Mathematical
Surveys
and
Monographs

Volume 185

Diffeology

Patrick Iglesias-Zemmour

American Mathematical Society

http://dx.doi.org/10.1090/surv/185

Diffeology

Mathematical
Surveys
and
Monographs

Volume 185

Diffeology

Patrick Iglesias-Zemmour

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 53Cxx, 53Dxx, 55Pxx, 55P35, 55Rxx, 55R65, 58A10, 58A40, 58Bxx.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-185

Library of Congress Cataloging-in-Publication Data

Iglesias-Zemmour, Patrick, 1953–
Diffeology / Patrick Iglesias-Zemmour.
pages cm. — (Mathematical surveys and monographs ; volume 185)
Includes bibliographical references.
ISBN 978-0-8218-9131-5 (alk. paper)
1. Global differential geometry. 2. Symplectic geometry. 3. Algebraic topology. 4. Differentiable manifolds. I. Title.

QA670.I35 2013
516.3′62—dc23

2012032894

10 9 8 7 6 5 4 3 2 1 18 17 16 15 14 13

לֹא עָלֶיךָ הַמְּלָאכָה לִגְמוֹר וְאֵין אַתָּה בֶן־חוֹרִין לִבָּטֵל מִמֶּנָּה

רבי טרפון, משנה אבות ב טז

Contents

Preface

At the end of the last century, differential geometry was challenged by theoretical physics: new objects were displaced from the periphery of the classical theories to the center of attention of the geometers. These are the irrational tori, quotients of the 2-dimensional torus by irrational lines, with the problem of quasi-periodic potentials, or orbifolds with the problem of singular symplectic reduction, or spaces of connections on principal bundles in Yang-Mills field theory, also groups and subgroups of symplectomorphisms in symplectic geometry and in geometric quantization, or coadjoint orbits of groups of diffeomorphisms, the orbits of the famous Virasoro group for example. All these objects, belonging to the outskirts of the realm of differential geometry, claimed their place inside the theory, as full citizens. Diffeology gives them satisfaction in a unified framework, bringing simple answers to simple problems, by being the right balance between rigor and simplicity, and pushing off the boundary of classical geometry to include seamlessly these objects in the heart of its concerns.

However, diffeology did not spring up on an empty battlefield. Many solutions have been already proposed to these questions, from functional analysis to noncommutative geometry, via smooth structures à la Sikorski or à la Frölicher. For what concerns us, each of these attempts is unsatisfactory: functional analysis is often an overkilling heavy machinery. Physicists run fast; if we want to stay close to them we need to jog lightly. Noncommutative geometry is uncomfortable for the geometer who is not familiar enough with the $\mathbf{C}^*$-algebra world, where he loses intuition and sensibility. Sikorski or Frölicher spaces miss the singular quotients. Perhaps most frustrating, none of these approaches embraces the variety of situations at the same time.

So, what's it all about? Roughly, a diffeology on an arbitrary set X declares, which of the maps from $\mathbf{R}^n$ to X are *smooth*, for all integers n. This idea, refined and structured by three natural axioms, extends the scope of classical differential geometry far beyond its usual targets. The smooth structure on X is then defined by all these *smooth parametrizations*, which are not required to be injective. This is what gives plenty of room for new objects, the quotients of manifolds for example, even when the resulting topology is vague. The examples detailed in the book prove that diffeology captures remarkably well the smooth structure of singular objects. But quotients of manifolds are not the sole target of diffeology, actually they were not even the first target, which was spaces of smooth functions, groups of diffeomorphisms. Indeed, these spaces have a natural *functional diffeology*, which makes the category Cartesian closed. But also, the theory is closed under almost all set-theoretic operations: products, sums, quotients, subsets etc. Thanks to these nice properties, diffeology provides a fair amount of applications and examples and offers finally a renewed perspective on differential geometry.

Also note the existence of a convenient *powerset diffeology*, defined on the set of all the subsets of a diffeological space. Thanks to this original diffeology, we get a clear notion of what is a smooth family of subsets of a diffeological space, without needing any model for the elements of the family. This powerset diffeology « encodes genetically » the smooth structure of many classical constructions without any exterior help. The set of the lines of an affine space, for example, inherits a diffeology from the powerset diffeology of the ambient space, and this diffeology coincides with its ordinary manifold diffeology, which is remarkable.

Moreover, every structural construction (homotopy, Cartan calculus, De Rham cohomology, fiber bundles etc.) renewed for this category, applies to all these derived spaces (smooth functions, differential forms, smooth paths etc.) since they are diffeological spaces too. This unifies the discourse in differential geometry and makes it more consistent, some constructions become more natural and some proofs are shortened. For example, since the space of smooth paths is itself a diffeological space, the Cartan calculus naturally follows and then gives a nice shortcut in the proof of homotopic invariance of the De Rham cohomology.

What about standard manifolds? Fortunately, they become a full subcategory. Then, considering manifolds and traditional differential geometry, diffeology does not subtract anything nor add anything alien in the landscape. About the natural question, "Why is such a generalization of differential geometry necessary, or for what is it useful?" the answer is multiple. First of all, let us note that differential geometry is already a generalization of traditional Greek Euclidean geometry, and the question could also be raised at this level. More seriously, on a purely technical level, considering many of the recent heuristic constructions coming from physics, diffeology provides a light formal rigorous framework, and that is already a good reason. Two examples:

Example 1. For a space equipped with a closed 2-form, diffeology gives a rigorous meaning to the moment maps associated with every smooth group action by automorphisms. It applies to every kind of diffeological space, it can be a manifold, a space of smooth functions, a space of connection forms, an orbifold or even an irrational torus. It works that way because the theory provides a unified coherent notion of differential forms, on all these kinds of spaces, and the tools to deal with them. In particular, such a general diffeological construction clearly reveals that the status of moment maps is high in the hierarchy of differential geometry. It is clearly a categorical construction which exceeds the ordinary framework of the geometry of manifolds: every closed 2-form on a diffeological space gets naturally a *universal moment map* associated with its group of automorphisms.

Example 2. Every closed 2-form on a simply connected diffeological space[1] is the curvature of a connection form on some diffeological principal bundle. The structure group of this bundle is the diffeological *torus of periods* of the 2-form, *i.e.*, the quotient of the real line by the group of periods of the 2-form. This construction is completely universal and applies to every diffeological space and to every closed 2-form, whether the form is integral or not. The only condition is that the group of periods is diffeologically discrete, that is, a strict subgroup of the real numbers. The construction of a prequantization bundle corresponds to the special case when the periods are a subgroup of the group generated by the Planck constant h or,

[1]The general case is a work in progress.

if we prefer, when the group of periods is generated by an integer multiple of the Planck constant.

The crucial point in these two constructions is that the quotient of a diffeological group — the group of momenta of the symmetry group by the holonomy for the first example, and the group of real numbers by the group of periods for the second — is naturally a nontrivial diffeological group whose structure is rich enough to make these generalizations possible. In this regard, the *contravariant approaches* — Sikorski or Frölicher differentiable spaces — are globally helpless because these crucial quotients are trivial, and this is irremedible. By respecting the internal (nontrivial) structure of these quotients, diffeology leads one to a good level of generality for such general constructions and statements. The reason is actually quite simple, the contravariant approaches define smooth structures by declaring which maps from X to $\mathbf{R}$ are smooth. Doing so, they capture only what looks like $\mathbf{R}$ — or a power of $\mathbf{R}$ — in X, killing everything else. The quotient of a manifold may not resemble $\mathbf{R}$ at all, if we wanted to capture its singularity, we would have to compare it with all kinds of standard quotients. *A contrario*, diffeology as a *covariant approach* assumes nothing about the resemblance of the diffeological space to some Euclidean space. It just declares what are the smooth families of elements of the set, and this is enough to retrieve the local aspect of the singularity, if it is it what we are interested in.

Another strong point is that diffeology treats simply and rigorously infinite-dimensional spaces without involving heavy functional analysis, where obviously it is not needed. Why would we involve deep functional analysis to show, for example, that every symplectic manifold is a coadjoint orbit of its group of automorphisms? It is so clear when we know that it is what happens when a Lie group acts transitively, and the group of symplectomorphisms acts transitively. In this case, and maybe others, diffeology does the job easily, and seems to be, here again, the right balance between rigor and simplicity. Recently A. Weinstein et al. wrote "For our purposes, spaces of functions, vector fields, metrics, and other geometric objects are best treated as diffeological spaces rather than as manifolds modeled on infinite-dimensional topological vector spaces" [**BFW10**].

NOTE A. The axiomatics of *Espaces différentiels*, which became later the diffeological spaces, were introduced by J.-M. Souriau in the beginning of the eighties [**Sou80**]. Diffeology is a variant of the theory of *differentiable spaces*, introduced and developed a few years before by K.T. Chen [**Che77**]. The main difference between these two theories is that Souriau's diffeology is more differential geometry oriented, whereas Chen's theory of differentiable spaces is driven by algebraic geometry considerations.

NOTE B. I began to write this textbook in June 2005. My goal was, first of all, to describe the basics of diffeology, but also to improve the theory by opening new fields inside, and by giving many examples of applications and exercises. If the basics of diffeology and a few developments have been published a long time ago now [**Sou80**] [**Sou84**] [**Don84**] [**Igl85**], many of the constructions appearing in this book are original and have been worked out during its redaction. This is what also explains why it took so long to complete. I chose to introduce the various concepts and constructions involved in diffeology from the simple to the complex, or from the particular to the more general. This is why there are repetitions, and some constructions, or proofs, can be shortened, or simplified. I included sometimes these simplifications as exercises at the end of the sections. In the examples treated, I tried to clearly separate what is the responsibility of the category and what is specific. I hope this will help for a smooth progression in the reading of this text.

NOTE C. By the time I wrote these words, and seven years after I began this project, a few physicists or mathematicians have shown some interest in diffeology, enough to write a few papers [**BaHo09**] [**Sta10**] [**Sch11**]. The point of view adopted in these papers is strongly categorical. Diffeology is a Cartesian closed category, complete and cocomplete. Thus, diffeology is an « interesting beast » from a pure categorical point of view. However, if I understand and appreciate the categorical point of view, it does not correspond to the way I apprehended this theory. I may not have commented clearly enough, or exhaustively, on the categorical aspects of the constructions and objects appearing there because my approach has been guided by my habits in classical differential geometry. I made an effort to introduce a minimum of new vocabulary or notation, to give the feeling that studying the geometry of a torus or of its group of diffeomorphisms, or the geometry of its quotient by an irrational line, is the same exercise, involving the same concepts and ideas, the same tools and intuition. I believe that the role of diffeology is to bring closer the objects involved in differential geometry, to treat them on an equal footing, respecting the ordinary intuition of the geometer. All in all, I no longer see diffeology as a replacement theory, but as the natural field of application of traditional differential geometry. But I judged, at the moment when I began this textbook, that diffeology was far enough from the main road to avoid moving too far away. Maybe it is not true anymore, and it is possible that, in a future revision of this book, I shall insist, or write a special chapter, on the categorical aspects of diffeology.

CONTENTS OF THE BOOK

Throughout its nine chapters, the contents of the book try to cover, from the point of view of diffeology, the main fields of differential geometry used in theoretical physics: differentiability, groups of diffeomorphisms, homotopy, homology and cohomology, Cartan differential calculus, fiber bundles, connections, and eventually some comments and constructions on what wants to be *symplectic diffeology*.

Chapter 1 presents the abstract constructions and definitions related to diffeology: objects are diffeologies, or diffeological spaces, and morphisms are *smooth maps*. This part contains all the categorical constructions: sums, products, subset diffeology, quotient diffeology, functional diffeology.

In Chapter 2 we shall discuss the local properties and related constructions, in particular: D-topology, generating families, local inductions or subductions, dimension map, modeling diffeology, in brief, everything related to local properties and constructions.

In Chapters 3 and 4, we shall see the notion of *diffeological vector spaces*, which leads to the definition of *diffeological manifolds*. Each construction is illustrated with several examples, not all of them coming from traditional differential geometry. In particular the examples of the infinite-dimensional sphere and the infinite-projective space are treated in detail.

Chapter 5 describes the diffeological theory of homotopy. It presents the definitions of connectedness, Poincaré's groupoid and fundamental groups, the definition of higher homotopy groups and relative homotopy. The exact sequence of the relative homotopy of a pair is established. Everything relating to functional diffeology of iterated spaces of paths or loops finds its place in this chapter.

Chapter 6 is about Cartan calculus: exterior differential forms and De Rham constructions, their generalization to the context of diffeology. Differential forms are defined and presented first on open subsets of real vector spaces, where everything is clearly explicit, and then carried over to diffeologies. Then, we shall see exterior derivative, exterior product, generalized Lie derivative, generalized Cartan formula, integration on chain, De Rham cohomology on diffeology, chain homotopy operator and obstructions to exactness of differential forms. We shall also see a very useful formula for the variation of the integral of differential forms on smooth chains. In particular, the generalization of Stokes' theorem; the homotopic invariance of De Rham cohomology, and the generalized Cartan formula are established by application of this formula.

Chapter 7 talks about *diffeological groups* and gives some constructions relative to objects associated with diffeological groups, for instance the space of its momenta, equivalence between right and left momenta, etc. Smooth actions of diffeological groups and natural coadjoint actions of diffeological groups on their spaces of momenta are defined.

Chapter 8 presents the theory of *diffeological fiber bundles*, defined by *local triviality along the plots* of the base space (not to be confused with the local triviality of topological bundles). It is more or less a rewriting of my thesis [**Igl85**]. We shall define principal and associated bundles, and establish the exact homotopy sequence of a diffeological fiber bundle. The construction of the universal covering and the construction of coverings by quotient is also a part of the theory, as well as the generalization of the monodromy theorem in the diffeological context. We shall also see, in this general framework, how we can understand connections, reductions, construction of the holonomy bundle and group. In the same vein, we shall represent any closed 1-form or 2-form on a diffeological space by a special structured fiber bundle, a groupoid.

In Chapter 9 we discuss *symplectic diffeology.* It is an attempt to generalize to diffeological spaces the usual constructions in symplectic geometry. This construction will use an essential tool, the *moment map*, or more precisely its generalization in diffeology. We have to note first that, if diffeology is perfectly adapted to describe covariant geometry, *i.e.*, the geometry of differential forms, pullbacks etc., it needs more work when it comes to dealing with contravariant objects, for example vectors. This is why it is better to introduce directly the space of momenta of a diffeological group, the diffeological equivalent of the dual of the Lie algebra, without referring to some putative Lie algebra. Then, we generalize the moment map relative to the action of a diffeological group on a diffeological space preserving a closed 2-form. This generalization also extends slightly the classical moment map for manifolds. Thanks to these constructions, we get the complete characterization of homogeneous diffeological spaces equipped with a closed 2-form ω. This theorem is an extension of the well-known Kirillov-Kostant-Souriau theorem. It applies to every kind of diffeological spaces, the ones regarded as singular by traditional differential geometry, as well as spaces of infinite dimensions. It applies to the exact/equivariant case as well as the not-exact/not-equivariant case, where exact here means *Hamiltonian*. In fact, the natural framework for these constructions is some equivariant cohomology, generalized to diffeology. This theory locates pretty well all the questions related to exactness versus nonexactness, equivariance versus nonequivariance, as well as the so-called Souriau *symplectic cohomology* [**Sou70**]. Incidentally, this definition of the moment map for diffeology gives a way for defining *symplectic diffeology*, without considering the *kernel* of a 2-form for a diffeological space, what can be problematic because of the contravariant nature of the kernel of a form. They are defined as diffeological spaces X, equipped with a closed 2-form ω which are homogeneous under some subgroup of the whole group of diffeomorphisms preserving ω, and such that the moment map is a covering. This definition can be considered as strong, but it includes a lot of various situations.[2] For example every connected symplectic manifold is symplectic in this meaning. Some refinements are needed to deal with some nonhomogeneous singular spaces like orbifolds for example, but this is still a work in progress. Many questions are still open in this new framework of symplectic diffeology. I discuss some of them when they appear throughut the book.

On the structure of the book

The book is made up of numbered chapters, each chapter is made of unnumbered sections. Each section is made of a series of numbered paragraphs, with a title which summarizes the content. Throughout the book, we refer to the numbered paragraphs as (art. X). Paragraphs may be followed by notes, examples, or a proof if the content needs one. This structure makes the reading of the book easy, one can decide to skip some proofs, and the title of each paragraph gives an idea about what the paragraph is about. Moreover, at the end of most of the sections there are one or more exercises related to their content. These exercises are here to familiarize the reader with the specific techniques and methods introduced by diffeology. We are forced, sometimes, to reconsider the way we think about things and change our methods accordingly. The solutions of the exercises are given at the end of the book in a special chapter. Also, at the end of the book there is a list of the main notations used. There is no index but a table of contents which includes the title of each paragraph, so it is easy to find the subject in which one is interested in, if it exists.

Acknowledgment

Thanks to Yael Karshon and François Ziegler for their encouragements to write this textbook on diffeology. I am especially grateful to Yael Karshon for our many discussions on orbifolds and smooth structures, and particularly for our discussions on « symplectic diffeology » which pushed me to think seriously about this subject.

I am very grateful to The Hebrew University of Jerusalem for its hospitality, specials thanks to Emmanuel Farjoun for his many invitations to work there. I started, and mainly carried out, this project in Jerusalem. Without my various

[2] I have often discussed the question of symplectic diffeological spaces with J.-M. Souriau. The definition given here seems to be a good answer to the question, because this moment map and the related construction of *elementary spaces* include the complete case of symplectic manifolds even if the action of the group of symplectomorphisms is not Hamiltonian, or the orbit is not linear but affine. But the few discussions I have had with Yael Karshon about the status of *symplectic orbifolds* will maybe lead to a refinement of the concept of symplectic diffeological space. It is however too early to conclude.

stays in Israel I would never have finished it. Givat Ram campus is a wonderful place where I enjoyed the studious and friendly atmosphere, and certainly its peaceful and intense scientific environment. Thanks also to David Blanc from the University of Haifa, to Misha Polyak and Yoav Moriah from the Technion, to Misha Katz from Bar-Ilan University, to Leonid Polterovich and Paul Biran, from Tel-Aviv University, for their care and invitations to share my ideas with them. Thanks also to Yonatan Barlev, Erez Nesrim and Daniel Shenfeld, from the Einstein Institute, for their participation in the 2006 seminar on Diffeology. Working together has been very pleasant and their remarks have been helpful.

Thanks to Paul Donato, my old accomplice in diffeology, who checked some of my claims when I needed a second look. Thanks also to the young guard: Jordan Watt, Enxin Wu and Guillaume Tahar, their enthusiasm is refreshing and their contribution somehow significant. Thanks also to the referee of the manuscript for his support and his clever suggestions which made me improve the content.

Thanks, obviously, to the CNRS that pays my salary, and gives me the freedom necessary to achieve this project. Thanks particularly to Jérôme Los for his support when I needed it.

Thanks also to Liliane Périchaud who helped me fixing misprints, to David Trotman for trying to make my English look English. Thanks eventually to the AMS publishing people for their patience and care, in particular to Jennifer Wright-Sharp for the wonderful editing job she did with the manuscript.

Patrick Iglesias-Zemmour
Jerusalem, September 2005 — Aix-en-Provence, August 2012

Laboratoire d'Analyse, Topologie et Probabilités du CNRS
39, rue F. Joliot-Curie
13453 Marseille, cedex 13
France
&
The Hebrew University of Jerusalem
Einstein Institute
Campus Givat Ram
91904 Jerusalem
Israel

`http://math.huji.ac.il/~piz/`
`piz@math.huji.ac.il`

Solutions to Exercises

☞ **Exercise 1, p. 6** (Equivalent axiom of covering). Consider the three axioms D1, D2, D3 (art. 1.5). From axiom D1 the constants maps cover X, thus D1' is satisfied. Hence, D1, D2, D3 imply D1', D2, D3. Conversely, consider D1', D2, and D3. Let x be a point of X. By D1' there exists a plot $P : U \to X$ such that x belongs to $P(U)$. Let r in U such that $P(r) = x$. Now let n be any integer. Let $\mathbf{r} : \mathbf{R}^n \to U$ be the constant parametrization mapping every point of $\mathbf{R}^n$ to r. The composition $P \circ \mathbf{r}$ is the constant parametrization $\mathbf{x}$ mapping $\mathbf{R}^n$ to x. Since $\mathbf{r}$ is smooth and thanks to D3, the parametrization $P \circ \mathbf{r}$ is a plot of X. Hence, D1 is satisfied and D1', D2, D3 imply D1, D2, D3. Therefore, the axioms D1, D2, D3 are equivalent to the axioms D1', D2, D3.

☞ **Exercise 2, p. 6** (Equivalent axiom of locality). Consider the three axioms D1, D2, D3 (art. 1.5). Let $P : U \to X$ be a parametrization. Assume that for any point r of U there exists an open neighborhood V_r of r such that $P_r = P \upharpoonright V_r$ belongs to $\mathcal{D}$. The family $(P_r)_{r\in U}$ is a compatible family of elements of $\mathcal{D}$ with P as supremum. Thanks to the axiom D2, P belongs to $\mathcal{D}$, and D2' is satisfied. Hence, D1, D2, D3 imply D1, D2', D3. Conversely, consider D1, D2' and D3. Now, let $\{P_i : U_i \to X\}_{i\in\mathcal{J}}$ be a family of compatible n-parametrizations, and let $P : U \to X$ be the supremum of the family. Let r be any point of U. By definition of P, there exists $P_i : U_i \to X$ with $r \in U_i$ such that $P \upharpoonright U_i = P_i$. Thus, the axiom D2' is satisfied. Hence, D1, D2', D3 imply D1, D2, D3. Therefore, the axioms D1, D2, D3 are equivalent to the axioms D1, D2', D3.

☞ **Exercise 3, p. 7** (Global plots and diffeology). Let $P : U \to X$ be an n-parametrization belonging to $\mathcal{D}$. For all points r in U there exists a real $\epsilon > 0$ such that the open ball $B(r, \epsilon)$, centered at r, of radius ϵ, is contained in U. Since the inclusion $B(r, \epsilon) \subset U$ is a smooth parametrization, the restriction $P \upharpoonright B(r, \epsilon)$ belongs to $\mathcal{D}$. Now, the following parametrization

$$\varphi : B(r,\epsilon) \to B(0,1), \quad \text{defined by} \quad \varphi : s \mapsto s' = \frac{1}{\epsilon}(s - r),$$

is a diffeomorphism. Next, let $\psi : B(0,1) \to \mathbf{R}^n$, and then $\psi^{-1} : \mathbf{R}^n \to B(0,1)$, given by

$$\psi(s) = \frac{s}{\sqrt{1 - \|s\|^2}} \quad \text{and} \quad \psi^{-1}(s') = \frac{s'}{\sqrt{1 + \|s'\|^2}}.$$

The parametrization ψ is a diffeomorphism. Hence, $\phi = \psi \circ \varphi$ is a diffeomorphism from $B(r, \epsilon)$ to $\mathbf{R}^n$. Then, thanks to the axiom of smooth compatibility, the global parametrization $(P \upharpoonright B(r, \epsilon)) \circ \phi^{-1} : \mathbf{R}^n \to X$ belongs to $\mathcal{D}$. By hypothesis, it also belongs to $\mathcal{D}'$. Thus, thanks again to the axiom of smooth compatibility, the parametrization $[(P \upharpoonright B(r, \epsilon)) \circ \phi^{-1}] \circ \phi = P \upharpoonright B(r, \epsilon)$ belongs to $\mathcal{D}'$. Now, P being the supremum of a compatible family of elements of $\mathcal{D}'$, thanks to the axiom of locality of diffeology, P is an element of $\mathcal{D}'$. Therefore, $\mathcal{D} \subset \mathcal{D}'$, exchanging $\mathcal{D}$ and $\mathcal{D}'$ gives $\mathcal{D}' \subset \mathcal{D}$, and finally $\mathcal{D} = \mathcal{D}'$.

Exercise 4, p. 8 (Diffeomorphisms between irrational tori). Let $P : U \to T_\alpha$ be a parametrization. Let us say that P *lifts locally along* π_α*, at the point* $r \in U$, if there exist an open neighborhood V of r and a smooth parametrization $Q : V \to \mathbf{R}$ such that $\pi_\alpha \circ Q = P \upharpoonright V$. Now, the property ($\ast$) writes P is a plot if it lifts locally along π_α, at every point of U.

1) Let us check, following (art. 1.11), that the property ($\ast$) defines a diffeology.

D1. Since π_α is surjective, for every point $\tau \in T_\alpha$ there exists $x \in \mathbf{R}$ such that $\tau = \pi_\alpha(r)$. Then, $\mathbf{x} : r \mapsto x$ is a lift in $\mathbf{R}$ of the constant parametrization $\boldsymbol{\tau} : r \mapsto \tau$.

D2. The axiom of locality is satisfied by the very definition of $\mathcal{D}$.

D3. Let $P : U \to T_\alpha$ satisfying ($\ast$). Let $F : U' \to U$ be a smooth parametrization. Let r' be a point of U', let $r = F(r')$, let V be an open neighborhood of r, and let Q be a smooth parametrization in $\mathbf{R}$ such that $\pi_\alpha \circ Q = P \upharpoonright V$. Let $V' = F^{-1}(V)$ and $Q' = Q \circ F$, defined on V'. Then, $\pi_\alpha \circ Q' = (P \circ F) \upharpoonright V'$.

2) Let us consider $f \in \mathcal{C}^\infty(T_\alpha, \mathbf{R})$. Since π_α and f are smooth, the map $F = f \circ \pi_\alpha$ belongs to $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ (art. 1.15). But since $\pi_\alpha(x+n+\alpha m) = \pi_\alpha(x)$, for every n, m in $\mathbf{Z}$, we also have $F(x+n+\alpha m) = F(x)$. Hence, F is smooth and constant on a dense subset of numbers $\mathbf{Z} + \alpha\mathbf{Z} \subset \mathbf{R}$ (for $x = 0$). Thus, F is constant and therefore f is constant. In other words, $\mathcal{C}^\infty(T_\alpha, \mathbf{R}) = \mathbf{R}$.

3) Let us consider a smooth map $f : T_\alpha \to T_\beta$. Note that, since π_α obviously satisfies ($\ast$), π_α is a plot of T_α. Then, by definition of differentiability (art. 1.14), $f \circ \pi_\alpha$ is a plot of T_β. Hence, for every real x_0 there exist an open neighborhood V of x_0 and a smooth parametrization $F : V \to \mathbf{R}$ such that $\pi_\beta \circ F = (f \circ \pi_\alpha) \upharpoonright V$. Since V is an open subset of $\mathbf{R}$ containing x_0, we can choose V as an interval centered at x_0. For all real numbers x and all pairs (n, m) of integers such that $x + n + \alpha m$ belongs to V, the identity $\pi_\beta \circ F = (f \circ \pi_\alpha) \upharpoonright V$ writes $\pi_\beta \circ F(x+n+\alpha m) = f \circ \pi_\alpha(x+n+\alpha m) = f \circ \pi_\alpha(x) = \pi_\beta \circ F(x)$. Thus, there exist two integers n' and m' such that

$$F(x+n+\alpha m) = F(x) + n' + \beta m'. \qquad (\spadesuit)$$

Since β is irrational, for every such x, n and m, the pair (n', m') is unique. There exists an interval $\mathcal{J} \subset V$ centered at x_0 and an interval $\mathcal{O}$ centered at 0 such that for every $x \in \mathcal{J}$ and for every $n + \alpha m \in \mathcal{O}$, $x + n + \alpha m \in V$. Since F is continuous and since $\mathbf{Z} + \alpha\mathbf{Z}$ is totally discontinuous, $n' + \beta m' = F(x+n+\alpha m) - F(x)$ is constant as function of x. But F is smooth, the derivative of the identity ($\spadesuit$), with respect to x, at the point x_0, gives $F'(x_0 + n + \alpha m) = F'(x_0)$. Then, since α is irrational, $\mathbf{Z} + \alpha\mathbf{Z} \cap \mathcal{O}$ is dense in $\mathcal{O}$ and since F' is continuous, $F'(x) = F'(x_0)$, for all $x \in \mathcal{J}$. Hence, F restricted to $\mathcal{J}$ is affine, there exist two numbers λ and μ such that

$$F(x) = \lambda x + \mu \quad \text{for all} \quad x \in \mathcal{J}. \qquad (\clubsuit)$$

Note that, by density of $\mathbf{Z} + \alpha\mathbf{Z}$, $\pi_\alpha(\mathcal{J}) = T_\alpha$. Hence F defines completely the function f. Now, applying ($\spadesuit$) to the expression ($\clubsuit$) of F, we get for all $n + \alpha m \in \mathcal{O}$

$$\lambda \times (n + \alpha m) \in \mathbf{Z} + \beta\mathbf{Z}. \qquad (\diamondsuit)$$

Let us show that actually ($\diamondsuit$) is satisfied for all $n + \alpha m$ in $\mathbf{Z} + \alpha\mathbf{Z}$. Let $\mathcal{O} = \,]-a, a[$, and let us take a not in $\mathbf{Z} + \alpha\mathbf{Z}$, even if we have to shorten $\mathcal{O}$ a little bit. Let $x \in \mathbf{Z} + \alpha\mathbf{Z}$, and $x > a$. There exists $N \in \mathbf{N}$ such that $0 < (N-1)a < x < Na$, and then $0 < x/N < a$. Now, by density of $\mathbf{Z} + \alpha\mathbf{Z}$ in $\mathbf{R}$, for all $\eta > 0$ there exists $y \in \mathbf{Z} + \alpha\mathbf{Z}$ such that $0 < x/N - y < \eta$. Choosing $\eta < a/N$ we have $0 < x - Ny < N\eta < a$, and $0 < y < x/N < a$. Thus, since $x - Ny \in \mathbf{Z} + \alpha\mathbf{Z} \cap \mathcal{O}$, $\lambda \times (x - Ny) = \lambda x - N \times (\lambda y) \in \mathbf{Z} + \beta\mathbf{Z}$. But $y \in \mathbf{Z} + \alpha\mathbf{Z} \cap \mathcal{O}$, thus $\lambda y \in \mathbf{Z} + \beta\mathbf{Z}$, and then $N \times (\lambda y) \in \mathbf{Z} + \beta\mathbf{Z}$, therefore $\lambda x \in \mathbf{Z} + \beta\mathbf{Z}$. Now, applying successively ($\diamondsuit$) to α and 1, we get $\lambda\alpha \in \mathbf{Z} + \beta\mathbf{Z}$ and $\lambda \in \mathbf{Z} + \beta\mathbf{Z}$. Let $\lambda\alpha = a + \beta b$ and $\lambda = c + \beta d$. If $\lambda \neq 0$, then $\alpha = (a + \beta b)/(c + \beta d)$.

4) Let us remark first that, since $\pi_\alpha(\mathcal{J}) = T_\alpha$, the map F, extended to the whole $\mathbf{R}$, still satisfies $\pi_\beta \circ F = f \circ \pi_\alpha$. Now, let us assume that f is bijective. Note that f surjective is equivalent to $\lambda \neq 0$. Let us express that f is injective: let $\tau = \pi_\alpha(x)$ and $\tau' = \pi_\alpha(x')$, if $f(\tau) = f(\tau')$, then $\tau = \tau'$, that is, $x' = x + n + \alpha m$, for some relative integers n and m. Using the lifting F, this is equivalent to if there exist two integers n' and m' such that $F(x') = F(x) + n' + \beta m'$, then there exist two integers n and m such that $x' = x + n + \alpha m$. But $F(x) = \lambda x + \mu$, with $\lambda \times (\mathbf{Z} + \alpha\mathbf{Z}) \subset \mathbf{Z} + \beta\mathbf{Z}$. Hence, the injectivity writes if $\lambda x' + \mu = \lambda x + \mu + n' + \beta m'$, then $x' = x + n + \alpha m$, which is equivalent to if $\lambda y \in \mathbf{Z} + \beta\mathbf{Z}$, then $y \in \mathbf{Z} + \alpha\mathbf{Z}$, and finally equivalent to $(1/\lambda) \times (\mathbf{Z} + \beta\mathbf{Z}) \subset \mathbf{Z} + \alpha\mathbf{Z}$. Now, let us consider the multiplication by λ, as a $\mathbf{Z}$-linear map, from the $\mathbf{Z}$-module $\mathbf{Z} + \alpha\mathbf{Z}$ to the $\mathbf{Z}$-module $\mathbf{Z} + \beta\mathbf{Z}$, defined in the respective bases $(1, \alpha)$ and $(1, \beta)$, by

$$\lambda \times 1 = c + d \times \beta \quad \text{and} \quad \lambda \times \alpha = a + b \times \beta.$$

The two modules being identified, by their bases, to $\mathbf{Z} \times \mathbf{Z}$, the multiplication by λ and the multiplication by $1/\lambda$ are represented by the matrices

$$\lambda \simeq L = \begin{pmatrix} c & a \\ d & b \end{pmatrix} \quad \text{and} \quad \frac{1}{\lambda} \simeq L^{-1}.$$

Thus, the matrix L is invertible as a matrix with coefficients in $\mathbf{Z}$, that is, $L = \mathrm{GL}(2, \mathbf{Z})$, or $ad - bc = \pm 1$.

