maz'ya and Shaposhnikova obtained sharp results on solutions in $W^{l,p}(\Omega)$ ([**141**], [**142**], [**143**]).

It proved to be possible to obtain substantial information on properties of elliptic boundary value problems without imposing a priori restrictions on the class of

domains, such as criteria of solvability and discreteness of spectrum formulated with the help of isoperimetric and isocapacitary inequalities, capacitary inner diameter and other potential theoretic terms (see Maz'ya [**99, 100, 104, 106**], Alvino, Cianchi, Maz'ya, Mercaldo [**10**], Cianchi, Maz'ya [**24**]). Wiener type criteria of regularity of a boundary point and pointwise estimates for solutions and their derivatives in unrestricted domains belong to another direction in the same area (see Maz'ya [**105**], Mayboroda, Maz'ya [**98**]).

In conclusion, we only list as key words some other classes of nonsmooth domains which appear in the studies of elliptic boundary value problems: nontangentially accessible domains, uniform domains, John domains, Jordan domains, Nikodym domains, Sobolev domains, extension domains etc.

Bibliography

[1] Adolfsson, V., *L^2-integrability of the second order derivatives for Poisson's equation in nonsmooth domains,* Math. Scand. **70** (1992) 146-160.

[2] Adolfsson, V., *L^p-integrability of the second order derivatives of Green potentials in convex domains.* Pacific J. Math. **159** (1993) 2, 201–225.

[3] Adolfsson, V., Jerison, D., *L^p-integrability of the second order derivatives for the Neumann problem in convex domains*, Indiana Univ. Math. J. **43** (1994) 1123-1138.

[4] Adolfsson, V, Pipher, J., *The inhomogeneous Dirichlet problem for Δ^2 in Lipschitz domains*, J. Funct. Anal. **159** (1998) 1, 137-190.

[5] Agmon, S., *Maximum theorems for solutions of higher order elliptic equations*, Bull. Amer. Math. Soc. **66** (1960) 77-80.

[6] Agmon, S., Douglis, A., Nirenberg, L., *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I,* Comm. Pure Appl. Math. **12** (1959) 623-727,

[7] Agmon, S., Douglis, A., Nirenberg, L., *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II,* Comm. Pure Appl. Math. **17** (1964) 35-92.

[8] Agmon, S., Nirenberg, L., *Properties of solutions of ordinary elliptic equations in Banach space,* Comm. Pure Appl. Math. **16** (1963) 121-163.

[9] Agranovich, M. S., Vishik, M. I., *Elliptic problems with parameter and parabolic problems of general type,* Uspekhi Mat. Nauk **19** (1964) 3, 53-161 (in Russian).

[10] Alvino, A., Cianchi, A., Maz'ya, V. G., Mercaldo, A., *Well-posed elliptic Neumann problems involving irregular data and domains*, to appear.

[11] Ammann, B., Nistor, V., *Weighted Sobolev spaces and regularity for polyhedral domains*, Comput. Methods Appl. Mech. Engrg. **196** (2007) 37-40, 3650–3659.

[12] Apushkinskaya, D. E., Nazarov, A. I., *The Dirichlet problem for quasilinear elliptic equations in domains with smooth closed edges*, Function theory and partial differential equations. J. Math. Sci. (New York) **105** (2001) 5, 2299–2318.

[13] Bers, L., John, F., Schechter, M., *Partial differential equations*, Lectures in Applied Mathematics, Interscience Publishers, New York - London - Sydney 1964.

[14] Borsuk, M., Kondratiev, V., *Elliptic boundary value problems of second order in piecewise smooth domains,* North-Holland Mathematical Library 69, Elsevier, Amsterdam 2006.

[15] Brown, R. M., *The mixed problem for Laplace's equation in a class of Lipschitz domains*, Comm. Partial Differential Equations 19 (1994), no. 7-8, 1217-1233.

[16] Brown, R., Shen, Z., *Estimates for the Stokes operator in Lipschitz domains*, Indiana Univ. J. **44** (1995) 1183-1206.

[17] Brown, R., Perry, P., Shen, Z., *On the dimension of the attractor for the nonhomogeneous Navier-Stokes equations in non-smooth domains*, Indiana Univ. Math. J. **49** (2000) 1, 81-112.

[18] Buffa, A., Costabel, M., Dauge, M., *Anisotropic regularity results for Laplace and Maxwell operators in a polyhedron*, C. R., Math., Acad. Sci. Paris 336 (2003) 7, 565-570.

[19] Carleman, T, *Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, Inaugural-Dissertation, Uppsala 1916.

[20] Chkadua, O., *Solvability and asymptotics of solutions of crack-type boundary-contact problems of the couple-stress elasticity.* Dedicated to the 100th birthday anniversary of Professor Victor Kupradze. Georgian Math. J. **10** (2003) 3, 427–465.

[21] Chkadua, O., Duduchava, R., *Asymptotics of solutions to some boundary value problems of elasticity for bodies with cuspidal edges*, Mem. Differential Equations Math. Phys. **15** (1998) 29-58.

[22] Chikouche, W., Mercier, D., Nicaise, S., *Regularity of the solution of some unilateral boundary value problems in polygonal and polyhedral domains*, Comm. Partial Differential Equations **28** (2003), 11-12, 1975–2001.

[23] Chikouche, W., Mercier, D., Nicaise, S., *Regularity of the solution of some unilateral boundary value problems in polygonal and polyhedral domains*, Comm. Partial Differential Equations **29** (2004), 1-2, 43-70.

[24] Cianchi, A., Maz'ya, V. G., *Neumann problems and isocapacitary inequalities*, J. Math. Pures Appl. **(9) 89** (2008) 1, 71-105.

[25] Costabel, M., Dauge, M., *General edge asymptotics of solutions of second order elliptic boundary value problems I, II,* Proc. Roy. Soc. Edinburgh **123A** (1993) 109-155, 157-184.

[26] Costabel, M., Dauge, M., *Singularités des équations de Maxwell dans un polyèdre*, C. R. Acad. Sci., Paris, Sér. I 324 (1997) 9, 1005-1010.

[27] Costabel, M., Dauge, M., *Maxwell and Lamé eigenvalues on polyhedra*, Math. Meth. Appl. Sci. 22 (1999), no. 3, 243–258.

