Elliptic Equations in Polyhedral Domains
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Historical remarks

1. Bibliographical notes to chapters

Chapter 1 (smooth domains and isolated singularities). In addition to the references given in Sections 1.1 and 1.2, we note that a historical survey on elliptic boundary value problems in domains with smooth boundaries and in domains with isolated singularities on the boundary can be found in the book [84], which contains many references related to this topic. Therefore, in what follows, we will refer only to works dealing with boundary singularities of positive dimension.

Chapter 2 (Dirichlet problem, nonintersecting edges). The material in this chapter is an adaption to the Dirichlet problem of a more general framework in the papers [118, 119, 120] by Maz'ya and Plamenevskii. The main difference is that, in contrast to these papers, we allow the right-hand side of the differential equation to belong to a weighted Sobolev or Hölder space of negative order. An earlier exposition of solvability and regularity results in Hilbert-Sobolev spaces of integer order was given by the same authors in [114]. A theory of the Dirichlet problem in Hilbert-Sobolev spaces of fractional order was developed by Dauge [31]. In [118, 119, 120], arbitrary elliptic equations supplied with different boundary conditions on the faces of a n-dimensional dihedron were considered.

In particular, as shown in [118], the boundary value problem is solvable in the weighted Sobolev space $V^{s,p}_\delta$ if the kernel and cokernel of the operator of the corresponding parameter-depending model problem in the plane cross-section angle (cf. Section 2.3) are trivial. This condition can be easily checked for the Dirichlet problem and for a broad class of strongly elliptic problems. In general, the algebraic verification of the triviality of the kernel and cokernel just mentioned is an open problem, but the answer is known in some special cases, see Maz'ya and Plamenevskii [112, 114, 115], Maz'ya [103], Komech [73], Eskin [51]. It is proved by Kozlov [77] that, under some requirements on the elliptic operator, one can achieve the triviality of the kernel of the model problem in question by prescribing a finite number of complementary conditions on the edge.

Note that the results in [118] were derived using an operator multiplier theorem for the Fourier transform. The approach in the present book goes up to the paper [119], where estimates of solutions in weighted Hölder spaces were obtained by means of point estimates for Green’s functions.

Various aspects of the elliptic theory for manifolds with edges (parametrices, Fredholm property, index) were studied in numerous works by Schulze and his collaborators by methods of the theory of pseudo-differential operators (see for example the monograph by Nazarovskii, Savin, Schulze and Sternin [154]).
Properties of the Dirichlet problem for the Laplacian stated in Subsections 2.6.6 and 2.8.6 are corollaries of the general Theorems 2.6.5 and 2.8.8. However, particular cases of these results were obtained previously by specific methods of the theory of second order elliptic equations with real coefficients. For instance, coercive estimates of solutions of the Dirichlet problem for second order elliptic equations in the weighted spaces $V^{2,2}_d$ were obtained by Kondrat’ev [75]. The paper [12] of Apushkinskaya and Nazarov is dedicated to Hölder estimates for solutions to the Dirichlet problem for quasilinear elliptic equations in domains with smooth closed edges of arbitrary dimension.

**Chapters 3 and 4 (Dirichlet problem in domains of polyhedral type).** Pointwise estimates for Green’s matrix of the Dirichlet problem for strongly elliptic equations of higher order were obtained in our paper [129]. In the same paper, one can find estimates of solutions in weighted $L_p$-Sobolev spaces similar to those in Sections 3.5 and 4.1. The Hölder estimates in Sections 3.6 and 4.2 were not published before.

In the paper [113] Maz’ya and Plamenevskii introduced a large class of multi-dimensional manifolds with edges of different dimensions intersecting under nonzero angles. This class of manifolds contains polyhedra in $\mathbb{R}^N$ as a very special case. A solvability theory for general elliptic boundary value problems on such manifolds in weighted $L_2$-Sobolev spaces was developed in [116] by an induction argument in dimensions of singular strata. It is assumed in this paper that kernels and cokernels of all model problems generated by edges of different dimensions are trivial, which is the case, in particular, for the Dirichlet problem. This material is reproduced in the book by Nazarov and Plamenevskii [160].