☞ **Exercise 5, p. 9** (Smooth maps on $\mathbf{R}/\mathbf{Q}$). This exercise is similar to Exercise 4, p. 8, with solution above.

1) Let us consider a smooth map $f : E_\mathbf{Q} \to E_\mathbf{Q}$. Since $\pi : \mathbf{R} \to E_\mathbf{Q}$ is a plot, by definition of differentiability (art. 1.14), $f \circ \pi$ also is a plot of $E_\mathbf{Q}$. Hence, for every real x_0 there exist an open neighborhood V of x_0 and a smooth parametrization $f : V \to \mathbf{R}$ such that $\pi \circ F = (f \circ \pi) \restriction V$. Since V is an open subset of $\mathbf{R}$ containing x_0, we can choose V to be an interval centered at x_0. For every real number x and every rational number q such that $x + q$ belongs to V, the identity $\pi \circ F = (f \circ \pi) \restriction V$ writes $\pi \circ F(x + q) = f \circ \pi(x + q) = f \circ \pi(x) = \pi \circ F(x)$. Thus, there exists a rational number q' such that $F(x + q) = F(x) + q'$. The rational number $q' = F(x + q) - F(x)$ is smooth in x and q, thus constant in x (see also Exercise 8, p. 14). Hence, taking the derivative of this identity, at the point x_0, with respect to x, we get $F(x_0 + q) = F'(x_0)$. Let us denote $\lambda = F'(x_0)$. Now, according to the continuity of F and the density of $\mathbf{Q}$ in $\mathbf{R}$, we have $F(x) = \lambda x + \mu$, where $\mu \in \mathbf{R}$. *A priori* F is defined on a smaller neighborhood $W \subset V$ of x_0, but by density of $\mathbf{Q}$ in $\mathbf{R}$ we get, as in Exercise 4, p. 8, $\pi(W) = E_\mathbf{Q}$. Thus, F can be extended to the whole $\mathbf{R}$ by the affine map $x \mapsto \lambda x + \mu$. Now, coming back to the condition $F(x + q) = F(x) + q'$, we get $\lambda(x + q) + \mu = \lambda x + \mu + q'$, that is, $\lambda q = q'$. Thus, λ is some number mapping any rational number into another, hence it is a rational number. Let us denote it by q. Therefore, the map f being defined by $\pi \circ F = f \circ \pi$, we get $f(\tau) = q\tau + \tau'$,

where $\tau' = \pi(\mu)$. Hence, any smooth map from $E_{\mathbf{Q}}$ to $E_{\mathbf{Q}}$ is affine. Note that f is a diffeomorphism if and only if $q \neq 0$.

2) With similar arguments as in the first question, we can check that any smooth map $f : T_\alpha \to E_{\mathbf{Q}}$ is the projection of an affine map $F : x \mapsto \lambda x + \mu$. But F needs to satisfy the condition $F(x + n + \alpha m) = F(x) + q$, where n and m are integers and q is a rational number. In particular, for $n = 1$ and $m = 0$, this gives $\lambda \in \mathbf{Q}$, and for $n = 0$ and $m = 1$, $\lambda\alpha \in \mathbf{Q}$. But since $\alpha \in \mathbf{R} - \mathbf{Q}$, this is satisfied only for $\lambda = 0$. And finally F is constant.

3) The identity $\mathcal{C}^\infty(E_{\mathbf{Q}}, \mathbf{R}) = \mathbf{R}$ is analogous to $\mathcal{C}^\infty(T_\alpha, \mathbf{R}) = \mathbf{R}$ of the second question of Exercise 4, p. 8. The second part of the question is similar to the second question of this exercise, inverting $\mathbf{Q}$ and $\mathbf{Z} + \alpha\mathbf{Z}$. The map F is affine, $F : x \mapsto \lambda x + \mu$, and for every $q \in \mathbf{Q}$, there exist two numbers $n, m \in \mathbf{Z}$ such that $F(x + q) = F(x) + n + \alpha m$. So, we get $\lambda q = n + \alpha m$. In particular, for $q = 1$, we get that $\lambda = a + \alpha b$, where $a, b \in \mathbf{Z}$. Hence, for any rational number q, $(a + \alpha b)q = n + \alpha m$, that is, $aq + bq\alpha = n + \alpha m$, or $(aq - n) + (bq - m)\alpha = 0$. Since $\alpha \in \mathbf{R} - \mathbf{Q}$, we get, for all $q \in \mathbf{Q}$, $aq \in \mathbf{Z}$ and $bq \in \mathbf{Z}$. But this implies that a and b are divisible by any integer, thus $a = 0$ and $b = 0$, and therefore $\lambda = 0$.

Exercise 6, p. 9 (Smooth maps on spaces of maps). We shall denote here the derivation map d/dx^k by ϕ_k.

1) The map $\phi_k : \mathcal{C}^\infty(\mathbf{R}) \to \mathcal{C}^\infty(\mathbf{R})$ is smooth if and only if, for every plot $P : U \to \mathcal{C}^\infty(\mathbf{R})$, the parametrization $\phi_k \circ P$ is a plot of $\mathcal{C}^\infty(\mathbf{R})$. According to the definition of the functional diffeology of $\mathcal{C}^\infty(\mathbf{R})$ (art. 1.13), the parametrization $\phi_k \circ P$ is a plot of $\mathcal{C}^\infty(\mathbf{R})$ if and only if the parametrization $(r, x) \mapsto (\phi_k \circ P(r))(x)$ is a smooth parametrization of $\mathbf{R}$, that is, if the parametrization

$$\psi_k : (r, x) \mapsto \left[\frac{d^k}{dx^k}(P(r))\right](x)$$

is smooth. But ψ_k is the k-partial derivative, with respect to x, of the parametrization $\mathbf{P} : (r, x) \mapsto P(r)(x)$,

$$\psi_k(r, x) = \frac{\partial^k \mathbf{P}}{\partial x^k}(r, x).$$

Since, by the very definition of the plots of $\mathcal{C}^\infty(\mathbf{R})$ (art. 1.13), the parametrization $\mathbf{P}$ is smooth, all of its partial derivatives are smooth. They are smooth with respect to the pair of variables (r, x), by the very definition of the class $\mathcal{C}^\infty$. Therefore, ψ_k is smooth and d^k/dx^k is a smooth map from $\mathcal{C}^\infty(\mathbf{R})$ to itself.

2) The map $\hat{x}$ is smooth if and only if, for every plot $P : U \to \mathcal{C}^\infty(\mathbf{R})$, the parametrization $\hat{x} \circ P : r \mapsto P(r)(x)$ is smooth. But, by the very definition of the functional diffeology (art. 1.13), the parametrization $\mathbf{P} : (r, x) \mapsto P(r)(x)$ is smooth. Since the map $\hat{x} \circ P$ is the composition $\mathbf{P} \circ j_x$, where j_x is the (smooth) inclusion $j_x : r \mapsto (r, x)$ from U to $U \times \mathbf{R}$, the parametrization $\hat{x} \circ P$ is smooth and $\hat{x}$ belongs to $\mathcal{C}^\infty(\mathcal{C}^\infty(\mathbf{R}), \mathbf{R})$. Now, note that for every integer k, the map $f \mapsto f^{(k)}(x)$ is just

$$f \mapsto f^{(k)} \mapsto \hat{x}(f^{(k)}) = \left[\hat{x} \circ \frac{d^k}{dx^k}\right](f),$$

that is, the composition of two smooth maps, therefore it is smooth (art. 1.15).

Hence, each component of the map

$$D_x^k : f \mapsto \left(\hat{x}(f), \left(\hat{x} \circ \frac{d}{dx}\right)(f), \ldots, \left(\hat{x} \circ \frac{d^k}{dx^k}\right)(f)\right)$$

is smooth, from $\mathcal{C}^\infty(\mathbf{R})$ to $\mathbf{R}$, and $D_x^k : \mathcal{C}^\infty(\mathbf{R}) \to \mathbf{R}^{k+1}$ is smooth.

3) From differential calculus in $\mathbf{R}^n$ we know that, for all $f \in \mathcal{C}^\infty(\mathbf{R})$, the map

$$F : x \mapsto \int_0^x f(t)dt$$

is continuous and smooth, with f as derivative. Since F is continuous and $F' = f$ is smooth, the primitive F is smooth. Now, $I_{a,b}(f) = F(b) - F(a)$. To prove that $I_{a,b}$ is smooth we have just to check that the map $I : f \mapsto F$ is smooth. Let $P : U \to \mathcal{C}^\infty(R)$ be a plot, we have

$$I \circ P(r) = x \mapsto \int_0^x P(r)(t)dt = \int_0^x \mathbf{P}(r,t)dt \quad \text{with} \quad \mathbf{P}(r,x) = P(r)(x).$$

Now, $I \circ P$ is a plot of $\mathcal{C}^\infty(\mathbf{R})$ if and only if the parametrization $(r,x) \mapsto (I \circ P(r))(x)$ is smooth, that is, if and only if the parametrization

$$\mathcal{P} : (r,x) \mapsto \int_0^x \mathbf{P}(r,t)\, dt$$

is smooth. But, by the very definition of the functional diffeology of $\mathcal{C}^\infty(\mathbf{R})$, the parametrization $\mathbf{P}$ is smooth. Thus, since the partial derivatives of $\mathbf{P}$ with respect to the variables r commute with the integration, on the one hand we have

$$\frac{\partial^n}{\partial r^n}\mathcal{P}(r,x) = \int_0^x \frac{\partial^n}{\partial r^n}\mathbf{P}(r,t)\, dt.$$

On the other hand, since the partial derivatives of $\mathbf{P}$, with respect to r or x, commute, we have, for $m \geq 1$

$$\frac{\partial^n \partial^m}{\partial r^n \partial x^m}\mathcal{P}(r,x) = \frac{\partial^n}{\partial r^n}\left[\frac{\partial^m}{\partial x^m}\int_0^x \mathbf{P}(r,t)\, dt\right] = \frac{\partial^n}{\partial r^n}\frac{\partial^{m-1}}{\partial x^{m-1}}\mathbf{P}(r,x).$$

Therefore, the parametrization $\mathcal{P}$ is smooth and $I, I_{a,b} \in \mathcal{C}^\infty(\mathcal{C}^\infty(\mathbf{R}))$.

4) Checking that the condition (art. 1.13, (♢)) defines a diffeology is straightforward. The same arguments as for (art. 1.11) can be used, or the general construction of subset diffeology, described in (art. 1.33). Now, the derivative is smooth and, restricted to $\mathcal{C}_0^\infty(\mathbf{R})$, is injective. The derivative also is surjective since $F : x \mapsto \int_0^x f(t)\, dt$ satisfies $F' = f$ and $F(0) = 0$. The inverse of d/dx is just the map I defined in the previous paragraph. Since we have seen that I is smooth, the derivative is a diffeomorphism. Moreover, it is a linear diffeomorphism.

☞ **Exercise 7, p. 14** (Locally constant parametrizations). Let us first assume that $P : U \to X$ is locally constant. Let r and r' be two connected points in U and γ be a path connecting r to r', that is, $\gamma \in \mathcal{C}^\infty(\mathbf{R}, U)$ with $\gamma(0) = r$ and $\gamma(1) = r'$.

a) The parametrization $p = P \circ \gamma$ is locally constant. Indeed, let $t \in \mathbf{R}$ be any point, and let $r = \gamma(t)$. Since P is locally constant, there exists an open neighborhood V of r such that $P \restriction V$ is a constant parametrization. Since γ is smooth, thus continuous, $W = \gamma^{-1}(V)$ is a domain and $p \restriction W = P \circ \gamma \restriction \gamma^{-1}(V)$ is constant.

b) The segment $[0,1]$ can be covered with a family of open intervals such that p is constant on each of them. Since $[0,1]$ is compact, there exists a finite subcovering

$\{I_k\}_{k=1}^N$ of $[0,1]$ such that $[0,1] \subset \bigcup_{k=1}^N I_k$ and for all $k = 1 \cdots N$, $p \upharpoonright I_k = \text{cst}$. Now, there exists an interval $I \in \{I_k\}_{k=1}^N$ such that $0 \in I$. Let $\mathcal{I}_1$ be the union of all the intervals of the family $\{I_k\}_{k=1}^N$ whose intersection with I is not empty. Since $\mathcal{I}_1$ is a union of open intervals containing 0, it is itself an open interval containing 0. If $\mathcal{I}_1 = I$, then the family was reduced to $\{I\}$ and we are done: I contains 0 and 1 and $p \upharpoonright I$ is constant. Now, let us assume that $\mathcal{I}_1 \neq I$. Note first that $p \upharpoonright \mathcal{I}_1$ is constant with value $p(0)$. Indeed, for every interval I_k containing 0, $p \upharpoonright I_k = [t \mapsto p(0)]$. Hence, replacing all the intervals I_k containing 0 by $\mathcal{J}_1$, we get a new finite covering of $[0,1]$ satisfying the same conditions as the previous one, but with a number of elements strictly less than N. Thus, after a finite number of steps we get a covering of $[0,1]$ made with a unique open interval on which p is constant. Therefore, $p(0) = p(1)$.

c) Conversely, let us assume that P is constant on every connected component of U. Let $r_0 \in U$, since U is open, there exists an open ball B centered at r_0 and contained in U. But, since B is path connected and contains r_0, B is contained in the connected component of r_0 in U. Thus, P is constant on B, that is, P is locally constant.

☞ **Exercise 8, p. 14** (Diffeology of $\mathbf{Q} \subset \mathbf{R}$). The fact that the plots of $\mathbf{R}$ with values in $\mathbf{Q}$ are a diffeology of $\mathbf{Q}$ is a slight adaptation of (art. 1.12), where $\mathbf{R}^2$ is replaced by $\mathbf{R}$ and the square by $\mathbf{Q}$. Now, let $P : U \to \mathbf{R}$ be a smooth map with values in $\mathbf{Q}$, let $r \in U$ and $P(r) = q$. Let us assume that P is not locally constant at the point r, that is, there exists a small ball B, centered at r, which does not contain any ball B', centered at r, on which P would be constant. Hence, there exists $r' \in B$ with $r' \neq r$ such that $q \neq q'$, where $q' = P(r')$. Next, let $f : t \mapsto r + t(r' - r)$, which can be defined on an open neighborhood of $[0,1]$. The map f sends $[0,1]$ onto the segment $[r, r']$, $f([0,1]) \subset B$. Thus, $Q = P \circ f$ is a real continuous function mapping $[0,1]$ onto $[q, q']$, with $Q(0) = q$ and $Q(1) = q'$. By the intermediate value theorem, f takes all the real values between q and q'. But there is always an irrational number between two distinct rational numbers. This contradicts the hypothesis that f takes only rational values. Therefore, there is no such plot P, and the only plots of $\mathbf{R}$ which take their values in $\mathbf{Q}$ are locally constant. We observe that we can replace $\mathbf{Q}$ by any countable subset, the intermediate value theorem will continue to apply. Let then $A \subset U$ be a countable subset of an n-domain, the composition of any plot P of A with the n coordinate projections $\mathrm{pr}_k : U \to \mathbf{R}$ is a plot of $\mathbf{R}$ taking its values in a countable subset, so locally constant. Therefore P is locally constant, and A is discrete.

☞ **Exercise 9, p. 14** (Smooth maps from discrete spaces). The proof is contained in (art. 1.20). Let X be a discrete diffeological space, let X' be some other diffeological space, and let $f : X \to X'$ be a map. Let P be a plot of X, that is, a locally constant map. The composition $f \circ P$ is thus locally constant, that is, a plot of the discrete diffeology. But the discrete diffeology is contained in every diffeology (art. 1.20), hence $f \circ P$ is a plot of X' and f is smooth.

☞ **Exercise 10, p. 14** (Smooth maps to coarse spaces). The proof is contained in (art. 1.21). Let X be some diffeological space, let X' be a coarse diffeological space, and et $f : X \to X'$ be a map. Let P be a plot of X. The composition $f \circ P$ is a parametrization of X'. Hence it is a plot of the coarse diffeology, since the coarse diffeology is the set of all the parametrizations (art. 1.21). Therefore f is smooth.

Exercise 11, p. 15 (Square root of the smooth diffeology). Let sq : $x \mapsto x^2$, and let us recall that a parametrization P is a plot of the diffeology $\mathcal{D} = \mathrm{sq}^*(\mathcal{C}^\infty_\star(\mathbf{R}))$ if and only if $\mathrm{sq} \circ \mathrm{P}$ is smooth. Since $\mathrm{sq} \in \mathcal{C}^\infty(\mathbf{R})$, for every smooth parametrization P of $\mathbf{R}$, the composite $\mathrm{sq} \circ \mathrm{P}$ is smooth. Therefore $\mathcal{C}^\infty_\star(\mathbf{R}) \subset \mathcal{D} = \mathrm{sq}^*(\mathcal{C}^\infty_\star(\mathbf{R}))$, the diffeology $\mathcal{D}$ is coarser than the smooth diffeology. Now, the map $|\cdot|$ satisfies $\mathrm{sq} \circ |\cdot| = \mathrm{sq}$. But sq is smooth, thus $|\cdot|$ is a plot of the diffeology $\mathcal{D}$. Actually, since $\mathcal{D}$ contains the parametrization $x \mapsto |x|$, which is not smooth, the diffeology $\mathcal{D}$ is strictly coarser than $\mathcal{C}^\infty_\star(\mathbf{R})$. Finally, for every smooth parametrization Q of $\mathbf{R}$, the parametrization $\mathrm{Q} \circ |\cdot|$ is a plot for the diffeology $\mathcal{D}$, thanks to the smooth compatibility axiom.

Exercise 12, p. 17 (Immersions of real domains). If $\mathrm{D}(f)(r)$ is injective at the point r, then the rank of f at the point r, denoted by $\mathrm{rank}(f)_r$, and equal by definition to the rank of the tangent linear map $\mathrm{D}(f)(r)$, is equal to n. Now, since the rank of smooth maps between real domains is semicontinuous below, $\mathrm{rank}(f)_r = n$ on some open neighborhood of r. Thus, by application of the rank theorem (see, for example, [**Die70a**, 10.3.1]) there exist an open neighborhood $\mathcal{O}$ of r, an open neighborhood $\mathcal{O}'$ of $f(r)$, a diffeomorphism φ from the open unit ball of $\mathbf{R}^n$ to $\mathcal{O}$, mapping 0 to r, and a diffeomorphism ψ from the open unit ball of $\mathbf{R}^m$ to $\mathcal{O}'$, mapping 0 to $f(r)$, such that $f \circ \varphi = \psi \circ j$, where $j : \mathbf{R}^n \to \mathbf{R}^m$ is the canonical induction from $\mathbf{R}^n$ to $\mathbf{R}^m$ $j(r_1, \ldots, r_n) = (r_1, \ldots, r_n, 0, \ldots, 0)$. Hence, $f \restriction \mathcal{O}$ is conjugate to an induction by two diffeomorphisms. Thus, $f \restriction \mathcal{O}$ is an induction.

Exercise 13, p. 17 (Flat points of smooth paths). Since γ is continuously differentiable and since $\gamma(t_n^0) = \gamma(t_{n+1}^0) = 0$, by application of Rolle's theorem [**Die70a**, 8.2, pb. 3], there exists a number $t_n^1 \in \left]t_n^0, t_{n+1}^0\right[$ such that $\gamma'(t_n^1) = 0$. Thus, the sequence t_n^1 converges to 0 and, by continuity, $\gamma'(0) = 0$. Now, by recursion, there exists a sequence of numbers $t_n^{k+1} \in \left]t_n^k, t_{n+1}^k\right[$, converging to 0, such that $\gamma^k(t_n^k) = 0$. Therefore, for any $k > 0$, $\gamma^k(0) = 0$ and γ is flat at 0.

Exercise 14, p. 17 (Induction of intervals into domains). 1) Let abs : $t \mapsto |t|$, and let $\mathrm{F} = f \circ \mathrm{abs}$. We have

$$\mathrm{F}(t) = f(-t), \text{ if } t < 0, \quad \mathrm{F}(0) = 0, \quad \text{and} \quad \mathrm{F}(t) = f(+t), \text{ if } t > 0.$$

The parametrization F is smooth on $]-\varepsilon, 0[$ and on $]0, +\varepsilon[$, because, restricted to these intervals, it is the composite of two smooth parametrizations. The only question is for $t = 0$. Next, since f is flat, for all integers p we have

$$\lim_{t \to 0^\pm} \frac{f(t)}{t^p} = 0 \quad \Rightarrow \quad \lim_{t \to 0^\pm} \frac{\mathrm{F}(t)}{t^p} = 0, \tag{$\diamondsuit$}$$

in particular for $p = 1$. Thus, F is derivable at 0 and $\mathrm{F}'(0) = 0$. But since $f'(0) = 0$, we also have

$$\begin{array}{rcccccl} \lim_{t \to 0^-} (\mathrm{F})'(t) & = & (-1) \lim_{t \to 0^-} f'(-t) & = & (-1) \times 0 & = & 0, \\ \lim_{t \to 0^+} (\mathrm{F})'(t) & = & (+1) \lim_{t \to 0^+} f'(+t) & = & (+1) \times 0 & = & 0. \end{array}$$

Thus, $\lim_{t \to 0^\pm} \mathrm{F}'(t) = \mathrm{F}'(0) = 0$. Hence, $\mathrm{F} \in \mathcal{C}^1(]-\varepsilon, +\varepsilon[, \mathbf{R}^n)$. Moreover, since F is $\mathcal{C}^1$, F' is derivable at 0 and its derivative is 0,

$$\mathrm{F}''(0) = \lim_{t \to 0^\pm} \frac{\mathrm{F}'(t)}{t} = \lim_{t \to 0^\pm} \frac{f(t)}{t^2} = 0.$$

Then, by recursion on p, and thanks to ($\diamondsuit$), we get $\mathrm{F} \in \mathcal{C}^p(]-\varepsilon, +\varepsilon[, \mathbf{R}^n)$, for all integers p. Thus, F is smooth. But since abs is not smooth, f is not an induction.

Now, composing with two translations at the source and at the target, the same proof applies for every point $t \in]-\varepsilon, +\varepsilon[$, and for every value $f(t)$. Therefore, an induction from $]-\varepsilon, +\varepsilon[$ to $\mathbf{R}^n$ is nowhere flat.

2) Since an induction $f :]-\varepsilon, +\varepsilon[\to \mathbf{R}^n$ is not flat at $t = 0$, there exists a smallest integer $k > 0$ such that, $f^{(j)}(0) = 0$ if $0 \leq j < k$, and $f^{(k)}(0) \neq 0$. If $k = 1$, then $p = 0$, and $\varphi = f$. Otherwise, if $k \geq 1$, then the Taylor expansion of f around 0 is reduced to

$$f(t) = t^p \times \varphi(t), \text{ with } \varphi(t) = t \times \int_0^1 \frac{(1-s)^p}{p!} f^{(p+1)}(st) ds, \text{ and } p = k - 1.$$

See, for example [**Die70a**, 8.14.3]. Since $f^{(k)}$ is smooth, the function φ is smooth and $\varphi'(0) = f^{(k)}(0)/k! \neq 0$.

Exercise 15, p. 17 (Smooth injection in the corner). Let us split the first question into two mutually exclusive cases.

1.A) If $0 \in \mathbf{R}$ is an isolated zero of γ, then there exists $\varepsilon > 0$ such that $\gamma(t) = 0$, and $t \in]-\varepsilon, +\varepsilon[$ implies $t = 0$. Since γ is continuous, γ maps $]-\varepsilon, 0[$ to the semiline $\{x\boldsymbol{e}_1 \mid x > 0\}$ or to the semiline $\{y\boldsymbol{e}_2 \mid y > 0\}$, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the vectors of the canonical basis of $\mathbf{R}^2$. But since γ is injective, these two cases are mutually exclusive. Without loss of generality, we can assume that $\gamma(]-\varepsilon, 0[) \subset \{x\boldsymbol{e}_1 \mid x > 0\}$ and $\gamma(]0, -\varepsilon[) \subset \{y\boldsymbol{e}_2 \mid y > 0\}$. Thus, for all $p > 0$, $\lim_{t \to 0^-} \gamma^{(p)}(t) = \alpha\boldsymbol{e}_1$ and $\lim_{t \to 0^+} \gamma^{(p)}(t) = \beta\boldsymbol{e}_2$. Hence, by continuity, $\alpha = \beta = 0$. Therefore, γ is flat at 0. Now, if γ is not assumed to be injective, then the parametrization $\gamma : t \to t^2\boldsymbol{e}_1$ is smooth, not flat, and satisfies $\gamma(0) = 0$.

1.B) If $0 \in \mathbf{R}$ is not an isolated zero of γ, then there exists a sequence $t_1^0 < \cdots < t_n^0 < \cdots$ of numbers, converging to 0, such that $\gamma(t_n^0) = 0$. The fact that γ is flat at 0 is the consequence of Exercise 13, p. 17.

2) First of all j is injective. Since the restriction of j on $]-\infty, 0[\ \cup \ ]0, +\infty[$ is smooth, the only problem is for $t = 0$. But the successive derivatives of j write

$$j^{(p)}(t) = \begin{pmatrix} q_-(t)\, e^{\frac{1}{t}} \\ 0 \end{pmatrix} \text{ if } t < 0, \quad \text{and} \quad j^{(p)}(t) = \begin{pmatrix} 0 \\ q_+(t)\, e^{-\frac{1}{t}} \end{pmatrix} \text{ if } t > 0,$$

where $q_\pm$ are two rational fractions. Then,

$$\lim_{t \to 0^\pm} q_\pm(t)\, e^{-\frac{1}{|t|}} = 0, \quad \text{thus} \quad \lim_{t \to 0^\pm} j^{(p)}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, j is smooth.

3) Let abs denote the map $t \mapsto |t|$. The map $j \circ \text{abs}$ is smooth but not abs. But, for an injection, to be an induction means that for a parametrization P, $j \circ P$ is smooth if and only if P is smooth. Thus, if j is a smooth injection, it is not an induction. This example is a particular case of Exercise 14, p. 17.

Exercise 16, p. 18 (Induction into smooth maps). First of all, f is injective. Let $\varphi \in \text{val}(f)$, then $f(x, v) = \varphi$, with $x = \varphi(0)$ and $v = \varphi'(0)$. Now let $\Phi : r \mapsto \varphi_r$ be a plot of $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^n)$ such that $\text{val}(\Phi) \subset \text{val}(f)$, thus $(r, t) \mapsto \varphi_r(t)$ is a smooth parametrization in $\mathbf{R}^n$. Now, $f^{-1} \circ \Phi(r) = (x = \varphi_r(0), v = (\varphi_r)'(0))$. Since the evaluation of a smooth function is smooth, $r \mapsto x$ is smooth. Next, since the derivative $(r, t) \mapsto (\varphi_r)'(t)$ is smooth and the evaluation is smooth, $r \mapsto v$ is smooth. Therefore, $f^{-1} : \mathrm{val}(f) \to \mathbf{R}^n \times \mathbf{R}^n$ is smooth, where val(f) is equipped with the subset diffeology of $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^n)$, thus f is an induction.

Exercise 17, p. 19 (Vector subspaces of real vector spaces). We shall apply the criterion (art. 1.31). First of all, since the vectors $b_1, \ldots, b_k$ are free, the map $\mathcal{B}$ is injective. Then, since $\mathcal{B}$ is linear, $\mathcal{B}$ is smooth from $\mathbf{R}^k$ to $\mathbf{R}^n$. Now, let $P : U \to \mathbf{R}^n$ be a smooth parametrization of $\mathbf{R}^n$ with values in $E = \mathcal{B}(\mathbf{R}^k)$. Thanks to the theorem of the incomplete basis, we can find $n-k$ vectors $b_{k+1}, \ldots, b_n$ and a map $M \in \mathrm{GL}(n, \mathbf{R})$ such that $M(b_i) = \boldsymbol{e}_i$, for $i = 1, \ldots, n$. Hence, $\mathcal{B}^{-1} \circ P = M \circ P$. Since $M \circ P$ is smooth, $\mathcal{B}^{-1} \circ P$ is smooth and $\mathcal{B}$ is an induction. Finally, every basis $\mathcal{B}$ of every k-subspace $E \subset \mathbf{R}^n$ realizes a diffeomorphism from $\mathbf{R}^k$ to E, equipped with the induced smooth diffeology.

Exercise 18, p. 19 (The sphere as diffeological subspace). The map f is clearly injective. The inverse is given by

$$f^{-1} : \{x' \in S^n \mid x' \cdot x > 0\} \to E \quad \text{with} \quad f^{-1}(x') = [\mathbf{1} - x\bar{x}]x',$$

where $[\mathbf{1} - x\bar{x}]$ is the orthogonal projector parallel to x. The notation $\bar{x}$ is for $x' \mapsto x \cdot x'$. Hence, since the map f^{-1} is the restriction of a linear map, thus smooth, to a subset, f^{-1} is smooth for the subset diffeology. And f is an induction, from the open ball $\{t \in E \mid \|t\| < 1\}$ to the semisphere $\{x' \in S^n \mid x' \cdot x > 0\}$.

Exercise 19, p. 20 (The pierced irrational torus). Let $\pi_\alpha : \mathbf{R} \to T_\alpha$ be the projection from $\mathbf{R}$ to its quotient $T_\alpha = \mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$; see Exercise 4, p. 8. By definition of the diffeology of T_α, this parametrization is a surjective plot of T_α. Now, by density of $\mathbf{Z} + \alpha\mathbf{Z}$ in $\mathbf{R}$, any open interval around $0 \in \mathbf{R}$ contains always a representative of every orbit of $\mathbf{Z} + \alpha\mathbf{Z}$. Hence, the plot π_α is not locally constant. Therefore, since a diffeology is discrete (art. 1.20) if and only if all its plots are locally constant, the diffeology of T_α is not discrete. Now let $\tau \in T_\alpha$ and $x \in \mathbf{R}$ such that $\pi_\alpha(x) = \tau$. Let $P : U \to T_\alpha - \tau$ be a plot, thus P is a plot of T_α such that $\tau \notin \mathrm{val}(P)$. Since P is a plot of T_α, for all $r_0 \in U$ there exist an open neighborhood V of r_0 and a smooth parametrization $Q : V \to \mathbf{R}$ such that $P \upharpoonright V = \pi_\alpha \circ Q$. But since $\tau \notin \mathrm{val}(P)$, $\mathrm{val}(Q) \cap (\mathbf{Z}+\alpha\mathbf{Z})(x) = \varnothing$, where $(\mathbf{Z}+\alpha\mathbf{Z})(x) = \{x + n + \alpha m \mid n, m \in \mathbf{Z}\}$, that is, $\mathrm{val}(Q) \subset \mathbf{R} - (\mathbf{Z} + \alpha\mathbf{Z})(x)$. Next, let $B \subset V$ be a ball centered at r_0, and let $r_1 \in B$ be any other point of B. Let $x_0 = Q(r_0)$, $x_1 = Q(r_1)$, and let us assume that $x_0 \leq x_1$; it would be equivalent to assume $x_0 \geq x_1$. Since Q is smooth and *a fortiori* continuous, the interval $[x_0, x_1]$ is contained in val(Q). But, since the orbit of x by $\mathbf{Z} + \alpha\mathbf{Z}$ is dense in $\mathbf{R}$, except if $x_1 = x_0$, the interval $[x_0, x_1]$ contains a point of the orbit $(\mathbf{Z} + \alpha\mathbf{Z})(x)$. Now, since by hypothesis Q avoids this orbit, $x_0 = x_1$. Hence, Q is locally constant, and thus P is locally constant. Therefore, the diffeology of $T_\alpha - \tau$ is discrete.

Exercise 20, p. 20 (A discrete image of $\mathbf{R}$). 1) The condition (♣) means that the parametrization is a plot of the *functional diffeology* (art. 1.13) or (art. 1.57). Then, let us consider the condition (♠).

D1. Let $\hat{\phi} : U \to \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ be the constant parametrization $\hat{\phi}(r) = \phi$. Hence, $\hat{\phi}(r) = \hat{\phi}(r_0) = \phi$ for every r_0 and every r in U. They coincide *a fortiori* outside any interval $[a, b]$.

D2. By the very definition the condition (♠) is local.

D3. Let $P : U \to \mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ satisfying (♠). Let $F : V \to U$ be a smooth parametrization. Let $s_0 \in V$ and $r_0 = F(s_0)$. Since P satisfies (♠), there exists an open ball B centered at r_0 and for every $r \in B$ there exists an interval $[a,b]$ such that $P(r)$ and $P(r_0)$ coincide outside $[a,b]$. Since F is smooth, the pullback $F^{-1}(B)$ is a domain containing s_0. Let us consider then an open ball $B' \subset F^{-1}(B)$ centered at s_0. For every $s \in B'$, $F(s) \in B$, thus there exists an interval $[a,b]$ such that $P \circ F(s) = P(r) \in B$ and $P \circ F(s_0) = P(r_0)$ coincide outside $[a,b]$. Therefore, $P \circ F$ satisfies (♠).