[28] Costabel, M., Dauge, M., *Singularities of electromagnetic fields in polyhedral domains*, Arch. Rational Mech. Anal. **151** (2000) 221-276.

[29] Costabel, M., Dauge, M., Nicaise, S., *Singularities of eddy current problems*, Math. Model. Numer. Anal. **37** (2003) 5, 807–831.

[30] Dauge, M., *Elliptic boundary value problems in corner domains - smoothness and asymptotics of solutions,* Lecture Notes in Mathematics, Vol. 1341, Springer-Verlag Berlin 1988.

[31] Dauge, M., *Problèmes de Neumann et de Dirichlet sur un polyèdre dans $\mathbb{R}^3$: regularité dans des espaces de Sobolev L_p*, C. R. Acad. Sci. Paris **307** Série I (1988) 27-32.

[32] Dauge, M., *Higher order oblique derivatives problems on polyhedral domains*, Comm. Partial Differential Equations **14** (1989) 1193-1227.

[33] Dauge, M., *Stationary Stokes and Navier-Stokes systems on two- and three-dimensional with corners. Part 1: Linearized equations,* SIAM J. Math. Anal. **20** (1989) 74-97.

[34] Dauge, M., *Neumann and mixed problems on curvilinear polyedra*, Integr. Equat. Oper. Th. **15** (1992) 227-261.

[35] Dauge, M., *Strongly elliptic problems near cuspidal points and edges*, Partial Differential Equations and Functional Analysis, 93-110. Progr. Nonlinear Differential Equations Appl. **22**, Birkhäuser Boston, MA, 1996.

[36] Dauge, M., *Singularities of corner problems and problems of corner singularities*, ESAIM, Proc. Vol. 6, 19-40 (1999).

[37] Deuring, P., von Wahl, W., *Strong solutions of the Navier-Stokes system in Lipschitz bounded domains*, Math. Nachr. **171** (1995) 111-148.

[38] Duduchava, R., Natroshvili, D., *Mixed crack type problem in anisotropic elasticity*, Math. Nachr. bf 191 (1998), 83–107.

[39] Duduchava, R., Wendland, W., *The Wiener-Hopf method for systems of pseudodifferential equations with an application to crack problems*, Integral Equations Operator Theory **23** (1995) 3, 294–335.

[40] Dimitrov, A., *On singularities in the solution of three-dimensional Stokes flow and incompressible elasticity problems with corners*, Int. J. Numer. Methods Eng. **60** (2004) 4, 773-801.

[41] Dimitrov, A., Andrä, H., Schnack, E., *Efficient computation of corner singularities in 3d-elasticity*, Int. J. Numer. Meth. Eng. **52** (2001) 8, 805-827.

[42] Dindoš, M., Mitrea, M., *The stationary Navier-Stokes system in nonsmooth manifolds: The Poisson problem in Lipschitz and C^1 domains*, Arch. Ration. Mech. Anal. **174** (2004) 1, 1-47.

[43] Duvaut, G, Lions, J.-L., *Les Inéquations en Mécanique et en Physique*, Dunod, Paris 1972.

[44] Ebmeyer, C., *Mixed boundary value problems for nonlinear elliptic systems in n-dimensional Lipschitzian domains*, Z. Anal. Anwendungen **18** (1999) 3, 539-555.

[45] Ebmeyer, C., Frehse, J., *Mixed boundary value problems for nonlinear elliptic equations in multi-dimensional non-smooth domains*, Math. Nachr. **203** (1999) 47-74.

[46] Ebmeyer, C., Frehse, J., *Steady Navier-Stokes equations with mixed boundary value conditions in three-dimensional Lipschitzian domains,* Math. Ann. **319** (2001) 349-381.

[47] Elschner, J., Kaiser, H.-C., Rehberg, J., Schmidt, G., *$W^{1,q}$ regularity results for elliptic transmission problems on heterogeneous polyhedra*, Math. Models Methods Appl. Sci. **17** 4, 593-615.

[48] Elschner, J., Rehberg, J., Schmidt, G., *Optimal regularity for elliptic transmission problems including C^1 interfaces*, Interfaces Free Bound **9**, 2, 233-252.

[49] Escobar, J. F., *Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate*, Comm. Pure Appl. Math. **43** (1990) 7, 857-883.

[50] Eskin, G. I., *Boundary value problems for elliptic pseudodifferential equations*, Translations of Mathematical Monographs, 52, Amer. Math. Soc., Providence, R.I., 1981.

[51] Eskin, G., *Boundary value problems for second-order elliptic equations in domains with corners*, Pseudodifferential operators and applications (Notre Dame, Ind., 1984), 105û131, Proc. Sympos. Pure Math., 43, Amer. Math. Soc., Providence, RI, 1985.

[52] Fromm, S. J., *Potential space estimates for Green potentials in convex domains*, Proc. Amer. Math. Soc. **119** (1993) 1, 225-233.

[53] Fromm, S. J., *Regularity of the Dirichlet problem in convex domains in the plane*, Michigan Math. J. **41** (1994) 3, 491–507.

[54] Fromm, S. J., Jerison, D., *Third derivative estimates for Dirichlet's problem in convex domains.* Duke Math. J. **73** (1994), 2, 257–268.

[55] Girault, V., Raviart, P.-A., *Finite element approximation of the Navier-Stokes equation*, Springer-Verlag 1979.

[56] Gohberg, I., Sigal, E. I., *Operator generalization of the theorem on the logarithmic residue and Rouché's theorem,* Mat. Sb.**84** (1971) 4, 607-629; English transl. in: Math. USSR Sb. **13** (1971).

[57] Grachev, N. V., Maz′ya, V. G., *Estimates for the Neumann problem in a polyhedral cone*, Preprint LiTH-MAT-R-91-28, University of Linköping 1991.

[58] Grisvard, P., *Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain*, Proc. 3rd Symp. Numer. Solut. Partial Differ. Equat., College Park 1975, 207-274 (1976).

[59] Grisvard, P., *Behaviour of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain,* Synspade 1975, Ed. Hubbard 1975.