A $L_2$-theory for the Dirichlet problem for general elliptic equations in three-dimensional polyhedral domains was also established in the papers by Lubuma, Nicaise [92] and Nicaise [164]. Some regularity results related to the Dirichlet problem for the Laplace equation in a polyhedral domain were obtained by Hanna and Smith [65], Grisvard [58, 60], Dauge [31], Ammann and Nistor [11]. Buffa, Costabel and Dauge [18] stated regularity assertions for the Laplace and Maxwell equations in isotropic and anisotropic weighted Sobolev spaces. The Dirichlet problem for the Lamé system (and for the Laplace equation as a particular case) in a broad class of piecewise smooth domains without cusps was investigated in detail by Maz’ya and Plamenevskii [124].

**Chapter 5 (Miranda-Agmon maximum principle).** The main results of this chapter were obtained in our papers [129] and [130], the Hölder estimates for the derivatives of Green’s matrix in convex polyhedral type domains presented in Subsection 5.1.5 were proved by Guzman, Leykekhman, Rossmann and Schatz [64].

The history starts with the estimate

\[(11.6.29) \quad ||u||_{C^{m-1}G} \leq c \left( \sum_{k=1}^{m} \left\| \frac{\partial^{k-1} u}{\partial n^{k-1}} \right\|_{C^{m-k} (\partial G)} + ||u||_{L_1(G)} \right),\]

for solutions of strongly elliptic equations $Lu = 0$ of order $2m > 2$ proved in the case of smooth boundaries by Miranda [144, 145] for two-dimensional and by Agmon
for higher-dimensional domains. Schulze [182, 183] justified analogous $C^k$ estimates for solutions of strongly elliptic systems and for more general boundary conditions $D^m g = f$ on $\partial G$, where $m \leq 2m - 1$.

Maz’ya and Plamenevskii [122] proved the estimate (11.6.29) for solutions of the biharmonic equation in a three-dimensional domain with conical vertices. As shown independently in Maz’ya, Rossman [130] and Pipher, Verchota [169], this estimate fails if the dimension is greater than 3. In [169, 170] Pipher and Verchota proved the estimate (11.6.29) for solutions of the biharmonic and polyharmonic equations in Lipschitz domains.

Chapter 6 (systems of second order, nonintersecting edges). The results of this chapter are borrowed from our paper [133]. Even when dealing only with the Dirichlet problem, we obtain new results in comparison with Chapter 2. Here the data and the solutions belong to a wider class of spaces with nonhomogeneous norms which include classical nonweighted Sobolev spaces. These spaces were earlier used in the paper [128] of Maz’ya and Rossmann, where general elliptic boundary value problems were considered under the assumption of the unique solvability of model problems in a plane cross-section angle.

The first treatment of the Neumann problem for the equation $\Delta u = 0$ in the presence of a smooth edge on the boundary was given as early as 1916 by Carleman [19], who used methods of potential theory. For the same problem see the works by Maz’ya and Plamenevskii [112, 115] and Solonnikov and Zajaczkowski [204], where solutions in the spaces $W^{1,2}_q$ were considered. Analogous results in the weighted Sobolev spaces $W^{1,p}_q$ and weighted Hölder spaces $C^{\alpha}_q$ were obtained in the preprint [190] by Solonnikov. Furthermore, the Green’s function for the Neumann problem was estimated in [190]. An $L_2$-theory for more general boundary value problems including the Neumann problem was developed in papers by Nazarov [155, 156], Rossman [177], Nazarov, Plamenevskii [158, 159] (see also the book of Maz’ya and Plamenevskii [160]). The elliptic oblique derivative problem in domains with nonintersecting edges was treated by Maz’ya and Plamenevskii [112].