Hence, the condition (♠) defines a diffeology. The two conditions (♣) and (♠) define the intersection of two diffeologies, that is, a diffeology (art. 1.22).

2) Let $f : \mathbf{R} \to \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$ be defined by $f(\alpha) = [x \mapsto \alpha x]$. Let $P : U \to \mathrm{val}(f) \subset \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$ be a plot for the diffeology defined by (♣) and (♠). Since f is injective, $f^{-1}(\phi) = \phi(1)$, there exists a unique real parametrization $r \mapsto \alpha(r)$, defined on U, such that $P(r) = [x \mapsto \alpha(r)x]$, actually $\alpha(r) = P(r)(1)$. Now let $r_0 \in U$, thanks to (♠) there exists an open ball B, centered at r_0 and for all $r \in B$, there exists an interval $[a,b]$ such that $P(r)$ and $P(r_0)$ coincide outside $[a,b]$, that is, for all $x \in \mathbf{R} - [a,b]$, $\alpha(r)x = \alpha(r_0)x$. We can choose $x \neq 0$, and thus $\alpha(r) = \alpha(r_0)$ for all $r \in B$. Therefore the plot P is locally constant, and this is the definition for $f(\mathbf{R})$ to be discrete. It follows that f is not smooth, since it is not locally constant, and therefore not a plot. But note that f is a plot for the diffeology defined only by (♣).

3) For $\mathcal{C}^\infty(\mathbf{R}^n)$, the condition (♠) must be replaced by the following:

(♠) For any $r_0 \in U$ there exists an open ball B, centered at r_0, and for every $r \in B$ there exists a compact $K \subset \mathbf{R}^n$ such that $P(r)$ and $P(r_0)$ coincide outside K.

Now, for the same kind of reason as for the second question, the injection $j : \mathrm{GL}(n,\mathbf{R}) \to \mathcal{C}^\infty(\mathbf{R}^n,\mathbf{R}^n)$ has a discrete image, and is not smooth. Indeed, two linear maps which coincide outside a compact coincide everywhere.

☞ **Exercise 21, p. 23** (Sum of discrete or coarse spaces). Let us consider the sum $X = \coprod_{i\in\mathcal{I}} X_i$ of discrete diffeological spaces. Let $P : U \to X$ be a plot. By definition of the sum diffeology, P takes locally its values in one of the X_i (art. 1.39). Let us say that $\mathrm{val}(P \upharpoonright V) \subset X_i$, where V is a subdomain of U. But since X_i is discrete, $P \upharpoonright V$ is locally constant. Hence, P itself is locally constant, that is, X is discrete. Now, let X be a discrete space and $\coprod_{x\in X}\{x\}$ be the sum of its elements. Every plot P of X is locally constant, hence P is locally a plot of $\coprod_{x\in X}\{x\}$. Thus, P is a plot of $\coprod_{x\in X}\{x\}$. Conversely, every plot P of $\coprod_{x\in X}\{x\}$ is locally constant. Thus, P is a plot of X. Therefore, every discrete space is the sum of its elements. Finally, let us consider the sum of two points $X = \{0\}\coprod\{1\}$. The diffeology of $\{0\}$ and $\{1\}$ is at the same time coarse and discrete. Let us consider the parametrization $P : \mathbf{R} \to X$ which maps each rational to the point 0 and each irrational to the point 1. This is a parametrization, thus a plot of the coarse diffeology of X (art. 1.21). But, since rational (or irrational) numbers are dense in $\mathbf{R}$, this parametrization is nowhere locally equal to 0 or equal to 1. Then, this parametrization is not a plot of the sum X. Hence, the sum X is not coarse. Sum of coarse spaces may be not coarse.

☞ **Exercise 22, p. 23** (Plots of the sum diffeology). Only the following method needs to be proved. Let $P : U \to X$ be a plot of the sum diffeology. For each index $i \in \mathcal{I}$, the set $U_i = P^{-1}(X_i)$ is open. Indeed, let $r \in U_i$, by definition of the sum diffeology, there exists an open neighborhood of r, let us say an open ball B centered at r, such that $P \upharpoonright B$ takes its values in U_i. Hence, U_i is the union of all these open

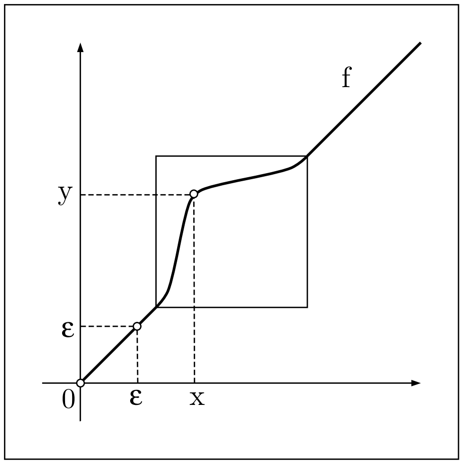

FIGURE Sol.1. The diffeomorphism f.

balls, thus U_i is a domain. Now, by construction the family $\{U_i\}_{i \in \mathcal{I}}$ is a partition of U, and for every index $i \in \mathcal{I}$, the restriction $P_i = P \upharpoonright U_i$ is a plot of X_i.

Exercise 23, p. 23 (Diffeology of $\mathbf{R} - \{0\}$). Let $P : U \to \mathbf{R} - \{0\}$ be a plot. Let $U_- = P^{-1}(]-\infty, 0[)$ and $U_+ = P^{-1}(]0, +\infty[)$, $U_\pm$ be two domains constituting a partition of U, indeed $U = U_- \cup U_+$ and $U_- \cap U_+ = \varnothing$. Let $P_- = P \upharpoonright U_-$ and $P_+ = P \upharpoonright U_+$. Then, P is the supremum of P_- and P_+. Therefore, the diffeology of $\mathbf{R} - \{0\}$ is the sum of the diffeologies of $]-\infty, 0[$ and $]0, +\infty[$; see Exercise 22, p. 23.

Exercise 24, p. 23 (Klein strata of $[0, \infty[$). Let $\varphi : [0, \infty[\to [0, \infty[$ be a diffeomorphism for the subset diffeology. Let us assume that $\varphi(0) \neq 0$. Since φ is bijective, there exists a point $x_0 > 0$ such that $\varphi(x_0) = 0$. Hence, there exists a closed interval centered at x_0, let us say $I = [x_0 - \varepsilon, x_0 + \varepsilon]$, $\varepsilon > 0$, such that $f = \varphi \upharpoonright I$ is positive, injective, continuous, and maps x_0 to 0. Let $a = f(x_0 - \varepsilon)$ and $b = f(x_0 + \varepsilon)$. We have $a > 0$ and $b > 0$, let us assume that $0 < a \leq b$ (it is not crucial). Now, since f is continuous, maps 0 to 0, and $x_0 + \varepsilon$ to b, for every point y between 0 and b there exists a point x between 0 and $x_0 + \varepsilon$ such that $f(x) = y$, and since $0 < a \leq b$, there exists $x_1 \in [0, x_0 + \varepsilon]$ such that $f(x_1) = a$. Thus, $a = f(x_0 - \varepsilon) = f(x_1)$ and $x_1 \neq x_0 - \varepsilon$. This is impossible, by hypothesis f is injective. Hence, $\varphi(0) = 0$. Therefore, the set $\{0\}$ is an orbit of $\mathrm{Diff}([0, \infty[)$, that is a Klein stratum. Next, since $\varphi(0) = 0$, $\varphi(]0, \infty[) =]0, \infty[$. Let $f = \varphi \upharpoonright]0, \infty[$, f is bijective, smooth and its inverse is also smooth. Thus, f is a diffeomorphism of $]0, \infty[$, and moreover $\lim_{x \to 0} = 0$. Let us now try to convince ourselves that any point x of $]0, \infty[$ can be mapped to any point y of $]0, \infty[$ by a diffeomorphism φ of $[0, \infty[$. Let us assume that $0 < x \leq y$. We claim that there exist a number $\varepsilon > 0$, and a diffeomorphism f of $]0, \infty[$ such that $f \upharpoonright]0, \varepsilon[$ is equal to the identity, and $f(x) = y$; see Figure Sol.1. The extension φ of f to $[0, \infty[$ defined by $\varphi(0) = 0$ is a diffeomorphism of $[0, \infty[$, got by gluing the identity on some interval $[0, \varepsilon[$ with f. Therefore $]0, \infty[$ is an orbit of $\mathrm{Diff}([0, \infty[)$, that is, a Klein stratum.

Exercise 25, p. 23 (Compact diffeology). First of all, let us remark that coinciding outside a compact in $\mathbf{R}$, or coinciding outside a closed interval is identical.

Now, a plot $P : U \to \mathbf{R}$ of the functional diffeology of $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ foliated by the relation $\sim$ is a plot of the functional diffeology of $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$ which takes locally its values in some class of $\sim$. Precisely, if and only if P fulfills the condition (♣) of Exercise 20, p. 20, and for every $r_0 \in U$ there exists an open neighborhood V of r_0, such that for all $r \in V$, $\mathrm{class}(P(r)) = \mathrm{class}(P(r_0))$, that is, for all $r \in V$ there exists a closed interval $[a, b]$ such that $P(r)$ and $P(r_0)$ coincide outside $[a, b]$. This is exactly the condition (♠) of Exercise 20, p. 20.

Exercise 26, p. 25 (Square of the smooth diffeology). Except for the plots with negative values, by definition of the pushforward of a diffeology (art. 1.43), a parametrization P of $\mathcal{D} = \mathrm{sq}_*(\mathcal{C}^\infty_\star(\mathbf{R}))$ writes locally $P(r) =_{\mathrm{loc}} Q(r)^2$, where $\mathrm{sq}(x) = x^2$. Hence, except for the plots with negative values (which are locally constant), every plot is locally the square of a smooth parametrization of R. Therefore, every plot of $\mathcal{D}$ is a plot of the smooth diffeology $\mathcal{C}^\infty_\star(\mathbf{R})$ of $\mathbf{R}$, thus $\mathcal{D}$ is finer than $\mathcal{C}^\infty_\star(\mathbf{R})$.

1) Let $P : U \to \mathbf{R}$ be a plot for $\mathcal{D}$. Let $r_0 \in U$ such that $P(r_0) < 0$. Since $P(r_0)$ does not belong to the set of values of sq, by application of the characterization of the plots of pushforwards of diffeologies (art. 1.43), P is locally constant around r_0. In particular, there exists an open ball B centered at r_0 such that $P \restriction B$ is constant.

2) Let $P : U \to \mathbf{R}$ be a plot for $\mathcal{D}$. Let $r_0 \in U$ such that $P(r_0) > 0$. Since $P(r_0)$ is in the set of values of sq, there exists a smooth parametrization Q of $\mathbf{R}$, defined on an open neighborhood V of r_0 such that $P(r) = Q(r)^2$. Now, since $P(r_0) > 0$, there exists an open ball B centered at r_0 such that $P \restriction B$ is strictly positive. Hence, $Q \restriction B$ does not vanish, thus Q keeps a constant sign on B and the map $\sqrt{P} : r \mapsto |Q(r)|$, defined on B, is smooth.

3) If $P(r_0) = 0$, then there exist an open neighborhood V of r_0 and a smooth parametrization $Q : V \to \mathbf{R}$ such that $P \restriction V = Q^2$. But $P(r_0) = 0$ implies $Q(r_0) = 0$. Thus, $D(P)(r_0) = 2Q(r_0) \times D(Q)(r_0) = 0$. Hence, the first derivative of P vanishes at r_0. Let u and v be two vectors of $\mathbf{R}^n$, with $n = \dim(P)$,

$$\begin{aligned} D^2(P)(r_0)(u)(v) &= D[r \mapsto 2Q(r) \times D(Q)(r)(u)](r_0)(v) \\ &= 2D(Q)(r_0)(v) \times D(Q)(r_0)(u) + 2Q(r_0) \times D^2(Q)(r_0)(u)(v) \\ &= 2D(Q)(r_0)(v) \times D(Q)(r_0)(u) \quad (\text{since } Q(r_0) = 0). \end{aligned}$$

Thus, since $H(v)(v) = 2[D(Q)(r_0)(v)]^2$, the Hessian $H = D^2(P)(r_0)$ is positive.

4) Since $f(x) = (x\sqrt{1-x})^2$, and $\sqrt{1-x}$ is smooth on $]-\infty, 1[$, $x\sqrt{1-x}$ is smooth, and f is a plot of $\mathcal{D}$.

Exercise 27, p. 27 (Subduction onto the circle). First of all, note that the map $\Pi : t \mapsto (\cos(t), \sin(t))$, from $\mathbf{R}$ to S^1, is smooth and surjective. Also note that, the function cos, restricted to the interval $]k\pi, \pi + k\pi[$, where $k \in \mathbf{Z}$, is a diffeomorphism onto $]0, 1[$. As well, the function sin, restricted to $]-\pi/2 + k\pi, \pi/2 + k\pi[$, is a diffeomorphism onto $]0, 1[$. See Figure Sol.2. For $k = 0$ the inverses are the standard functions acos and asin. Let us denote, for now, the inverses of these restrictions by

$$\mathrm{acos}_k :]0, 1[\to]k, \pi + k[\quad \text{and} \quad \mathrm{asin}_k :]0, 1[\to]-\pi/2 + k, \pi/2 + k[.$$

Now, let $P : U \to S^1$ be a smooth parametrization of S^1, that is, $P(r) = (x(r), y(r))$, where x and y are smooth real parametrizations and $x(r)^2 + y(r)^2 = 1$. Let $r_0 \in U$. We shall distinguish four cases.

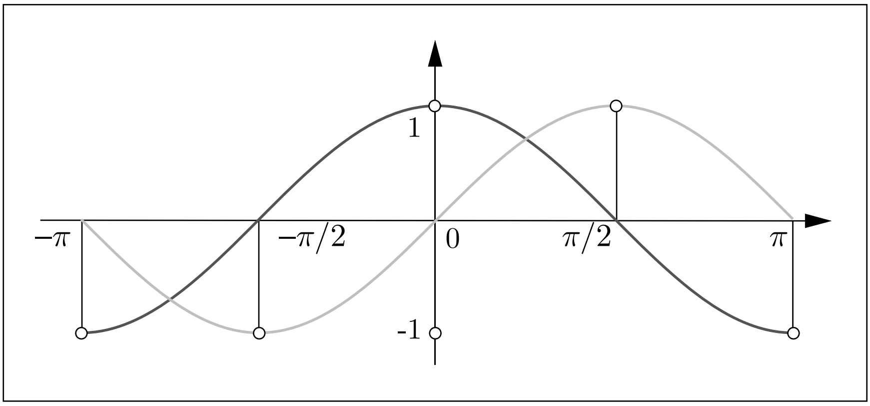

FIGURE Sol.2. The functions sine and cosine.

a) $y(r_0) \in]0,+1[$. Locally, $r \mapsto \text{acos}(y(r))$ is a local lifting of P along Π.

b) $y(r_0) \in]-1,0[$. Locally, $r \mapsto \text{acos}_1(y(r))$ is a local lifting of P along Π.

c) $x(r_0) = +1$. Locally, $r \mapsto \text{asin}(y(r))$ is a local lifting of P along Π.

d) $x(r_0) = -1$. Locally, $r \mapsto \text{asin}_1(y(r))$ is a local lifting of P along Π.

Therefore, $\Pi : \mathbf{R} \to S^1$ is a subduction. In other words, Π is strict (art. 1.54), and its factorization identifies naturally the quotient $\mathbf{R}/2\pi\mathbf{Z}$ with $S^1 \subset \mathbf{R}^2$.

Exercise 28, p. 27 (Subduction onto diffeomorphisms). Let us begin first by noting that, for all $f, g \in G$, $(g \circ f)'(x) \neq 0$, because $(g \circ f)'(x) = g'(f(x))f'(x)$, $f'(x) \neq 0$ and $g'(x) \neq 0$. Note next that, since $f'(x) \neq 0$ for all $x \in \mathbf{R}$, $f'(x)$ has a constant sign. Thus, if $f'(x) > 0$, then f is strictly increasing and $f(x + 2\pi) = f(x) + 2\pi$, else if $f'(x) < 0$, then f is strictly decreasing and $f(x + 2\pi) = f(x) - 2\pi$. Note that generally, for all $k \in \mathbf{Z}$, $f(x + 2\pi k) = f(x) + 2\pi k$ if f is increasing, and $f(x + 2\pi k) = f(x) - 2\pi k$ if f is decreasing. Now, it is immediate that, in any case, increasing or decreasing, for all $g, f \in G$, $(g \circ f)(x + 2\pi) = (g \circ f)(x) \pm 2\pi$. Moreover, for all $f \in G$, f is unbounded. Therefore, $f \in \text{Diff}(\mathbf{R})$, and G is a subgroup of $\text{Diff}(\mathbf{R})$. Now, let us check that $(\diamondsuit)$ defines a diffeology of $\text{Diff}(S^1)$. Let $r \mapsto f$ be a constant parametrization, for all plots Q of S^1, $(r, s) \mapsto f(Q(r)) = f \circ Q(r)$ is a plot of S^1 since f and Q are smooth. Axiom D1 is checked. Now, let $P : U \to \text{Diff}(S^1)$ be a parametrization such that for all $r \in U$ there exists an open neighborhood W of r such that $P \restriction W$ satisfies $(\diamondsuit)$. For all plots $Q : V \to S^1$, $(r, s) \mapsto (P \restriction W)(r)(Q(s))$, defined on $W \times V$, is smooth. Thus, the parametrization $(r, s) \mapsto P(r)(Q(s))$ is locally smooth, therefore smooth. Axiom D2 is checked. Let $P : U \to \text{Diff}(S^1)$ satisfying $(\diamondsuit)$, and let $F : W \to U$ be a smooth parametrization. Let $Q : V \to S^1$ be a plot, then $(t, s) \mapsto (r = F(t), s) \mapsto P(r)(Q(s)) = (P \circ F)(t)(Q(s))$ is smooth since it is the composite of smooth maps. Axiom D3 is checked. Therefore, $(\diamondsuit)$ defines a diffeology of $\text{Diff}(S^1)$. Let us assume $f' > 0$. Note that $\Pi \circ f : \mathbf{R} \to S^1$ is smooth and surjective. Next, $(\Pi \circ f)(x + 2\pi) = \Pi(f(x) + 2\pi) = (\cos(f(x) + 2\pi), \sin(f(x) + 2\pi)) = (\cos(f(x)), \sin(f(x))) = \Pi \circ f(x)$. Thus $\Pi \circ f(x)$ is 2π-periodic, therefore there exists a function $\varphi : S^1 \to S^1$ defined by $\varphi(z) = \Pi \circ f(x)$ for every $x \in \mathbf{R}$ such that $z = (\cos(x), \sin(x))$.

$$\begin{tikzcd} & \mathbf{R} \arrow[r, "f"] \arrow[d, "\Pi"] & \mathbf{R} \arrow[d, "\Pi"] \\ V \arrow[ur, "Q"] \arrow[r, "P \restriction V"'] & S^1 \arrow[r, "\varphi"'] & S^1 \end{tikzcd}$$

Let $P : U \to S^1$ be a plot, since Π is a subduction (Exercise 27, p. 27) for all $r \in U$ there exist an open neighborhood V of r and a smooth parametrization Q of $\mathbf{R}$ such that $\Pi \circ Q = (P \restriction V)$. Thus, $\varphi \circ (P \restriction V) = (\varphi \circ \Pi \circ Q) \restriction V = \Pi \circ f \circ Q$, that is, a composite of smooth maps. Thus $\varphi \circ (P \restriction V)$ is a plot of S^1, and therefore $\varphi \circ P$. Now, thanks to $f(x + 2\pi) = f(x) + 2\pi$, by denoting $y = f(x)$ and composing with f^{-1}, we get $f^{-1}(y + 2\pi) = f^{-1}(y) + 2\pi$. Hence, there exists a surjective smooth map $\bar\varphi : S^1 \to S^1$ such that $\Pi \circ f^{-1} = \bar\varphi \circ \Pi$. Composing the two identities we get $\varphi \circ \bar\varphi = \bar\varphi \circ \varphi = \mathbf{1}_{S^1}$, thus $\bar\varphi = \varphi^{-1}$. Since φ and $\bar\varphi$ are smooth, φ is a diffeomorphism of S^1. Next, let $\psi = \Phi(g)$, $\Pi \circ g = \psi \circ \Pi$. Thus, $\Pi \circ (g \circ f) = (\psi \circ \varphi) \circ \Pi$, therefore $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$, and $\Phi : G \to \mathrm{Diff}(S^1)$ is a homomorphism. Let $f \in \ker(\Phi)$, that is, $\Pi \circ f = \Pi$, then for all x, $f(x) = x + 2\pi k(x)$ with $k(x) \in \mathbf{Z}$, but $x \mapsto k(x) = f(x) - x$ is smooth, hence $k(x) = k$ is constant. The case $f' < 0$ is analogous. Therefore, $\ker(\Phi) = \{x \mapsto x + 2\pi k \mid k \in \mathbf{Z}\} \simeq \mathbf{Z}$. Now, let us show that Φ is surjective. Let $\varphi \in \mathrm{Diff}(S^1)$, $\varphi \circ \Pi$ is smooth and we admitted that there exists a smooth lift $f : \mathbf{R} \to \mathbf{R}$ such that $\Pi \circ f = \varphi \circ \Pi$ (it is actually a consequence of the monodromy theorem (art. 8.25)).

a) Necessarily $f(x + 2\pi) = f(x) + 2\pi k(x)$, and for the same reason as just above, $k(x) = k$ is constant. Note that k is necessarily nonzero, f cannot be periodic since a periodic function has necessarily a point where f' vanishes, and this is impossible since f projects onto a diffeomorphism of the circle. Then, note that $(1/k)f(x+2\pi) = (1/k)f(x) + 2\pi$. Thus, there exists a smooth function $\psi : S^1 \to S^1$ such that $\psi(z) = \Pi((1/k)f(x))$, $z = \Pi(x)$. Defining $\hat{k}(z) = \Pi(kx)$, we get $\phi(z) = \hat{k} \circ \psi(x)$. But ψ and $\hat{k}$ are surjective and $\hat{k}$ is not injective if $k \neq \pm 1$, thus $k = \pm 1$ and $f(x + 2\pi) = f(x) \pm 2\pi$. Considering φ^{-1}, the same argument gives $g \in \mathcal{C}^\infty(\mathbf{R})$ such that $\varphi^{-1} \circ \Pi = \Pi \circ g$ and $g(x + 2\pi) = g(x) \pm 2\pi$.

b) Now, $\Pi \circ (f \circ g) = \Pi \circ (g \circ f) = \Pi$, and hence $f \circ g$, as well as $g \circ f$, belongs to $\ker(\Phi)$, that is, $g \circ f(x) = x + 2\pi\ell$ and $f \circ g(x) = x + 2\pi\ell'$, $\ell, \ell' \in \mathbf{Z}$. Then, since $(g \circ f)'(x) = g'(f(x))f'(x) = 1$, f and g are both strictly increasing or strictly decreasing. Next, left composition of $f \circ g(x) = x + 2\pi\ell'$ with g gives $\ell' = \ell$ if f and g are increasing, and $\ell' = -\ell$ otherwise. In both cases, let $\bar{f}(x) = g(x) - 2\pi\ell$, then $\bar{f}$ still satisfies $\varphi^{-1} \circ \Pi = \Pi \circ \bar{f}$, but now $f \circ \bar{f} = \bar{f} \circ f = \mathbf{1}_{\mathbf{R}}$. Therefore $\bar{f} = f^{-1}$, $f \in G$ and Φ is surjective.

Now, let us consider a plot $P : r \mapsto \varphi_r$ of $\mathrm{Diff}(S^1)$ defined on an open ball B. Thus, $(r, x) \mapsto \varphi_r(\Pi(x))$ is a smooth parametrization of S^1, defined on $\mathrm{B} \times \mathbf{R}$. We admitted that there exists a smooth lift $(r, x) \mapsto f_r(x)$, from $\mathrm{B} \times \mathbf{R}$ to $\mathbf{R}$ along Π, that is, $\Pi \circ f_r(x) = \varphi_r \circ \Pi(x)$. Applying a) to this situation, we get a smooth function $r \mapsto k_r$, defined on B to $\mathbf{Z}$ such that $2\pi k_r = f_r(x + 2\pi) - f_r(x)$. Thus, $k_r = k$ is constant. By continuing same reasoning, we get two plots $r \mapsto f_r$ and $r \mapsto g_r$ such that $f_r(x+2\pi) = f_r(x) \pm 2\pi$, $g_r(x+2\pi) = g_r(x) \pm 2\pi$ and $\Pi \circ (f_r \circ g_r) = \Pi \circ (g_r \circ f_r) = \Pi$, where the sign $+$ or $-$ is constant on B. Then, we get similarly $g_r \circ f_r(x) = x + 2\pi\ell_r$

and $f_r \circ g_r(x) = x + 2\pi\ell'_r$, $\ell_r, \ell'_r \in \mathbf{Z}$. For the same reason as previously, $\ell_r = \ell$ and $\ell'_r = \ell'$ are constant, and $\ell' = \pm\ell$ according to the situation. Thus, the change $\bar{f}_r = g_r - 2\pi\ell$ still defines a plot, and $\bar{f}_r = f_r^{-1}$. Therefore, thanks to b) we deduce that $f_r \in G$ for all r, and satisfies: $\Pi \circ f_r = \varphi_r \circ \Pi$ and $\Pi \circ f_r^{-1} = \varphi_r \circ \Pi$. Thus we get a plot $r \mapsto f_r$ covering $r \mapsto \varphi_r$, that is, $\Phi(f_r) = \varphi_r$, for all $r \in B$. Finally, considering a general plot $r \mapsto \varphi_r$, defined on some real domain, every point is the center of some open ball, and applying what we just checked, we can locally lift smoothly this plot in G along Φ. Therefore, Φ is a subduction. Moreover, the kernel of Φ being $\mathbf{Z}$, as we have seen above, the fiber of the projection Φ are the orbits of the action of $\mathbf{Z}$ on G, that is, $k(f) = f + 2\pi k$, $k \in \mathbf{Z}$ and $f \in G$.

For the fifth question, let us note that, since $(x, a) \mapsto x + a$ is clearly smooth, the map $a \mapsto T_a = [x \mapsto x + a]$ is smooth. Conversely if $r \mapsto T_{a(r)}$ is a plot of G, then $a(r) = T_{a(r)}(0)$ and thus $r \mapsto a(r)$ is smooth. Therefore $a \mapsto T_a$ is an induction and the image of $\mathbf{R}$ in G is diffeomorphic to $\mathbf{R}$. Now, since Φ is a subduction, the map $\phi : \mathbf{R}/2\pi\mathbf{Z} \to \mathrm{Diff}(S^1)$, defined by $\phi(\mathrm{class}(a)) = \Phi(T_a)$ is an induction. Since $\mathbf{R}/2\pi\mathbf{Z}$ is diffeomorphic to S^1, thanks to Π, the image of $\mathbf{R}$ by $a \mapsto \Phi(T_a)$, equipped with the subset diffeology, is diffeomorphic to the circle.

☞ **Exercise 29, p. 31** (Quotients of discrete or coarse spaces). Let $X_\circ$ be a discrete diffeological space: the plots of X are locally constant parametrizations (art. 1.20). Let $\sim$ be any equivalence relation on X, $Q = X/\sim$, and let $\pi : X \to Q$ be the projection. By definition of the quotient diffeology (art. 1.43), a plot $P : U \to Q$ lifts locally along a plot of X. Hence, each local lift of P is locally constant. Therefore, P is locally constant and Q is discrete.

☞ **Exercise 30, p. 31** (Examples of quotients). Let us consider the diffeology of the circle defined in (art. 1.11). The circle $S^1 \subset \mathbf{C}$ is obviously in bijection with the classes of the equivalence relation $t \sim t'$ if and only if $t' = t + k$, with $k \in \mathbf{Z}$, that is, $\mathrm{class}(t) \mapsto \exp(2\pi i t) = \cos(2\pi t) + i\sin(2\pi t)$. Then, thanks to the uniqueness of quotients (art. 1.52), to get the identification $S^1 \simeq \mathbf{R}/\mathbf{Z}$ we just have to check that the map $t \mapsto \exp(2\pi i t)$ is a subduction, but the subduction has been proved in Exercise 27, p. 27. Regarding the irrational torus T_α in Exercise 4, p. 8, or the quotient $\mathbf{R}/\mathbf{Q}$ of Exercise 5, p. 9, the diffeology defined by (✱) is, by definition, the quotient diffeology. Concerning the diffeology of $\mathrm{Diff}(S^1)$ defined in Exercise 28, p. 27, we have seen that the map $G \to \mathrm{Diff}(S^1)$ is surjective, and set theoretically $\mathrm{Diff}(S^1) \sim G/\mathbf{Z}$, where $\mathbf{Z} \sim \ker(\Phi)$. Since Φ is a subduction, by uniqueness of quotients we get that diffeologically $\mathrm{Diff}(S^1) \simeq G/\mathbf{Z}$.

☞ **Exercise 31, p. 31** (The irrational solenoid). 1) let us check that the map $q : (x, y) \mapsto (p(x), p(y))$, where $p(t) = (\cos(2\pi t), \sin(2\pi t))$, is strict. First of all, this map is clearly smooth since cos and sin are smooth. Now, according to the definition, q is strict if and only if the map

$$\mathrm{class}(x, y) \mapsto ((\cos(2\pi x), \sin(2\pi x)), (\cos(2\pi y), \sin(2\pi y)))$$

is an induction, from $\mathbf{R}^2/\mathbf{Z}^2$ to $\mathbf{R}^2 \times \mathbf{R}^2$, where $\mathrm{class} : \mathbf{R}^2 \to \mathbf{R}^2/\mathbf{Z}^2$ denotes the natural projection. We know already that $\Pi : t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is strict (Exercise 27, p. 27), and q is just the product $\Pi \times \Pi$. Thus, a plot $\Phi : U \to S^1 \times S^1 \subset \mathbf{R}^2 \times \mathbf{R}^2$ is just a pair of plots P and Q from U to S^1, which can be individually smoothly lifted locally along Π, and give a local lift of q itself. Therefore, q is strict. Now, since α is irrational, $q_\alpha = q \restriction \Delta_\alpha$ is injective. Indeed, for

$t, t' \in \mathbf{R}$, $q(t, \alpha t) = q(t', \alpha t')$ means $(\cos(2\pi t'), \sin(2\pi t')) = (\cos(2\pi t), \sin(2\pi t))$ and $(\cos(2\pi\alpha t'), \sin(2\pi\alpha t')) = (\cos(2\pi\alpha t), \sin(2\pi\alpha t))$, that is, $t' = t + k$ and $\alpha t' = \alpha t + k'$ with $k, k' \in \mathbf{Z}$, that gives $\alpha k - k' = 0$, thus $k = k' = 0$ and $t' = t$.

2) Let $\Phi : U \to \mathcal{S}_\alpha \subset S^1 \times S^1 \subset \mathbf{R}^2 \times \mathbf{R}^2$ be a plot, with $\Phi(r) = (P(r), Q(r))$. Since q is strict, for all $r \in U$, there exists locally a smooth lift $r' \mapsto (x(r'), y(r'))$ in $\mathbf{R}^2$, defined on a neighborhood V of r, such that $q(x(r'), y(r')) = (P(r'), Q(r'))$ Thus, $q(x(r'), y(r')) \in \mathcal{S}_\alpha$ for all $r' \in V$. But, $r' \mapsto (x(r'), \alpha x(r')) \in \Delta_\alpha \subset \mathbf{R}^2$ is smooth, and $q(x(r'), \alpha x(r'))$ belongs to $\mathcal{S}_\alpha$ too. Therefore, there exists $r' \mapsto k(r') \in \mathbf{Z}$ such that $y(r') = \alpha x(r') + k(r')$, that is, $k(r') = y(r') - x(r')$. Thus, $r' \mapsto k(r')$ is smooth and takes its values in $\mathbf{Z}$, hence $k(r') = k$ constant. Then, $r' \mapsto (x(r'), y(r') - k)$ is a plot of $\mathcal{S}_\alpha$ with $q(x(r'), y(r') - k) = (P(r'), Q(r'))$, thus $q_\alpha : \Delta_\alpha \to \mathcal{S}_\alpha$ is an injective subduction, that is, a diffeomorphism from Δ_α to $\mathcal{S}_\alpha$, and therefore an induction.