[60] Grisvard, P., *Elliptic problems in nonsmooth domains,* Monographs and Studies in Mathematics **21** Pitman, Boston 1985.

[61] Grisvard, P., *Edge behavior of the solution of an elliptic problem*, Math. Nachr. **132** (1987) 281-299.

[62] Grisvard, P., *Singularities in boundary value problems*, Masson, Paris 1992.

[63] Grüter, M., Widman, K.-O., *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982) 303-342.

[64] Guzman, J., Leykekhman, D., Rossmann, J., Schatz, A. H., *Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods*, Numerische Mathematik **112** (2009) 221-243

[65] Hanna, M. S., Smith, K. T., *Some remarks on the Dirichlet problem in piecewise smooth domains*, Comm. Pure Appl. Math. **20** (1967) 575-593.

[66] Jakab, T., Mitrea, I., Mitrea, M., *Sobolev estimates for the Green potential associated with the Robin-Laplacian in Lipschitz domains satisfying a uniform exteriour ball condition*, Sobolev Spaces in mathematics II, Applications in Analysis and Partial Differential Equations, International Mathematical Series, Vol. 9, Springer, 2008.

[67] Jerison, D. S., Kenig, C. E., *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130** (1995) 161-219.

[68] Kadlec, J., *The regularity of the solution of the Poisson problem in a domain whose boundary is similar to that of a convex domain*, Czechoslovak. Math. J. **14 (89)** (1964) 386-393 (in Russian).

[69] Kalex, H.-U., *Zur Lösungstheorie für die Gleichungen der wärmegekoppelten Strömung zäher, incompressibler Fluide mit temperaturabhängigen Materialkoeffizienten in zweidimensionalen Gebieten mit nichtglatten Randdaten*, Thesis, Techn. Univ. Dresden 1990.

[70] Kalex, H.-U., *On the flow of a temperature dependent Bingham fluid in non-smooth bounded two-dimensional domains*, Z. Anal. Anwendungen **11** (1992) 4, 509-530.

[71] Kenig, C. E., *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics **83**, AMS, Providence, RI, 1994.

[72] Knees, D., *On the regularity of weak solutions of quasilinear elliptic transmission problems on polygedral domains*, Z. Anal. Anwendungen **23** (2004) 509-546.

[73] Komech, A. I., *Elliptic boundary value problems on manifolds with piecewise smooth boundary*, Mat. Sb. **92** (1973) 89-134 (in Russian).

[74] Kondrat'ev, V. A., *Boundary value problems for elliptic equations in domains with conical or angular points,* Trudy Moskov. Mat. Obshch. **16** (1967) 209-292, English transl. in: Trans. Moscow Math. Soc. **16** (1967) 227-313.

[75] Kondrat'ev, V. A., *On the smoothness of solutions of the Dirichlet problem for elliptic equations of second order in a piecewise smooth domain*, Differ. Uravn. **6** (1970) 10, 1831-1843 (in Russian).

[76] Kondrat'ev, V. A., *Singularities of the solutions of the Dirichlet problem for elliptic equations of second order in a neighborhood of an edge*, Differ. Uravn. **13** (1977) 11, 2026-2032 (in Russian).

[77] Kozlov, V. A., *A theorem on a strong zero for an elliptic boundary value problem in a corner*, Mat. Sbornik **180** (1989), 6, 831–849; English transl. in: Math. USSR Sbornik **67** (1990) 1, 283–302.

[78] Kozlov, V. A., *Behavior of solutions to the Dirichlet problem for elliptic systems in convex domains*, Comm. Partial Differential Equations **34** (2009) 24-51.

[79] Kozlov, V. A., Maz′ya, V. G., *Spectral properties of operator bundles generated by elliptic boundary value problems in a cone*, Funktsional. Anal. i Prilozhen. **22** (1988), 38-46; English transl. in: Funct. Anal. Appl. **22** (1988) 2, 114-121.

[80] Kozlov, V. A., Maz'ya, V. G., *On the spectrum of an operator pencil generated by the Neumann problem in a cone*, Algebra i Analiz **3** (1991), 111-131; English transl. in: St. Petersburg Math. J. **3** (1992) 2, 333-353.

[81] Kozlov, V. A., Maz'ya, V. G., *Differential equations with operator coefficients*, Springer Monographs in Mathematics, Springer Berlin - Heidelberg - New York 1999.

[82] Kozlov, V. A., Maz'ya, V. G., *Asymptotic formula for solutions to elliptic equations near the Lipschitz boundary*, Ann. Mat. Pura ed Appl. **184** (2005), 185-213.

[83] Kozlov, V. A., Maz'ya, V. G., Rossmann, J., *Conic singularities of solutions to problems in hydrodynamics of a viscous fluid with a free surface,* Math. Scand. **83** (1998) 103-141.

[84] Kozlov, V. A., Maz'ya, V. G., Rossmann, J., *Elliptic boundary value problems in domains with point singularities*, Mathematical Surveys and Monographs **52**, Amer. Math. Soc., Providence, Rhode Island 1997.

[85] Kozlov, V. A., Maz'ya, V. G., Rossmann, J., *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs **85**, Amer. Math. Soc., Providence, Rhode Island 2001.

[86] Kozlov, V. A., Maz'ya, V. G., Schwab, C., *On singularities of solutions to the Dirichlet problem of hydrodynamics near the vertex of a cone*, J. Reine Angew. Math. **456** (1994) 65-97.

[87] Kratz, W., *On the maximum modulus theorem for Stokes function*, Appl. Anal. **58** (1995) 293-302.

[88] Kress, R., *Linear integral equations*, Springer-Verlag Berlin - Heidelberg - New York - London - Paris - Tokyo - Hong Kong 1989.

[89] Ladyzhenskaya, O. A., *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York 1969.

[90] Leguillon, D., Sanchez-Palencia, E., *Computation of singular sulutions in elliptic problems and elasticity*, Masson, John Wiley, Paris 1987.

[91] Lions, J.-L., Magenes, E., *Problèmes aux limites non homogènes et applications*, Dunod, Paris 1968.

[92] Lubuma, J. M.-S., Nicaise, S., *Dirichlet problems in polyhedral domains. I: Regularity of the solutions*, Math. Nachr. **168** (1994) 243-261.