Nazarov and Sweers [161] investigated the $W^{2,2}$-solvability of the biharmonic equation with prescribed boundary value of the solution and its Laplacian in a three-dimensional domain with variable opening at the edge, where some interesting effects arise for a critical opening.

If the domain is smooth and the role of an edge is played by a smooth surface of codimension 1 in the boundary separating different boundary conditions, another approach to mixed problems based on the Wiener-Hopf method was used starting in the 1960s (see the monograph by Eskin [50]). A similar approach proved to be effective in the study of boundary value problems for domains with two-dimensional cracks and interior cuspidal edges (see Duduchava and Wendland [39], Duduchava and Natroshvili [38], Chkadua [20], Chkadua, Duduchava [21] et al.). In particular in [39], the Wiener-Hopf method was developed for systems of boundary pseudo-differential equations which allowed to manage without the factorization of corresponding matrix symbols and to investigate the asymptotics of the solution to the crack problem in an anisotropic medium.

Exterior cuspidal edges which require different methods were studied by Dauge [35], Schulze, Tarkhanov [187], Rabinovich, Schulze, Tarkhanov [173, [5] for higher-dimensional domains. Schulze [182, 183] justified analogous $C^k$ estimates for solutions of strongly elliptic systems and for more general boundary conditions $D^m g = f$ on $\partial G$, where $m \leq 2m - 1$. Maz’ya and Plamenevskii [122] proved the estimate (11.6.29) for solutions of the biharmonic equation in a three-dimensional domain with conical vertices. As shown independently in Maz’ya, Rossman [130] and Pipher, Verchota [169], this estimate fails if the dimension is greater than 3. In [169, 170] Pipher and Verchota proved the estimate (11.6.29) for solutions of the biharmonic and polyharmonic equations in Lipschitz domains.

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Exterior cuspidal edges which require different methods were studied by Dauge [35], Schulze, Tarkhanov [187], Rabinovich, Schulze, Tarkhanov [173,
Chapters 7 and 8 (second order systems in domains of polyhedral type). These chapters contain a somewhat extended exposition of the results obtained by the authors in [133, 134, 135]. New features in comparison with Chapters 3 and 4 are the use of nonhomogeneous Sobolev and Hölder norms, and the inclusion of the Neumann problem.

The Neumann problem for the Laplace equation in a polyhedral cone was earlier studied in the preprint [57] of Grachev and Maz’ya, where the authors obtained estimates for the solutions in weighted Sobolev and Hölder spaces and pointwise estimates of the Green’s matrix. Daube [34] proved regularity assertions for solutions of the Neumann problem for second order elliptic equations with real coefficients in nonweighted $L_p$-Sobolev spaces. Regularity results in weighted $L_2$-Sobolev spaces for general self-adjoint systems were proved by Nazarov and Plamenevskii [157]. The behavior of the solution of the Neumann problem for the Lamé system near the vertex of a polyhedron is studied in the book by Grisvard [62].

Mixed boundary value problems for the Laplace equation with Dirichlet and Neumann conditions are considered e.g. in the above mentioned works by Daube [34] and Grisvard [62]. The same problems were studied by Ebmeier [44], Ebmeier and Frehse [45] for nonlinear second order equations in $N$-dimensional domains, $N \geq 3$, with piecewise smooth boundaries. Nicaise [163] obtained regularity results for solutions of mixed boundary value problems to the Lamé system in $L_2$-Sobolev spaces.

Maz’ya [102, 103] and Daube [32] studied oblique derivative problems in domains of polyhedral type. Transmission problems in polyhedral domains were handled in papers by Costabel, Daube, Nicaise [29], Chikouche, Mercier, Nicaise [22, 23], Knees [72], Elschnier, Rehberg, Schmidt [48], Elschnier, Kaiser, Rehberg, Schmidt [47].