3) We use the identification given by the factorization $h : \mathbf{R}^2/\mathbf{Z}^2 \to S^1 \times S^1$, of the strict map $q : \mathbf{R}^2 \to S^1 \times S^1$. Then, the quotient $(S^1 \times S^1)/\mathcal{S}_\alpha = h(\mathbf{R}^2/\mathbf{Z}^2)/\mathcal{S}_\alpha$, is equivalent to $\mathbf{R}^2/[\mathbf{Z}^2(\Delta_\alpha)]$ where the equivalence relation is defined by the action of the subgroup $\mathbf{Z}^2(\Delta_\alpha)$, that is, the set of $(x+n, \alpha x+m)$ with $x \in \mathbf{R}$ and $(n, m) \in \mathbf{Z}^2$. Let $\rho : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $\rho(x, y) = (0, y - \alpha x)$, it is obviously a projector, $\rho \circ \rho = \rho$, and clearly $\text{class} \circ \rho = \text{class}$, with $\text{class} : \mathbf{R}^2 \to \mathbf{R}^2/[\mathbf{Z}^2(\Delta_\alpha)]$. Now, let $X' = \text{val}(\rho)$, that is, $X' = \{0\} \times \mathbf{R}$. The restriction to X' of the equivalence relation, defined by the action of $\mathbf{Z}^2(\Delta_\alpha)$ on $\mathbf{R}^2$, is defined by the action of $\mathbf{Z}^2$, $(n, m) : (0, y) \mapsto (0, y+m-\alpha n)$. Therefore, the quotient $(S^1 \times S^1)/\mathcal{S}_\alpha$ is equivalent to $X'/(\mathbf{Z} + \alpha\mathbf{Z})$ (art. 1.53, Note), that is, equivalent to $\mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z}) = T_\alpha$.

☞ **Exercise 32, p. 31** (A minimal powerset diffeology). Let $\mathfrak{P}(X)^*$ be the set of all the nonempty subsets of X, thus $\mathfrak{P}(X) = \{\varnothing\} \cup \mathfrak{P}(X)^*$. Let $\mathcal{D}$ be the set of parametrizations of $\mathfrak{P}(X)^*$ defined as follows.

(♡) A parametrization $P : U \to \mathfrak{P}(X)^*$ belongs to $\mathcal{D}$ if, for all $r \in U$, there exist an open neighborhood $V \subset U$ of r and a plot $Q : V \to X$ such that, for all $r' \in V$, $Q(r') \in P(r')$.

Let us check that $\mathcal{D}$ is a diffeology. Let $P : r \mapsto A \in \mathfrak{P}(X)^*$ be a constant parametrization, and let $x \in A$. The constant parametrization $Q : r \mapsto x$ satisfies $Q(r) \in P(r)$. The covering axiom is thus satisfied. The locality axiom is satisfied by construction. Now, let $P : U \to \mathfrak{P}^*(X)$ belong to $\mathcal{D}$, and let $F : U' \to U$ be a smooth parametrization. Let $r' \in U'$ and $r = F(r')$, let $V \subset U$ be a neighborhood of r, and let $Q : V \to X$ be a plot such that $Q(s) \in P(s)$ for all $s \in V$, according to (♡). Let $V' = F^{-1}(V)$, since F is smooth, thus continuous, V' is an open neighborhood of r'. Now, $Q' = Q \circ F$ is a plot of X, and satisfies $Q'(s) = (Q \circ F)(s) \in (P \circ F)(s)$ for all $s \in V'$. Thus, the smooth compatibility axiom is satisfied and $\mathcal{D}$ is a diffeology of $\mathfrak{P}(X)^*$. Next, we consider $\mathfrak{P}(X)$ as the diffeological sum of the singleton $\{\varnothing\}$ and the diffeological space $\mathfrak{P}(X)^*$, equipped with $\mathcal{D}$. Then, let us consider an equivalence relation $\sim$ on X. The subset $X/\sim = \text{class}(X)$ is contained in $\mathfrak{P}(X)^*$, since no class is empty. Let $P : U \to X/\sim$ be a plot of $\mathfrak{P}(X)^*$. For each $r \in U$, let us choose $x_r \in X$ such that $\text{class}(x_r) = P(r)$, that is, $x_r \in P(r)$. By definition of $\mathcal{D}$, there exists — defined on a neighborhood of each point of U — a plot Q of X such that $Q(r) \in P(r)$, that is, $Q(r) \in \text{class}(x_r)$. Thus, $\text{class}(Q(r)) = \text{class}(x_r) = P(r)$, hence, Q is a local smooth lift of P along class, and that is the definition of the quotient diffeology on $X/\sim$.

☞ **Exercise 33, p. 31** (Universal construction). First of all, let us note that $\mathrm{ev} : \mathcal{N} \to X$ is surjective. Then, every plot $P : U \to X$ lifts naturally by $\mathbf{P} : r \mapsto (P, r)$ in $\mathcal{N}$, along ev: $\mathrm{ev} \circ \mathbf{P} = P$. Therefore $\mathrm{ev} : \mathcal{N} \to X$ is a subduction and X is the diffeological quotient of $\mathcal{N}$ by the relation $(P, r) \sim (P', r')$ if and only if $P(r) = P'(r')$. Next, let us assume that the map $\sigma : x \mapsto ([0 \mapsto x], 0) \in \mathcal{N}$ is smooth. Thus, for every plot $P : U \to X$, $\sigma \circ P : r \mapsto ([0 \mapsto P(r)], 0)$ is a plot of $\mathcal{N}$. By definition of the diffeological sum (art. 1.39), for all $r \in U$, there exists an open neighborhood V of r such that $[0 \mapsto P(r)]$ is constant on V, that is, $P(r) = x$ for some $x \in X$ and for all $r \in V$, but this is the definition of the discrete diffeology (art. 1.20). Therefore, if the set of 0-plots is a smooth section of $\mathrm{ev} : \mathcal{N} \to X$, then X is discrete, and conversely.

☞ **Exercise 34, p. 31** (Strict action of SO(3) on $\mathbf{R}^3$). If $X = 0 \in \mathbf{R}^3$, then obviously $\mathrm{R}(0) : \mathrm{SO}(3) \mapsto 0$ is a trivial subduction. Let us assume that $X \neq 0$, the orbit $\mathrm{SO}(3) \cdot X$ is the sphere of vectors X' with norm $\rho = \|X\|$. By construction, SO(3) preserves the norm. Now, let $r \mapsto X_r$ be a plot of $\mathbf{R}^3$ such that $\|X_r\| = \rho$. Let $u_r = X_r/\rho$, thus $r \mapsto u_r$ is a plot of the unit sphere $S^2 \subset \mathbf{R}^3$, for the subset diffeology. Let r_0 be a point in the domain of this plot, and there exists a vector w not collinear with u_{r_0}. The parametrization $r \mapsto w_r = [\mathbf{1}_{\mathbf{R}^3} - u_r u_r^t]\, w$ is smooth, where u_r^t is the transpose of the vector u_r, and $[\mathbf{1}_{\mathbf{R}^3} - u_r u_r^t]$ is the projector orthogonal to u_r. The real function $\nu : r \mapsto \|w_r\|$ is smooth, and since w is not collinear with u_{r_0}, $\nu(r_0) \neq 0$. Thus, there exists a (possibly small) open ball B, centered at r_0, such that for all $r \in \mathrm{B}$, $\nu(r) \neq 0$. Therefore, the parametrization $r \mapsto v_r = w_r/\nu(r)$, defined on B, is a plot of the sphere S^2 satisfying $v_r \perp u_r$. Next, let $N_r = [u_r\ v_r\ u_r \wedge v_r]$ be the matrix made by juxtaposing the three column vectors, the symbol $\wedge$ denoting the vector product. By construction, $r \mapsto N_r$ is smooth and $N_r \in \mathrm{SO}(3)$. Now, $N_r \boldsymbol{e}_1 = u_r$, where $\boldsymbol{e}_1$ is the first vector of the canonical basis of $\mathbf{R}^3$. By the same way, we can find a unit vector v, orthogonal to $u = X/\rho$, such that $M = [u\ v\ u \wedge v] \in \mathrm{SO}(3)$, and thus $M\boldsymbol{e}_1 = u$. Hence, the parametrization $r \mapsto M_r = N_r M^t$ is smooth, takes its values in SO(3) and satisfies $M_r X = X_r$. Therefore, the orbit map R(X) is strict.

NOTE. This is a particular case of a more general theorem: for a Lie group acting smoothly on a manifold, which is Hausdorff and second countable, the orbit map is always strict [**IZK10**].

☞ **Exercise 35, p. 33** (Products and discrete diffeology). Let us equip the product $X = \prod_{i \in \mathcal{I}} X_i$ with discrete diffeology. Thus, every plot $r \mapsto x$ of X is locally constant, and then any composite $r \mapsto x_i$ is locally constant too, thus smooth (first axiom of diffeology). This is an example, related to the discussion (art. 1.25), where the interesting set of diffeologies on X — the ones such that the projections π_i are smooth — is trivially bounded below. The supremum of this family, which is a maximum (the product diffeology), is therefore the distinguished diffeology. However, it is not the only reason for which that diffeology is interesting; see (art. 1.56). If we consider the sum diffeology of the family (art. 1.39), that is, $X' = \coprod_{i \in \mathcal{I}} X_i$, and if we equip X' with the coarse diffeology, then the canonical injections $j_i : X_i \to X'$, defined by $j_i(x) = (i, x)$, are smooth, simply because any map to a coarse space is smooth. In that case, the set of diffeologies such that the injections are smooth is bounded above by the coarse diffeology, the distinguished diffeology is thus the infimum of that family, that is, the sum diffeology. It is not surprising that products and coproducts are dual constructions of each other.

☞ **Exercise 36, p. 33** (Products of coarse or discrete spaces). Let us consider the product $X = \prod_{i \in \mathcal{I}} X_i$ of coarse spaces. Let P be any parametrization of X. Since for every projection π_i, the composition $\pi_i \circ P$ is a parametrization of X_i, the parametrization P is a plot of the product. Therefore, the product diffeology is coarse. Now, let $X = \prod_{i \in \mathcal{I}} X_i$ be a finite product of discrete spaces, and let $N = \#\mathcal{I}$. Let $P : U \to X$ be a plot. For every $i \in \mathcal{I}$, $\pi_i \circ P$ is locally constant. Let $r_0 \in U$, so there exist N open neighborhoods V_i of r_0 and N points x_i, such that $x_i \in X_i$ and $\pi_i \circ P \restriction V_i = [r \mapsto x_i]$. Hence, since $\#\mathcal{I}$ is finite, $V = \bigcap_{i \in \mathcal{I}} V_i$, V is still an open neighborhood of r_0, and $P \restriction V$ is constant, equal to $s = [i \mapsto (i, x_i)]$. Therefore, P is locally constant and X is discrete. Next, let us consider an arbitrary product $X = \prod_{i \in \mathcal{I}} X_i$ of discrete spaces. We cannot apply the previous method, since an arbitrary intersection of domains may be not open. Then we shall use the result of Exercise 7, p. 14. Let $P : U \to X$ be a plot. By definition of the product diffeology, for all $i \in \mathcal{I}$, $\pi_i \circ P$ is a plot of X_i, that is, locally constant. Now, let r_0 be any point of U, thanks to Exercise 7, p. 14, $\pi_i \circ P$ is constant on the path connected component V of r_0. Thus, for every $i \in \mathcal{I}$, $\pi_i \circ P \restriction V = [r \mapsto x_i]$, where $x_i = \pi_i \circ P(r_0)$. Hence, $P \restriction V = [i \mapsto (i, x_i)]$ is a constant parametrization. Therefore, P is locally constant and X is discrete.

☞ **Exercise 37, p. 33** (Infinite product of $\mathbf{R}$ over $\mathbf{R}$). The sum $X = \coprod_{t \in \mathbf{R}} \mathbf{R}$ is the set of pairs (t, s), with t and s in $\mathbf{R}$. Set theoretically, X is the product $\mathbf{R} \times \mathbf{R}$. Thus, a plot P of X is a pair (T, S) of parametrizations of $\mathbf{R}$, defined on some common domain U, such that for every $r_0 \in U$ there exist an open neighborhood V of r_0, a real t_0, with $T \restriction V = [r \mapsto t_0]$ and $S \restriction V \in \mathcal{C}^\infty(V, \mathbf{R})$. Now, let $X = \prod_{t \in \mathbf{R}} \mathbf{R}$. By definition (art. 1.55), X is the set of maps $[t \mapsto (t, s)]$ such that $s \in \mathbf{R}$. Thus, set theoretically, X is equivalent to $\mathrm{Maps}(\mathbf{R}, \mathbf{R})$, the set of maps from $\mathbf{R}$ to $\mathbf{R}$. Also, an element of X can be regarded as an indexed family $x = (x_t)_{t \in \mathbf{R}}$. A plot $P : U \to X$ is any parametrization $P : r \mapsto (x_t(r))_{t \in \mathbf{R}}$ such that for every $t \in \mathbf{R}$, the parametrization x_t is a plot of $\mathbf{R}$, that is, a smooth parametrization in $\mathbf{R}$.

☞ **Exercise 38, p. 33** (Graphs of smooth maps). Let us assume first that $f : X \to X'$ is such that $\mathrm{pr}_X : \mathrm{Gr}(f) \to X$ is a subduction. Let $P : U \to X$ be some plot, and let $r_0 \in U$. Since $\mathrm{pr}_X \restriction \mathrm{Gr}(f)$ is a subduction, there exist an open neighborhood V of r_0 and a plot $Q : V \to \mathrm{Gr}(f)$ such that $\mathrm{pr}_X \circ Q = P \restriction V$. Thus, $Q(r) = (P(r), f(P(r)))$ for every $r \in V$. But, since Q is a plot of $\mathrm{Gr}(f) \subset X \times X'$, $(f \circ P) \restriction V$ is a plot of X', by definition of the product and the subset diffeologies. So, $f \circ P$ is locally a plot of X', thus $f \circ P$ is a plot of X'. Therefore f is smooth. Conversely, let $f : X \to X'$ be a smooth map, and let $P : U \to X$ be a plot. Then, $f \circ P$ is a plot of X', and $Q : r \mapsto (P(r), f \circ P(r))$ is a plot of $X \times X'$. But $\mathrm{val}(Q) \subset \mathrm{Gr}(f)$, so Q is a plot of $\mathrm{Gr}(f)$, for the subset diffeology. Moreover $\mathrm{pr}_X \circ Q = P$, so Q is a lifting of P along pr_X. Therefore, $\mathrm{pr}_X \restriction \mathrm{Gr}(f)$ is a subduction.

☞ **Exercise 39, p. 34** (The 2-torus). In the solution of Exercise 31, p. 31, we have seen that a plot $\Phi : U \to S^1 \times S^1 \subset \mathbf{R}^2 \times \mathbf{R}^2$ is just a pair of plots (P, Q) of $S^1 \subset \mathbf{R}^2$, with $\mathrm{def}(P) = \mathrm{def}(Q) = U$, that is, by definition, a plot for the product diffeology for a finite family of spaces. Also note that the standard diffeology on $\mathbf{R}^n$ is the product diffeology of n copies of $\mathbf{R}$.

☞ **Exercise 40, p. 39** (The space of polynomials). 1) To prove that the map $j_n : (\mathbf{R}^m)^{n+1} \to \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^m)$ is an induction, we apply the criterion stated in (art. 1.31).

a) The map j_n is injective. Indeed, a polynomial is characterized by its coefficients.

b) The map j_n is smooth. Let $P : U \to (\mathbf{R}^m)^{n+1}$ be a plot, that is, $P : r \mapsto P(r) = (P_0(r), \ldots, P_n(r))$, where the P_i are smooth parametrizations of $\mathbf{R}^m$ (art. 1.55). Now, the map j_n is smooth if and only if, for every smooth parametrization $\tau : V \to \mathbf{R}$, the parametrization $(r, s) \mapsto P_0(r) + \tau(s)P_1(r) + \cdots + \tau(s)^n P^n(r)$ is a smooth parametrization of $\mathbf{R}^m$. But this is the case, since it is a sum of products of smooth parametrizations.

c) The map j_n^{-1} is smooth. Let $P : U \to \mathrm{Pol}_n(\mathbf{R}^m) \subset \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^m)$ be a plot. Now, let $P_k(r) \in \mathbf{R}^m$ be the coefficients of $P(r)$, $r \in U$, such that $P(r)(t) = P_0(r) + tP_1(r) + \cdots + t^n P^n(r)$ for all t in $\mathbf{R}$. Or, in other words, such that $j_n^{-1} \circ P(r) = (P_0(r), \ldots, P_n(r))$. But the coefficients $P_k(r)$ are

$$P_0(r) = P(r)(0) \quad \text{and} \quad P_k(r) = \frac{1}{k!} \left. \frac{d^k P(r)(t)}{dt^k} \right|_{t=0}, \ k = 1, \ldots, n,$$

and P being a plot of $\mathrm{Pol}_n(\mathbf{R}^m)$, the parametrization $\mathbf{P} : (r, t) \mapsto P(r)(t)$ is smooth. Hence, each coefficient P_k is a partial derivative of a smooth parametrization,

$$\left. \frac{d^k P(r)(t)}{dt^k} \right|_{t=0} = \left. \frac{\partial^k \mathbf{P}(r, t)}{\partial t^k} \right|_{t=0}.$$

Therefore, P_k is a smooth parametrization of $\mathbf{R}^m$. Thus $j_n^{-1} \circ P$ is a plot of $(\mathbf{R}^m)^{n+1}$, and j_n^{-1} is smooth. In conclusion, the space $\mathrm{Pol}_n(\mathbf{R}, \mathbf{R}^m)$, equipped with the functional diffeology, inherited from $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^m)$, is diffeomorphic to the real vector space $(\mathbf{R}^m)^{n+1}$.

2) Let ω be a domain in $(\mathbf{R}^m)^{n+1}$, and Ω be the subset of $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^n)$ defined by

$$\Omega = \left\{ f \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^m) \;\middle|\; \left(\frac{f(0)}{0!}, \frac{f'(0)}{1!}, \frac{f''(0)}{2!}, \ldots, \frac{f^{(n)}(0)}{n!} \right) \in \omega \right\}.$$

By construction, every polynomial $[t \mapsto x_0 + tx_1 + \cdots + t^n x_n]$, with coefficients $x = (x_0, x_1, \ldots, x_n)$ in ω, belongs to Ω. More precisely, $j_n(\omega) = \Omega \cap \mathrm{Pol}_n(\mathbf{R}, \mathbf{R}^n)$. Now, let $P : U \to \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^m)$ be some plot, we have

$$P^{-1}(\Omega) = \left\{ r \in U \;\middle|\; \left(P(r)(0), \frac{1}{1!} \left. \frac{\partial P(r,t)}{\partial t} \right|_{t=0}, \ldots, \frac{1}{n!} \left. \frac{\partial^n P(r,t)}{\partial t^n} \right|_{t=0} \right) \in \omega \right\}.$$

Since P is smooth, the various partial derivatives are smooth and then continuous. Hence, the following map $\phi : U \to (\mathbf{R}^m)^{n+1}$, defined by

$$\phi : r \mapsto \left(P(r, 0), \frac{1}{1!} \left. \frac{\partial P(r,t)}{\partial t} \right|_{t=0}, \ldots, \frac{1}{n!} \left. \frac{\partial^n P(r,t)}{\partial t^n} \right|_{t=0} \right),$$

is continuous. Therefore, $P^{-1}(\Omega)$ is the preimage of the domain ω by the continuous map ϕ, thus a domain. The proof is complete.

☞ **Exercise 41, p. 39** (A diffeology for the space of lines). A polynomial f of degree 1, from $\mathbf{R}$ to $\mathbf{R}^n$, is a map $f : t \mapsto x + tv$, where $(x, v) \in (\mathbf{R}^n)^2$. The image of f is an (affine) line of $\mathbf{R}^n$ if and only if $v \neq 0$. The coefficients x and v are called the *origin* for x, since $x = f(0)$, and the *velocity* for v, since $v = f'(0)$. Hence, defining this subspace $\mathrm{PL}(\mathbf{R}^n)$ of polynomials as the space of *parametrized lines* of $\mathbf{R}^n$ makes sense.

1) Let $f = [t \mapsto x + tv]$ and $g = [t \mapsto x' + tv']$ be two lines having the same image in $\mathbf{R}^n$, that is, $f(\mathbf{R}) = g(\mathbf{R})$. So, $x' = g'(0) \in f(\mathbf{R})$, thus there exists a number b such that $x' = x + bv$. Now, since $f(\mathbf{R}) = g(\mathbf{R})$ the derivative of f and g are proportional. Thus, there exists a number a such that $v' = av$. But since v and v' are not zero, $a \neq 0$. Hence, $g(t) = x + bv + atv = x + (at + b)v$, and then $g(t) = f(at + b)$. Conversely if $g(t) = f(at + b)$, since $a \neq 0$, it is clear that $g(\mathbf{R}) = f(\mathbf{R})$.

2) The set $(a, b) : t \mapsto at + b$ of transformations of $\mathbf{R}$, where a and b are real numbers such that $a \neq 0$, is the affine group, denoted by $\mathrm{Aff}(\mathbf{R})$. It is isomorphic to the group of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} at + b \\ 1 \end{pmatrix}.$$

The action of the affine group on the space of lines defined in the first question, $(a, b)(f) = [t \mapsto f(at + b)]$, is the composition $(a, b)(f) = f \circ (a, b)$. It is in fact an anti-action, since $(a, b)[(a', b')(f)] = f \circ (a', b') \circ (a, b) = [(a', b') \circ (a, b)](f)$.

3) Thanks to Exercise 40, p. 39, we know that the inclusion $(x, v) \mapsto [t \mapsto x + tv]$ is an induction from $(\mathbf{R}^n)^2$ to $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^n)$. Hence $\mathrm{PL}(\mathbf{R}^n)$, equipped with the functional diffeology, is diffeomorphic to $\mathbf{R}^n \times (\mathbf{R}^n - \{0\})$.

4) Thus, the equivalence relation defining the oriented trajectory of the parametrized line is the following, $(x, v) \sim (x + bv, av)$ where $(a, b) \in \mathrm{Aff}_+(\mathbf{R})$, that is, $a, b \in \mathbf{R}$ and $a \neq 0$. Now, the map ρ consists into two maps. The second one $v \mapsto u$ is well defined and smooth since $v \neq 0$. The first one is the orthogonal projector to u or, which is equivalent, to v. In other words $r = [\mathbf{1}_n - u\bar{u}]x$, where $\bar{u}$ is the covector $\bar{u} : w \mapsto u \cdot w$. Therefore, ρ is a smooth map from $\mathbf{R}^n \times (\mathbf{R}^n - \{0\})$ into itself. Finally, the image of ρ is clearly the subset of $\mathbf{R}^n \times \mathbf{R}^n$ made up with the pairs of vectors (r, u) such that $u \cdot r = 0$ and $\|u\| = 1$, which is equivalent to TS^{n-1}, as it has been defined. Now, let f and g be the lines defined respectively by (x, v) and (x', v'). Let us assume that $\rho(x, v) = \rho(x', v') = (r, u)$. So, $u = v/\|v\| = v'/\|v'\|$. Thus, there exists $a > 0$ such that $v' = av$. Then, $[\mathbf{1}_n - u\bar{u}]x = [\mathbf{1}_n - u\bar{u}]x' = r$ implies that the orthogonal projection to u of $x' - x$ is zero, hence $x' - x$ is proportional to u, or which is equivalent, to v. Thus, there exists a number b such that $x' = x + bv$. Therefore, if $\rho(x', v') = \rho(x, v)$, then there exists an element $(a, b) \in \mathrm{Aff}_+(\mathbf{R})$ such that $g = (a, b)(f)$, and the lines defined by (x', v') and (x, v) have the same oriented trajectory. The converse is as clear as the direct way. Moreover, if v is unitary and x is orthogonal to v, then $\rho(x, v) = (v, x)$. Hence $\rho \circ \rho = \rho$. Therefore, the map ρ satisfies the conditions of (art. 1.53). Its image, equivalent to TS^{n-1}, is diffeomorphic to the quotient space $[\mathbf{R}^n \times (\mathbf{R}^n - \{0\})]/\mathrm{Aff}_+(\mathbf{R})$, that is, diffeomorphic to $\mathrm{UL}_+(\mathbf{R}^n) = \mathrm{PL}(\mathbf{R}^n)/\mathrm{Aff}_+(\mathbf{R})$. Considering the lines in $\mathbf{R}^2$, the space TS^1 describes the oriented unparametrized lines, a point $(x, u) \in \mathsf{TS}^1$ describes the line passing through x and directed by u. Hence, the set of unparametrized and nonoriented lines is equivalent to the quotient $\mathsf{TS}^1/\{\pm 1\}$, where $\varepsilon(x, u) = (x, \varepsilon u)$, $\varepsilon \in \{\pm 1\}$. Thanks to the diffeomorphism $(x, u) \mapsto (u, r = x \cdot Ju)$ from TS^1 to $S^1 \times \mathbf{R}$, where J is the $\pi/2$ positive rotation, the action of $\{\pm 1\}$ transmutes into $\varepsilon(u, r) = (\varepsilon u, \varepsilon r)$. The quotient is a realization of the Möbius strip.

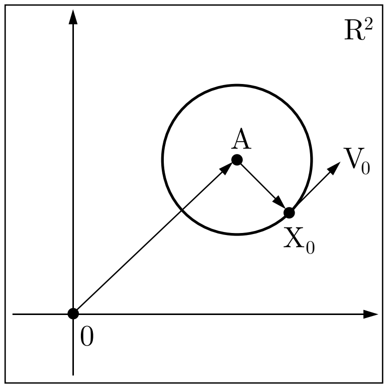

FIGURE Sol.3. Initial conditions of the ODE.

☞ **Exercise 42, p. 40** (A diffeology for the set of circles). We shall describe the set of circles in the plane $\mathbf{R}^2$ as the trajectories of the solutions of an ordinary differential equation. Let us set up the adequate differential equation. Let C be the circle centered at the point A, with radius r, where $A \in \mathbf{R}^2$ and $r \in [0, \infty[$. Let $\mathrm{R}(\theta)$ be the rotation with angle θ, and let $X_0 \in \mathbf{R}^2$ such that $r = \|X_0\|$ (Figure Sol.3). The circle C can be described by

$$\mathrm{C} = \{A + \mathrm{R}(\omega t)X_0 \in \mathbf{R}^2 \mid t \in \mathbf{R}\},$$

where $\omega \in \mathbf{R}$ and $\omega \neq 0$. Thus, the circle C is the set of values of the map

$$t \mapsto X(t) = A + \mathrm{R}(\omega t)X_0, \quad \text{with} \quad t \in \mathbf{R}. \tag{$\diamondsuit$}$$

These functions are the solutions of the ordinary differential equation

$$\ddot{X}(t) + \omega^2 X(t) = \mathrm{cst}. \tag{$\heartsuit$}$$

We call the *trajectory* of the *curve* $[t \mapsto X(t)] \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^2)$ the set of its values, that is, $\mathrm{traj}(X) = \{X(t) \mid t \in \mathbf{R}\}$. We must not confuse the trajectory $\mathrm{traj}(X) \subset \mathbf{R}^2$ and the curve $X \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^2) \subset \mathbf{R} \times \mathbf{R}^2$. Precisely, $\mathrm{traj}(X) = \mathrm{pr}_2(\mathrm{Gr}(X))$, where the graph $\mathrm{Gr}(X)$ of X is equivalent to X. As well as the lines of Exercise 41, p. 39, are the trajectories of the solutions of the differential equation $\ddot{X}(t) = \mathrm{cst}$, the circles are the trajectories of the solutions of the differential equation ($\heartsuit$). Thus, an exercise about the structure of the set of circles could be the following.

Let $\mathrm{Sol}(\heartsuit)$ *be the space of solutions of the ordinary differential equation* ($\heartsuit$), *equipped with the functional diffeology induced by* $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^2)$. *Show that the trajectories of the solutions are the circles, centered somewhere for some radius. Describe the spaces of circles, equipped with the quotient diffeology* $\mathrm{Sol}(\heartsuit)/\mathrm{traj}$, *where two solutions are identified by their trajectories.*

☞ **Exercise 43, p. 47** (Generating tori). The map $\pi : t \mapsto (\cos(t), \sin(t))$ from $\mathbf{R}$ to $S^1 \subset \mathbf{R}^2$ is a generating family for S^1. For $X = T_\alpha$ or $X = \mathbf{R}/\mathbf{Q}$, a generating family can be chosen to be the natural projections $\mathrm{class} : \mathbf{R} \to X$.

☞ **Exercise 44, p. 47** (Global plots as generating families) Let $P : U \to X$ be an n-plot of X, with n a positive integer (for $n = 0$ there is nothing to prove). Let $r_0 \in U$. There exists $\varepsilon > 0$ such that the open ball $B(r_0, \varepsilon)$ is contained in

U. Now, it is a standard result that the ball $B(r_0, \varepsilon)$ is diffeomorphic to $\mathbf{R}^n$; let $\varphi : B(r_0, \varepsilon) \to \mathbf{R}^n$ be such a diffeomorphism. Thus, $\psi = (P \upharpoonright B(r_0, \varepsilon)) \circ \varphi^{-1}$ is defined on $\mathbf{R}^n$ with values in X, and since ψ is the composite of a plot with a smooth map, it is a plot of X, a global plot. Then, $P \upharpoonright B(r_0, \varepsilon) = \psi \circ \varphi$, where $\psi \in \mathcal{P}$ and φ is a smooth parametrization of $\mathbf{R}^n$. Thus, the diffeology of X is generated by its global plots.

☞ **Exercise 45, p. 47** (Generating the half-line). The pullback $j^*(\{\mathbf{1_R}\})$ is the set of parametrizations $F : U \to [0, \infty[$ such that $j \circ F$ is constant or there exist an element $F' \in \{\mathbf{1_R}\}$ and a smooth parametrization $\phi : U \to \operatorname{def}(F')$ with $j \circ F = F' \circ \phi$. Thus, $F' = \mathbf{1_R}$, $\operatorname{def}(F') = \mathbf{R}$, and then $F = \phi$. Therefore, F is any smooth parametrization of $\mathbf{R}$ with values in $[0, \infty[$, and $j^*(\{\mathbf{1_R}\})$ is the whole diffeology of the half-line.

☞ **Exercise 46, p. 47** (Generating the sphere). Let $P : U \to S^n$ be a smooth parametrization, $r_0 \in U$, $x_0 = P(r_0)$, and $E_0 = x_0^\perp$. The real function $r \mapsto x_0 \cdot P(r)$ is smooth and satisfies $x_0 \cdot P(r_0) = 1$. There exists then a small open ball B_0, centered at r_0, such that for all $r \in B_0$, $x_0 \cdot P(r) > 0$. Thus, $P(B_0)$ is contained in the values of the map f_0, associated with the point r_0, of Exercise 18, p. 19. Now, let $S_0 : \mathcal{B} \to E_0$ be defined by $S_0(s_1, \ldots, s_n) = \sum_{i=1}^n s_i u_i$, then $F_0 = f_0 \circ S_0$. Since f_0 is an induction, $\phi = f_0^{-1} \circ (P \upharpoonright B_0)$ is a smooth map from B_0 to E_0, and $\psi = S_0^{-1} \circ \phi$ a smooth parametrization of $\mathbf{R}^n$. But $\psi = S_0^{-1} \circ f_0^{-1} \circ (P \upharpoonright B_0) = F_0^{-1} \circ (P \upharpoonright B_0)$, thus $(P \upharpoonright B_0) = F_0 \circ \psi$, where ψ is smooth. Therefore, the plots F are a generating family for the sphere S^n.

☞ **Exercise 47, p. 47** (When the intersection is empty). First of all, since $[x \mapsto x] \neq [x \mapsto 2x]$, $\mathcal{F} \cap \mathcal{F}' = \varnothing$. Thus, $\langle \mathcal{F} \cap \mathcal{F}' \rangle = \langle \varnothing \rangle = \mathcal{D}_\circ(\mathbf{R})$ (art. 1.67). Now, $\mathcal{F}$ is the family made up just with the identity of $\mathbf{R}$, so it generates the usual diffeology, $\langle \mathcal{F} \rangle = \mathcal{C}^\infty_\star(\mathbf{R})$. But, for any smooth parametrization P of $\mathbf{R}$, $2 \times P$ is smooth and $P = 2 \times Q$, with Q equal to the smooth parametrization $P/2$. Thus, $\langle \mathcal{F} \rangle = \langle \mathcal{F}' \rangle$ and $\langle \mathcal{F} \rangle \cap \langle \mathcal{F}' \rangle = \langle \mathcal{F} \rangle = \mathcal{C}^\infty_\star(\mathbf{R})$, and $\langle \mathcal{F} \cap \mathcal{F}' \rangle \neq \langle \mathcal{F} \rangle \cap \langle \mathcal{F}' \rangle$.

