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Renormalization and Effective Field Theory

Kevin Costello



American Mathematical Society

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Providence, Rhode Island

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This book tells mathematicians about an amazing subject invented by physicists and it tells physicists how a master mathematician must proceed in order to understand it. Physicists who know quantum field theory can learn the powerful methodology of mathematical structure, while mathematicians can position themselves to use the magical ideas of quantum field theory in “mathematics” itself. The retelling of the tale mathematically by Kevin Costello is a beautiful tour de force.

—Dennis Sullivan

This book is quite a remarkable contribution. It should make perturbative quantum field theory accessible to mathematicians. There is a lot of insight in the way the author uses the renormalization group and effective field theory to analyze perturbative renormalization; this may serve as a springboard to a wider use of those topics, hopefully to an eventual nonperturbative understanding.

—Edward Witten



Quantum field theory has had a profound influence on mathematics, and on geometry in particular. However, the notorious difficulties of renormalization have made quantum field theory very inaccessible for mathematicians. This book provides complete mathematical foundations for the theory of perturbative quantum field theory, based on Wilson’s ideas of low-energy effective field theory and on the Batalin–Vilkovisky formalism. As an example, a cohomological proof of perturbative renormalizability of Yang–Mills theory is presented.

An effort has been made to make the book accessible to mathematicians who have had no prior exposure to quantum field theory. Graduate students who have taken classes in basic functional analysis and homological algebra should be able to read this book.



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