The conditions ensuring the solvability and regularity of solutions near the vertices depend on information about eigenvalues of the operator pencils $\mathcal{A}_k(\lambda)$ and $\mathcal{A}_j(\lambda)$ introduced in Section 8.1. Information of this nature is collected in the book by Kozlov, Maz’ya and Rossman [85]. The pencil generated by the Neumann problem for elliptic differential operators of arbitrary order was investigated by Kozlov and Maz’ya [80]. Assuming that the cone is convex, it was shown by Escobar [49] and in another way by Maz’ya [107] that the first positive eigenvalue of the pencil $\delta + \lambda(\lambda + N - 2)$ with zero Neumann conditions satisfies the sharp inequality $\lambda_1 \geq 1$. Earlier Daube [34] found a rougher estimate $\lambda_1 > (\sqrt{5} - 1)/2$ in the three-dimensional case.

For special problems and special domains, eigenvalues of operator pencils generated by the Neumann problem were calculated numerically by Leguillon and Sanchez-Palencia [90], Dimitrov [40], Dimitrov, Andrá and Schnack [41] et al.

Chapter 9 (Stokes and Navier-Stokes systems, nonintersecting edges). This chapter is an extended version of our paper [136] concerning the mixed boundary value problem for the Stokes system in a dihedron. Some related results can be found in the earlier paper of Solonnikov [189] and Maz’ya, Plamenevskii and Stupelis [125], where the Dirichlet problem and a particular mixed boundary
value problem were studied in connection with a nonlinear hydrodynamical problem with free boundary. A detailed exposition of the results obtained in [125] can be found in Stupelis [193]. In contrast to [136], the paper [125] deals with solutions in weighted Sobolev and Hölder spaces with homogenous norms.

**Chapters 10 and 11 (Stokes and Navier-Stokes systems, domains of polyhedral type).** These chapters contain results obtained by the authors in [136]–[140] and [179]. The starting point for the development of this theory was the paper by Maz’ya and Plamenevskiǐ [124] dedicated to the Dirichlet problem.

The inequality (11.3.3) for the eigenvalues of the pencil generated by the Dirichlet problem for the Stokes system obtained by Maz’ya and Plamenevskiǐ in [123] was the first result of this nature. More estimates for the eigenvalues can be found in the paper by Dauge [33]. A detailed analysis of these eigenvalues including a variational principle for real ones was developed by Kozlov, Maz’ya and Schwaß [86] (see also the book [85]). The only paper, where the eigenvalues corresponding to the Neumann were touched upon, is that of Kozlov and Maz’ya [79]. Some results corresponding to various mixed type problems were obtained by Kozlov, Maz’ya and Rossmann [83] (see also the book [85]).

The results in Section 11.5 have a long history which begins with Odquist’s inequality

\[ \|u\|_{L^\infty(G)} \leq c \|u\|_{L^\infty(\partial G)} \]

for the solutions of the Stokes system (11.6.1) (see [166]). A proof of this inequality for domains with smooth boundaries is given e.g. in the book by Ladyzhenskaya [89]. We refer also to the papers of Maz’ya and Kresin [108], Naumann [153], Kratz [87] and Maremonti [93]. Using point estimates of the Green’s matrix, Maz’ya and Plamenevskiǐ [123, 124] proved this inequality for solutions of the Stokes system in three-dimensional domains with conical points and in domains of polyhedral type.

For the nonlinear problem (11.6.1), (11.6.2), Solonnikov [191] showed that solutions satisfy the estimate (11.6.21) with a certain unspecified function \( F \) if the boundary \( \partial G \) is smooth. An estimate of this form can be also deduced from the results in a paper of Maremonti and Russo [94]. Maz’ya and Plamenevskiǐ [124] proved for domains of polyhedral type that the solution \( u \) of (11.6.1), (11.6.2) with finite Dirichlet integral is continuous in \( \overline{G} \) if \( h \) is continuous on \( \partial G \). However, the paper [124] contains no estimates for the maximum modulus of \( u \). In our paper [140], we proved the inequality (11.6.21) for domains of polyhedral type and obtained the representation \( F(t) = c_0 t^\nu e^{c_1 t^\nu} \) for the function \( c \).