☞ **Exercise 48, p. 50** (Has the set $\{0, 1\}$ dimension 1?). Since $\{\pi\}$ is a generating family, the dimension of $\{0, 1\}_\pi$ is less or equal than 1, $\dim\{0, 1\}_\pi \leq 1$. Now, since the plot π is not locally constant — by density of the rational, or irrational, numbers in $\mathbf{R}$ — the space $\{0, 1\}_\pi$ is not discrete. Hence, $\dim\{0, 1\}_\pi \neq 0$ (art. 1.81), and then $\dim\{0, 1\}_\pi = 1$. This example shows how strongly the dimension of a diffeological space is related to its diffeology and not to some set theoretic considerations on the underlying set. A space consisting in a finite number of points can have an indiscrete diffeology. Remark that, in topology too, a finite set of points can be indiscrete.

☞ **Exercise 49, p. 50** (Dimension of tori). By definition, the projection $\pi : \mathbf{R} \to \mathbf{R}/\Gamma$ is a subduction (art. 1.46). But since $\mathbf{R}$ is a real domain, π is a plot of the quotient, and $\mathcal{F} = \{\pi\}$ is a generating family for $\mathbf{R}/\Gamma$, thus $\dim(\mathcal{F}) = 1$. Hence, as a direct consequence of the definition (art. 1.78) — or as a consequence of (art. 1.82), since $\dim(\mathbf{R}) = 1$ — $\dim(\mathbf{R}/\Gamma) \leq 1$. Now, if $\dim(\mathbf{R}/\Gamma) = 0$, then the diffeology of the quotient is generated by the constant parametrizations. Since the projection π is a plot, it lifts locally at the point 0 in the constant plot $\mathbf{0} : \mathbf{R} \to \{0\}$, but since $\mathbf{R}$ is pathwise connected, the lift is global (see Exercise 7, p. 14), and $\pi = [0] \circ \mathbf{0}$, where $[0] : \{0\} \to \mathbf{R}/\Gamma$ maps 0 to $[0] = \pi(0)$. Thus, since π is surjective, $\mathbf{R}/\Gamma = \{[0]\}$ and $\Gamma = \mathbf{R}$. Therefore, if $\Gamma \subset \mathbf{R}$ is a strict subgroup, that is, $\Gamma \neq \mathbf{R}$, we have necessarily

$\dim(\mathbf{R}/\Gamma) = 1$. In particular, this applies to the circle $S^1 \simeq \mathbf{R}/\mathbf{Z}$ (see Exercise 27, p. 27), or to the irrational tori $\mathbf{R}/\sum_{i=1\dots N} \alpha_i \mathbf{Z}$, where the α_i are some numbers, independent over $\mathbf{Q}$; see Exercise 43, p. 47. It also applies to $\mathbf{R}/\mathbf{Q}$; see Exercise 5, p. 9. Thus, $\dim(S^1) = \dim(\mathbf{R}/\mathbf{Q}) = \dim(T_\alpha) = 1$.

Exercise 50, p. 50 (Dimension of $\mathbf{R}^n/\mathrm{O}(n,\mathbf{R})$). Let $\Delta_n = \mathbf{R}^n/\mathrm{O}(n,\mathbf{R})$, $n \in \mathbf{N}$, equipped with quotient diffeology.

1) Let us denote by $\pi_n : \mathbf{R}^n \to \Delta_n$ the projection from $\mathbf{R}^n$ onto its quotient. Since, by the very definition of Ê $\mathrm{O}(n,\mathbf{R})$, $\|x'\| = \|x\|$ if and only if $x' = $ ÊAx, with $A \in \mathrm{O}(n,\mathbf{R})$, there exists a bijection $f : \Delta_n \to [0,\infty[$ such that $f \circ \pi_n = \nu_n$, where $\nu_n(x) = \|x\|^2$. Now, thanks to the uniqueness of quotients (art. 1.52), f is a diffeomorphism between Δ_n equipped with the quotient diffeology and $[0,\infty[$, equipped with the pushforward of the standard diffeology of $\mathbf{R}^n$ by the map ν_n. Now, let us denote by $\mathcal{D}_n$ the pushforward of the standard diffeology of $\mathbf{R}^n$ by ν_n. The space $([0,\infty[, \mathcal{D}_n)$ is a representation of Δ_n.

$$\begin{tikzcd}
\mathbf{R}^n \arrow[d, "\pi_n"'] \arrow[dr, "\nu_n"] & \\
\Delta_n \arrow[r, "f"'] & {[0,\infty[}
\end{tikzcd}$$

2) Let us denote by 0_k the zero of $\mathbf{R}^k$. Next, let us assume that the plot ν_n can be lifted at the point 0_n along a p-plot $P : U \to \Delta_n$, with $p < n$. Let $\phi : V \to U$ be a smooth parametrization such that $P \circ \phi = \nu_n \restriction V$. We can assume without loss of generality that $P(0_p) = 0$ and $\phi(0_n) = 0_p$. If it is not the case, we compose P with a translation mapping $\phi(0_n)$ to 0_p. Now, since P is a plot of Δ_n, it can be lifted locally at the point 0_p along ν_n. Let $\psi : W \to \mathbf{R}^n$ be a smooth parametrization such that $0_p \in W$ and $\nu_n \circ \psi = P \restriction W$. Let us introduce $V' = \phi^{-1}(W)$. We have then the following commutative diagram.

$$\begin{tikzcd}
 & W \arrow[d, "P \restriction W"] \arrow[dr, "\psi"] & \\
V' \arrow[ur, "\phi \restriction V'"] \arrow[r, "\nu_n \restriction V'"'] & {[0,\infty[} & \mathbf{R}^n \arrow[l, "\nu_n"]
\end{tikzcd}$$

Now, denoting by $F = \psi \circ \phi \restriction V'$, we get $\nu_n \restriction V' = \nu_n \circ F$, with $F \in \mathcal{C}^\infty(V', \mathbf{R}^n)$, $0_n \in V'$ and $F(0_n) = 0_n$, that is,

$$\|x\|^2 = \|F(x)\|^2.$$

The derivative of this identity gives

$$x \cdot \delta x = F(x) \cdot D(F)(x)(\delta x), \text{ for all } x \in V' \text{ and for all } \delta x \in \mathbf{R}^n.$$

The second derivative, computed at the point 0_n, where F vanishes, gives then

$$1_n = M^t M \quad \text{with} \quad M = D(F)(0_n),$$

where M^t is the transposed matrix of M. But $D(F)(0_n) = D(\psi)(0_p) \circ D(\phi)(0_n)$. Let us denote $A = D(\psi)(0_p)$ and $B = D(\phi)(0_n)$, $A \in L(\mathbf{R}^p, \mathbf{R}^n)$ and $B \in L(\mathbf{R}^n, \mathbf{R}^p)$. Thus $M = AB$ and the previous identity $\mathbf{1}_n = M^t M$ becomes $\mathbf{1}_n = B^t A^t AB$. But the rank of B is less or equal to p which is, by hypothesis, strictly less than n, which would imply that the rank of $\mathbf{1}_n$ is strictly less than n. And this is not true: the rank of $\mathbf{1}_n$ is n. Therefore, the plot ν_n cannot be lifted locally at the point 0_n by a p-plot of Δ_n with $p < n$.

3) The diffeology of Δ_n, represented by $([0,\infty[, \mathcal{D}_n)$, is generated by ν_n. Hence, $\mathcal{F} = \{\nu_n\}$ is a generating family for Δ_n. Therefore, by definition of the dimension of diffeological spaces (art. 1.78), $\dim(\Delta_n) \leq n$. Let us assume that $\dim(\Delta_n) = p$ with $p < n$. Then, since ν_n is a plot of Δ_n it can be lifted locally, at the point 0_n, along an element P' of some generating family $\mathcal{F}'$ for Δ_n. The family $\mathcal{F}'$ satisfies $\dim(\mathcal{F}') = p$. But, by definition of the dimension of generating families (art. 1.77), we get $\dim(P') \leq p$, that is, $\dim(P') < n$. This is not possible, thanks to the second question. Therefore, $\dim(\Delta_n) = n$. Now, since the dimension is a diffeological invariant (art. 1.79), $\Delta_n = \mathbf{R}^n/\mathrm{O}(n, \mathbf{R})$ is not diffeomorphic to $\Delta_m = \mathbf{R}^m/\mathrm{O}(m, \mathbf{R})$ when $n \neq m$.

☞ **Exercise 51, p. 50** (Dimension of the half-line). First of all, let us remark that all the maps $\nu_n : \mathbf{R}^n \to \Delta_\infty$, defined by $\nu_n(x) = \|x\|^2$, are plots of Δ_∞. Indeed, these ν_n are smooth parametrizations of $\mathbf{R}$ and take their values in $[0,\infty[$. Now, let us assume that $\dim(\Delta_\infty) = N < \infty$. Hence for any integer n, the plot ν_n lifts locally at the point 0_n along some p-plot of Δ_∞, with $p \leq N$. Let us choose now $n > N$. Then, there exist a smooth parametrization $f : U \to \mathbf{R}$ such that $\mathrm{val}(f) \subset [0,\infty[$, that is, f is a p-plot of Δ_∞, and a smooth parametrization $\phi : V \to U$ such that $f \circ \phi = \nu_n \restriction V$.

$$\begin{tikzcd} & U \arrow[d, "f"] \\ V \arrow[ur, "\phi"] \arrow[r, "\nu_n \restriction V"'] & {[0,\infty[} \end{tikzcd}$$

We can assume, without loss of generality, that $0_p \in U$, $\phi(0_n) = 0_p$, which implies $f(0_p) = 0$. Now, let us follow the method of Exercise 50, p. 50. The first derivative of ν_n at a point $x \in V' = \phi^{-1}(V)$ is given by

$$x = D(f)(\phi(x)) \circ D(\phi)(x).$$

Since f is smooth, positive, and $f(0) = 0$, we have in particular $D(f)(0_p) = 0$. Now, considering this property, the second derivative, computed at the point 0_n, gives, in matricial notation,

$$\mathbf{1}_n = M^t H M, \text{ where } M = D(\phi)(0) \text{ and } H = D^2(f)(0),$$

where M^t is M transposed, and H is the Hessian of ϕ at the point 0_n, a symmetric bilinear map. The matrix M represents the tangent map of f at 0_p. Now, since we chose $n > N$ and assumed $p \leq N$, we have $p < n$. Thus the map M has a nonzero kernel and then $M^t H M$ is degenerate, which is impossible since $\mathbf{1}_n$ is nondegenerate. Therefore, the dimension of Δ_∞ is unbounded, that is, infinite.

☞ **Exercise 52, p. 53** (To be a locally constant map). Let $\gamma \in \mathcal{C}^\infty(\mathbf{R}, X)$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. For all $t \in [0,1]$ there exists a superset V_t of $\gamma(t)$ such that $f \upharpoonright V_t$ is a local smooth map, according to the definition (art. 2.1). In particular $\mathcal{I}_t = \gamma^{-1}(V_t)$ is a 1-domain containing t and it satisfies $f(\gamma(\mathcal{I}_t)) = \text{cst}$. The $\mathcal{I}_t$ are a covering of the segment $[0,1]$ which is compact. Then, adapting the arguments of Exercise 7, p. 14, to $f \circ \gamma$, we get $f(x_0) = f(x_1)$. Therefore, f is constant on the connected components of X.

☞ **Exercise 53, p. 56** (Diffeomorphisms of the square). A diffeomorphism from the square must send corner into corner, because every smooth map into a corner must be flat (see Exercise 15, p. 17), which is not the case for the other points of the square. Thus, we can associate with every diffeomorphism φ of the square a permutation $\sigma = h(\varphi)$ of the set of four corners. The map h is clearly a homomorphism. But φ is a diffeomorphism which also permutes the edges of the square in a coherent way with σ, the image of connected edges to a corner must be connected to the image of this corner. Eventually, the image of h is the dihedral group with eight elements, generated by the rotation of angle $\pi/4$ and a reflection by an axis of symmetry.

☞ **Exercise 54, p. 56** (Smooth D-topology). Let $U \subset \mathbf{R}^n$ be a domain, that is, an ordinary open subset of $\mathbf{R}^n$. Let $A \subset U$ be open in U, that is, A open in $\mathbf{R}^n$. Let $P : V \to U$ be a plot of U, that is, any smooth parametrization. Since smooth parametrizations are continuous maps for the standard topology, the pullback $P^{-1}(A)$ is open. Then, any open set of U, for the usual topology, is D-open. Conversely, let $A \subset U$ be D-open, the identity map 1_U being a plot of U, the subset $1_U^{-1}(A) = A$ is open. Then, any D-open set of U is open for the usual topology. Thus, the standard topology and the D-topology of smooth domains coincide.

☞ **Exercise 55, p. 56** (D-topology of irrational tori). Let $\pi : \mathbf{R} \to T_\Gamma$ be the natural projection. The set T_Γ is equipped with the quotient diffeology (art. 1.50). Let $A \subset T_\Gamma$ be a nonempty D-open. Since the projection π is smooth, it is D-continuous (art. 2.9). Thus, $\pi^{-1}(A)$ is a D-open in $\mathbf{R}$, that is, $\pi^{-1}(A)$ is a domain (Exercise 54, p. 56). Let $\tau \in A$ and $x \in \mathbf{R}$ such that $\pi(x) = \tau$. So, $\pi^{-1}(A)$ contains x and its whole orbit by the action of Γ. Let us denote by $\mathcal{O}$ this orbit, thus $\pi^{-1}(A)$ is an open neighborhood of $\mathcal{O}$. But Γ being dense in $\mathbf{R}$, the orbit $\mathcal{O}$ also is dense, and $\pi^{-1}(A)$ is an open neighborhood of a dense subset of $\mathbf{R}$. Therefore, $\pi^{-1}(A) = \mathbf{R}$ and $A = T_\Gamma$. Therefore, the only nonempty D-open set of T_Γ is T_Γ itself. The D-topology of T_Γ is coarse. Now, a full functor is a functor surjective on the arrows [**McL71**]. Since the D-topology of T_Γ is coarse, any map from T_Γ to T_Γ is D-continuous. But we know by Exercise 5, p. 9, that all maps from $T_\mathbf{Q}$ to $T_\mathbf{Q}$ are not smooth, just the affine ones. Hence the D-topology functor is not surjective on the arrows, that is, not full.

☞ **Exercise 56, p. 57** ($\mathbf{Q}$ is discrete but not embedded in $\mathbf{R}$). Let us recall that $\mathbf{Q}$ is discrete in $\mathbf{R}$ (Exercise 8, p. 14), that is, the subset diffeology is discrete. The D-topology of $\mathbf{R}$ is the smooth topology Exercise 54, p. 56 and, since $\mathbf{Q} \subset \mathbf{R}$ is discrete, the D-topology of $\mathbf{Q}$ is discrete (art. 2.11). But, since any nonempty open set of the topology induced by $\mathbf{R}$ on $\mathbf{Q}$ contains always an infinite number of points (it is generated by the intersections of open intervals and $\mathbf{Q}$), the induced D-topology is not discrete. Then, $\mathbf{Q}$ is not embedded in $\mathbf{R}$.

Exercise 57, p. 57 (Embedding $\mathrm{GL}(n, \mathbf{R})$ in $\mathrm{Diff}(\mathbf{R}^n)$). Let us recall that the plots of the functional diffeology of $\mathrm{Diff}(\mathbf{R}^n)$ (art. 1.61) are the parametrizations $P : U \to \mathrm{Diff}(\mathbf{R}^n)$ such that

$$[(r,x) \mapsto P(r)(x)] \text{ and } [(r,x) \mapsto P(r)^{-1}(x)] \text{ belong to } \mathcal{C}^\infty(U \times \mathbf{R}^n, \mathbf{R}^n).$$

1) The diffeology of $\mathrm{GL}(n, \mathbf{R})$ induced by $\mathrm{Diff}(\mathbf{R}^n)$ coincides with the ordinary diffeology.

The plots of the standard diffeology of $\mathrm{GL}(n, \mathbf{R})$ are the parametrizations $P : U \to \mathrm{GL}(n, \mathbf{R})$ such that every component P_{ij} is smooth, that is,

$$P_{ij} : r \mapsto \langle \boldsymbol{e}_j \mid P(r)(\boldsymbol{e}_i)\rangle \in \mathcal{C}^\infty(U, \mathbf{R}), \text{ for all } i, j = 1, \ldots, n,$$

where we have denoted by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ the vectors of the canonical basis of $\mathbf{R}^n$, and by $\langle \cdot \mid \cdot \rangle$ the ordinary scalar product of $\mathbf{R}^n$. Now, for each $(r,x) \in U \times \mathbf{R}^n$, $P(r)(x) = P(r)(\sum_{i=1}^n x^i \boldsymbol{e}_i) = \sum_{i=1}^n x^i P(r)(\boldsymbol{e}_i) = \sum_{i=1}^n x^i P_{ij}(r)\boldsymbol{e}_j$. If all the components P_{ij} of the parametrization P are smooth, then $(r,x) \mapsto \sum_{i=1}^n x^i P_{ij}(r)\boldsymbol{e}_j$ is smooth, and the map $(r,x) \mapsto P(r)(x)$ is smooth. Since the determinant of $P(r)$ never vanishes, the same holds for $(r,x) \mapsto P(r)^{-1}(x)$. Therefore, P is a plot of the functional diffeology. Conversely, if P is a plot of the functional diffeology — that is, the parametrization $(r,x) \mapsto P(r)(x)$ is smooth — then, restricting this map to $x = \boldsymbol{e}_i$, we get that the map $r \mapsto P(r)(\boldsymbol{e}_i) = \sum_{j=1}^n P_{ij}(r)\boldsymbol{e}_i$ is smooth. So, by contracting this parametrization to the vector $\boldsymbol{e}_j$, we get that all the matrix components P_{ij} are smooth. Thus, the inclusion $\mathrm{GL}(n, \mathbf{R}) \hookrightarrow \mathrm{Diff}(\mathbf{R}^n)$ is an induction.

2) The inclusion $\mathrm{GL}(n, \mathbf{R}) \hookrightarrow \mathrm{Diff}(\mathbf{R}^n)$ is an embedding.

We have to show that the topology of $\mathrm{GL}(n, \mathbf{R})$ induced by the D-topology of $\mathrm{Diff}(\mathbf{R}^n)$ coincides with the D-topology of $\mathrm{GL}(n, \mathbf{R})$, that is, the topology induced by its inclusion into $\mathbf{R}^{n\times n}$. Let $B(\mathbf{1}_n, \varepsilon)$ be the open ball in $\mathrm{GL}(n, \mathbf{R})$ centered at the identity $\mathbf{1}_n$, with radius ε. Let Ω_ε be the set of all diffeomorphisms defined by

$$\Omega_\varepsilon = \{f \in \mathrm{Diff}(\mathbf{R}^n) \mid D(f)(0) \in B(\mathbf{1}_n, \varepsilon)\},$$

where $D(f)(0)$ is the tangent linear map of f at the point 0. Now, let us prove the following.

a) The set Ω_ε is open for the D-topology of $\mathrm{Diff}(\mathbf{R}^n)$.

Let $P : U \to \mathrm{Diff}(\mathbf{R}^n)$ be a plot, that is, $[(r,x) \mapsto P(r)(x)] \in \mathcal{C}^\infty(U \times \mathbf{R}^n, \mathbf{R}^n)$. The pullback of Ω_ε by P is the set of $r \in U$ such that the tangent map $D(P(r))(0)$ is in the ball $B(\mathbf{1}_n, \varepsilon)$, formally,

$$P^{-1}(\Omega_\varepsilon) = \{r \in U \mid D(P(r))(0) \in B(\mathbf{1}_n, \varepsilon)\}.$$

Considering P as a smooth map defined on $U \times \mathbf{R}^n$, $D(P(r))(0)$ is the partial derivative of P, with respect to the second variable, computed at the point $x = 0$. The map $[r \mapsto D(P(r))(0)]$ is then continuous, by definition of smoothness. Hence, the pullback of Ω_ε by this map is open. Because the imprint of this open set on $\mathrm{GL}(n, \mathbf{R})$ is exactly the ball $B(\mathbf{1}_n, \varepsilon)$, we deduce that any open ball of $\mathrm{GL}(n, \mathbf{R})$ centered at $\mathbf{1}_n$ is the imprint of a D-open set of $\mathrm{Diff}(\mathbf{R}^n)$.

b) Every open of $\mathrm{GL}(n, \mathbf{R})$ is the imprint of a D-open set of $\mathrm{Diff}(\mathbf{R}^n)$.

By using the group operation on $\mathrm{GL}(n, \mathbf{R})$ and since any open set of $\mathrm{GL}(n, \mathbf{R})$ is a union of open balls, every open subset of $\mathrm{GL}(n, \mathbf{R})$ is the imprint of some D-open subset of $\mathrm{Diff}(\mathbf{R}^n)$. Therefore, $\mathrm{GL}(n, \mathbf{R})$ is embedded in $\mathrm{Diff}(\mathbf{R}^n)$.

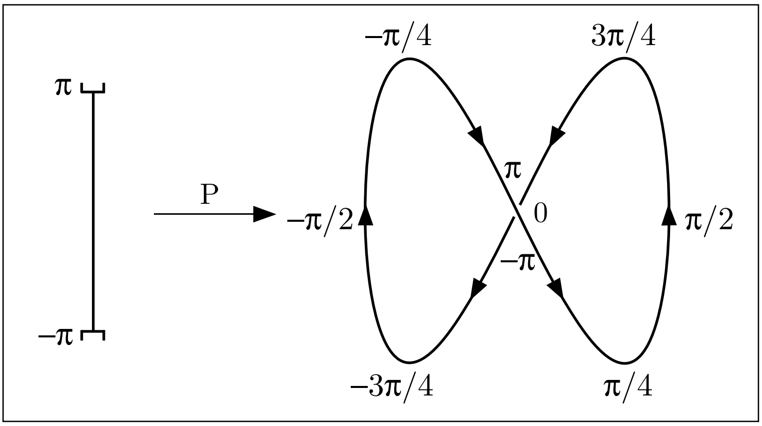

FIGURE Sol.4. The plot P.

Exercise 58, p. 57 (The irrational solenoid is not embedded). The solenoid is the subgroup $[\alpha] = \{(1, e^{i2\pi\alpha t}) \mid t \in \mathbf{R}\} \subset T^2$, with $\alpha \in \mathbf{R} - \mathbf{Q}$. It is the image of the induction $j : t \mapsto (1, e^{i2\pi\alpha t})$, from $\mathbf{R}$ to T^2, Exercise 31, p. 31. Because $[\alpha]$ is dense in T^2, the pullback of any open disc of T^2 by j is an infinite disjoint union of intervals of $\mathbf{R}$. Thus, an open interval $]a, b[\subset \mathbf{R}$ cannot be the preimage of an open subset of T^2. Therefore, the solenoid is not embedded in T^2.

Exercise 59, p. 58 (The infinite symbol). 1) If $j(t) = j(t')$, $t \neq t'$, and $t, t' \in]-\pi, \pi[$, then $t' - t = \pm\pi/2$ and $t' - t = \pm\pi/4$. Thus, $t = t'$ and thus the map j is injective.

2) The drawing of P (Figure Sol.4) clearly shows that j and P have the same image in $\mathbf{R}^2$. But the precise reason is given by (♣) in 3).

3) Comparing the figure of j and the figure of P, we see clearly that

$$j^{-1} \circ P(]-\pi/4, \pi/4[) =]-\pi, -3\pi/4[\cup \{0\} \cup]3\pi/4, \pi[.$$

The map $j^{-1} \circ P$ has a continuity gap at $t = 0$, so $j^{-1} \circ P$ is not continuous, *a fortiori* not smooth. But, we have precisely:

$$j^{-1} \circ P(t) = \begin{cases} -t - \pi & t \in]-\pi/4, 0[\\ 0 & t = 0 \\ -t + \pi & t \in]0, \pi/4[. \end{cases} \qquad (\clubsuit)$$

Hence, P is a plot of $\mathbf{R}^2$ with values in $j(]-\pi, \pi[)$, but $j^{-1} \circ P$ is not smooth. Therefore, by application of (art. 1.31), the injection j is not an induction.

4) The map j is an immersion, and its derivative never vanishes on $]-\pi, \pi[$. Thus, as an application of Exercise 12, p. 17, it is a local induction everywhere.

Exercise 60, p. 60 (Quotient by a group of diffeomorphisms). 1) Let $P : U \to \mathcal{Q}$ be a plot. By definition of the quotient diffeology of G/π, for all $r_0 \in U$ there exist an open neighborhood V of r_0 and a plot $\gamma : V \to G$ such that $P(r) = \gamma(r)(x)$ for all $r \in V$. Hence, $P \restriction V = [r \mapsto (r, x) \mapsto \gamma(r)(x)]$, but $[r \mapsto (r, x)]$ is clearly smooth, and $[(r, x) \mapsto \gamma(r)(x)]$ is smooth by the very definition of the functional diffeology. Thus, $P \restriction V$ is a plot of the subset diffeology. Therefore j is smooth. The quotient diffeology of $G(x)$ is finer than its subset diffeology.

2) Let $P : U \to \mathcal{Q}$ be a plot, and let $r \in U$ and $g \in G$ such that $\pi(g) = P(r)$, that is, $g(x) = P(r)$. By definition of the quotient diffeology there exist an open neighborhood V of r and a plot $\gamma : V \to G$ such that $P \upharpoonright V = \pi \circ \gamma$. Let $g' = \gamma(r)$, then $\pi(g') = \pi(g) = P(r)$, that is, $g(x) = g'(x)$ or $g'^{-1}(g(x)) = x$. Now, let us define on V, $\gamma' = [s \mapsto \gamma(s) \circ g'^{-1} \circ g]$. On the one hand, we have $\gamma'(s)(x) = \gamma(s)(g'^{-1}(g(x)))$, but $g'^{-1}(g(x)) = x$, thus $\gamma'(s)(x) = \gamma(s)(x)$, that is, $\pi \circ \gamma' = \pi \circ \gamma = P \upharpoonright V$. On the other hand we have $\gamma'(r) = \gamma(r) \circ g'^{-1} \circ g$, but $g' = \gamma(r)$, thus $\gamma'(r) = g$. Since, by definition of the functional diffeology of G (art. 1.61), composition and inversion are smooth, the parametrization γ' is a plot of G. Moreover, γ' satisfies the conditions $P \upharpoonright V = \pi \circ \gamma'$ and $\gamma'(r) = g$. Therefore π is a local subduction.

3) By definition of generating families (art. 1.66), a plot P of the Tahar rug $\mathcal{T}$ writes locally $[r \mapsto (t(r), c)]$ or $[r \mapsto (c, t(r))]$, where c is some constant and t is a smooth real function. Now, let $u = (a, b) \in \mathbf{R}^2$, the composition $T_u \circ P$ writes locally either $[r \mapsto (t'(r), c')]$, with $t'(r) = t(r) + a$ and $c' = c + b$, or $[r \mapsto (c', t'(r))]$, with $t'(r) = t(r) + b$ and $c' = c + a$. Thus, T_u is smooth. Then, since $(T_u)^{-1} = T_{-u}$, T_u is a diffeomorphism of $\mathcal{T}$. Now, let $r \mapsto u(r) = (a(r), b(r))$ be a parametrization of $\mathbf{R}^2$ such that $r \mapsto T_{u(r)}$ is a plot for the functional diffeology. Composed with the 1-plots $t \mapsto (t, c)$, where c runs over $\mathbf{R}$, we must get a plot $(r, t) \mapsto (t + a(r), c + b(r))$ of $\mathcal{T}$, that is, a plot which is locally of the first or the second kind. Hence, either $(r, t) \mapsto t + a(r)$ is locally constant, or $(r, t) \mapsto c + b(r)$. But $(r, t) \mapsto t + a(r)$ is not locally constant because of its dependency on t, thus $(r, t) \mapsto c + b(r)$ is locally constant, that is, $b(r) =_{\rm loc} b$, for some $b \in \mathbf{R}$. In the same way, composing with the 1-plots $t \mapsto (c, t)$, we get that $a(r) =_{\rm loc} a$. Therefore, $r \mapsto T_{u(r)}$ is locally constant, that is, the group of translations, equipped with the functional diffeology, is discrete. Finally, the action of the translations on $\mathcal{T}$ is free, the orbit of $(0, 0)$ is $\mathcal{T}$, and since the diffeology of $\mathcal{T}$ is not the discrete diffeology, we get an example, for the first question, where the diffeology of $\mathcal{Q}$ is strictly finer than the one of $\mathcal{O}$.

☞ **Exercise 61, p. 60** (A not so strong subduction). As an application of (art. 1.52), the underlying set of $\mathcal{Q}$ can be represented by the half-line $[0, \infty[$ equipped with the image of the diffeology of $\mathbf{R} \coprod \mathbf{R}^2$ by the map $p : x \mapsto \|x\|^2$ see Figure Sol.5. Let 0 and $(0, 0)$ be the zeros of $\mathbf{R}$ and $\mathbf{R}^2$, and let P be the plot $p \upharpoonright \mathbf{R}^2$. Then, $P(0, 0) = 0 \in [0, \infty[$. Let us assume now that P lifts locally at $(0, 0)$ along p, by a plot f such that $f(0, 0) = 0$. Thus, f takes its values in $\mathbf{R}$ and $p \circ f = P$, that is, $f(a, b)^2 = a^2 + b^2$, at least locally. Since f is continuous and vanishes only at $(0, 0)$, and since the complementary of $(0, 0)$ in $\mathbf{R}^2$ is connected, f keeps a constant sign, thus $f(a, b) = \pm\sqrt{a^2 + b^2}$. But none of these two cases is smooth at $(0, 0)$. Therefore, p is not a local subduction at the point 0.

☞ **Exercise 62, p. 60** (A powerset diffeology). 1) Let us check that the parametrizations defined by (♣) are a diffeology.

D1. Let $P : U \to \mathfrak{P}(X)$ be the constant parametrization $r \mapsto A \subset X$. Let $r_0 \in U$, and let $Q_0 \in \mathcal{D}$ such that $\mathrm{val}(Q_0) \subset A = P(r_0) = P(r)$ for all $r \in U$. Let $Q : U \to \mathcal{D}$ given by $Q(r) = Q_0$, for all $r \in \mathcal{D}$. This is a constant family of plots of X, hence smooth. Thus, P satisfies the condition (♣). Hence, the constant parametrizations satisfy (♣).

D2. The locality axiom is satisfied by construction: (♣) is a local property.

D3. Let us consider a parametrization $P : U \to \mathfrak{P}(X)$ satisfying (♣). Let $F : U' \to U$ be a smooth parametrization. Let $P' = P \circ F$. Let $r'_0 \in U'$ and $r_0 = F(r'_0)$. By

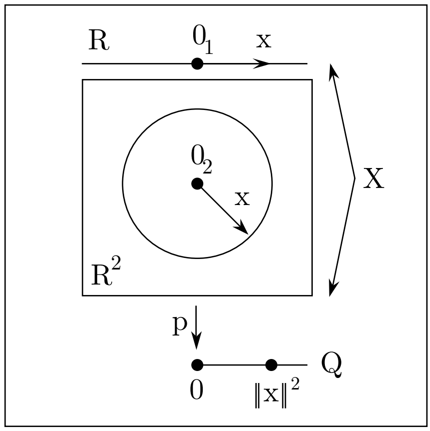

FIGURE Sol.5. The quotient $\mathcal{Q}$.

hypothesis, for every $Q_0 \in \mathcal{D}$ such that val$(Q_0) \subset P(r_0) = P'(r'_0)$, there exist an open neighborhood V of r_0 and a smooth family of plots $Q : V \to \mathcal{D}$ such that $Q(r_0) = Q_0$ and val$(Q(r)) \subset P(r)$, for all $r \in V$. Let us then define $V' = F^{-1}(V)$ and $Q' = Q \circ F$. Since F is smooth, V' is a domain, and since Q is a smooth family of plots, so is $Q \circ F$. Thus, for every $r'_0 \in U'$, for every $Q_0 \in \mathcal{D}$ such that val$(Q_0) \subset P'(r'_0)$, we found a smooth family of plots Q' such that $Q'(r'_0) = Q(r_0) = Q_0$ and val$(Q'(r')) =$ val$(Q \circ F(r')) =$ val$(Q(r)) \subset P(r) = P'(r')$, where $r = F(r')$. Therefore, $P' = P \circ F$ satisfies (♣).