[93] Maremonti, P., *On the Stokes equations: The maximum modulus theorem*, Math. Models Methods Appl. Sci. **10** (2000) 7, 1047-1072.

[94] Maremonti, P., Russo, R., *On existence and uniqueness of classical solutions to the stationary Navier-Stokes equations and to the traction problem of linear elastostatics*, Russo, R. (ed.) Classical problems in mechanics, Rome: Aracne Editrice. Quad. Mat. **1** (1997) 171-251.

[95] Mayboroda, S., Mitrea, M., *Sharp estimates for Green potentials on non-smooth domains*, Math. Res. Lett. **11** (2004) 4, 481-492.

[96] Mayboroda, S., Mitrea, M., *The Poisson problem for the Lamé system on low-dimensional Lipschitz domains*, Intgral Methods in Science an Engineering (2006) 137-160.

[97] Mayboroda, S., Maz'ya, V., *Boundedness of the Hessian of a biharmonic function in a convex domain*, Comm. Partial Diff. Eq. **33** (2008), no 8.

[98] Mayboroda, S., Maz'ya, V., *Boundedness of the gradient and Wiener test for the solutions of the biharmonic equation*, to appear in Invent. Math.

[99] Maz'ya, V. G., *On Neumann's problem in domains with nonregular boundaries*, Sibirsk. Matem. Zh. **9** (1968) 6, 1322-1350.

[100] Maz'ya, V. G., *On weak solutions of the Dirichlet and Neumann problems*, Trans. Moscow Math. Soc. **20** (1969) 135-172.

[101] Maz'ya, V. G., *On the coercivity of the Dirichlet problem in a domain with nonregular boundary*, Izv. Vyssh. Uchebn. Zaved. Matem. **4** (1973) 54-76.

[102] Maz'ya, V. G., *On a problem with oblique derivative for a domain of polyhedral type*, Dokl. Akad. Nauk SSSR **211** (1973) 1; English transl. in: Soviet Math Dokl. **14** (1973) 4.

[103] Maz'ya, V. G., *On the oblique derivative problemin a domain with edges of different dimensions*, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **7** (1973) 34-39; English transl. in: Vestnik Leningrad. Univ. Math. **6** (1979) 148-154.

[104] Maz'ya, V. G., *Sobolev spaces*, Springer 1985.

[105] Maz'ya, V. G., *The Wiener test for higher order elliptic equations*, Duke Math. J. **115** (2002) 3, 479-512.

[106] Maz'ya, V. G., *Analytic criteria in the qualitative spectral analysis of the Schrödinger operator, Spectral theory and mathematical physics* a Festschrift in honor of Barry Simon's 60th birthday, Proc. Sympos. Pure Math., Part 1, Amer. Math. Soc., Providence, RI , 76 , 2006,257-288 .

[107] Maz'ya, V. G., *Boundedness of the gradient of the solution to the Neumann-Laplace problem in a convex domain*, C.R. Acad. Sci. Paris, Ser. I **347** (2009) 517-520.

[108] Maz'ya, V. G., Kresin, G. I., *On the maximum principle for strongly elliptic and parabolic second order systems with constant coefficients*, Mat. Sb. **125** (1984), 458-480; English transl. in: Math. USSR-Sb. **53** (1986), 457-479.

[109] Maz'ya, V. G., Mitrea, M., Shaposhnikova, T., *The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients*, to appear.

[110] Maz'ya, V. G., Nazarov, S. A., Plamenevskiĭ, B. A. *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten,* Vol. 1, 2, Akademie-Verlag, Berlin 1991.

[111] Maz'ya, V. G., Netrusov, Yu. V., Poborchi, *Boundary values of functions from Sobolev spaces in some non-Lipschitzian domains*, Algebra i Analiz **11** (1999) 1, 141-170; English transl. in: St. Petersburg Math. J. **11** (2000) 1, 107-128.

[112] Maz'ya, V. G., Plamenevskiĭ, B. A., *Problem with oblique derivative in a domain with piecewise smooth boundary,* Funktsional. Anal. i Prilozh. **5** (1971) 3, 102-103; English transl. in: Functional Anal. Appl. **5** (1971).

[113] Maz'ya, V. G., Plamenevskiĭ, B. A. *A certain class of manifolds with singularities*, Izv. Vyssh. Uchebn. Zaved. Matematika **11** (1972) 46-52 (in Russian).

[114] Maz'ya, V. G., Plamenevskiĭ, B. A., *On elliptic boundary value problems in domains with piecewise smooth boundary*, Trudy Simp. Mekh. Sploshnoi Sredy 1971, Metsniereba, Tbilissi 1973, 171-182 (in Russian).

[115] Maz'ya, V. G., Plamenevskiĭ, B. A., *On boundary value problems for a second order elliptic equation in a domain with edges*, Vestnik Leningrad. Univ. Mat. Mekh. Astron. **1** (1975) 102-108; English transl. in: Vestnik Leningrad Univ. Math. **8** (1980) 99-106.

[116] Maz'ya, V. G., Plamenevskiĭ, B. A., *Elliptic boundary value problems on manifolds with singularities*, Probl. Mat. Anal. **6** 85-142, Izdat. Leningrad. Univ. 1977.

[117] Maz'ya, V. G., Plamenevskiĭ, B. A., *Estimates in L_p and Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary,* Math. Nachr. **81** (1978) 25-82, English transl. in: Amer. Math. Soc. Transl., Vol. 123 (1984) 1-56.

[118] Maz'ya, V. G., Plamenevskiĭ, B. A., *L_p estimates for solutions of elliptic boundary value problems in domains with edges,* Tr. Moskov. Mat. Obs. **37** (1978) 49-93; English transl. in: Trans. Moscow. Math. Soc. **37** (1980) 49-97.

[119] Maz'ya, V. G., Plamenevskiĭ, B. A., *Estimates of the Green function and Schauder estimates for solutions of elliptic boundary value problems in a dihedral angle*, Sibirsk. Mat. Zh. **19** (1978) 5, 1065-1082 (in Russian).