**2. Bibliographical notes to other related material**

The whole theme of elliptic boundary value problems in nonregular domains is so rich that obviously we could touch upon only a small part of it. In order to illustrate the variety of results in this area, we give here some references related to topics outside of this book without aiming at complete satisfaction to a certain extent.

**Asymptotics of solutions near edges and vertices.** The asymptotic expansions of solutions near boundary singularities are not treated in this book, but
this theme was thoroughly studied simultaneously with solvability properties and became a broad area of research. The asymptotics of solutions of the Dirichlet problem for elliptic equations of second order in a neighborhood of an edge was described by Kondrat’ev [76] and Nikishkin [165] and for the Laplace equation by Grisvard [61]. Asymptotic formulas for solutions to general elliptic boundary value problems were proved by Maz’ya and Plamenevskii [114], Maz’ya and Rossman [127, 128], Dauge [30], Nazarov and Plamenevskii [160]. It was assumed in the last works that the edges do not contain “critical” points, i.e. that there is no bifurcation in singularities. The case of critical edge points was discussed in the papers by Rempp and Schulze [175], and Schulze [184, 185]. Explicit asymptotic formulas for such cases were derived by Costabel and Dauge [25], Maz’ya and Rossman [132]. The asymptotics of solutions near polyhedral vertices was studied by von Petersdorff and Stephan [202] and Dauge [36] for second order equations. The last paper is a masterful survey of the area. We also mention a comprehensive study of singularities of solutions to the Maxwell equation by Costabel and Dauge [26, 27, 28].

**Lipschitz graph and other domains.** Needless to say, there are other areas in the theory of elliptic boundary value problems differing both by classes of domains and the methods of research. First of all, there exists a rich theory dealing with Lipschitz graph boundaries and based on refined methods of harmonic analysis. We refer only to the survey monograph by Kenig [71] and more recent works by Adolphsson, Pipher [4], Brown, Perry, Shen [17], Brown [15], Brown, Shen [16], Deuring, von Wahl [37], Dindoš, Mitrea [42], Ebmeyer [44], Ebmeyer, Frehse [45, 46], Jakab, Mitrea, Mitrea [66], Jerison, Kenig [67], Mayboroda, Mitrea [95, 96], Mitrea [146], Mitrea, Monniaux [147], Mitrea, Taylor [148]–[151], Pipher, Verchota [168, 171], Shen [180, 181] and Verchota [198, 199].

Successful attempts to apply these methods, which are based on the so-called Rellich’s identity, to non-Lipschitz graph polyhedral domains in \(\mathbb{R}^3\) and \(\mathbb{R}^4\) were undertaken by Verchota [197], Verchota and Vogel [200, 201], Venouziou and Verchota [196].

Asymptotic formulas for solutions of the Dirichlet problem for strongly elliptic equations of arbitrary order near the Lipschitz graph boundary were found by Kozlov and Maz’ya [82]. The same boundary value problem with data in Besov spaces was treated in Maz’ya, Mitrea and Shaposhnikova [109] under an assumption on the boundary formulated in terms of the space BMO. Sharp conditions of the \(W^{2,2}\)-solvability of the Dirichlet problem for the Laplace equation in a domain in \(C^1\) but not in \(C^2\) were derived in Maz’ya [101].

Additional information was derived for boundary value problems in arbitrary convex domains (Kadlec [68], Adolphsson [1, 2], Adolphsson, Jerison [3], Fromm [52, 53], Fromm, Jerison [54], Kozlov, Maz’ya [82], Maz’ya [107], Mayboroda, Maz’ya [97]).

Introducing classes of Lipschitz graph domains characterized in terms of Sobolev multipliers, Maz’ya and Shaposhnikova obtained sharp results on solutions in \(W^{1,p}(\Omega)\) ([141], [142], [143]).

