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For Ann and Sheila

Contents

Preface

This book concerns the theory of unipotent elements in simple algebraic groups over algebraically closed or finite fields, and nilpotent elements in the corresponding simple Lie algebras. These topics have been an important area of study for decades, with applications to representation theory, character theory, the subgroup structure of algebraic groups and finite groups, and the classification of the finite simple groups. Even detailed information on centralizers is important. For example, information regarding the component groups of centralizers is useful in studying representations of Weyl groups.

There is a great deal of literature on unipotent and nilpotent elements, and many beautiful general results have been proved. In addition to the general theory, there are many situations where precise information on conjugacy classes is of great importance, such as class representatives and precise centralizers. Here the literature is less satisfactory.

More than anything else, our reason for writing this book is that we believe that the information on centralizers is of sufficient importance that it deserves a single source, where results are presented completely in all characteristics, and with consistent notation. In particular the detailed tables of results for exceptional algebraic and finite groups in Chapter 22 should be easily understandable and usable by readers, and likewise tables for some low-dimensional classical groups in Chapter 8.

This is our aim and our approach to this, while using ideas from the literature, is in many parts new. Our results go beyond what is currently known in several ways. For example, the literature on centralizers of unipotent and nilpotent elements in classical groups and Lie algebras in characteristic 2 is not complete, and we obtain complete information. We establish a number of new structural results on centralizers, their embeddings in certain parabolic subgroups, and how the reductive part of the centralizer is embedded in the ambient group.

The book is divided into 22 chapters. The first is an introduction to the topic and overview of the results in the book, and the second contains a number of basic results on algebraic groups that will be used throughout; some of these are standard, others less so, but proofs are provided in most cases. Our results for classical groups are proved in Chapters 3–6. Chapter 3 concerns the case where the characteristic of the underlying field is "good" (meaning that it is not 2 for symplectic and orthogonal groups), and the analysis is fairly short and elementary. This is not the case for characteristic 2, covered in Chapters 4,5 and 6. Here our approach is for the most part new, as are many of the results, and takes substantial effort. In Chapter 7, these results are applied to give corresponding results on classes and centralizers in finite classical groups, and some tables illustrating our results for various classical groups of dimension up to 10 are given in Chapter 8.

The remainder of the book, Chapters 9–22, is devoted to the exceptional groups G_2, F_4, E_6, E_7 and E_8. A key feature of our approach is that we first focus on the classes and centralizers of nilpotent elements, and then use these results to deal with the unipotent elements. This approach has the advantage that our theory for nilpotent elements e has a number of structural features that are not present for unipotent elements, such as the existence of a naturally defined 1-dimensional torus acting on the 1-space spanned by e, and an associated parabolic subgroup, which turns out to contain the centralizer of e. The main results for nilpotent elements are stated in Chapter 9, and proved in the following seven chapters. Unipotent elements are then handled in Chapters 17–20. Finally, Chapter 21 contains proofs of some of our general results on the structure and embedding of centralizers, together with various corollaries of our main results; and Chapter 22 has detailed tables of classes and centralizers in the exceptional algebraic groups, and also in the associated finite groups of Lie type.

It will be apparent even from this brief discussion that in this book we are focussing almost exclusively on the classification and centralizer structure of unipotent and nilpotent classes. There are many other issues concerning these classes which are of great interest in algebraic group theory, algebraic geometry and representation theory. We shall not touch upon these subjects directly, although a number of proofs do require a certain amount of representation theory.

This book does not contain an introduction to the theory of algebraic groups; neither does it contain definitions and basic properties of the simple groups. Nevertheless, we have written it with the intention of being comprehensible to graduate students and researchers who have a basic knowledge of these topics.

We would like to thank Tim Burness for reading the manuscript and suggesting many corrections, Ross Lawther for double-checking some of the calculations in Chapter 13, and Donna Testerman and Bob Guralnick for helpful comments.