[120] Maz'ya, V. G., Plamenevskiĭ, B. A., *Schauder estimates of solutions of elliptic boundary value problems in domains with edges on the boundary,* Partial Differential Equations (Proc. Sem. S. L. Sobolev, 1978, No. 2) Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1978, 69-102; English transl. in: Amer. Math. Soc. Transl. **123** (1984) 141-169.

[121] Maz'ya, V. G., Plamenevskiĭ, B. A., *On the asymptotics of the fundamental solutions of elliptic boundary value problems in regions with conical points,* Probl. Mat. Anal. **7** (1979) 100-145, English transl. in: Sel. Math. Sov. **4** (1985) 4, 363-397.

[122] Maz'ya, V. G., Plamenevskiĭ, B. A., *On the maximum principle for the biharmonic equation in a domain with conical points*, Izv. Vyssh. Uchebn. Zaved. **2** (1981) 52-59 (in Russian).

[123] Maz'ya, V. G., Plamenevskiĭ, B. A., *On properties of solutions of three-dimensional problems of elasticity theory and hydrodynamics in domains with isolated singular points* (in Russian), Dinamika Sploshn. Sredy **50** (1981) 99-121; English transl. in: Amer. Math. Soc. Transl. **123** (1984) 109-123.

[124] Maz'ya, V. G., Plamenevskiĭ, B. A., *The first boundary value problem for classical equations of mathematical physics in domains with piecewise smooth boundaries*, Part 1: Z. Anal. Anwendungen **2** (1983) 4, 335-359, Part 2: Z. Anal. Anwendungen **2** (1983)6, 523-551 (in Russian).

[125] Maz'ya, V. G., Plamenevskiĭ, B. A., Stupyalis, L. I., *The three-dimensional problem of steady-state motion of a fluid with a free surface*, Diff. Uravn. i Primen.-Trudy Sem. Protsessy Optimal. Upravl. **23** (1979); English transl. in: Amer. Math. Soc. Transl. **123** (1984) 171-268.

[126] Maz'ya, V. G, Poborchi, S., *Differentiable Functions on Bad Domains*, World Scientific, 1997, XVIII.

[127] Maz′ya, V. G., Rossmann, J., *Über die Lösbarkeit und die Asymptotik der Lösungen elliptischer Randwertaufgaben in Gebieten mit Kanten I–III*, Preprint P-MATH 07/84, 30/84, 31/84 Institut für Math., Akademie der Wissenschaften der DDR, Berlin 1984.

[128] Maz′ya, V. G., Rossmann, J., *Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten*, Math. Nachr. **138** (1988) 27-53.

[129] Maz'ya, V. G., Rossmann, J., *On the Agmon-Miranda maximum principle for solutions of elliptic equations in polyhedral and polygonal domains*, Ann. Global Anal. Geom. **9** (1991) 3, 253-303.

[130] Maz'ya, V. G., Rossmann, J., *On the Agmon-Miranda maximum principle for solutions of strongly elliptic equations in domains of $\mathbb{R}^n$ with conical ponts*, Ann. Global Anal. Geom. **10** (1992) 125-150.

[131] Maz'ya, V. G., Rossmann, J., *On a problem of Babuška (Stable asymptotics of solutions to the Dirichlet problem for elliptic equations of second order in domains with angular points*, Math. Nachr. **155** (1992) 199-220.

[132] Maz′ya, V. G., Rossmann, J., *On the behaviour of solutions to the Dirichlet problem for second order elliptic equations near edges and polyhedral vertices with critical angles*, Z. Anal. Anwendungen **13** (1994) 1, 19-47.

[133] Maz'ya, V. G., Rossmann, J., *Point estimates for Green's matrix to boundary value problems for second order elliptic systems in a polyhedral cone*, Z. Angew. Math. Mech. **82** (2002) 5, 291-316.

[134] Maz'ya, V. G., Rossmann, J., *Weighted L_p estimates of solutions of boundary value problems for second order elliptic systems in polyhedral domains*, Z. Angew. Math. Mech. **83** (2003) 7, 435-467.

[135] Maz'ya, V. G., Rossmann, J., *Schauder estimates for solutions to boundary value problems for second order elliptic systems in polyhedral domains*, Applicable Analysis **83** (2004) 3, 271-308.

[136] Maz'ya, V. G., Rossmann, J., *Pointwise estimates for Green's kernel of a mixed boundary value problem to the Stokes system in a polyhedral cone*, Math. Nachr. **278** (2005) 15, 1766-1810.

[137] Maz'ya, V. G., Rossmann, J., *Schauder estimates for solutions to a mixed boundary value problem for Stokes system in polyhedral domains*, Math. Meth. Appl. Sci. **29** (2006) 9, 965-1017.

[138] Maz'ya, V. G., Rossmann, J., *L_p estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains*, Math. Nachr. **280** (2007) 7, 751-793.

[139] Maz'ya, V. G., Rossmann, J., *Mixed boundary value problems for the Navier-Stokes system in polyhedral domains*, Arch. Rational Mech. Anal. **194** (2009) 2, 669-712.

[140] Maz'ya, V. G., Rossmann, J., *A maximum modulus estimate for solutions of the Navier-Stokes system in domains of polyhedral type*, Math. Nachr. **282** (2009) 3, 459-469.

[141] Maz'ya, V. G., Shaposhnikova, T., *Theory of multipliers in spaces of differentiable functions*, Pitman 1985.

[142] Maz'ya, V. G., Shaposhnikova, T., *Theory of Sobolev Multipliers with Applications to Differential and Integral Operators*, Grundlehren der Mathematischen Wissenschaften, vol. 337, Springer 2008.

[143] Maz'ya, V. G., Shaposhnikova, T., *Higher regularity in the classical layer potential theory for Lipschitz domains*, to appear.

[144] Miranda, C., *Formule die maggiorazione e teorema di esistenza per le funzioni biarmoniche in due variabili*, Giorn. Math. Battaglini bf 78 (1949), 97-118.

[145] Miranda, C., *Teorema del massimo modulo u teorema di esistenza e di unicita per il problema Dirichlet relative alle equazioni ellitiche in due variabili*, Ann. Mat. Pura Appl. **46** (1958), 265-311.