It proved to be possible to obtain substantial information on properties of elliptic boundary value problems without imposing a priori restrictions on the class of
domains, such as criteria of solvability and discreteness of spectrum formulated with the help of isoperimetric and isocapacitary inequalities, capacitary inner diameter and other potential theoretic terms (see Maz’ya [99, 100, 104, 106], Alvino, Cianchi, Maz’ya, Mercaldo [10], Cianchi, Maz’ya [24]). Wiener type criteria of regularity of a boundary point and pointwise estimates for solutions and their derivatives in unrestricted domains belong to another direction in the same area (see Maz’ya [105], Mayboroda, Maz’ya [98]).

In conclusion, we only list as key words some other classes of nonsmooth domains which appear in the studies of elliptic boundary value problems: nontangentially accessible domains, uniform domains, John domains, Jordan domains, Nikodym domains, Sobolev domains, extension domains etc.
Bibliography


List of Symbols

Chapter 1

\( \mathbb{R} \) set of real numbers
\( \mathbb{C} \) set of complex numbers
\( \partial_{x_j}, D_{x_j} \) derivatives, 9
\( \partial^2_{x_j}, D^2_{x_j} \) higher order derivatives, 9
\( C^0(\Omega) \) set of functions with bounded, continuous derivatives of order \( l \), 10
\( C^{l,\sigma}(\Omega) \) Hölder space, 10
\( L_p(\Omega) \) Lebesgue space, 10
\( W^{1,p}(\Omega), W^{1,p}_{0}(\Omega) \) Sobolev spaces, 10
\( W^{1,1-p,p}(\partial\Omega) \) trace space, 10
\( L(x, D_x) \) linear differential operator, 11
\( B(x, D_x) \) differential operator, 11
\( L^+ \) principal part of \( L \), 11
\( \ker\mathcal{A} \) kernel of the operator \( \mathcal{A} \), 13
\( \mathcal{R}(\mathcal{A}) \) range of \( \mathcal{A} \), 13
\( K \) cone or angle, 16
\( \Omega \) subdomain of the unit sphere, 16
\( \rho = |x| \) distance from the origin, 16
\( \partial\Omega \) boundary of \( \Omega \), 17
\( C^{\infty}_0(\overline{\mathcal{K}\setminus\{0\}}) \) set of infinitely differentiable functions with compact support vanishing near the origin, 17
\( V^{1,p}_\delta(K) \) weighted Sobolev space, 17
\( V^{1,1-p,p}_\delta(\partial\mathcal{K}\setminus\{0\}) \) trace space, 18

Chapter 2

\( K \) two-dimensional wedge, 24
\( x' = (x_1, x_2) \) point in \( K \), 24
\( r, \varphi \) polar coordinates, 24
\( \theta \) opening angle of \( K \), 24
\( \gamma^\pm \) sides of \( K \), 24
\( D = K \times \mathbb{R} \) dihedron, 24
\( \Gamma^\pm = \gamma^\pm \times \mathbb{R} \) faces of \( D \), 24
\( M \) edge of \( D \), 24
\( V^{1,p}_\delta(K) \) weighted Sobolev space, 24
\( V^{1,p}_\delta(D) \) weighted Sobolev space, 24
\( C^{\infty}_0(\overline{D\setminus M}) \) set of infinitely differentiable functions with compact support in \( \overline{D\setminus M} \), 24
\( V^{1,p}_\delta(D) \) weighted Sobolev space, 28
\( (\cdot, \cdot)_D \) scalar product in \( L_2(D) \), 28
\( V^{1,1-p,p}_\delta(D) \) weighted Sobolev space, 28
\( V^{1,1-p,p}_\delta(\Gamma^\pm) \) trace space, 30
\( L(D_x) \) differential operator, 32
\( n \) outer unit normal vector, 32
\( L^+(D_x) \) formally adjoint operator, 37
\( L(D_{x^\epsilon}, \eta) \) parameter-dependent operator, 39