2) Consider now the relation $\mathcal{R}$ from $\mathfrak{P}(X)$ to $\mathcal{D}$ defined by the inclusion

$$\mathcal{R} = \{(A, Q) \in \mathfrak{P}(X) \times \mathcal{D} \mid \mathrm{val}(Q) \subset A\}.$$

Let $P : U \to \mathfrak{P}(X)$ be a parametrization regarded as the relation

$$P = \{(r, A) \in U \times \mathfrak{P}(X) \mid A = P(r)\}.$$

The composite $\mathcal{R} \circ P$, also denoted by $P^*(\mathcal{R})$, is then given by

$$P^*(\mathcal{R}) = \{(r, Q) \in U \times \mathcal{D} \mid \mathrm{val}(Q) \subset P(r)\}.$$

The parametrization P is a plot of the powerset diffeology if and only if the first projection $\mathrm{pr}_1 : P^*(\mathcal{R}) \to U$ is everywhere a local subduction. Now, if $f : X \to Y$ is a map between two diffeological spaces, f is smooth if and only if the first projection from $\mathbf{f} = \{(x, y) \in X \times Y \mid y = f(x)\}$ to X is a subduction, which is equivalent to being a local subduction, in this case. Note that this construction gives us an idea of the difference between a *smooth relation* and a diffeological space X to another X'. Indeed, let $\mathcal{R} \subset X \times X'$ be a relation from X to X', we declare $\mathcal{R}$ smooth if for every plot P in def$(\mathcal{R}) = \mathrm{pr}_1(\mathcal{R}) \subset X$, for every $r \in U$ and every $(P(r), x') \in \mathcal{R}$, there exists a plot $Q : V \to X'$, with $V \subset U$, such that $(P(r'), Q(r')) \in \mathcal{R}$ for all $r' \in V$ and $Q(r) = x'$. In other words, $\mathrm{pr}_1 : \mathcal{R} \to$ def$(\mathcal{R})$ is a local subduction everywhere, where $\mathcal{R}$ and def$(\mathcal{R})$ are equipped with the subset diffeology. With this terminology, back to the powerset diffeology, a parametrization $P : U \to \mathfrak{P}(X)$ is a plot if the composite $\mathcal{R} \circ P$ is a smooth relation from U to $\mathcal{D}$.

3) Let us check now that the map $j : x \mapsto \{x\}$ is an induction. We consider the criterion for (art. 1.31). First of all let us remark that j is injective. Next, let us

check that the map j is smooth. Let $P : U \to X$ be a plot. Thus, $j \circ P(r) = \{P(r)\}$. Let $r_0 \in U$ and $Q_0 \in \mathcal{D}$ such that $\operatorname{val}(Q_0) \subset j \circ P(r_0) = \{P(r_0)\}$. So, $\operatorname{val}(Q_0)$ is the point $P(r_0)$ of X, that is, Q_0 is the constant plot $s \mapsto P(r_0)$. Let us then define, for every $r \in U$, $Q(r)$ as the constant plot $[s \mapsto P(r)]$, with $\operatorname{def}(Q(r)) = \operatorname{def}(Q_0)$. Since, for every $r \in U$, the parametrization $[(r, s) \mapsto Q(r)(s) = P(r)]$, defined on $U \times \operatorname{def}(Q_0)$, is clearly a plot of X, then Q is a smooth family of plots of X. Therefore, $j \circ P$ is a plot of $\mathfrak{P}(X)$, and j is smooth. Then, let us check that the map $j^{-1} : j(X) \to X$ is smooth. Let $P : U \to j(X)$ be a plot of the powerset diffeology. First of all, for every $r \in U$ there exists a unique point $q(r) \in X$ such that $P(r) = \{q(r)\}$. Then, since P is a plot of $\mathfrak{P}(X)$, for all $r_0 \in U$, for every plot Q_0 of X such that $\operatorname{val}(Q_0) \subset P(r_0) = \{q(r_0)\}$, there exist an open neighborhood V of r_0, and a smooth family of plots Q of X, such that $\operatorname{val}(Q(r)) \subset P(r)$, for all $r \in V$. Then, let us choose the 0-plot $Q_0 : \mathbf{R}^0 \to X$, with $Q_0(0) = q(r_0)$, that is, $\operatorname{val}(Q_0) = P(r_0)$. Thus, Q is necessarily a smooth family of 0-plots (see (art. 1.63)). But $\operatorname{val}(Q(r)) \subset P(r) = \{q(r)\}$ means exactly that $Q(r)(0) = q(r)$. Hence, $q \upharpoonright V = [r \mapsto Q(r)(0)]$ is a plot of X. Therefore, $q = j^{-1} \circ P$ is a plot of X, and j^{-1} is smooth. Finally, thanks to the criterion (art. 1.31), j is an induction from X into $\mathfrak{P}(X)$.

4) Let us show, now, that the Tzim-Tzum $\mathcal{T}$ is a plot of the powerset diffeology of $\mathfrak{P}(\mathbf{R}^2)$. Let $t_0 \in \mathbf{R}$ and consider $\mathcal{T}_{t_0}$. Let $Q_0 : U \to \mathcal{T}_{t_0}$ be a plot. If $t_0 < 0$, then we can choose $Q(t)(r) = Q_0(r)$ for $t \in]3t_0/2, t_0/2[$. Q is a smooth family of plots of $\mathbf{R}^2$ such that $Q(t_0) = Q_0$ and $\operatorname{val}(Q(t)) \subset \mathcal{T}(t) = \mathbf{R}^2$. For $t_0 = 0$, we choose $Q(t)(r) = (e^t + t/\|Q_0(r)\|)Q_0(r)$. Since $\mathcal{T}(0) = \mathbf{R}^2 - \{0\}$, $Q_0(r) \neq 0$ for all r, $Q(t)$ is well defined and Q is a smooth family of plots of $\mathbf{R}^2$. Next, note first that $Q(0) = Q_0$. Then, for $t \geq 0$, $\|Q(t)(r)\| = t + e^t\|Q_0(r)\| > t$, since $\|Q_0(r)\| > 0$, thus $\operatorname{val}(Q(t)) \subset \mathcal{T}(t)$. For $t < 0$ there is nothing to check since $\mathcal{T}(t) = \mathbf{R}^2$. Now, if $t_0 > 0$, we can choose $Q(t)(r) = [1 + (t - t_0)/t_0]Q_0(r)$. We have $Q(t_0) = Q_0$, then $\|Q(t)(r)\| = |1 + (t - t_0)/t_0|\|Q_0(r)\|$. But, $\|Q_0(r)\| > t_0$ implies $\|Q(t)(r)\| > |1 + (t - t_0)/t_0|t_0 = |t|$. Thus, if $t \geq 0$, then $\|Q(t)(r)\| > t$, for all r, that is, $\operatorname{val}(Q(t)) \subset \mathcal{T}(t)$. We exhausted all the cases, and therefore $\mathcal{T}$ is a plot of the powerset diffeology of $\mathfrak{P}(\mathbf{R}^2)$. As we can see, there is a blowing up for $t = 0$, the space opens up, and an empty bubble appears and grows with it. This is the reason for which we named this plot Tzim-Tzum.

Exercise 63, p. 61 (The powerset diffeology of the set of lines). First of all let us note that, given a line $\mathbf{D} \in \operatorname{Lines}(\mathbf{R}^n)$, the solution of the equation $j(u, x) = \mathbf{D}$, with $(u, x) \in TS^{n-1}$, has exactly two solutions:

$$u = \pm\frac{r - r'}{\|r - r'\|} \quad \text{and} \quad x = [\mathbf{1} - u\bar{u}]r,$$

where r and r' are any two different points of $\mathbf{D}$ and $[\mathbf{1} - u\bar{u}]$ is the projector orthogonal to u. By definition, $\bar{u}(v) = u \cdot v$, then $[\mathbf{1} - u\bar{u}](v) = v - (u \cdot v)u$. Let us prove now that the map j is smooth. Let $P : s \mapsto (u(s), x(s))$ be a plot of TS^{n-1}, defined on some domain U, that is, P is a smooth parametrization of $\mathbf{R}^n \times \mathbf{R}^n$ with values in TS^{n-1}. Let $P' = j \circ P : s \mapsto x(s) + \mathbf{R}u(s)$. We want to show that P' is a plot of $\mathfrak{P}(\mathbf{R}^n)$. Let then $s_0 \in U$, $u_0 = u(s_0)$, $x_0 = x(s_0)$, $\mathbf{D}_0 = P'(s_0) = j(u_0, x_0)$ and $Q_0 : W \to \mathbf{D}_0$ be some smooth parametrization in $\mathbf{D}_0$. Since Q_0 is a plot of $\mathbf{D}_0$, for any $w \in W$, $Q_0(w) - x_0$ is proportional to u_0, that is, $Q_0(w) - x_0 = \tau u_0$. But since $u_0 \cdot x_0 = 0$, $Q_0(w) - x_0 = \tau u_0$ implies $\tau = u_0 \cdot (Q_0(w) - x_0) = u_0 \cdot Q_0(w)$. Hence, defining $\tau(w) = u_0 \cdot Q_0(w)$, we get $Q_0(w) = x_0 + \tau(w)u_0$, where $\tau \in \mathcal{C}^\infty(W, \mathbf{R})$.

Let us define now

$$Q = [s \mapsto [w \mapsto x(s) + \tau(w)u(s)]], \quad \text{where} \quad s \in U \quad \text{and} \quad w \in W.$$

Since x, u and τ are smooth, Q is a plot of the diffeology of $\mathbf{R}^n$ and satisfies $Q(s_0) = Q_0$. Hence, P' is a plot of $\mathfrak{P}(\mathbf{R}^n)$. Therefore, j is smooth. Let us now prove that j is a subduction onto its image, that is, onto the space Lines$(\mathbf{R}^n)$. Let $P : U \to \text{Lines}(\mathbf{R}^n)$ be a plot and $s_0 \in U$. Since $P(s_0)$ is a line of $\mathbf{R}^n$, there exists $(u_0, x_0) \in TS^{n-1}$ such that $P(s_0) = x_0 + \mathbf{R}u_0$. Let $Q_0 = [t \mapsto x_0 + tu_0]$, with $t \in \mathbf{R}$, Q_0 is a plot of $\mathbf{R}^n$ such that $\text{val}(Q_0) \subset P(s_0)$. Hence, since P is a plot for the powerset diffeology of $\mathbf{R}^n$, there exist an open neighborhood V of s_0, and a plot Q of the smooth diffeology of $\mathbf{R}^n$, such that $Q(s_0) = [t \mapsto x_0 + tu_0]$ and $\text{val}(Q(s)) \subset P(s)$. Let us choose $t = 0 \in \text{def}(Q(s_0))$, since Q is a plot of the smooth diffeology of $\mathbf{R}^n$, there exists an open neighborhood W of s_0 and there exists $\varepsilon > 0$ such that $(t, s) \mapsto Q(s)(t)$, defined on $W \times]-\varepsilon, +\varepsilon[$, is smooth. Let us then define, for all $s \in W$,

$$\nu(s) = \left.\frac{\partial Q(s)(t)}{\partial t}\right|_{t=0}.$$

Since Q is smooth, the parametrization ν is smooth. We have $\nu(s_0) = u_0 \neq 0$. Thus, there exists an open neighborhood W' of s_0 on which ν does not vanish. Therefore, the map

$$u : s \mapsto \frac{\nu(s)}{\|\nu(s)\|}$$

is smooth on W'. Moreover, by construction, $u(s)$ directs the line $P(s)$. Now, let

$$x(s) = Q(s)(0) - [u(s) \cdot Q(s)(0)]u(s).$$

Since $Q(s)(0) \in P(s)$ and $u(s)$ directs $P(s)$, the point $x(s)$ belongs to $P(s)$, and by construction $u(s) \cdot x(s) = 0$. So, the parametrization of TS^{n-1} defined by $\phi : s \mapsto (u(s), x(s))$ is smooth and satisfies $j \circ \phi = P$. Hence, ϕ is a local lift of P along j, defined on W'. Combined with the surjectivity and the differentiability of j, this is the criterion for j to be a subduction (art. 1.48) from TS^{n-1} onto its image Lines$(\mathbf{R}^n)$. Therefore, the set of lines is diffeomorphic to the quotient $TS^{n-1}/\{\pm 1\}$, where -1 acts by reversing the orientation, that is, $\pm(u, x) = (\pm u, x)$.

☞ **Exercise 64, p. 64** (The diffeomorphisms of the half-line). Let us prove first that any diffeomorphism f of $\Delta_\infty = [0, \infty[\subset \mathbf{R}$ satisfies the three points.

1) Since the dimension map is invariant under diffeomorphism (art. 2.24) and since the origin is the only point where the dimension is infinite, as shown in Exercise 51, p. 50, f fixes the origin, $f(0) = 0$.

2) Since $f(0) = 0$ and f is a bijection, we have $f(]0, \infty[) =]0, \infty[$. Now, since the restriction of a diffeomorphism to any subset is a diffeomorphism of this subset onto its image, for the subset diffeology (art. 1.33), we have $f \restriction]0, \infty[\in \text{Diff}(]0, \infty[)$. Let us recall that the induced diffeology on the open interval is the standard diffeology (art. 1.9). Moreover, since $f(0) = 0$, restricted to $]0, \infty[$, f is necessarily strictly increasing.

3) Since f is smooth, by the very definition of differentiability (art. 1.14), for any smooth parametrization P of the interval $[0, \infty[$, the composite $f \circ P$ is smooth, in particular for $P = [t \mapsto t^2]$. Hence, the map $\varphi : t \mapsto f(t^2)$ defined on $\mathbf{R}$ with values in $[0, \infty[$ is smooth. Now, by theorem 1 of [**Whi43**], since φ is smooth, f can be extended to an open neighborhood of $[0, \infty[$ by a smooth function. Hence, f

is continuously differentiable at the origin. Moreover since f is a diffeomorphism of Δ_∞, what has been said for f can be said for f^{-1}. And, since $(f^{-1})'(0) = 1/f'(0)$ and f is increasing, we have $f'(0) > 0$. Conversely, a function f satisfying the three above conditions can be extended by a smooth function to some open neighborhood of $[0, \infty[$. Hence, f is the restriction to $[0, \infty[$ of a smooth function $\tilde{f}$ such that $\tilde{f}(0) = 0$, $\tilde{f}$ is strictly increasing on $[0, \infty[$, and $\tilde{f}'(0) > 0$, then f is smooth for the subset diffeology as well as its inverse, that is, f is a diffeomorphism of Δ_∞.

☞ **Exercise 65, p. 66** (Vector space of maps into $\mathbf{K}^n$). Let us recall that a parametrization $P : U \to E = \mathcal{C}^\infty(X, \mathbf{K})$ is a plot for the functional diffeology if and only if, for every plot $Q : V \to X$, the parametrization $[(r, s) \mapsto P(r)(Q(s))]$ is a plot of $\mathbf{K}^n$ (art. 1.57). Let (P, P') be a plot of the product $E \times E$ (art. 1.55). The parametrization $[r \mapsto P(r) + P'(r)]$ satisfies, for any plot $Q : V \to X$, $[(r, s) \mapsto (P(r) + P'(r))(Q(s)) = P(r)(Q(s)) + P'(r)(Q(s))] = [(r, s) \mapsto (P(r)(Q(s)), P'(r)(Q(s))) \mapsto P(r)(Q(s)) + P'(r)(Q(s))]$. But this is the composite of two plots, thus a plot. Hence, the addition in E is smooth. Now, for any $\lambda \in \mathbf{K}$, $[(r, s) \mapsto \lambda \times P(r)(Q(s))] = [(r, s) \mapsto (\lambda, P(r)(Q(s))) \mapsto \lambda \times P(r)(Q(s))]$ also is the composite of two plots, thus a plot. Therefore, the space $\mathcal{C}^\infty(X, \mathbf{K})$, equipped with the functional diffeology, is a diffeological vector space.

☞ **Exercise 66, p. 71** (Smooth is fine diffeology). By the very definition of smooth parametrizations in $\mathbf{K}^n$, every plot $P : U \to E$ splits over the canonical basis $(\boldsymbol{e}_i)_{i=1}^n$, that is, $P : r \mapsto \sum_{i=1}^n P_i(r)\boldsymbol{e}_i$, where $P_i \in \mathcal{C}^\infty(U, \mathbf{K})$, $i = 1, \ldots, n$.

☞ **Exercise 67, p. 71** (Finite dimensional fine spaces). Let $r_0 \in U$ be any point. By definition of the fine diffeology, there exist an open neighborhood V of r_0, a finite family $\{\phi_\alpha\}_{\alpha\in A}$ of smooth parametrizations of $\mathbf{K}$, defined on V, and a family $\{u_\alpha\}_{\alpha\in A}$ of vectors of E, such that $P \restriction V : r \mapsto \sum_{\alpha\in A} \phi_\alpha(r)u_\alpha$. Now, let $\mathcal{B} = \{\boldsymbol{e}_i\}_{i=1}^n$ be a basis of E. Each u_α writes $\sum_{i=1}^n u_\alpha^i \boldsymbol{e}_i$, where the u_α^i belong to $\mathbf{K}$. Hence, $P \restriction V : r \mapsto \sum_{\alpha\in A} \sum_{i=1}^n \phi_\alpha(r)u_\alpha^i \boldsymbol{e}_i = \sum_{i=1}^n \phi_i(r)\boldsymbol{e}_i$, where $\phi_i(r) = \sum_{\alpha\in A} \phi_\alpha(r)u_\alpha^i$. But the ϕ_i are still smooth parametrizations of $\mathbf{K}$, thus $P \restriction V : r \mapsto \sum_{i=1}^n \phi_i(r)\boldsymbol{e}_i$ with $\phi_i \in \mathcal{C}^\infty(V, \mathbf{K})$. Now, let V and V' two such domains on which the plot P writes $P \restriction V : r \mapsto \sum_{i=1}^n \phi_i(r)\boldsymbol{e}_i$ with $\phi_i \in \mathcal{C}^\infty(V, \mathbf{K})$, and $P \restriction V' : r \mapsto \sum_{i=1}^n \phi'_i(r)\boldsymbol{e}_i$ with $\phi'_i \in \mathcal{C}^\infty(V', \mathbf{K})$. Let $r \in V \cap V'$, we have $P(r) = P \restriction V(r) = P \restriction V'(r)$, that is, $\sum_{i=1}^n \phi_i(r)\boldsymbol{e}_i = \sum_{i=1}^n \phi'_i(r)\boldsymbol{e}_i$, or $\sum_{i=1}^n (\phi_i(r) - \phi'_i(r))\boldsymbol{e}_i = 0$, but since $\mathcal{B} = \{\boldsymbol{e}_i\}_{i=1}^n$ is a basis of E, $\phi_i(r) = \phi'_i(r)$. Therefore, the ϕ_i have a unique smooth extension on U such that $P(r) = \sum_{i=1}^n \phi_i(r)\boldsymbol{e}_i$, for all $r \in U$, and the ϕ_i belong to $\mathcal{C}^\infty(U, \mathbf{K})$. Finally, since linear isomorphisms between fine vector spaces are smooth isomorphisms (art. 3.9), the basis $\mathcal{B}$ realizes a smooth isomorphism from $\mathbf{K}^n$ to E.

☞ **Exercise 68, p. 71** (The fine topology). The diffeology of E is generated by the linear injections $j : \mathbf{K}^n \to E$ (art. 3.8), where n runs over $\mathbf{N}$, hence Ω is D-open if and only if its preimage by each of these injections is open in $\mathbf{K}^n$. Or, equivalently, if the intersection of Ω with any vector subspace F, of finite dimension, is open for the smooth topology of F.

☞ **Exercise 69, p. 74** (Fine Hermitian vector spaces). Let E be a fine diffeological real vector space. Let $\cdot$ be any Euclidean product. Let $(P, P') : U \to E \times E$ be a plot, that is, P and P' are plots of E. Let $r_0 \in U$. There exist two local families, $(\phi_\alpha, u_\alpha)_{\alpha\in A}$ and $(\phi'_{\alpha'}, u'_{\alpha'})_{\alpha'\in A'}$, defined on some open neighborhood V of r_0, such that $P(r) = \sum_{\alpha \in A} \phi_\alpha(r) u_\alpha$ and $P'(r) = \sum_{\alpha' \in A'} \phi'_{\alpha'}(r) u'_{\alpha'}$, for all r in V (art. 3.7). Thus, $P(r) \cdot P'(r) = \sum_{\alpha \in A} \sum_{\alpha' \in A'} \phi_\alpha(r)\phi'_{\alpha'}(r) u_\alpha \cdot u'_{\alpha'}$, for all $r \in V$. But this is a finite linear combination of smooth parametrizations of $\mathbf{R}$, thus smooth. Now, since the map $r \mapsto P(r) \cdot P'(r)$ is locally smooth, it is smooth. Therefore, the Euclidean product $\cdot$ is smooth, and $(E, \cdot)$ is an Euclidean diffeological vector space. The same arguments hold for the Hermitian case.

Exercise 70, p. 74 (Finite dimensional Hermitian spaces). Let E be a Euclidean diffeological vector space of dimension n. Let us denote by $\mathcal{D}$ its diffeology. Let $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ be an orthonormal basis of E. Let $P : U \to E$ be a plot of E. For every $r \in U$, $P(r) = \sum_{k=1}^{n} (\boldsymbol{e}_k \cdot P(r))\boldsymbol{e}_k$, since, by hypothesis, the maps $P_k : r \mapsto \boldsymbol{e}_k \cdot P(r)$ are smooth, the plot P is a plot of the fine diffeology. Hence, the diffeology $\mathcal{D}$ is finer that the fine diffeology, but the fine diffeology is the finest vector space diffeology. Therefore $\mathcal{D}$ is the fine diffeology. The same argument holds for the Hermitian case. The (art. 3.2) states that the coarse diffeology is always a vector space diffeology. For finite dimensional spaces, the existence of a smooth Euclidean structure reduces the set of vector space diffeologies to the unique fine diffeology. In other words, there exists only one kind of finite Euclidean, or Hermitian, diffeological vector space of dimension n, the class of $(\mathbf{R}^n, \cdot)$, or $(\mathbf{C}^n, \cdot)$.

Exercise 71, p. 74 (Topology of the norm and D-topology). Let B be the open ball, for the topology of the norm, centered at $x \in E$, and with radius ε. Let $P : U \to E$ be some plot. The preimage of B by the plot P is the preimage of $]-\infty, \varepsilon^2[$ by the map $r \mapsto \|x - P(r)\|^2$, but this map is smooth, then D-continuous (art. 2.9). Hence, the ball B is D-open. Thus, thanks to the differentiability of translations and dilations, any open set for the topology of the norm is D-open. In other words, the topology of the norm is finer than the D-topology.

Exercise 72, p. 74 (Banach's diffeology). Let us denote by $\mathcal{C}^\infty_E$ the set of class $\mathcal{C}^\infty$ parametrizations of E. Let us check first that $\mathcal{C}^\infty_E$ is a diffeology.

D1. Let $x \in E$ and $\mathbf{x} : r \mapsto x$ be the constant parametrization with value x. Then, $D(\mathbf{x})(r) = 0$ for all r. Therefore, $\mathcal{C}^\infty_E$ contains the constants.

D2. Belonging to $\mathcal{C}^\infty_E$ is by definition a local property.

D3. Let $P : U \to E$ be an element of $\mathcal{C}^\infty_E$ and $F : V \to U$ be a smooth parametrization. For the real domains equipped with the usual Euclidean norm, to be smooth and to be of class $\mathcal{C}^\infty$, in the sense of Banach spaces, coincide. Hence, $D(P \circ F)(s) = D(P)(r) \circ D(F)(s)$, where $r = F(s)$, and for any $s \in V$. Since $D(P)(r)$ and $D(F)(s)$ are together of class $\mathcal{C}^\infty$, the composite also is of class $\mathcal{C}^\infty$, and $P \circ F$ belongs to $\mathcal{C}^\infty_E$.

Therefore, $\mathcal{C}^\infty_E$ is a diffeology. Now, the fact that this diffeology is a vector space diffeology comes from the linear properties of the tangent map: $D(P + P')(r) = D(P)(r) + D(P')(r)$ and $D(\lambda \times P)(r) = \lambda \times D(P)(r)$. Next, let E and F be two Banach spaces, equipped with the Banach diffeology. Let $f : E \to F$ be a map, smooth for the Banach diffeology. Then f takes plots of E to plots of F, in particular smooth curves, since to be a smooth curve for the Banach diffeology means exactly to be Banach-smooth. Thus, thanks to Boman's theorem, f is Banach-smooth. Conversely, let f be Banach-smooth. Since every n-plot $P : U \to E$ is by definition Banach-smooth, the composite $f \circ P$ is Banach-smooth, that is, a plot of F. Hence, f is smooth for the Banach diffeology. Therefore, the functor which associates with every Banach space its diffeology is a full faithful functor.

☞ **Exercise 73, p. 75** ($\mathcal{H}_\mathbf{C}$ isomorphic to $\mathcal{H}_\mathbf{R} \times \mathcal{H}_\mathbf{R}$). Since for $Z = X + iY$, $Z_k^* Z_k = X_k^2 + Y_k^2$, the map ψ is an isometry. The bijectivity and the linearity of ψ are obvious. And since the diffeology is fine, to be linear implies to be smooth (art. 3.9). Therefore, ψ is a smooth linear isomorphism.

☞ **Exercise 74, p. 84** (The irrational torus is not a manifold). Let us assume that $T_\alpha = \mathbf{R}/[\mathbf{Z} + \alpha\mathbf{Z}]$ is a manifold. We know that the dimension of the torus is 1 (Exercise 49, p. 50). Thus, there should exist a family of inductions from some 1-domains to $\mathbf{R}$, satisfying the criterion (art. 4.6). Let $j : I \to \mathbf{R}$ be such an induction, where I is some interval. Since $j(I)$ cuts each orbit of $\mathbf{Z} + \alpha\mathbf{Z}$ in at most one point, and since each orbit is dense, if $j(I)$ is not empty, then $j(I)$ is just a point. But then it is not injective an cannot be an induction. Hence, T_α is not a manifold. Note that, since the D-topology contains only one nonempty D-open, T_α itself (art. 55), and since the values of any local diffeomorphism is D-open, if T_α would be a manifold it would be diffeomorphic to some 1-domain. But this cannot be for, roughly speaking, the same reasons as above.

☞ **Exercise 75, p. 84** (The sphere as paragon). 1) The map F_x is injective, its inverse is given by

$$F_x^{-1} : S_x^n \to E_x \quad \text{with} \quad u \mapsto v = \frac{1}{1 + \bar{u}x}\,[1 - x\bar{x}]\,u,$$

where $\bar{x}$ is the transposed of x, that is, the linear map from $\mathbf{R}^{n+1}$ into $\mathbf{R}$ defined by $\bar{x}y = x \cdot y$ for any $y \in \mathbf{R}^{n+1}$, and $x\bar{x}$ is the map $x\bar{x} : u \mapsto (\bar{x}u)x = (x \cdot u)x$. Since F_x is a sum and product of smooth maps (the denominator $1 + \|v\|^2$ never vanishes), it is clearly smooth. The image of F_x is the sphere S^n deprived of the point $-x$. Let us denote $S_x^n = \mathrm{val}(F_x) = S^n - \{-x\}$. The map F_x^{-1}, restricted to S_x^n, is clearly smooth, because it is the restriction of a smooth map defined on the domain $\Omega_x = \{u \in \mathbf{R}^{n+1} \mid u \cdot x \neq -1\}$ to the subspace S_x^n. Moreover S_x^n is open for the D-topology of S^n. Indeed, the pullback $P^{-1}(S_x^n)$, by any plot P of S^n, is equal to the pullback by $P^{-1}(\Omega_x)$, where P is regarded as a plot of $\mathbf{R}^{n+1}$. Since P is a plot of the smooth diffeology of $\mathbf{R}^{n+1}$, and Ω_x is a domain of $\mathbf{R}^{n+1}$, $P^{-1}(\Omega_x)$ is a domain. Therefore, thanks to (art. 2.10), F_x is a local diffeomorphism mapping E_x onto S_x^n.

2) Thus, $\bigcup_{x \in S^n} F_x(E_x) = S^n$, and for every $x \in S^n$, F_x is a local diffeomorphism with E_x. But $E_x \sim \mathbf{R}^n$, hence there exists a family of local diffeomorphisms from $\mathbf{R}^n$ to S^n whose values cover S^n. Therefore S^n is a manifold of dimension n.

3) The maps F_N and F_{-N} are local diffeomorphisms from $\mathbf{R}^n = N^\perp$ to S^n. Moreover, $\mathrm{val}(F_N) \cup \mathrm{val}(F_{-N}) = S^n$. Hence, $\{F_N, F_{-N}\}$ is a generating family of the diffeology of S^n (art. 4.2), that is, an atlas.

☞ **Exercise 76, p. 92** (The space of lines in $\mathcal{C}^\infty(\mathbf{R}, \mathbf{R})$). 1) It is immediate to check that if (f_1, g_1) and (f_2, g_2) belong to $\mathcal{E} \times (\mathcal{E} - \{0\})$ and define the same line L, then necessarily $f_2 = f_1 + \lambda g_1$ and $g_2 = \mu g_1$, where $\lambda \in \mathbf{R}$ and $\mu \in \mathbf{R} - \{0\}$. Now, let $L = \{f + sg \mid s \in \mathbf{R}\}$ and $r \in \mathbf{R}$. If $g(r) \neq 0$, then let $\beta = g/g(r)$ and $\alpha = f - f(r)\beta$. Hence, $\alpha(r) = 0$, $\beta(r) = 1$ and $L = F_r(\alpha, \beta)$. The fact that $g(r) \neq 0$ is a property of the line L, indeed for any other pair (f', g') defining the same line, $g' = \mu g$ with $\mu \neq 0$, and then $g'(r) \neq 0$. Let us denote this space, defined by $g(r) \neq 0$, by $\mathrm{Lines}_r(\mathcal{E})$. Thus, $\mathrm{val}(F_r) = \mathrm{Lines}_r(\mathcal{E})$. Note next that F_r is injective. Indeed, if (α_1, β_1) and (α_2, β_2) belong to $\mathcal{E}_r^0 \times \mathcal{E}_r^1$ and define the same line, then

$\beta_2 = \mu\beta_1$, which implies $\mu = 1$, and $\alpha_2 = \alpha_1 + \lambda\beta_1$, which implies $\lambda = 0$. Therefore F_r is a bijection from $\mathcal{E}_r^0 \times \mathcal{E}_r^1$ onto its image $\mathrm{Lines}_r(\mathcal{E})$. Moreover, since F_r is the restriction of a smooth map to $\mathcal{E}_r^0 \times \mathcal{E}_r^1 \subset \mathcal{E} \times (\mathcal{E} - \{0\})$, it is a smooth injection. Let us check now that F_r is an induction. Let $u \mapsto L_u$ be a plot of $\mathrm{Lines}_r(\mathcal{E})$, there exists locally a smooth parametrization $u \mapsto (f_u, g_u)$ in $\mathcal{E} \times (\mathcal{E} - \{0\})$, with $g_u(r) \neq 0$, such that $L_u = \{f_u + s g_u \mid s \in \mathbf{R}\}$. The map $(f_u, g_u) \mapsto (\alpha_u, \beta_u)$, defined by $\alpha_u = f_u - f_u(r)\beta_u$ and $\beta_u = g_u/g_u(r)$, being smooth, the parametrization $u \mapsto (\alpha_u, \beta_u)$ is a plot of $\mathcal{E}_r^0 \times \mathcal{E}_r^1$ such that $L_u = F_r(\alpha_u, \beta_u)$. Therefore, F_r is an induction. Finally, let us check that $\mathrm{Lines}_r(\mathcal{E})$ is D-open. Thanks to (art. 2.12), we just need to check that the subset $\mathcal{O}_r \subset \mathcal{E} \times (\mathcal{E} - \{0\})$ of (f, g) such that $g(r) \neq 0$ is D-open. Let $P : u \mapsto (f_u, g_u)$ be a plot of $\mathcal{E} \times (\mathcal{E} - \{0\})$, and let $\phi(u) = g_u(r)$. Since $(u, r) \mapsto g_u(r)$ is smooth, ϕ is smooth. Thus, $P^{-1}(\mathcal{O}_r) = \{u \in \mathrm{def}(P) \mid g_u(r) \neq 0\} = \phi^{-1}(\mathbf{R} - 0)$, is open. Therefore, $\mathcal{O}_r$ is D-open and then F_r is a local diffeomorphism from $\mathcal{E}_r^0 \times \mathcal{E}_r^1$ to $\mathrm{Lines}(\mathcal{E})$, with values $\mathrm{Lines}_r(\mathcal{E})$.

2) Since for every line L there exists (f, g) such that $L = \{f + sg \mid s \in \mathbf{R}\}$, with $g \neq 0$, there exists some $r \in \mathbf{R}$ such that $g(r) \neq 0$, thus the union of all the subsets $\mathrm{Lines}_r(\mathcal{E})$ covers $\mathrm{Lines}(\mathcal{E})$. Now, every $\mathcal{E}_r^0$ is isomorphic to $\mathcal{E}_0^0$, by $\alpha \mapsto \alpha \circ T_r = [r' \mapsto \alpha(r' + r)]$, and $\mathcal{E}_r^1$ also is isomorphic (as an affine space) to $\mathcal{E}_0^0$, by $\beta \mapsto \beta \circ T_r - 1$. Therefore, $\mathrm{Lines}(\mathcal{E})$ is a diffeological manifold modeled on $\mathcal{E}_0^0 \times \mathcal{E}_0^0$, where $\mathcal{E}_0^0 = \{f \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}) \mid f(0) = 0\}$.

