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Quadrature Theory

The Theory of Numerical Integration on a Compact Interval

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Preface

Methods for the approximate calculation of definite integrals are covered in every book on numerical analysis. Our intention here is to provide a complementary treatment of this topic by presenting a coherent theory that encompasses many deep and elegant results as well as a large number of interesting (solved and open) problems.

The inclusion of the word "theory" in the title highlights our emphasis on concepts rather than numerical recipes. Thus, no computer programs and only a few numerical examples are given in the book. The focus on theory does not, however, mean that we pass over concrete practical problems, merely that we choose to restrict our attention to problems for which a guaranteed result can be obtained in a systematic manner. Systematic analyses of this kind rely on certain properties of the integrand, over and beyond the knowledge of finitely many function values. Such additional information about the integrand (called "co-observations") forms the central organizing principle for our theory, and distinguishes our book from other texts on quadrature. A wide variety of co-observations are examined in this monograph, as we believe such information will be very useful for solving problems in practical contexts.

While quadrature theory is often viewed as a branch of numerical analysis, its influence extends much further: it has been the starting point of many far-reaching generalizations in various directions, as well as a testing ground for new ideas and concepts; in fact, in many instances the extensions seem more "natural" than the original motivating problem. We shall discuss such generalizations, although the classical problem will remain our guiding star throughout the book.

Working on quadrature has given us great pleasure over the years, and we hope we can convey our enthusiasm for the subject to the readers of this book.

The mathematical prerequisites for engaging with this text are knowledge (at the level taught in most undergraduate courses) of linear algebra, advanced calculus and real analysis.

We thank our wives for their patience and assistance in many ways.

We are grateful to Alice Yew for her help in editing our manuscript.

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Symbols

$\lfloor x \rfloor$	largest integer less than or equal to x	
$\lceil x \rceil$	smallest integer greater than or equal to x	
$f\big\|_X$	restriction of the function f to the set X	
$U^\perp$	orthogonal complement of a subspace U	
$(u)_+^r$	truncated rth power	50
$\\|f\\|$	supremum norm of a function f	29
$\\|L\\|$	norm of a linear functional L	29
$\partial\mathcal{G}$	boundary of a domain $\mathcal{G}$ in the complex plane	
$A(\eta, Q)$	the coefficient of $f(\eta)$ in Q	40
B_ν	Bernoulli polynomial	325
B_ν^*	Bernoulli function	326
$\mathbb{C}$	the set of complex numbers	
$C[X]$	the space of continuous functions	29
$\mathrm{conv}(M)$	convex hull of $M \subset \mathbb{R}^m$	34
$\mathcal{C}$	co-observation	7
$\mathcal{C}_M^{(r)}$	co-observation defined by the rth derivative being bounded	47
$\mathcal{C}_M^{\mathrm{Var}}$	co-observation comprising functions of bounded variation	48
$\mathcal{C}_M^{\mathrm{Var},r}$	co-observation comprising functions whose rth derivatives are of bounded variation	49
$\mathcal{C}_\omega^{\mathrm{mc}}$	co-observation defined by a modulus of continuity	48
$\widetilde{\mathcal{C}}_M^{(r)}$	co-observation comprising periodic functions from $\mathcal{C}_M^{(r)}$	49
$\mathcal{C}_M^{\mathrm{Sa},r}$	Sard's co-observation	73
$\mathcal{C}_M^{\mathrm{Da}}$	Davis's co-observation	78

$\mathcal{C}_M(\mathcal{G})$	co-observation comprising functions holomorphic in $\mathcal{G}$	82
$\mathcal{C}_r^{\mathrm{Kr}}$	Kress's co-observation	266
$\mathcal{C}_M^{\mathrm{FP}}$	co-observation comprising convex functions	93
$\deg Q$	degree of exactness of the quadrature rule Q	101
$\dim U$	dimension of a linear space U	
$\mathrm{dist}(f, U)$	best approximation error	31
dvd	divided difference	121
Δ	difference operator	106
E_n	Stieltjes polynomial	201
$\mathcal{E}_r$	ellipse in the complex plane	77
I	functional to be estimated	1, 47
$\mathrm{ind}(Q)$	index of a quadrature rule Q	42
$\mathrm{Info}(f)$	available information about f	8
intpol	interpolation polynomial	100
K_s	sth Peano kernel	53
$\mathcal{K}_r$	Favard constant	63
$\mathrm{Ker}\, O$	kernel of a mapping O	
L_λ	Laplace coefficient	105
$\mathrm{Lip}_M\, \alpha$	the set of Lipschitz continuous functions	220
$\mathbb{N}$	the set of positive integers	
O	observation	7
$\omega(f;\delta)$	modulus of continuity	48
P_n	Legendre polynomial	319
$\mathbb{P}_n$	space of algebraic polynomials of degree at most n	
q_n	normalized orthogonal polynomial	315
Q	quadrature rule, estimation rule	1, 13
$\underset{\vee}{Q}$, $\hat{Q}$	best lower bound and best upper bound for I	8
$\mathbf{Q}(O)$	set of all estimation rules based on the observation O	26
$\mathbf{Q}^{\mathrm{gen}}$	set of all estimation rules	13
$\mathbf{Q}_n$	set of all quadrature rules with n nodes	4

$\mathbf{Q}^+(I,U)$	set of positive estimation rules for I that are exact for all $u \in U$	40
Q_n^{best}	best quadrature rule	26
Q^{opt}	optimal estimation rule	3, 13
Q^{so}	strongly optimal estimation rule	8
Q^{η}	Krein rule with node η	43
Q_n^{G}	Gaussian rule	100
$Q_n^{\text{Gr},r}$	Gregory rule	237
Q_n^{Kro}	Kronrod rule	202
Q_n^{Lo}	Lobatto rule	153
Q_n^{Mi}	midpoint rule	10
$Q_n^{\text{Ra},a}$, $Q_n^{\text{Ra},b}$	Radau rule	151, 152
$Q_{2^{m+\sigma}+1}^{\text{Ro}(\sigma)}$	Romberg rule	246
Q_{2m+1}^{Si}	Simpson's rule	230
Q_n^{Tr}	trapezoidal rule	10
$\mathbb{R}$	the set of real numbers	
R	remainder functional	1, 16
ρ_n^{best}	infimum of worst-case errors over n-dimensional observations	26
ρ^{intr}	intrinsic error	8
ρ^{opt}	error of a standard estimation framework	8
$\rho(Q)$	(worst-case) error of Q	13
$\rho(Q,\mathcal{C})$	(worst-case) error of Q in the class $\mathcal{C}$	1
$\Re z$, $\Im z$	real and imaginary parts of a complex number z	
$\mathcal{S}_r(\xi_1,\dots,\xi_m)$	set of spline functions	39
T_n	Chebyshev polynomials of the first kind	318
$\mathbb{T}_m$	space of trigonometric polynomials of degree at most m	137
U_n	Chebyshev polynomials of the second kind	319
$\operatorname{Var} f$	total variation of a function f	
$\operatorname{Var} Q$	variance of a quadrature rule Q	291
$\mathbb{Z}$	the set of integers	

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