Chapter 3

\( L(D_x) \) differential operator, 89
\( \mathcal{K} \) cone in \( \mathbb{R}^3 \), 90
\( M_1, \ldots, M_d \) edges of \( \mathcal{K} \), 90
\( \Gamma_1, \ldots, \Gamma_d \) faces of \( \mathcal{K} \), 90
\( \Omega = \mathcal{K} \cap S^2 \) subdomain of the unit sphere \( S^2 \), 90
\( \gamma_1, \ldots, \gamma_d \) sides of \( \Omega \), 90
\( S \) set of singular boundary points, 90
\( V^{1,p}_{\delta,\sigma}(K) \) weighted Sobolev space, 90
\( \rho(x) \) distance from the vertex of \( \mathcal{K} \), 90
\( r_k(x) \) distance from the edge \( M_k \), 90
\( r(x) \) distance from \( S \), 90
\( V^{1,p}_{\delta,\sigma}(K) \) weighted Sobolev space, 90
\( (\cdot, \cdot)_K \) scalar product in \( L_2(K) \), 91
\( V^{1,1-p,p}_{\delta,\sigma}(K) \) weighted Sobolev space, 91
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\( n^\pm \) outer unit normal to \( \Gamma^\pm \), 215
\( d^\pm \) numbers of the set \( \{0, 1\} \), 215
\( B^\pm(D_x) \) differential operator in the boundary conditions, 215
\( L^1,2(D) \) function space, 215
\( H_D \) subspace of \( L^{1,2}(D) \), 215
\( \mathcal{H}_D \) dual space of \( \mathcal{H}_D \), 215
\( (\cdot, \cdot)_D \) scalar product in \( L^2(D)^d \), 215
\( L_0(\lambda), B_0^\pm(\lambda) \) parameter-dependent differential operators, 216
\( x' = (x_1, x_2) \) 217
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\( A(\lambda) \) operator pencil, 217
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\( L^+(D_x) \) formally adjoint differential operator to \( L(D_x) \), 217
\( C^+(D_x) \) differential operator, 217
\( \mathcal{L}^a_0(\lambda), \mathcal{C}_a^0(\lambda) \) parameter-depending differential operators, 217
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\( K \) two-dimensional angle, 218
\( \gamma^\pm \) sides of \( K \), 218
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\( B^+(D_{x^+}, \xi) \) parameter-depending differential operators, 218
\( b_K(\cdot, \cdot, \xi) \) parameter-depending sesquilinear form, 218
\( \langle \cdot, \cdot \rangle_K \) scalar product in \( L_2(K) \), 218
\( A_k \) operator of the boundary value problem, 219
\( M \) edge of the dihedron, 223
\( r(x) = |x'| \) distance from the edge, 223
\( L_{1,p}^p(D) \) weighted Sobolev space, 223
\( W_{1,p}^p(D) \) weighted Sobolev space, 223
\( W_{1,p}^p(\mathbb{R}) \) Sobolev-Slobodetskii space, 223
\( \circ \) average of \( u \) with respect to the angle \( \varphi \), 224
\( E \) extension operator, 226
\( \mathbb{R}_+^2 = (0, \infty) \times \mathbb{R} \) half-plane, 231
\( V_{1,p}^p(\mathbb{R}_+^2) \) weighted Sobolev space, 231
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\( p_k(u) \) Taylor polynomial of \( u \), 236
\( \mathbb{R} = (0, \infty) \) half-axis, 237
\( W_{1,p}^p(\mathbb{R}_+) \) weighted Sobolev space, 237
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\( L_{1-1/p,p}^{1-1/p,p} \) trace space, 237
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\( \sigma(u) \) stress tensor, 261
\( \varepsilon(u) \) strain tensor, 261
\( \theta \) opening of the angle (dihedron), 262
\( G(x, \xi) \) Green’s matrix, 262
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\( \Gamma_1, \ldots, \Gamma_d \) faces of \( K \), 290
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\( A_{l,k} \) coefficients of \( L(D_x) \), 290
\( I_0, I_1 \) sets of indices, 290