[146] Mitrea, M., *Boundary value problems for Dirac operators and Maxwell's equations in non-smooth domains*, Math. Meth. Appl. Sci. **25** (2002) no. 16-18, 1355-1369.

[147] Mitrea, M., Monniaux, S., *The regularity of the Stokes operator and the Fujita-Kato approach to the Navier-Stokes initial value prpblem in Lipschitz domains*, J. Funct. Anal. **254** (2008) 6, 1522-1574.

[148] Mitrea, M., Taylor, M., *Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev-Besov space results and the Poisson problem*, J. Funct. Anal. **176** (2000) 1-79.

[149] Mitrea, M., Taylor, M., *Navier-Stokes equations on Lipschitz domains in Riemannian manifolds*, Math. Ann.**321** (2001) 4, 955-987.

[150] Mitrea, M., Taylor, M., *Sobolev and Besov space estimates for solutions to second order PDE on Lipschitz domains in manifolds with Dini or Hölder continuous metric tensors*, Comm. Partial Differential Equations **30** (2005) 1-37.

[151] Mitrea, M., Taylor, M., *The Poisson problem in weighted Sobolev spaces on Lipschitz domains*, Indiana Univ. Math. J. **55** (2006) 1063-1089.

[152] Morrey, C. B., *Multiple integrals in the calculus of variations*, Springer Berlin - Heidelberg - New York 1966.

[153] Naumann, J. *On a maximum principle for weak solutions of the stationary Stokes system*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. IV. Ser. 15, No. 1 (1988) 149-168.

[154] Nazaikinskiĭ, V. E., Savin, A. Yu, Schulze, B.-W., Sternin, B. Yu., *Elliptic theory on singular manifolds*, Differential and Integral Equations and Their Applications, 7. Chapman Hall/CRC, Boca Raton, FL, 2006.

[155] Nazarov, S. A., *Estimates near the edge of the solution of the Neumann problem for an elliptic system*, Vestnik Leningrad. Univ. **1** (1988) 1, 37-42.

[156] Nazarov, S. A., *The polynomial property of self-adjoint elliptic boundary value problems and an algebraic description of their attributes*, Uspekhi Mat. Nauk **54** (1999) 5, 77-142; English transl. in: Russian Mat. Surveys **54** (1999) 5, 947-1014.

[157] Nazarov, S. A., Plamenevskiĭ, B. A., *Neumann problem for self-adjoint systems in a domain with piecewise smooth boundary*, Trudy Leningrad. Mat. Obshch. **1** (1991) 174-211.

[158] Nazarov, S. A., Plamenevskiĭ, B. A., *Elliptic problems with the radiation condition on edges of the boundary*, Mat. Sb. **183** (1992) 13-44; English transl. in: Russian Acad. Sci. Sb. Math. **77** (1994) 1, 149-176.

[159] Nazarov, S. A., Plamenevskiĭ, B. A., *Self-adjoint elliptic problems with the radiation condition on edges of the boundary*, Algebra i Analiz **4** (1992) 196-225; English transl. in: St. Petersburg Math. J. **4** (1993) 3, 569-594.

[160] Nazarov, S. A., Plamenevskiĭ, B. A., *Elliptic problems in domains with piecewise smooth boundaries,* De Gruyter Expositions in Mathematics **13**, Berlin-New York 1994.

[161] Nazarov, S. A., Sweers, G., *Boundary value problems for the biharmonic equation and the iterated Laplacian in a three-dimensional domain with an edge*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 336 (2006), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 37, 153–198, 276–277; English transl. in: J. Math. Sci. (N. Y.) 143 (2007), no. 2, 2936–2960.

[162] Nicaise, S., *Polygonal interface problems,* Verlag Peter Lang GmbH, Frankfurt/Main 1993.

[163] Nicaise, S., *Regularity of the weak solution of the Lamé system in nonsmooth domains*, Bandle, C. (ed.) et al., Progress in partial differential equations: calculus of variations, applications. Pitman Res. Notes Math. Ser. 267 (1993) 272-284.

[164] Nicaise, S., *Regularity of the solutions of elliptic systems in polyhedral domains*, Bull. Belg. Math. Soc. **4** (1997) 411-429.

[165] Nikishkin, V. A., *Singularities of the solutions of the Dirichlet problem for elliptic differential equations of second order in a neighborhood of an edge*, Vestnik Moskov. Univ. Ser. Mat. **34** (1979) 51-62 (in Russian).

[166] Odquist, F. K. G., *Über die Randwertaufgaben der Hydrodynamik zäher Flüssigkeiten*, Math. Zeitschrift **32** (1930) 329-375.

[167] Orlt, M., Sändig, A.-M., *Regularity of viscous Navier-Stokes flows in nonsmooth domains*, Costabel, M. (ed.) et al., Boundary value problems and integral equations in nonsmooth domains, Proceedings of the conference at the CIRM, Luminy, France, May 3-7, 1993, Marcel Dekker, New York, Lect. Notes Pure Appl. Math. **167** (1995) 185-201.

[168] Pipher, J., Verchota, G. C., *The Dirichlet problem in L_p for the biharmonic equation on Lipschitz domains*, Amer. J. Math. **114** (1992) 923-972.

[169] Pipher, J., Verchota, G. C., *A maximum principle for biharmonic functions in Lipschitz and C^1 domains*, Comment. Math. Helv. **68** (1993) 3, 385-414.

[170] Pipher, J., Verchota, G. C., *Maximum principles for the polyharmonic equation on Lipschitz domains*, Potential Analysis **4** (1995) 6, 615-636.

[171] Pipher, J., Verchota, G. C., *Dilation invariant estimates and the boundary Gårding inequality for higher order elliptic operators*, Ann. of Math. **142** (1995) 1, 1-38.

[172] Plamenevskiĭ, B. A., *Ellipic boundary value problems in domains with piecewise smooth boundary*, Partial Differential Equations IX, Encyclopedia of Mathematical Sciences, Vol. 79 (M.S. Agranovich, Yu.V. Egorov, M.A. Shubin Eds.) Springer Berlin - Heidelberg 1997.

[173] Rabinovich, V., Schulze, B.-W., Tarkhanov, N., *Elliptic boundary value problems in domains with intersecting cuspidal edges*, Dokl. Akad. Nauk **375** (2000) 6, 737-741.

