Subgroup Complexes
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Stephen D. Smith
To my mother, Anna Elizabeth Yust Smith Kirn
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Preface and Acknowledgments

As will be indicated in a moment in the Introduction, this book is primarily intended as an exposition—which hopes to bring a wider audience into contact with an area of research that I have enjoyed working in, over many years.

But of course during those years, I gained much of my own experience by benefiting from the knowledge of very many colleagues. So in this preface, I would first like to take the opportunity to thank them—apologizing in advance to anyone I may have left out. (Of course the reader will see the work of these experts emerging, as the later exposition in the book proceeds.)

Some personal acknowledgments. My introduction to the methods of finite geometry dates mainly to my collaboration with Mark Ronan, beginning around 1979. I also learned a great deal about geometries from Bill Kantor, Jon Hall, Don Higman, Ernie Shult, Francis Buekenhout, and Bruce Cooperstein.

During the 1980s, many experts in finite group theory, motivated partly by the work of Tits on buildings, became interested in geometries underlying simple groups. I particularly benefited from long-term contact with Michael Aschbacher, Franz Timmesfeld, and Geoff Robinson.

Discussions with Peter Webb and Jacques Thévenaz were instrumental in leading me into the more specifically topological methods underlying subgroup complexes; and in effect led to my later collaboration with Dave Benson. Many other topologists helped educate me in their area; particular Alejandro Adem, Jim Milgram, Bill Dwyer, Bob Oliver, and Jesper Grodal. Especially in recent years it has been a pleasure to discuss developments made by John Maginnis and Silvia Onofrei.

Also during the 1970s and 1980s, many combinatorialists (notably Stanley) were also developing similar techniques for the combinatorics of posets (partially ordered sets). Some of my initial contacts with that area were around 1981 with Jim Walker and Bob Proctor. Soon thereafter I began a particularly valuable ongoing correspondence with Anders Björner. Over the years I have also profited from discussions with other experts—notably Volkmar Welker, Michelle Wachs, and John Shareshian.

And of course we also learn from our students: It was a pleasure to work with Peter Johnson, Andrew Mathas, Matt Bardoe, Kristin Umland, and Phil Grizzard—who wrote their theses with me at the University of Illinois at Chicago (UIC), in aspects of this general research area. I also had some involvement in the thesis work of Tony Fisher under George Glauberman, and of Paul Hewitt under Jon Hall.
In a similar vein, it was a pleasure to work in this area with several postdoctoral scholars at UIC: namely Alex Ryba, Satoshi Yoshiara, and Masato Sawabe; and indeed with Yoav Segev, even before completion of his Ph.D.

**The more specific history of this book.** I first collected much of the present material while on sabbatical at Notre Dame, in preparation for a Fall 1990 graduate course there: Math 671, *Subgroup Complexes*.

During Fall 1994, I revised and expanded those old notes, to use as the text for the UIC graduate course Math 532 (Topics in Algebra): *Subgroup Complexes*. I would like to thank the students in that course for their questions and corrections, and for their general interest: Matt Bardoe, Joe Fields, Venketraman Ganesan, Julianne Rainbolt, and Kristin Umland.

A preliminary draft of the book was provisionally accepted for *Surveys of the AMS* in 1995. At that time, I received many detailed and very helpful suggestions from various colleagues, particularly Satoshi Yoshiara and Jacques Thévenaz, which strongly influenced the overall structure of the final version.

However, the book went to the back burner for some years, when I was involved in more urgent collaborations on books with Michael Aschbacher, Dave Benson, Richard Lyons, and Ron Solomon; and I have only managed to complete this book recently. (I particularly thank Sergei Gelfand and his staff at the American Mathematical Society, for their patience with me during this lengthy delay.)

During July 2005, the material of the book was again used as a text—for the summer graduate seminar Math 593 at UIC. Again I thank the students in the course for their willingness to assist me in the final revision process: Hossein Andikfar, Chris Atkinson, Chris Cashen, Phil Grizzard, Jason Karcher, Dean Leonardi, Jing Tao, and Klaus Weide. Their suggestions in particular led me to try to make a clearer distinction between the more elementary exposition, and the more advanced examples. This essentially resulted in the “optional tracks” for reading the book, described below in the Introduction.

I received helpful suggestions on the final (2011) draft of the book from a number of colleagues, including Matt Bardoe, Anders Björner, Jesper Grodal, Jon Hall, Bill Kantor, Ian Leary, Silvia Onofrei, Geoff Robinson, Masato Sawabe, Jacques Thévenaz, Rebecca Waldecker, Satoshi Yoshiara, and Peter Webb. I also thank the anonymous referees contacted by the AMS.

**Institutional acknowledgments.** Parts of this book were developed during several sabbatical periods at Caltech, as well as at Notre Dame and U. Illinois–Urbana. I am also grateful to All Souls College-Oxford, for a Visiting Fellowship during Hilary Term 2009, when some of the final work was carried out.

My overall work has been partially supported over the years by summer grants, first from NSF and more recently from NSA.

**Dedication.** Of course the support and encouragement of my wife Judy Baxter have been unflagging.

Finally I'd like to formally dedicate this book to my mother, Anna Elizabeth Yust Smith Kirn: who at various times earlier in my career asked when I was going to write a book (as opposed to the usual journal articles).

So, although several other books have actually appeared since I started this one, I'm finally in a position to say: Well, Mom—here it is.
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This book is intended as an overview of a research area that combines geometries for groups (such as Tits buildings and generalizations), topological aspects of simplicial complexes from $p$-subgroups of a group (in the spirit of Brown, Quillen, and Webb), and combinatorics of partially ordered sets. The material is intended to serve as an advanced graduate-level text and partly as a general reference on the research area. The treatment offers optional tracks for the reader interested in buildings, geometries for sporadic simple groups, and $G$-equivariant equivalences and homology for subgroup complexes.