3) The space of maps from $\{1, 2\}$ to $\mathbf{R}$ is diffeomorphic to $\mathbf{R}^2$. Let $f \simeq (x_1, x_2)$ and $g \simeq (u_1, u_2)$, $g \neq 0$ means that $(u_1, u_2) \neq (0, 0)$. Now, there are four spaces $\mathcal{E}_r^i$, where $r = 1, 2$ and $i = 0, 1$, that is, $\mathcal{E}_1^0 = \{(0, x_2) \mid x_2 \in \mathbf{R}\}$, $\mathcal{E}_1^1 = \{(1, u_2) \mid x_2 \in \mathbf{R}\}$, $\mathcal{E}_2^0 = \{(x_1, 0) \mid x_1 \in \mathbf{R}\}$, $\mathcal{E}_2^1 = \{(u_1, 1) \mid u_1 \in \mathbf{R}\}$. Thus,

$$F_1\left(\begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \begin{pmatrix} 1 \\ u_2 \end{pmatrix}\right) = \left\{\begin{pmatrix} s \\ x_2 + su_2 \end{pmatrix} \mid s \in \mathbf{R}\right\},$$

$$F_2\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 \\ 1 \end{pmatrix}\right) = \left\{\begin{pmatrix} x_1 + su_1 \\ s \end{pmatrix} \mid s \in \mathbf{R}\right\}.$$

The chart F_1 maps $\mathbf{R}^2$ to the subspace of lines not parallel to the x_2-axis and the chart F_2 maps $\mathbf{R}^2$ to the subspace of lines not parallel to the x_1-axis. We have seen that the set of unparametrized and nonoriented lines in $\mathbf{R}^2$ is diffeomorphic to the Möbius strip (Exercise 41, p. 39, question 4), thus the set $\mathcal{A} = \{F_1, F_2\}$ is an atlas of this famous manifold.

☞ **Exercise 77, p. 93** (The Hopf S^1-bundle). Let $J : \mathcal{S}_\mathbf{C} \to \mathcal{H}_\mathbf{C}$ be the natural inclusion. Since $J(z \times Z) = z \times J(Z)$, the injection J projects onto a map $j : \mathcal{S}_\mathbf{C}/U(1) \to \mathcal{P}_\mathbf{C} = \mathcal{H}_\mathbf{C}^\star/\mathbf{C}^\star$, according to the following commuting diagram, where $\pi_\mathcal{S}$ and $\pi_\mathcal{H}$ are the natural projections. Since J is smooth, and since $\pi_\mathcal{S}$ is a subduction, j is smooth. The map j is obviously injective. Then, since for every $Z \in \mathcal{H}_\mathbf{C}^\star$, $Z/\|Z\| \in \mathcal{S}_\mathbf{C}$, j is surjective. Now, since the map $\rho : Z \mapsto Z/\|Z\|$ from $\mathcal{H}_\mathbf{C}^\star$ to $\mathcal{S}_\mathbf{C}$ is smooth ($Z \neq 0$), j^{-1} is smooth. Indeed, a plot P of $\mathcal{P}_\mathbf{C}$ lifts locally to $\mathcal{H}_\mathbf{C}^\star$. Composing the local lift with ρ we get a lift in $\mathcal{S}_\mathbf{C}$. Therefore, j is a diffeomorphism.

$$\begin{tikzcd} \mathcal{S}_{\mathbf{C}} \arrow[r, "J"] \arrow[d, "\pi_{\mathcal{S}}"'] & \mathcal{H}^{\star}_{\mathbf{C}} \arrow[d, "\pi_{\mathcal{H}}"] \\ \mathcal{S}_{\mathbf{C}}/\mathrm{U}(1) \arrow[r, "j"'] & \mathcal{P}_{\mathbf{C}} = \mathcal{H}^{\star}_{\mathbf{C}}/\mathbf{C}^{\star} \end{tikzcd}$$

Now, let us transpose this construction to $\mathcal{H}_{\mathbf{R}} \times \mathcal{H}_{\mathbf{R}}$. The sphere $\mathcal{S}_{\mathbf{C}}$ is diffeomorphic to the sphere $\mathcal{S} = \left\{(X, Y) \in \mathcal{H}_{\mathbf{R}} \times \mathcal{H}_{\mathbf{R}} \mid \|X\|^2 + \|Y\|^2 = 1\right\}$. The group $\mathrm{U}(1) = \{z = \cos(\theta) + i\sin(\theta) \mid \theta \in \mathbf{R}\}$ is equivalent to the group

$$\mathrm{SO}(2, \mathbf{R}) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \;\middle|\; \theta \in \mathbf{R} \right\}.$$

With this identification, the action of $\mathrm{U}(1)$ on $\mathcal{S}_{\mathbf{C}}$ transmutes to the following action of $\mathrm{SO}(2, \mathbf{R})$ on $\mathcal{S}$,

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Hence, the projective space $\mathcal{P}_{\mathbf{C}}$ also is diffeomorphic to $\mathcal{S}/\mathrm{SO}(2, \mathbf{R})$.

Exercise 78, p. 93 ($\mathrm{U}(1)$ as subgroup of diffeomorphisms). First of all, let us note that j is an injective homomorphism (a monomorphism), $j(zz') = j(z) \circ j(z')$, and $j(z)^{-1} = j(z^{-1})$. Moreover, j is $\mathbf{C}$-linear, thus smooth (art. 3.9). Now, let $P : U \to \mathrm{U}(1)$ be a parametrization such that $j \circ P$ is smooth for the functional diffeology of $\mathrm{GL}(\mathcal{H}_{\mathbf{C}})$. Let us apply the criterion of (art. 3.12) to the plot $j \circ P$. Let $r_0 \in U$, let $Z \in \mathcal{H}_{\mathbf{C}}$, $Z \neq 0$, and let $F = \mathbf{C}Z \subset \mathcal{H}_{\mathbf{C}}$ be the complex line generated by Z. There exist an open neighborhood V of r_0 and a finite dimensional subspace $F' \subset \mathcal{H}_{\mathbf{C}}$ such that: $(j \circ P) \upharpoonright V \in \mathrm{L}(F, F')$ and $r \mapsto (j \circ P(r)) \upharpoonright F$ is a plot of $\mathrm{L}(F, F')$. But clearly $F' = F$. Now, let $\mathcal{Z} : \mathbf{C} \to F$ be a basis, a $\mathbf{C}$-linear isomorphism. In this basis, the parametrization $r \mapsto j(P(r))$ becomes the multiplication by $P(r)$, that is, $\mathcal{Z}^{-1} \circ j(P(r)) \circ \mathcal{Z} : z \mapsto P(r)z$. Thus, $[r \mapsto [z \mapsto P(r)z]]$ being smooth means just that $P : U \to \mathrm{U}(1)$ is smooth. Therefore j is an induction. In other words, the multiplication by an element of $\mathrm{U}(1)$ is a subgroup of $\mathrm{U}(\mathcal{H})$ isomorphic, as diffeological group (art. 7.1), to $\mathrm{U}(1)$.

Exercise 79, p. 99 (Reflexive diffeologies). 1) The coarsest diffeology $\mathcal{D}$ is the intersection of the pullbacks of the smooth diffeology $\mathcal{C}^{\infty}_{\star}(\mathbf{R})$ of $\mathbf{R}$, by the elements of the family $\mathcal{F}$,

$$\mathcal{D} = \bigcap_{f \in \mathcal{F}} f^*(\mathcal{C}^{\infty}_{\star}(\mathbf{R})).$$

The plots of this diffeology are explicitly defined by,

($\diamondsuit$) $P : U \to X$ belongs to $\mathcal{D}$ if and only if, for all $f \in \mathcal{F}$, $f \circ P \in \mathcal{C}^{\infty}(U, \mathbf{R})$.

Indeed, by definition of the pullback diffeology, $f^*(\mathcal{C}^{\infty}_{\star}(\mathbf{R}))$ is the coarsest diffeology such that f is smooth (art. 1.26). Thus, every diffeology $\mathcal{D}$ such that $\mathcal{F} \subset \mathcal{D}(X, \mathbf{R})$ is contained in $f^*(\mathcal{C}^{\infty}_{\star}(\mathbf{R}))$, and therefore, is contained in their intersection over the $f \in \mathcal{F}$. Since the intersection of any family of diffeologies is a diffeology (art. 1.22), this intersection is a diffeology, and by construction, the coarsest. About the finest diffeology, we know that for the discrete diffeology on X, $\mathcal{C}^{\infty}(X_{\circ}, \mathbf{R}) = \mathrm{Maps}(X, \mathbf{R})$; see Exercise 9, p. 14.

2) Let $\mathcal{D}$ be the diffeology subordinated to $\mathcal{F}$, and $\mathcal{D}'$ be the diffeology subordinated to $\mathcal{D}(X,\mathbf{R})$. By construction, $\mathcal{F} \subset \mathcal{D}(X,\mathbf{R})$ and $\mathcal{D}(X,\mathbf{R}) \subset \mathcal{D}'(X,\mathbf{R})$, then $\mathcal{F} \subset \mathcal{D}'(X,\mathbf{R})$, but $\mathcal{D}$ is the coarsest diffeology such that $\mathcal{F} \subset \mathcal{D}(X,\mathbf{R})$, thus $\mathcal{D}' \subset \mathcal{D}$. Then, by definition of $\mathcal{D}(X,\mathbf{R})$, for all $\phi \in \mathcal{D}(X,\mathbf{R})$, for all plots $P \in \mathcal{D}$, $\phi \circ P$ is smooth. But this is exactly the condition $(\diamondsuit)$ for P to belong to $\mathcal{D}'$, so $\mathcal{D} \subset \mathcal{D}'$. Therefore $\mathcal{D}' = \mathcal{D}$, and the diffeology subordinated to $\mathcal{F}$ is reflexive.

3) Let X be a manifold, let $\mathcal{D}$ be its diffeology, and let $n = \dim(X)$. Let us prove first that for every $x_0 \in X$ and every local smooth function $f : \mathcal{O}' \to \mathbf{R}$, defined on an D-open neighborhood $\mathcal{O}'$ of x_0, there exists a smooth function $\bar{f} : X \to \mathbf{R}$ which coincides with f on a D-open neighborhood $\mathcal{O} \subset \mathcal{O}'$ of x_0. Indeed, let $F : V \to X$ be a chart of X, and $\xi_0 \in V$, such that $F(\xi_0) = x_0$. We can assume that $F(V) \subset \mathcal{O}'$. There exist two balls $\mathcal{B} \subset \mathcal{B}' \subset V$, centered at ξ_0, and a smooth real function $\varepsilon : V \to \mathbf{R}$ such that ε is equal to 1 on $\mathcal{B}$ and equal to 0 outside $\mathcal{B}'$. Then, the local real function $x \mapsto \varepsilon(F(x)) \times f(x)$, defined on $F(V) \subset \mathcal{O}'$, can be extended smoothly on X by 0. This extension $\bar{f}$ coincides with f on $\mathcal{O} = F(\mathcal{B})$. Now, let $P : U \to X$ be an element of $\mathcal{D}'$, $r_0 \in U$, and $x_0 = P(r_0)$. Let $F : V \to X$ be a chart, such that $F(\xi_0) = x_0$. The cochart $F^{-1} : x \mapsto (\phi_1(x), \ldots, \phi_n(x))$ is made with local smooth functions ϕ_i. Thus, the $\bar{\phi}_i \circ P$ are smooth, by definition of $\mathcal{D}'$. Then, there exists a small neighborhood $W \subset U$ of r_0 such that the $\phi_i \circ (P \restriction W)$ are smooth. By construction, $P \restriction W = F \circ Q$, where $Q : W \to \mathrm{def}(F)$ is the smooth parametrization $Q : r \mapsto (\phi_1 \circ P(r), \ldots, \phi_n \circ P(r))$. Hence, locally P belongs to $\mathcal{D}$, which implies $P \in \mathcal{D}$. Therefore, $\mathcal{D}' \subset \mathcal{D}$ and X is reflexive.

4) We know that $\mathcal{C}^\infty(T_\alpha, \mathbf{R})$ is reduced to the constants when $\alpha \notin \mathbf{Q}$; see Exercise 4, p. 8. Thus, the subordinated diffeology to $\mathcal{C}^\infty(T_\alpha, \mathbf{R})$ is the coarse diffeology, since the composite of a constant map with any parametrization is constant. But we also know that T_α is not trivial. Therefore T_α is not reflexive.

☞ **Exercise 80, p. 99** (Frölicher spaces). We assume X reflexive. If $c \in \mathcal{C} = \mathcal{C}^\infty(\mathbf{R}, X)$ and $f \in \mathcal{F} = \mathcal{C}^\infty(X,\mathbf{R})$, then, by definition of $\mathcal{C}^\infty(X,\mathbf{R})$, $f \circ c \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$. This gives, at the same time, $\mathcal{C} \subset \mathfrak{C}(\mathcal{F})$ and $\mathcal{F} \subset \mathfrak{F}(\mathcal{C})$. Next, let $c \in \mathfrak{C}(\mathcal{F})$, that is, for all $f \in \mathcal{F}$, $f \circ c \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$, but that means that $c \in \mathcal{D}(\mathbf{R}, X)$, where $\mathcal{D}$ denotes the diffeology subordinated to $\mathcal{F}$ (Exercise 79, p. 99). But since X is reflexive, $\mathcal{D}(\mathbf{R}, X) = \mathcal{C}^\infty(\mathbf{R}, X)$, and then $c \in \mathcal{C}^\infty(\mathbf{R}, X)$. Thus, if $c \in \mathfrak{C}(\mathcal{F})$ then $c \in \mathcal{C}$, that is, $\mathfrak{C}(\mathcal{F}) \subset \mathcal{C}$. Therefore, $\mathfrak{C}(\mathcal{F}) = \mathcal{C}$. Consider now $f \in \mathfrak{F}(\mathcal{C})$, that is, $f \in \mathrm{Maps}(X,\mathbf{R})$ such that for all $c \in \mathcal{C}^\infty(\mathbf{R}, X)$, $f \circ c \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$. Let $P \in \mathcal{C}^\infty(\mathbf{R}^n, X)$. Since P is a plot, for all $\gamma \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R}^n)$, $P \circ \gamma \in \mathcal{C}^\infty(\mathbf{R}, X)$, thus $f \circ P \circ \gamma \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$, let $F = f \circ P : \mathbf{R}^n \to \mathbf{R}$, then $F \circ \gamma \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R})$, for all $\gamma \in \mathcal{C}^\infty(\mathbf{R},\mathbf{R}^n)$. By application of Boman's theorem, we get $F \in \mathcal{C}^\infty(\mathbf{R}^n,\mathbf{R})$, that is, $P \circ F \in \mathcal{C}^\infty(\mathbf{R}^n,\mathbf{R})$. Then, after localization (see Exercise 44, p. 47) it comes that for every plot P of X, $f \circ P$ is smooth, that is, $f \in \mathcal{C}^\infty(X,\mathbf{R})$. Thus $\mathfrak{F}(\mathcal{C}) \subset \mathcal{F}$, and then $\mathfrak{F}(\mathcal{C}) = \mathcal{F}$. Therefore, for every reflexive diffeological space X, $\mathcal{C} = \mathcal{C}^\infty(\mathbf{R}, X)$ and $\mathcal{F} = \mathcal{C}^\infty(X,\mathbf{R})$ satisfy the Frölicher condition.

☞ **Exercise 81, p. 111** (Connecting points). We know that X is diffeomorphic to $\mathcal{C}^\infty(\{0\}, X)$ (art. 1.64). Thus, the functor π_0 (art. 5.10) gives an isomorphism.

☞ **Exercise 82, p. 111** (Connecting segments). If x and x' are connected, then they satisfy obviously the condition of the exercise. Conversely, let $\sigma :]a', b'[\to X$ such that $\sigma(a) = x$, $\sigma(b) = x'$ and $a' < a < b < b'$. Let $f(t) = (b-a)t + a$, then $f(0) = a$ and $f(1) = b$. Thus, $\sigma \circ f(0) = x$ and $\sigma \circ f(1) = x'$. Composing then $\sigma \circ f$

with the smashing function λ (art. 5.5), we get a stationary path $\gamma = \sigma \circ f \circ \lambda$ such that $\gamma(0) = x$ and $\gamma(1) = x'$.

☞ **Exercise 83, p. 111** (Contractible space of paths). The map $\rho : s \mapsto [\gamma \mapsto \gamma_s = [t \mapsto \gamma(st)]]$, is a path in $\mathcal{C}^\infty(\text{Paths}(X, x, \star))$ such that $\rho(0)$ is the constant map with value the constant path $\mathbf{x} : t \mapsto x$, and $\rho(1)$ is the identity on Paths$(X, x, \star)$. Therefore, ρ is a deformation retraction of Paths$(X, x, \star)$ to $\mathbf{x}$. Next, the same map ρ, defined on Paths$(X, A, \star)$, gives a deformation retraction from Paths$(X, A, \star)$ to the constant paths in A, which gives a homotopy equivalence.

☞ **Exercise 84, p. 111** (Deformation onto stationary paths). We shall check that $f : \gamma \mapsto \gamma^\star$, from Paths(X) to stPaths(X), and the inclusion j : stPaths(X) $\to$ Paths(X) are homotopic inverses of each other (art. 5.10). Thus, we have to check that $f = j \circ f : \text{Paths}(X) \to \text{Paths}(X)$ is homotopic to $\mathbf{1}_{\text{Paths}(X)}$, and $f \restriction \text{stPaths}(X) = f \circ j : \text{stPaths}(X) \to \text{stPaths}(X)$ is homotopic to $\mathbf{1}_{\text{stPaths}(X)}$. Let us consider

$$f_s : \gamma \mapsto \gamma_s \quad \text{with} \quad \gamma_s : t \mapsto \gamma[\lambda(s)\lambda(t) + (1 - \lambda(s))t],$$

where λ is the smashing function (Figure 5.1). The map $s \mapsto f_s$ is clearly a smooth homotopy from $f_0 = \mathbf{1}_{\text{Paths}(X)}$ to $f_1 = f$. Now, we shall check that for all $\gamma \in$ stPaths(X), $f_s(\gamma) \in$ stPaths(X), for all s. Let $x = \gamma(0)$ and $x' = \gamma(1)$. Then, let $\varepsilon' > 0$ such that $\gamma(t) = x$ for all $t \leq \varepsilon'$, and $\gamma(t) = x'$ for all $t \geq 1 - \varepsilon'$. First of all, let $\varepsilon'' = \inf(\varepsilon, \varepsilon')$, so for all $t \leq \varepsilon''$, $\gamma(t) = x$ and $\lambda(t) = 0$, and for all $t \geq 1 - \varepsilon''$, $\gamma(t) = x'$ and $\lambda(t) = 1$.

A) If $t \leq \varepsilon''$, then $\lambda(t) = 0$, and $\lambda(s)\lambda(t) + (1 - \lambda(s))t = (1 - \lambda(s))t$ but $0 \leq 1 - \lambda(s) \leq 1$. Thus, if $t \leq 0$, then $(1 - \lambda(s))t \leq 0 < \varepsilon''$, and if $0 < t \leq \varepsilon''$, then $(1 - \lambda(s))t \leq t \leq \varepsilon''$. Hence, $\gamma_s(t) = x$.

B) If $t \geq 1 - \epsilon''$, then $\lambda(t) = 1$ and $\lambda(s)\lambda(t) + (1 - \lambda(s))t = \lambda(s) + (1 - \lambda(s))t$. If $t \geq 1$, then, since $0 \leq \lambda(s) \leq 1$, $1 \leq \lambda(s) \times 1 + (1 - \lambda(s))t \leq t$, thus $\gamma_s(t) = x'$. If $0 < 1 - t \leq \varepsilon''$, then $\lambda(s) + (1 - \lambda(s))t = t + (1 - t)\lambda(s) \geq t \geq 1 - \varepsilon''$, thus $\gamma_s(t) = x'$.

Therefore, $f_s \in \mathcal{C}^\infty(\text{stPaths}(X), \text{stPaths}(X))$ and $s \mapsto f_s$ and j are homotopic inverses of each other. Thus, Paths(X) and stPaths(X) are homotopy equivalent. Now, since $f_s(\gamma)(0) = \gamma(0)$ and $f_s(\gamma)(1) = \gamma(1)$, for all s, this equivalence also holds for the diffeology foliated by the projection ends.

☞ **Exercise 85, p. 112** (Contractible quotient). The deformation retraction $\rho : s \mapsto [z \mapsto sz]$ from $\mathbf{C}$ to $\{0\}$ is equivariant by the action of $\mathbf{Z}_m$, that is, $\rho(s) \circ \zeta_k = \zeta_k \rho(s)$, for all s. Thus, there exists a smooth map $r(s) : \mathbf{C}/\mathbf{Z}_m \mapsto \mathbf{C}/\mathbf{Z}_m$ such that $\text{class} \circ \rho(s) = r(s) \circ \text{class}$ for all s, where class : $\mathbf{C} \to \mathbf{C}/\mathbf{Z}_m$ is the projection. Moreover, considering $\bar\rho : \mathbf{R} \times \mathbf{C} \to \mathbf{R} \times \mathbf{C}$, defined by $\bar\rho(s, z) = \rho(s)(z)$, and the action of $\mathbf{Z}_m$ on $\mathbf{R} \times \mathbf{C}$ acting trivially on the first factor and accordingly on the second, we get a smooth map $\bar r : \mathbf{R} \times \mathbf{C}/\mathbf{Z}_m \to \mathbf{R} \times \mathbf{C}/\mathbf{Z}_m$, defining a deformation retraction of the quotient $\mathbf{C}/\mathbf{Z}_m$ to class(0).

☞ **Exercise 86, p. 112** (Locally contractible manifolds). Let $m \in M$, where M is a manifold. By definition of what is a manifold, there exists a chart $F : U \to M$ mapping some point r to m. Let $\Omega \subset M$ be an open neighborhood of m, its preimage by F is an open neighborhood of r. Then, there exists a small ball $B \subset F^{-1}(\Omega)$ centered at the point r. Since F is a local diffeomorphism, the image $F(B) \subset \Omega$ is a D-open neighborhood of m, and since B is contractible, $F(B)$ is contractible. Therefore, M is locally contractible.

☞ **Exercise 87, p. 123** (Homotopy of loops spaces). Fix $x \in X$, and consider the map $\mathrm{pr}_x : \mathrm{comp}_x(\ell) \mapsto \mathrm{comp}(\ell)$, where $\mathrm{comp}_x(\ell) \in \pi_0(\mathrm{Loops}(X, x)) = \pi_1(X, x)$ is the connected component of ℓ in $\mathrm{Loops}(X, x)$ and $\mathrm{comp}(\ell) \in \pi_0(\mathrm{Loops}(X))$ is the connected component of ℓ in $\mathrm{Loops}(X)$. This map is well defined. Indeed, if ℓ and ℓ' are fixed-ends homotopic, then they are *a fortiori* free-ends homotopic, or if we prefer, since the injection $\mathrm{Loops}(X, x) \subset \mathrm{Loops}(X)$ is smooth, it induces a natural map $\mathrm{pr}_x : \pi_0(\mathrm{Loops}(X, x)) \to \pi_0(\mathrm{Loops}(X))$.

First, let us check that this map is surjective. Let $\ell' \in \mathrm{Loops}(X)$ and $x' = \ell'(0)$, since X is connected there exists a path γ connecting x to x'. We can consider ℓ' and γ stationary since we know that every path is fixed-ends homotopic to a stationary path (art. 5.5). Let us consider $\ell = \gamma \vee \ell' \vee \bar{\gamma}$ with $\bar{\gamma}(t) = \gamma(1-t)$, thus $\ell \in \mathrm{Loops}(X, x)$. Now, let $\gamma_s(t) = \gamma(s + t(1-s))$, γ_s is a path connecting $\gamma_s(0) = \gamma(s)$ to $\gamma_s(1) = x'$, and $\ell_s = \gamma_s \vee \ell' \vee \bar{\gamma}_s$ is a free-ends homotopy connecting $\ell_0 = \gamma \vee \ell' \vee \bar{\gamma} = \ell$ to $\ell_1 = \mathbf{x}' \vee \ell' \vee \mathbf{x}'$, where $\mathbf{x}'$ is the constant path at x'. Now, $\mathbf{x}' \vee \ell' \vee \mathbf{x}'$ is homotopic to ℓ', then ℓ and ℓ' belong to the same connected component in $\mathrm{Loops}(X)$, thus $\mathrm{comp}(\ell) = \mathrm{comp}(\ell')$, that is, $\mathrm{pr}_x(\mathrm{comp}_x(\ell)) = \mathrm{comp}(\ell')$.

Next, let k_0 and k_1 in $\pi_1(X, x)$, and let us prove that $\mathrm{pr}_x(k_0) = \mathrm{pr}_x(k_1)$ implies $k_0 = \tau \cdot k_1 \cdot \tau^{-1}$ for some $\tau \in \pi_1(X, x)$. Let $k_0 = \mathrm{comp}_x(\ell_0)$ and $k_1 = \mathrm{comp}_x(\ell_1)$, with ℓ_0 and ℓ_1 in $\mathrm{Loops}(X, x)$. Let us assume that $\mathrm{pr}_x(k_0) = \mathrm{pr}_x(k_1)$, that is, ℓ_0 and ℓ_1 are free-ends homotopic, in other words, $\mathrm{comp}(\ell_0) = \mathrm{comp}(\ell_1)$. Let $s \mapsto \ell_s$ be a free-ends homotopy, thus $\ell' : s \mapsto \ell_s(0) = \ell_s(1)$ is a loop based at x. Indeed, $\ell'(0) = \ell_0(0) = x$ and $\ell'(1) = \ell_1(0) = x$, that is, $\ell' \in \mathrm{Loops}(X, x)$. Then, $\ell'_s : t \mapsto \ell'(st)$ is a path, connecting $\ell'_s(0) = \ell'(0) = \ell_0(0) = x$ to $\ell'_s(1) = \ell'(s) = \ell_s(0) = \ell_s(1)$. Hence, $\sigma_s = \ell'_s \vee \ell_s \vee \bar{\ell}'_s$ is a loop based at x, for all s. Thus, σ is a fixed-ends homotopy connecting $\sigma_0 = \mathbf{x} \vee \ell_0 \vee \mathbf{x}$ to $\sigma_1 = \ell' \vee \ell_1 \vee \bar{\ell}'$, *i.e.*, $\mathrm{comp}_x(\sigma_0) = \mathrm{comp}_x(\sigma_1)$. Since σ_0 is fixed-ends homotopic to ℓ_0, and since $\ell' \in \mathrm{Loops}(X, x)$, that writes again $\mathrm{comp}_x(\ell_0) = \mathrm{comp}_x(\ell' \vee \ell_1 \vee \bar{\ell}')$, that is, $k_0 = \tau \cdot k_1 \cdot \tau^{-1}$ with $\tau = \mathrm{comp}_x(\ell')$. Therefore, k_0 and k_1 are conjugate.

Conversely, let us check that $\mathrm{pr}_x(k) = \mathrm{pr}_x(\tau \cdot k \cdot \tau^{-1})$, where k and τ belong to $\pi_1(X, x)$. Let $k = \mathrm{comp}_x(\ell)$ and $\tau = \mathrm{comp}_x(\ell')$, with ℓ and ℓ' in $\mathrm{Loops}(X, x)$. Then, $\mathrm{pr}_x(k) = \mathrm{comp}(\ell)$ and $\mathrm{pr}_x(\tau \cdot k \cdot \tau^{-1}) = \mathrm{comp}(\ell' \vee \ell \vee \bar{\ell}')$. Let us define $\gamma_s : t \mapsto \ell'(s + t(1-s))$, γ_s is a path in X satisfying $\gamma_s(0) = \ell'(s)$ and $\gamma_s(1) = \ell'(1) = x$. Then, since $\gamma_s(1) = \ell(0) = x$ and $\ell(1) = \bar{\gamma}_s(0) = x$, $\sigma_s = \gamma_s \vee \ell \vee \bar{\gamma}_s$ is well defined. Now, $\sigma_s(0) = \gamma_s(0) = \ell'(s)$ and $\sigma_s(1) = \bar{\gamma}_s(1) = \gamma_s(0) = \ell'(s)$, thus $\sigma_s \in \mathrm{Loops}(X)$. Next, $\sigma_0 = \gamma_0 \vee \ell \vee \bar{\gamma}_0 = \ell' \vee \ell \vee \bar{\ell}'$ and $\sigma_1 = \gamma_1 \vee \ell \vee \bar{\gamma}_1 = \mathbf{x} \vee \ell \vee \mathbf{x}$. Hence, we got a path $s \mapsto \sigma_s$ in $\mathrm{Loops}(X)$ connecting $\ell' \vee \ell \vee \bar{\ell}'$ to $\mathbf{x} \vee \ell \vee \mathbf{x}$, that is, $\mathrm{comp}(\ell' \vee \ell \vee \bar{\ell}') = \mathrm{comp}(\mathbf{x} \vee \ell \vee \mathbf{x})$, but since $\mathrm{comp}(\mathbf{x} \vee \ell \vee \mathbf{x}) = \mathrm{comp}(\ell)$, $\mathrm{comp}(\ell' \vee \ell \vee \bar{\ell}') = \mathrm{comp}(\ell)$. Therefore, $\mathrm{pr}_x(\tau \cdot k \cdot \tau^{-1}) = \mathrm{pr}_x(k)$.

Eventually, the map $\mathrm{pr}_x : \pi_1(X, x) \to \pi_0(\mathrm{Loops}(X))$ projects onto a bijection between the set of conjugacy classes of $\pi_1(X, x)$ and $\pi_0(\mathrm{Loops}(X))$. Note that the fact that not all the paths, in the proof, are *a priori* stationary, can be addressed by using the smashing function introduced in (art. 5.5) wherever we need. Note also that, if the group $\pi_1(X, x)$ is Abelian, then $\pi_0(\mathrm{Loops}(X)) = \pi_1(X, x)$, in particular when $X = \mathsf{G}$ is a diffeological group.

Let us consider now the inclusion $\mathrm{Loops}(X, x) \subset \mathrm{Loops}(X)$ and let us choose $\ell \in \mathrm{Loops}(X, x)$. The short exact sequence of morphisms of pointed spaces, described in (art. 5.18), applied to this situation, writes

$$\begin{cases} \hat{0}_\# : (\pi_0(\mathrm{Paths}(\mathrm{Loops}(X), \mathrm{Loops}(X,x), \ell), [t \mapsto \ell])) \to (\pi_0(\mathrm{Loops}(X,x)), \ell), \\ i_\# : (\pi_0(\mathrm{Loops}(X,x)), \ell) \to (\pi_0(\mathrm{Loops}(X)), \ell). \end{cases}$$

This exercise tells us that $\mathrm{val}(\hat{0}_\#)$, which coincides with $\ker(i_\#)$ — the subset of the components of $\mathrm{Loops}(X,x)$ which can be connected, through $\mathrm{Loops}(X)$, to ℓ — is the subset of classes of loops of X, pointed at x, conjugated in $\pi_1(X,x) = \pi_0(\mathrm{Loops}(X,x))$ with the class of ℓ. Note in particular that the conjugacy class of the class of the constant loop $\mathbf{x} = [t \mapsto x]$ is reduced to the class of $\mathbf{x}$, thus the intersection of the connected component of $\mathbf{x}$ in $\mathrm{Loops}(X)$ with $\mathrm{Loops}(X,x)$ is reduced to the connected component of $\mathbf{x}$ in $\mathrm{Loops}(X,x)$.

☞ **Exercise 88, p. 139** (Antisymmetric 3-form). By antisymmetry, $A_{jki} = -A_{jik} = +A_{ijk}$, as well, $A_{kij} = -A_{ikj} = +A_{ijk}$. Hence, for any triple of indices, $A_{ijk} = (1/3)[A_{ijk} + A_{jki} + A_{kij}]$. Now, A is the zero tensor if and only if all its coordinates are equal to zero, that is, for every triple of indices $A_{ijk} + A_{jki} + A_{kij} = 0$.

