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The Ubiquitous Quasidisk

**Frederick W. Gehring
Kari Hag**



American Mathematical Society

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With contributions by
Ole Jacob Broch



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Preface

In August 1981 the first author gave a short course of six lectures on function theory at the NATO Advanced Study Institute in Montreal. The object was to point out the surprising connections which quasidisks—by definition the images of a disk under a quasiconformal mapping of the Riemann sphere—have with various branches of analysis and geometry.

The written account, *Characteristic properties of quasidisks* (97 triple-spaced pages), was published by the University Press of the University of Montreal in 1982 and became quite popular. Thus the notes were out of print after a few years. In the meantime the number of characterizing properties increased, and in the late 1990s we decided to write a book with the tentative title *The Ubiquitous Quasidisk* as an expanded version of the “Montreal notes”.

This book will hopefully be an inspiration for graduate students in geometric function theory. More specifically, the book could be a candidate for the text of a semester-long second-year graduate course on selected topics in the field. The texts by Ahlfors [7] and by Lehto and Virtanen [117] on quasiconformal mappings provide valuable reference literature. A more recent account for additional material is the book by Astala, Iwaniec, and Martin [16].

Our mathematical descendant Ole Jacob Broch has been of invaluable help in writing up the manuscript. He was assisted by Geir Arne Hjelle, another former student, who transformed most of the hand drawn figures to computer pictures in a way that we think helps to preserve the spirit of the “Montreal notes”. We would also like to thank Per Hag and Olli Martio who have read the manuscript and made valuable suggestions.

Frederick W. Gehring
Kari Hag
Ann Arbor/Trondheim
September 2011

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This book focuses on gathering the numerous properties and many different connections with various topics in geometric function theory that quasidisks possess. A quasidisk is the image of a disk under a quasiconformal mapping of the Riemann sphere. In 1981 Frederick W. Gehring gave a short course of six lectures on this topic in Montreal and his lecture notes “Characteristic Properties of

Quasidisks” were published by the University Press of the University of Montreal. The notes became quite popular and within the next decade the number of characterizing properties of quasidisks and their ramifications increased tremendously. In the late 1990s Gehring and Hag decided to write an expanded version of the Montreal notes. At three times the size of the original notes, it turned into much more than just an extended version. New topics include two-sided criteria. The text will be a valuable resource for current and future researchers in various branches of analysis and geometry, and with its clear and elegant exposition the book can also serve as a text for a graduate course on selected topics in function theory.

Frederick W. Gehring (1925–2012) was a leading figure in the theory of quasiconformal mappings for over fifty years. He received numerous awards and shared his passion for mathematics generously by mentoring twenty-nine Ph.D. students and more than forty postdoctoral fellows.

Kari Hag received her Ph.D. under Gehring’s direction in 1972 and worked with him on the present text for more than a decade.



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