[174] Rabinovich, V., Schulze, B.-W., Tarkhanov, N., *Boundary value problems in oscillating cuspidal wedges*, Rocky Mountain J. Math. 34 (2004) 4, 1399–1471.

[175] Rempel, S., Schulze, B.-W., *Branching of asymptotics for elliptic operators on manifolds with edges*, Partial differential equations, Banach Cent. Publ. **19** (1987) 231-236.

[176] Roitberg, Ya., *Elliptic boundary value problems in the spaces of distributions*, Kluwer Academic Publihers Dordrecht-Boston-London 1996.

[177] Rossmann, J., *The asymptotics of the solutions of linear elliptic variational problems in domains with edges,* Z. Anal. Anwendungen **9** (1990) 3, 565-575.

[178] Rossmann, J., *On two classes of weighted Sobolev-Slobodetskiĭ spaces in a dihedral angle*, Partial Differential Equations, Banach Center Publications **27** (1992) 399-424.

[179] Rossmann, J., *Hölder estimates for Green's matrix of the Stokes system in convex polyhedra*, Around the Research of Vladimir Maz'ya II. Partial Differential Equations, pp. 315-336, Springer (2010).

[180] Shen, Z., *A note on the Dirichlet problem for the Stokes system in Lipschitz domains*, Proc. Amer. Math. Soc. **123** (1995) 801-811.

[181] Shen, Z., *Necessary and sufficient conditions for the solvability of the L_p Dirichlet problem on Lipschitz domains*, Math. Ann. **336** (2006) 697-725.

[182] Schulze, B.-W., *Abschätzungen in Normen gleichmäßiger Konvergenz für elliptische Randwert-Probleme*, Math. Nachr. **67** (1975) 303-315.

[183] Schulze, B.-W., *On a priori estimates in uniform norms for strongly elliptic systems*, Sibirsk. Mat. Zh. **16** (1975) 2, 384-393 (in Russian).

[184] Schulze, B.-W., *Regularity with continuous and branching asymptotics for elliptic operators on manifolds with edges*, Integral Equations Operator Theory **11** (1988) 557-602.

[185] Schulze, B.-W., *Elliptic regularity with continuous and branching edge asymptotics*, Internat. Ser. Numer. Math. **94** (1990) 275-284.

[186] Schulze, B.-W., *Pseudo-differential operators on manifolds with singularities,* Studies in Mathematics and Its Applications. 24. North-Holland, Amsterdam etc., 1991.

[187] Schulze, B.-W., Tarkhanov, N. *Ellipticity and parametrices on domains with cuspidal edges*, Contemp. Math. **242** Amer. Math. Soc., Providence 1999, 217-256.

[188] Solonnikov, V. A., *On Green's matrices for elliptic boundary. II*, Proc. Steklov Inst. Math. **116** (1971) 187-226.

[189] Solonnikov, V. A., *The solvability of the three-dimensional problem with a free boundary for a system o stationary Navier-Stokes equations*, Zap. Nauchn. Sem. Leningrad. Otd. Math. Inst. **84** (1979) 252-285.

[190] Solonnikov, V. A., *Estimates of solutions of the Neumann problem for a second order elliptic equation in domains with erdges on the boundary*, Preprint P-4-83, Leningrad. Otdel. Mat. Inst. Steklov (LOMI), 1983.

[191] Solonnikov, V. A., *On a maximum modulus estimate of the solution of the stationary problem for the Navier-Stokes equations*, Zap. Nauchn. Sem. S.-Petersburg Otdel. Mat. Inst. Steklov (POMI) **249** (1997) 294-302; English transl. in: J. Math. Sci. (New York) **101** (2000) 5, 3563-3569.

[192] Stein, E. M., *Singular integrals an differentiability properties of functions*, Princeton University Press, Princeton, New Jersey 1970.

[193] Stupelis, L., *Navier-Stokes equations in irregular domains*, Mathematics and its Applications, 326. Kluwer Academic Publishers Group, Dordrecht, 1995.

[194] Temam, R., *Navier-Stokes equations, theory and numerical analysis*, North-Holland, Amsterdam 1977

[195] Triebel, H., *Interpolation theory, function spaces, differential operators*, Deutscher Verlag der Wissenschaften Berlin 1978.

[196] Venouziou, M., Verchota, G. C., *The mixed problem for harmonic functions in polyhedra of* $\mathbb{R}^3$, Preprint math.AP 0803.0957v1 arXiv.org (2008).

[197] Verchota, G., *The use of Rellich identities on certain nongraph boundaries*, Harmonic analysis and boundary value problems (Fayetteville, AR, 2000), 127–138, Contemp. Math., 277, Amer. Math. Soc., Providence, RI, 2001.

[198] Verchota, G. C., *The biharmonic Neumann problem in Lipschitz domains*, Acta Math. **194** (2005) 217-279.

[199] Verchota, G., *Counterexamples and uniqueness for* $L_p(\partial\Omega)$ *oblique derivative problems*, J. Funct. Anal. **245** (2007) 2, 413-437.

[200] Verchota, G. C., Vogel, A. L., *A multidirectional Dirichlet problem*, J. Geom. Anal. **13** (2003) 3, 495-520.

[201] Verchota, G. C., Vogel, A. L., *The multidirectional Neumann problem in* $\mathbb{R}^4$, Math. Ann. **335** (2006) 3, 571-644.

[202] von Petersdorff, T., Stephan, E. P., *Decomposition in edge and corner singularities for the solution of of the Dirichlet problem for the Laplacian in a polyhedron*, Math. Nachr. **149** (1990) 71-104.

[203] Wloka, J. *Partial differential equations*, Cambridge University Press 1987.

[204] Zajackowski, W., Solonnikov, V. A., *On the Neumann problem for second order elliptic equations in domains with edges on the boundary*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **127** (1983) 7-48 (in Russian).