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Glossary of symbols

$\tilde{A}_i$, subsystem A_i of short roots, 11
$A_n^\epsilon(q)$, $D_n^\epsilon(q)$, $E_6^\epsilon(q)$, 19
Alt_n, alternating group of degree n, 12
$Aut(G)$, automorphism group of G, 17
B_G, Borel subgroup, 11
$C_3(a_1)$, 129, 269
$C_G(T,e) = C_G(T) \cap C_G(e)$, 5
$C_V(T,e)$, 158
$C_V(e)$, annihilator space of e, 153
$d\tau$, differential of τ, 41
$D(m)$, indecomposable module, 86
$D_n(a_i)$, distinguished class in D_n, 58, 90, 110
Dih_{2n}, dihedral group of order $2n$, 12
e, nilpotent element, 3
e, u corresponding nilpotent and unipotent elements, 287
e_α, root vector in $L(G)$, 11
$e_{c_1 \dots c_r}$, notation for $e_{c_1\alpha_1+\dots+c_r\alpha_r}$, 11
$e_{ij\dots}$, notation for $e_{\alpha_i+\alpha_j+\dots}$, 11
$f_\alpha = e_{-\alpha}$, 11
$G(q)$, finite group of Lie type, 1
G_σ, fixed point group of σ in G, 114
G_τ, fixed point group of τ, 9
$h_\alpha(c)$, element of maximal torus T_G, 11
$\mathrm{Inndiag}(G(q))$, 352
J_i, Jordan block, 39
K, algebraically closed field, 1
$L(G)$, Lie algebra of G, 1
$L(G)(q)$, Lie algebra over $\mathbb{F}_q$, 22
$L(G)_i$, $L(Q)_i$, 136
$L(G)_{\geq i}$, $L(Q)_{\geq i}$, 136
$L(Q)^{(i)}$, 12
$L(Q)_k$, 31
$L(Q)_{\geq k}$, 31
$L(Q^{(\geq i)})$, 12
$[m;l]$, a χ-function, 59
$M_1/M_2/\cdots$, notation for a module, 12
P, parabolic subgroup, 4
P^-, opposite parabolic, 11
$P_{ij\dots}$, parabolic subgroup, 25
$Q^{(\geq i)}/Q^{(\geq i+1)}$, i^{th} level of Q, 12
$Q_{\geq 2}$, 4
$Q_{\geq k}$, 31, 136
$R_u(X)$, unipotent radical of X, 9
s^x, image of s under x, 13
s_α, reflection in the root α, 11
$SL_n(K)$, 9
$SO_n(K)$, $O_n(K)$, 9
S^2V, symmetric square of V, 42
S_x, fixed points of x in S, 13
$Sp_{2n}(K)$, 9
Sym_n, symmetric group of degree n, 12
T, 1-dimensional torus, 4
T-labelling, 133
$T(G)_1$, tangent space at the identity, 35
T_G, maximal torus of G, 9
$T_X(\lambda)$, tilting module of high weight λ, 11
u, unipotent element, 4
U_α, root subgroup, 11
U_i, connected unipotent group of dimension i, 9
$U_{c_1 \dots c_r}$, notation for $U_{c_1\alpha_1+\dots+c_r\alpha_r}$, 11
$U_{ij\dots}$, notation for $U_{\alpha_i+\alpha_j+\dots}$, 11
$V \downarrow Y$, restriction of V to Y, 13
$V(m), W(m)$, indecomposables for u, 59, 91
$V(m), W(m), W_l(m)$, indecomposables for e, 59, 65, 66
$V_X(\lambda)$ (or just λ), irreducible KX-module of high weight λ, 11
$W(G)$, Weyl group of G, 11
$W_X(\lambda)$, Weyl module of high weight λ, 11
$X.Y$, extension of X by Y, 12
Z_p, cyclic group of order p, 5
Δ-module, 142
$\Delta(\lambda;\mu)$, 141
$\Pi(G)$, system of fundamental roots, 9
$\Sigma(G)$, root system, 9
$\alpha_{ij\dots}$, notation for $\alpha_i + \alpha_j + \dots$, 11
χ_V, χ-function, 59
κ, map from unipotents to nilpotents, 94
λ_i, fundamental dominant weight, 11
ω, semilinear map on $L(G)$, 258
σ, Frobenius morphism, 114, 258
σ_q, q-field morphism, 19
$\wedge^2 V$, alternating square of V, 42

Index

This book concerns the theory of unipotent elements in simple algebraic groups over algebraically closed or finite fields, and nilpotent elements in the corresponding simple Lie algebras. These topics have been an important area of study for decades, with applications to representation theory, character theory, the subgroup structure of algebraic groups and finite groups, and the classification of the finite simple groups.

The main focus is on obtaining full information on class representatives and centralizers of unipotent and nilpotent elements. Although there is a substantial literature on this topic, this book is the first single source where such information is presented completely in all characteristics. In addition, many of the results are new—for example, those concerning centralizers of nilpotent elements in small characteristics. Indeed, the whole approach, while using some ideas from the literature, is novel, and yields many new general and specific facts concerning the structure and embeddings of centralizers.

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