\( d_k \) numbers of the set \( \{0, 1\} \), 290
\( L^{1,2}(\Omega) \) function space, 290
\( L^{1/2,2}(\Gamma_j) \) trace space, 291
\( b_k(\cdot, \cdot) \) sesquilinear form, 291
\( \mathcal{H}_K \) subspace of \( L^{1,2}(K) \), 291
\( \langle \cdot, \cdot \rangle_k \) scalar product in \( L_2(K) \), 291
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\( A_k(\lambda) \) operator pencil, 291
\( \delta_{+}^{(k)}, \delta_{-}^{(k)} \) positive real numbers, 292
\( \mathcal{H}_\Omega \) subspace of \( W^{1,2}(\Omega)' \), 292
\( \gamma_j \) sides of \( \Omega \), 292
\( a(\cdot, \cdot; \lambda) \) parameter-dependent sesquilinear form, 292
\( \mathfrak{A}(\lambda) \) operator pencil, 292
\( \delta_{+}^{(k)}, \delta_{-}^{(k)} \) positive real numbers, 292
\( \mathcal{H}_\Omega \) subspace of \( W^{1,2}(\Omega)' \), 292
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\( a(\cdot, \cdot; \lambda) \) parameter-dependent sesquilinear form, 292
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\( r_k(x) \) distance from the edge \( M_k \), 295
\( S \) set of singular boundary points, 295
\( r(x) \) distance from \( S \), 295
\( W_{\beta,0}^{l,p}(K) \) weighted Sobolev space, 295
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\( \mathcal{A}_3 \) operator of the boundary value
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$N(x, D_x)$ conormal derivative, 355  
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$d_j$ numbers of the set $\{0, 1\}$, 356  
$B_j(D_x)$ operator in the boundary condition, 356  
$b(\cdot, \cdot)$ sesquilinear form, 356  
$\mathcal{H}$ subspace of $W^{1,2}(\mathcal{G})^s$, 356  
$\mathcal{G}$ domain of polyhedral type, 356  
$\Gamma_j$ faces of $\mathcal{G}$, 356  
$M_k$ edges of $\mathcal{G}$, 356  
$x^{(i)}$ vertices of $\mathcal{G}$, 356  
$\mathcal{S}$ set of singular boundary points, 356  
$\tau_k(x)$ distance from $M_k$, 357  
$\rho_j(x)$ distance from $x^{(i)}$, 357  
$r(x)$ distance from $\mathcal{S}$, 357  
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$W_{\beta,\delta}^{1,p}(\mathcal{G})$ weighted Sobolev space, 357  
$W_{\beta,\delta}^{1-1/p,p}(\Gamma_j)$ trace space, 357  
$L_{\beta,\delta}(x, D_x)$ principal part of $L(x, D_x)$, 358  
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$D$ dihedron, 381  
$K$ two-dimensional angle, 381  
$\theta$ opening of the angle $K$, 381  
$\Gamma^+ \Gamma^-$ faces of $D$, 381  
$u$ outward normal vector, 381  
$u_n$ normal component of $u$, 381  
$u_t$ tangent component of $u$, 381  
$\varepsilon(u)$ strain tensor, 381  
$\varepsilon(n)u = \varepsilon(u)n$, 381  
$d^+, d^-$ integer numbers, 383  
$S^\pm, N^\pm$ operators in the boundary conditions on $\Gamma^\pm$, 383  
$b_{D}(\cdot, \cdot)$ bilinear form, 383  
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$\mu_+$ positive real number, 410  
$\mathbb{R}^3_+$ half-space, 412  
$G^+(x, \xi)$ Green’s matrix in $\mathbb{R}^3_+$, 412  
$G(x, \xi)$ Green’s matrix in $\mathbb{R}^3$, 413  
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$G(x, \xi)$ Green’s matrix of the problem in a dihedron, 425

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$d_j$ integer numbers, 444  
$V^{1,2}_{\delta,\lambda}(\mathcal{K})$ weighted Sobolev space, 444  
$\mathcal{V}^{1,2}_{\delta,\lambda}(\mathcal{K})$ weighted Sobolev space, 444
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