List of Symbols

Chapter 1

$\mathbb{R}$ set of real numbers
$\mathbb{C}$ set of complex numbers
∂_{x_j}, D_{x_j} derivatives, 9
∂_x^α, D_x^α higher order derivatives, 9
$C^l(\Omega)$ set of functions with bounded, continuous derivatives of order l, 10
$C^{l,\sigma}(\Omega)$ Hölder space, 10
$L_p(\Omega)$ Lebesgue space, 10
$W^{l,p}(\Omega)$, $\overset{\circ}{W}{}^{l,p}(\Omega)$ Sobolev spaces, 10
$W^{l-1/p,p}(\partial\Omega)$ trace space, 10
$L(x, D_x)$ linear differential operator, 11
$B(x, D_x)$ differential operator, 11
L° principal part of L, 11
$\ker \mathcal{A}$ kernel of the operator $\mathcal{A}$, 13
$\mathcal{R}(\mathcal{A})$ range of $\mathcal{A}$, 13
$\mathcal{K}$ cone or angle, 16
Ω subdomain of the unit sphere, 16
$\rho = |x|$ distance from the origin, 16
$\partial\Omega$ boundary of Ω, 17
$C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$ set of infinitely differentiable functions with compact support vanishing near the origin, 17
$V_\beta^{l,p}(\mathcal{K})$ weighted Sobolev space, 17
$V_\beta^{l-1/p,p}(\partial\mathcal{K}\backslash\{0\})$ trace space, 18

Chapter 2

K two-dimensional wedge, 24
$x' = (x_1, x_2)$ point in K, 24
r, φ polar coordinates, 24
θ opening angle of K,
$\gamma^\pm$ sides of K, 24
$\mathcal{D} = K \times \mathbb{R}$ dihedron, 24
$\Gamma^\pm = \gamma^\pm \times \mathbb{R}$ faces of $\mathcal{D}$, 24
M edge of $\mathcal{D}$, 24
$V_\delta^{l,p}(K)$ weighted Sobolev space, 24,
$V_\delta^{l,p}(\mathcal{D})$ weighted Sobolev space, 24
$C_0^\infty(\overline{\mathcal{D}}\backslash M)$ set of infinitely differentiable functions with compact support in $\overline{\mathcal{D}}\backslash M$, 24
$\overset{\circ}{V}{}_\delta^{l,p}(\mathcal{D})$ weighted Sobolev space, 28
$(\cdot,\cdot)_{\mathcal{D}}$ scalar product in $L_2(\mathcal{D})$, 28
$V_\delta^{-l,p}(\mathcal{D})$ weighted Sobolev space, 28
$V_\delta^{l-1/p,p}(\Gamma^\pm)$ trace space, 30
$L(D_x)$ differential operator, 32
n outer unit normal vector, 32
$L^+(D_x)$ formally adjoint operator, 37
$L(D_{x'}, \eta)$ parameter-dependent operator, 39
$E_\delta^{l,p}(K)$ weighted Sobolev space, 39
$A(\lambda)$, $A^+(\lambda)$ operator pencils, 40
δ_+, δ_- positive real numbers, 40
$\overset{\circ}{E}{}_\delta^{l,p}(K)$ weighted Sobolev space, 42
$E_\delta^{-l,p}(K)$ weighted Sobolev space, 42
$E_{\beta,\delta}^{0,p}(K)$ weighted Lebesgue space, 42
A_δ operator of a model problem, 50
$G(x, \xi)$ Green's matrix, 58
I_ℓ identity matrix, 59
$G^*(x, \xi)$ adjoint Green's matrix, 59
$N_\delta^{l,\sigma}(K)$ weighted Hölder space, 72
$N_\delta^{l,\sigma}(\mathcal{D})$ weighted Hölder space, 72
$N_\delta^{l,\sigma}(\Gamma^\pm)$ weighted Hölder space, 72
$L_\delta^\infty(\mathcal{D})$ weighted function space, 76

Chapter 3

$L(D_x)$ differential operator, 89
$\mathcal{K}$ cone in $\mathbb{R}^3$, 90
$M_1, \ldots, M_d$ edges of $\mathcal{K}$, 90
$\Gamma_1, \ldots, \Gamma_d$ faces of $\mathcal{K}$, 90
$\Omega = \mathcal{K} \cap S^2$ subdomain of the unit sphere S^2, 90
$\gamma_1, \ldots, \gamma_d$ sides of Ω, 90
$\mathcal{S}$ set of singular boundary points, 90
$V_{\beta,\delta}^{l,p}(\mathcal{K})$ weighted Sobolev space, 90
$\rho(x)$ distance from the vertex of $\mathcal{K}$, 90
$r_k(x)$ distance from the edge M_k, 90
$r(x)$ distance from $\mathcal{S}$, 90
$\overset{\circ}{V}{}_{\beta,\delta}^{l,p}(\mathcal{K})$ weighted Sobolev space, 90
$(\cdot,\cdot)_{\mathcal{K}}$ scalar product in $L_2(\mathcal{K})$, 91
$V_{\beta,\delta}^{-l,p}(\mathcal{K})$ weighted Sobolev space, 91
$V_{\beta,\delta}^{l-1/p,p}(\Gamma_j)$ trace space, 94
θ_k angle at the edge M_k, 96

Chapter 4

Chapter 5

Chapter 6

Chapter 7

Chapter 8

Chapter 9

Chapter 10

Chapter 11

List of Examples

Index

This is the first monograph which systematically treats elliptic boundary value problems in domains of polyhedral type. The authors mainly describe their own recent results focusing on the Dirichlet problem for linear strongly elliptic systems of arbitrary order, Neumann and mixed boundary value problems for second order systems, and on boundary value problems for the stationary Stokes and Navier–Stokes systems. A feature of the book is the systematic use of Green's matrices. Using estimates for the elements of these matrices, the authors obtain solvability and regularity theorems for the solutions in weighted and non-weighted Sobolev and Hölder spaces. Some classical problems of mathematical physics (Laplace and biharmonic equations, Lamé system) are considered as examples. Furthermore, the book contains maximum modulus estimates for the solutions and their derivatives.

The exposition is self-contained, and an introductory chapter provides background material on the theory of elliptic boundary value problems in domains with smooth boundaries and in domains with conical points.

The book is destined for graduate students and researchers working in elliptic partial differential equations and applications.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-162

AMS *on the* Web
www.ams.org

ISBN 978-0-8218-4983-5

9 780821 849835

SURV/162