☞ **Exercise 89, p. 140** (Expanding the exterior product). The formula for the exterior product of 1-forms is given in (art. 6.15). We get first $[\mathrm{Ext}(b)(c)](x_2)(x_3) = b(x_2)c(x_3) - b(x_3)c(x_2)$. Then,

$$\begin{aligned} \mathrm{Ext}(a)(\mathrm{Ext}(b)(c))(x_1)(x_2)(x_3) &= a(x_1)[\mathrm{Ext}(b)(c)](x_2)(x_3) \\ &- a(x_2)[\mathrm{Ext}(b)(c)](x_1)(x_3) \\ &- a(x_3)[\mathrm{Ext}(b)(c)](x_2)(x_1) \\ &= a(x_1)[b(x_2)c(x_3) - b(x_3)c(x_2)] \\ &- a(x_2)[b(x_1)c(x_3) - b(x_3)c(x_1)] \\ &- a(x_3)[b(x_2)c(x_1) - b(x_1)c(x_2)]. \end{aligned}$$

Developing the factors gives immediately that

$$\mathrm{Ext}(a)(\mathrm{Ext}(b)(c))(x_1)(x_2)(x_3) = \sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma)\, a(x_{\sigma(1)})b(x_{\sigma(2)})c(x_{\sigma(3)}).$$

☞ **Exercise 90, p. 140** (Determinant and isomorphisms). First of all, if the v_i are linearly independent, then they form a basis $\mathcal{B}$, and $\mathrm{vol} = c\ \mathrm{vol}_\mathcal{B}$, with $c \neq 0$. Then, $\mathrm{vol}(v_1)\cdots(v_n) = c\ \mathrm{vol}_\mathcal{B}(v_1)\cdots(v_n) = c \neq 0$. The contraposition of this sentence is, if $\mathrm{vol}(v_1)\cdots(v_n) = 0$, then the v_i are not linearly independent. Therefore, $\mathrm{vol}(v_1)\cdots(v_n) = 0$ if and only if the v_i are linearly independent. Next, let us assume that M has a nonzero kernel. Let $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_k)$ be a basis of $\ker(M)$. Thanks to the incomplete basis theorem, there exists a basis $\mathcal{B} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of E, with $\dim(E) = n$. Then, by definition of the determinant, $\det(M) = \mathrm{vol}_\mathcal{B}(M\boldsymbol{e}_1)\cdots(M\boldsymbol{e}_n)$ (art. 6.19, (♡)). But $M\boldsymbol{e}_1 = 0$, thus $\det(M) = 0$. Conversely, let $\det(M) = 0$, then for any basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$, $\mathrm{vol}(M_1)\cdots(M_n) = 0$, where $M_i = M\boldsymbol{e}_i$. Hence, the M_i are not linearly independent. There exists then a family of numbers λ_i, not all zero, such that $\sum_{i=1}^n \lambda_i M_i = 0$, that is, $M(\sum_{i=1}^n \lambda_i \boldsymbol{e}_i) = 0$. Hence, $v = \sum_{i=1}^n \lambda_i \boldsymbol{e}_i \neq 0$ and $Mv = 0$. Therefore, $\ker(M) \neq \{0\}$ and M is not a linear isomorphism.

☞ **Exercise 91, p. 140** (Determinant is smooth). Once a basis is chosen, the determinant of M is a multilinear combination of the matrix coefficients of M

(art. 6.19, (♠)), thus the determinant is smooth. The variation of the determinant is given by application of the formula (art. 6.19, (♡)).

$$
\begin{aligned}
\delta[\det(M)] &= \delta[\mathrm{vol}_{\mathcal{B}}(M\boldsymbol{e}_1)\cdots(M\boldsymbol{e}_n)] \\
&= \sum_{i=1}^{n} \mathrm{vol}_{\mathcal{B}}(M\boldsymbol{e}_1)\cdots(\delta M\boldsymbol{e}_i)\cdots(M\boldsymbol{e}_n) \\
&= \sum_{i=1}^{n} \mathrm{vol}_{\mathcal{B}}(M\boldsymbol{e}_1)\cdots(MM^{-1}\delta M\boldsymbol{e}_i)\cdots(M\boldsymbol{e}_n) \\
&= \det(M) \times \sum_{i=1}^{n} \mathrm{vol}_{\mathcal{B}}(\boldsymbol{e}_1)\cdots(M^{-1}\delta M\boldsymbol{e}_i)\cdots(\boldsymbol{e}_n).
\end{aligned}
$$

But $M^{-1}\delta M\boldsymbol{e}_i = \sum_{j=1}^{n}[M^{-1}\delta M]_i^j\boldsymbol{e}_j$. Thus,

$$
\begin{aligned}
\mathrm{vol}_{\mathcal{B}}(\boldsymbol{e}_1)\cdots(M^{-1}\delta M\boldsymbol{e}_i)\cdots(\boldsymbol{e}_n) &= \mathrm{vol}_{\mathcal{B}}(\boldsymbol{e}_1)\cdots\left(\sum_{j=1}^{n}[M^{-1}\delta M]_i^j\boldsymbol{e}_j\right)\cdots(\boldsymbol{e}_n) \\
&= \sum_{j=1}^{n}[M^{-1}\delta M]_i^j\,\mathrm{vol}_{\mathcal{B}}(\boldsymbol{e}_1)\cdots(\boldsymbol{e}_j)\cdots(\boldsymbol{e}_n) \\
&= [M^{-1}\delta M]_i^i.
\end{aligned}
$$

Therefore,

$$
\delta[\det(M)] = \det(M) \times \sum_{i=1}^{n}[M^{-1}\delta M]_i^i = \det(M)\mathrm{Tr}(M^{-1}\delta M).
$$

Exercise 92, p. 140 (Determinant of a product). Let $\mathcal{B} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ be a basis of E. According to (art. 6.19), $\det(MN) = \mathrm{vol}_{\mathcal{B}}(MN\boldsymbol{e}_1)\cdots(MN\boldsymbol{e}_n) = \det(M)\,\mathrm{vol}_{\mathcal{B}}(N\boldsymbol{e}_1)\cdots(N\boldsymbol{e}_n) = \det(M)\det(N)$. Next, $\det(s \times M) = \det((s \times \mathbf{1}_n)M) = \det(s \times \mathbf{1}_n)\det(M)$. A direct computation shows that $\det(s \times \mathbf{1}_n) = s^n$. Therefore, $\det(s \times M) = s^n \times \det(M)$.

Exercise 93, p. 146 (Coordinates of the exterior derivative). Let us decompose ω in a basis, $\omega(x) = \sum_{i<j<\cdots<k} \omega_{ij\cdots k}(x)\,\boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \cdots \wedge \boldsymbol{e}^k$. By definition of the exterior derivative (art. 6.24),

$$
d\omega(x) = \sum_{i<j<\cdots<k}\sum_{l=1}^{n} \partial_l\omega_{ij\cdots k}(x)\,\boldsymbol{e}^l \wedge \boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \cdots \wedge \boldsymbol{e}^k.
$$

The monomial $(d\omega)_{ijk\cdots l}\,\boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \boldsymbol{e}^k \wedge \cdots \wedge \boldsymbol{e}^l$, with $i < j < k < \cdots < l$, is then obtained by grouping the terms containing the indices $ijk\cdots l$, that is,

$$
\begin{aligned}
(d\omega)_{ij\cdots kl}\,\boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \boldsymbol{e}^k \wedge \cdots \wedge \boldsymbol{e}^l &= \partial_i\omega_{jk\cdots l}\,\boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \boldsymbol{e}^k \wedge \cdots \wedge \boldsymbol{e}^l \\
&+ \partial_j\omega_{ik\cdots l}\,\boldsymbol{e}^j \wedge \boldsymbol{e}^i \wedge \boldsymbol{e}^k \wedge \cdots \wedge \boldsymbol{e}^l \\
&+ \partial_k\omega_{ij\cdots l}\,\boldsymbol{e}^k \wedge \boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \cdots \wedge \boldsymbol{e}^l \\
&+ \cdots \\
&+ \partial_l\omega_{ijk\cdots}\,\boldsymbol{e}^l \wedge \boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \boldsymbol{e}^k \wedge \cdots .
\end{aligned}
$$

Now, let us consider a term $\partial_k\omega_{ij\cdots l}\,\boldsymbol{e}^k \wedge \boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \cdots \wedge \boldsymbol{e}^l$, where the index k is at the rank m in $ijk\cdots l$. If $m = 1$, then $k = i$ and we do not change anything. If $m = 2$, then $k = j$, we transpose the first two covectors to change $\boldsymbol{e}^j \wedge \boldsymbol{e}^i \wedge \boldsymbol{e}^k \wedge \cdots \wedge \boldsymbol{e}^l$

into $\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \wedge \cdots \wedge \mathbf{e}^l$ and we do not touch the indices of $\omega_{ik\cdots l}$ in the partial derivative. If $m \geq 2$, then we perform $m-1$ transpositions of the covectors to reorder $\mathbf{e}^k \wedge \mathbf{e}^i \wedge \mathbf{e}^j \wedge \cdots \wedge \mathbf{e}^l$ into $\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \wedge \cdots \wedge \mathbf{e}^l$ and, thanks to the antisymmetry of the coefficient $\omega_{ij\cdots l}$, we perform $m-2$ transpositions to send the index i, which is at the rank 1, to the rank $m-1$, changing $\omega_{ij\cdots l}$ into $\omega_{ji\cdots l}$. Then, the cost for these transpositions is $(-1)^{m-1} \times (-1)^{m-2} = (-1)^{2m-3} = -1$. Therefore, $(d\omega)_{ijk\cdots\ell} = \partial_i \omega_{jk\cdots\ell} - \partial_j \omega_{ik\cdots\ell} - \partial_k \omega_{ji\cdots\ell} - \cdots - \partial_\ell \omega_{jk\cdots i}$.

☞ **Exercise 94, p. 147** (Integral of a 3-form on a 3-cube). Let α be a 3-form on $\mathbf{R}^3$, $\alpha = f(x_1, x_2, x_3)\, dx_1 \wedge dx_2 \wedge dx_3$. Let $x \mapsto F[x_1, x_2](x)$ be a primitive of $x \mapsto f(x_1, x_2, x)$, let $x \mapsto F[x_1](x)(x_3)$ be a primitive of $x \mapsto F[x_1, x](x_3)$, and let $x \mapsto F(x)(x_2)(x_3)$ be a primitive of $x \mapsto F[x](x_2)(x_3)$. The integral of α on a 3-cube $C = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ is given by

$$\begin{aligned}
\int_C \alpha &= \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \int_{a_3}^{b_3} dx_3\, f(x_1, x_2, x_3) \\
&= \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \{F[x_1, x_2](b_3) - F[x_1, x_2](a_3)\} \\
&= \int_{a_1}^{b_1} dx_1 \{F[x_1](b_2)(b_3) - F[x_1](a_2)(b_3)\} \\
&- \int_{a_1}^{b_1} dx_1 \{F[x_1](b_2)(a_3) - F[x_1](a_2)(a_3)\} \\
&= \{F(b_1)(b_2)(b_3) - F(a_1)(b_2)(b_3)\} \\
&- \{F(b_1)(a_2)(b_3) - F(a_1)(a_2)(b_3)\} \\
&- \{F(b_1)(b_2)(a_3) - F(a_1)(b_2)(a_3)\} \\
&+ \{F(b_1)(a_2)(a_3) - F(a_1)(a_2)(a_3)\}.
\end{aligned}$$

☞ **Exercise 95, p. 158** (Functional diffeology of 0-forms). Let $\phi : V \to \Omega^0(X)$ be a plot for the functional diffeology (art. 6.29), that is, for all n-plots $P : U \to X$, $n \in \mathbf{N}$, the parametrization $(s, r) \mapsto \phi(s)(P)(r)$ belongs to $\mathcal{C}^\infty(V \times U, \Lambda^0(\mathbf{R}^n))$. But $\Omega^0(X) = \mathcal{C}^\infty(X, R)$, thus $\phi(s)(P)(r) = \phi(s)(P(r))$, and since $\Lambda^0(\mathbf{R}^n) = \mathbf{R}$, $[(s, r) \mapsto \phi(s)(P(r))] \in \mathcal{C}^\infty(V \times U, \mathbf{R})$. But this is the very definition of the plots for the functional diffeology of $\mathcal{C}^\infty(X, \mathbf{R})$ (art. 1.57).

☞ **Exercise 96, p. 158** (Differential forms against constant plots). Let $r_0 \in U$, and let V be an open neighborhood of r_0 such that $P \upharpoonright V = [r \mapsto x_0 = P(r_0)]$. Then, $P \upharpoonright V = [r \mapsto 0 \mapsto x_0]$, where $[r \mapsto 0]$ is the only map from V to $\mathbf{R}^0 = \{0\}$. By application of the compatibility axiom we have $\alpha(P \upharpoonright V) = [r \mapsto 0]^*(\alpha(\hat{x}_0))$, where $\hat{x}_0$ is the 0-plot $0 \mapsto x_0$, but $\alpha(\hat{x}_0) = 0$, thus $\alpha(P \upharpoonright V) = 0$. Therefore, since $\alpha(P)$ vanishes locally everywhere, $\alpha(P) = 0$.

☞ **Exercise 97, p. 159** (The equi-affine plane). First of all, note that $\alpha(\gamma)$ is trilinear and smooth. Thus, $\alpha(\gamma)$ is a smooth covariant 3-tensor on $\operatorname{def}(\gamma)$. Now, let us consider a plot $P : U \to \mathbf{R}^2$ for the wire diffeology. Then, for all $r_0 \in \operatorname{def}(P)$ there exist an open neighborhood V of r_0, a smooth map $q \in \mathcal{C}^\infty(V, \mathbf{R})$, an arc γ in $\mathbf{R}^2$, defined on some interval, such that $P \upharpoonright V = \gamma \circ q$, as shown in Figure Sol.6. Now, we have to check that if we have two such decompositions, satisfying $\gamma' \circ q' = \gamma \circ q$, then $q^*(\alpha(\gamma)) = q'^*(\alpha(\gamma'))$. Let $r \in U$ and $\delta r, \delta' r, \delta'' r \in \mathbf{R}^n$, where $n = \dim(U)$,

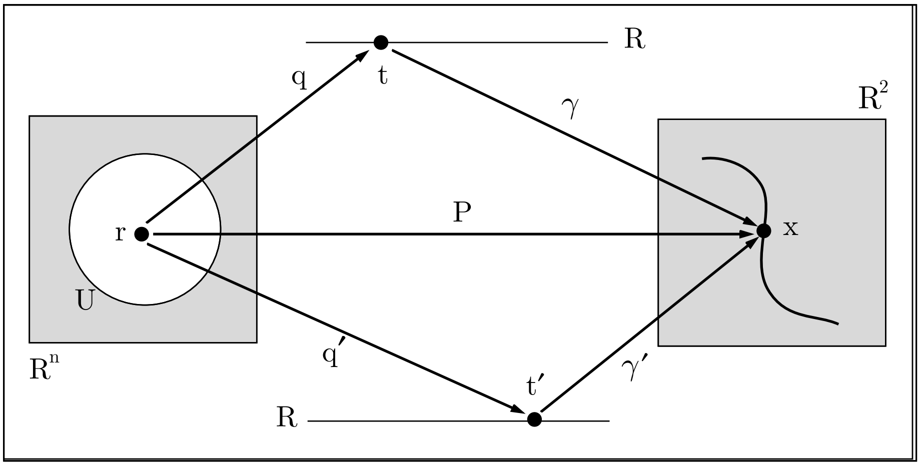

FIGURE Sol.6. Compatibility condition.

we have

$$q^*(\alpha(\gamma))_r(\delta r, \delta' r, \delta'' r) = \alpha(\gamma)_t(\delta t, \delta' t, \delta'' t),$$

with

$$t = q(r) \quad \text{and} \quad \delta t = \frac{\partial t}{\partial r}(\delta r) \quad \text{etc.},$$

and *mutatis mutandis* for γ' and q'. Next, let us derive the condition $\gamma' \circ q' = \gamma \circ q$. We get

$$D(\gamma \circ q)(r)(\delta r) = D(\gamma' \circ q')(r)(\delta r), \quad \text{that is,} \quad \dot\gamma(t)\delta t = \dot\gamma'(t')\delta t', \qquad (\diamondsuit)$$

and for the second derivative,

$$D^2(\gamma \circ q)(r)(\delta' r)(\delta'' r) = \ddot\gamma(t)(\delta' t)(\delta'' t) + \dot\gamma(t)\frac{\partial^2 t}{\partial r^2}(\delta' r)(\delta'' r).$$

We have then,

$$\ddot\gamma(t)(\delta' t)(\delta'' t) + \dot\gamma(t)\frac{\partial^2 t}{\partial r^2}(\delta' r)(\delta'' r) = \ddot\gamma'(t')(\delta' t')(\delta'' t') + \dot\gamma'(t')\frac{\partial^2 t'}{\partial r^2}(\delta' r)(\delta'' r),$$

and therefore,

$$\begin{aligned} \omega(\dot\gamma(t)(\delta t), \ddot\gamma(t)(\delta' t)(\delta'' t)) \quad = \quad & \omega\Big(\dot\gamma'(t')(\delta t'), \ddot\gamma'(t')(\delta' t')(\delta'' t') \\ & + \quad \dot\gamma'(t')\frac{\partial^2 t'}{\partial r^2}(\delta' r)(\delta'' r) - \dot\gamma(t)\frac{\partial^2 t}{\partial r^2}(\delta' r)(\delta'' r)\Big). \end{aligned}$$

But $\dot\gamma(t)$ and $\dot\gamma'(t')$ are collinear $(\diamondsuit)$ and ω is antisymmetric, thus the second order derivatives in r disappear from the right hand side, and we get finally

$$\omega(\dot\gamma(t)(\delta t), \ddot\gamma(t)(\delta' t)(\delta'' t)) = \omega(\dot\gamma'(t')(\delta t'), \ddot\gamma'(t')(\delta' t')(\delta'' t')),$$

that is, $q^*(\alpha(\gamma)) = q'^*(\alpha(\gamma'))$. Thus, α is the expression, in the generating family of the wire plane, of a covariant 3-tensor.

☞ **Exercise 98, p. 159** (Liouville 1-form of the Hilbert space). 1) Let us develop the restriction of P' to V,

$$P' \restriction V : r \mapsto \sum_{\alpha' \in A'} \lambda'_{\alpha'}(r)(X'_{\alpha'}, Y'_{\alpha'}).$$

Thus,

$$P \restriction V = P' \restriction V \;\Rightarrow\; \sum_{\alpha \in A} \lambda_\alpha X_\alpha = \sum_{\alpha' \in A'} \lambda'_{\alpha'} X'_{\alpha'} \text{ and } \sum_{\alpha \in A} \lambda_\alpha Y_\alpha = \sum_{\alpha' \in A'} \lambda'_{\alpha'} Y'_{\alpha'}.$$

Let

$$\Lambda(P \restriction V) = \Big(\sum_{\alpha \in A} \lambda_\alpha X_\alpha\Big) \cdot \Big(\sum_{\beta \in A} d\lambda_\beta Y_\beta\Big) - \Big(\sum_{\alpha \in A} \lambda_\alpha Y_\alpha\Big) \cdot \Big(\sum_{\beta \in A} d\lambda_\beta X_\beta\Big).$$

Then,

$$\begin{aligned} \Lambda(P \restriction V) - \Lambda(P' \restriction V) &= \Big(\sum_{\alpha \in A} \lambda_\alpha X_\alpha\Big) \cdot \Big(\sum_{\alpha'' \in A''} d\lambda''_{\alpha''} Y''_{\alpha''}\Big) \\ &- \Big(\sum_{\alpha \in A} \lambda_\alpha Y_\alpha\Big) \cdot \Big(\sum_{\alpha'' \in A''} d\lambda''_{\alpha''} X''_{\alpha''}\Big), \end{aligned}$$

where A'' is the following reordering of the two sets of indices A and A', with $\lambda''_{\alpha''}$ $X''_{\alpha''}$ and $Y''_{\alpha''}$ following this reordering. Let $A = \{1, \ldots, a\}$ and $A' = \{1, \ldots, a'\}$. We denote $A'' = \{1, \ldots, a''\}$ such that $a'' = a + a'$, with

$$\begin{array}{llll} \lambda''_{\alpha''} = \lambda_\alpha & \text{if } 1 \le \alpha'' \le a & \text{and } \lambda''_{\alpha''} = \lambda'_{\alpha'} & \text{if } a+1 \le \alpha'' \le a+a', \\ Y''_{\alpha''} = Y_\alpha & \text{if } 1 \le \alpha'' \le a & \text{and } Y''_{\alpha''} = -Y'_{\alpha'} & \text{if } a+1 \le \alpha'' \le a+a', \\ X''_{\alpha''} = X_\alpha & \text{if } 1 \le \alpha'' \le a & \text{and } X''_{\alpha''} = -X'_{\alpha'} & \text{if } a+1 \le \alpha'' \le a+a'. \end{array}$$

With this reordering we get

$$\sum_{\alpha \in A} \lambda_\alpha Y_\alpha = \sum_{\alpha' \in A'} \lambda'_{\alpha'} Y'_{\alpha'} \quad \Rightarrow \quad \sum_{\alpha'' \in A''} \lambda''_{\alpha''} Y''_{\alpha''} = 0,$$

$$\sum_{\alpha \in A} \lambda_\alpha X_\alpha = \sum_{\alpha' \in A'} \lambda'_{\alpha'} X'_{\alpha'} \quad \Rightarrow \quad \sum_{\alpha'' \in A''} \lambda''_{\alpha''} X''_{\alpha''} = 0.$$

Let us project these vectors on each factor $\mathbf{R}$ by the projection pr_k. We get

$$\sum_{\alpha'' \in A''} \lambda''_{\alpha''} Y''_{\alpha''} = 0 \quad \Rightarrow \quad \sum_{\alpha'' \in A''} \lambda''_{\alpha''} Y''_{\alpha'',k} = 0,$$

$$\sum_{\alpha'' \in A''} \lambda''_{\alpha''} X''_{\alpha''} = 0 \quad \Rightarrow \quad \sum_{\alpha'' \in A''} \lambda''_{\alpha''} X''_{\alpha'',k} = 0.$$

But $X''_{\alpha'',k}$ and $Y''_{\alpha'',k}$ are just real numbers, then both $\sum_{\alpha'' \in A''} \lambda''_{\alpha''} X''_{\alpha'',k}$ and $\sum_{\alpha'' \in A''} \lambda''_{\alpha''} Y''_{\alpha'',k}$ are smooth functions of $r \in V$. Since these functions vanish identically, their derivatives, with respect to r, vanish too. We get then

$$\sum_{\alpha'' \in A''} d\lambda''_{\alpha''} Y''_{\alpha'',k} = 0 \quad \Rightarrow \quad \sum_{\alpha'' \in A''} d\lambda''_{\alpha''} Y''_{\alpha''} = 0,$$

$$\sum_{\alpha'' \in A''} d\lambda''_{\alpha''} X''_{\alpha'',k} = 0 \quad \Rightarrow \quad \sum_{\alpha'' \in A''} d\lambda''_{\alpha''} X''_{\alpha''} = 0.$$

Finally, $\Lambda(P \restriction V) = \Lambda(P' \restriction V)$.

2) Let us consider a covering U_i of U such that the plot P, restricted to each U_i, is the sum of a finite linear combination of vectors, with smooth parametrizations as

coefficients. Let i and j be two indices of the covering, and let $P_i = P \upharpoonright U_i$. By the previous statement we have

$$\Lambda(P_i) \upharpoonright U_i \cap U_j = \Lambda(P_j) \upharpoonright U_i \cap U_j.$$

Because a differential form is local (art. 6.36), there exists a 1-form $\Lambda(P) = \sup\{\Lambda(P_i)\}_i$, defined on U such that $\Lambda(P) \upharpoonright U_i = \Lambda(P_i)$.

3) We still need to show that the map Λ is a 1-form on $\mathcal{H}_{\mathbf{R}} \times \mathcal{H}_{\mathbf{R}}$, that is, to check that for every plot $P : U \to \mathcal{H}_{\mathbf{R}} \times \mathcal{H}_{\mathbf{R}}$, and for any smooth parametrization $F : U' \to U$, $\Lambda(P \circ F) = F^*(\Lambda(P))$. Let $r_0' \in U'$, $r_0 = F(r_0')$, and

$$P \upharpoonright V : r \mapsto \sum_{\alpha \in A} \lambda_\alpha(r)(X_\alpha, Y_\alpha),$$

as usual. Let us define now $V' = F^{-1}(V)$ and $\lambda'_\alpha = \lambda_\alpha \circ F$. We have

$$\begin{aligned}
\Lambda(P \circ F \upharpoonright V')_{r'}(\delta r') &= \sum_{\alpha,\beta \in A} X_\alpha \cdot Y_\beta \; \lambda'_\alpha(r') d\lambda'_\beta(r')(\delta r') \\
&= \sum_{\alpha,\beta \in A} X_\alpha \cdot Y_\beta \; \lambda_\alpha(F(r')) d\lambda_\beta(F(r'))(D(F_{r'})(\delta r')) \\
&= \sum_{\alpha,\beta \in A} X_\alpha \cdot Y_\beta \; \lambda_\alpha(r) d\lambda_\beta(r)(\delta r),
\end{aligned}$$

with $r = F(r')$ and $\delta r = D(F)_{r'}(\delta r')$. But that is the definition of the pullback. Therefore, $\Lambda(P \circ F \upharpoonright V') = F^*(\Lambda(P \upharpoonright V))$. Then, since this is true locally, and since it is a local property, it is true globally and $\Lambda(P \circ F) = F^*(\Lambda(P))$.

☞ **Exercise 99, p. 159** (The complex picture of the Liouville form). The identity is obtained just by developing the computation as follows,

$$\begin{aligned}
(Z \cdot dZ - dZ \cdot Z)(P) &= \sum_{\alpha \in A} \lambda^*_\alpha (X_\alpha - iY_\alpha) \sum_{\beta \in A} d\lambda_\beta (X_\beta + iY_\beta) \\
&- \sum_{\alpha \in A} d\lambda^*_\alpha (X_\alpha - iY_\alpha) \sum_{\beta \in A} \lambda_\beta (X_\beta + iY_\beta) \\
&= \sum_{\alpha,\beta \in A} \lambda^*_\alpha d\lambda_\beta [X_\alpha X_\beta + Y_\alpha Y_\beta + i(X_\alpha Y_\beta - Y_\alpha X_\beta)] \\
&- \sum_{\alpha,\beta \in A} d\lambda^*_\alpha \lambda_\beta [X_\alpha X_\beta + Y_\alpha Y_\beta + i(X_\alpha Y_\beta - Y_\alpha X_\beta)] \\
&= \sum_{\alpha,\beta \in A} (X_\alpha X_\beta + Y_\alpha Y_\beta)(\lambda^*_\alpha d\lambda_\beta - d\lambda^*_\alpha \lambda_\beta) \\
&+ i \sum_{\alpha,\beta \in A} (X_\alpha Y_\beta - Y_\alpha X_\beta)(\lambda^*_\alpha d\lambda_\beta - d\lambda^*_\alpha \lambda_\beta).
\end{aligned}$$

But $\sum_{\alpha,\beta\in A}(X_\alpha X_\beta + Y_\alpha Y_\beta)(\lambda_\alpha^* d\lambda_\beta - d\lambda_\alpha^* \lambda_\beta) = 0$ for symmetry reasons. Hence, developing, for each index, $\lambda_\alpha = a_\alpha + ib_\alpha$, we get

$$\begin{aligned}
&(Z\cdot dZ - dZ\cdot Z)(P)\\
&\quad = i \sum_{\alpha,\beta\in A} (X_\alpha Y_\beta - Y_\alpha X_\beta)(\lambda_\alpha^* d\lambda_\beta - d\lambda_\alpha^* \lambda_\beta)\\
&\quad = i \sum_{\alpha,\beta\in A} (X_\alpha Y_\beta - Y_\alpha X_\beta)(a_\alpha da_\beta - a_\beta da_\alpha + b_\alpha db_\beta - b_\beta db_\alpha)\\
&\quad - \sum_{\alpha,\beta\in A} (X_\alpha Y_\beta - Y_\alpha X_\beta)(a_\alpha db_\beta + a_\beta db_\alpha - b_\alpha da_\beta - b_\beta da_\alpha).
\end{aligned}$$

But the second term of the right hand side vanishes for symmetry reasons. Thus, it remains

$$\begin{aligned}
(Z\cdot dZ - dZ\cdot Z)(P) &= i \sum_{\alpha,\beta\in A} (X_\alpha Y_\beta - Y_\alpha X_\beta)(a_\alpha da_\beta - a_\beta da_\alpha)\\
&+ i \sum_{\alpha,\beta\in A} (X_\alpha Y_\beta - Y_\alpha X_\beta)(b_\alpha db_\beta - b_\beta db_\alpha).
\end{aligned}$$

Let us now come back to the map $\Phi : Z \mapsto (X, Y)$, identifying $\mathcal{H}_{\mathbf{C}}$ and $\mathcal{H}_{\mathbf{R}} \times \mathcal{H}_{\mathbf{R}}$. The plot $\Phi \circ P$ writes necessarily $\Phi \circ P(r) = \sum_{j\in\mathcal{J}} \mu_j(r)(X_j, Y_j)$. Then, developing $\sum_{\alpha\in A} \lambda_\alpha Z_\alpha$, we obtain the family $(\mu_j, (X_j, Y_j))_{j\in\mathcal{J}}$ as the union of two families,

$$(\mu_j, (X_j, Y_j))_{j\in\mathcal{J}} = (a_\alpha, (X_\alpha, Y_\alpha)_{\alpha\in A} \bigcup (b_\alpha, (-Y_\alpha, X_\alpha)_{\alpha\in A}.$$

Applying the form Λ to $\Phi \circ P$, for this family, we get

$$\begin{aligned}
\Lambda(\Phi\circ P) &= \sum_{\alpha,\beta\in A} (X_\alpha Y_\beta - Y_\alpha X_\beta)(a_\alpha da_\beta - a_\beta da_\alpha)\\
&+ \sum_{\alpha,\beta\in A} (-Y_\alpha X_\beta + X_\alpha Y_\beta)(b_\alpha db_\beta - b_\beta db_\alpha).
\end{aligned}$$

Comparing the last two expressions we get $\Lambda(\Phi \circ P) = (1/2i)(Z \cdot dZ - dZ \cdot Z)(P)$. Thanks to locality (art. 6.36), this equality is still satisfied for any plot of $\mathcal{H}_{\mathbf{R}} \times \mathcal{H}_{\mathbf{R}}$. Hence, we can conclude that

$$\Phi^*(\Lambda) = \frac{1}{2i}[Z\cdot dZ - dZ\cdot Z].$$

☞ **Exercise 100, p. 160** (The Fubini-Study 2-form). We use the notations of Exercise 99, p. 159, and $j(z)$ denotes the multiplication $Z \mapsto zZ$.

1) We have

$$j(z)^*(\varpi) = \frac{1}{2i}[(zZ)\cdot d(zZ) - d(zZ)\cdot(zZ)].$$

But since $z \in U(1)$, $z^*z = 1$, thus $(zZ)\cdot d(zZ) = Z\cdot dZ$, as well $d(zZ)\cdot(zZ) = dZ\cdot Z$. Therefore, $j(z)^*(\varpi) = \varpi$.

2) Let $P : \mathcal{O} \to \mathcal{S}_{\mathbf{C}}$ and $P' : \mathcal{O} \to \mathcal{S}_{\mathbf{C}}$ be two plots such that $\pi \circ P = \pi \circ P'$. By definition of π, there exists a unique parametrization $\zeta : \mathcal{O} \to U(1)$ such that $P'(r) = \zeta(r) \times P(r)$. We have to check that ζ is smooth. Since P and P' are plots of the fine diffeology, for every $r_0 \in \mathcal{O}$ there exist two local families (art. 3.7) $(\lambda_\alpha, Z_\alpha)_{\alpha\in A}$ and $(\lambda'_{\alpha'}, Z'_{\alpha'})_{\alpha'\in A'}$, defined on some open neighborhood V of r_0, such that $P(r) =_{\mathrm{loc}} \sum_{\alpha\in A} \lambda_\alpha(r) Z_\alpha$ and $P'(r) =_{\mathrm{loc}} \sum_{\alpha'\in A'} \lambda_{\alpha'}(r) Z'_{\alpha'}$. Moreover, the Z_α

and the $Z'_{\alpha'}$ can be chosen as bases of the vector subspaces, F and F′, they generate (art. 3.8). Then, let us consider a basis $\mathcal{Z}'' = (Z''_{\alpha''})_{\alpha'' \in A''}$ of the sum $F'' = F + F'$. The condition $P'(r) = \zeta(r) \times P(r)$ writes in the basis $\mathcal{Z}''$,

$$\sum_{\beta'' \in A''} \lambda'_{\beta''}(r) Z''_{\beta''} = \zeta(r) \times \sum_{\beta'' \in A''} \lambda_{\beta''}(r) Z_{\beta''},$$

where the $\lambda_{\beta''}(r)$ and $\lambda'_{\beta''}(r)$ are the coordinates of $P(r)$ and $P'(r)$ in the basis $\mathcal{Z}''$. Thus, for every $\beta'' \in A''$, we have

$$\lambda'_{\beta''}(r) = \zeta(r) \times \lambda_{\beta''}(r).$$

But $P(r)$ and $P'(r)$ never vanish. There exists then some index $\beta'' \in A''$ for which $\lambda'_{\beta''}(r_0) \neq 0$. Since the function $\lambda'_{\beta''}$ is a smooth parametrization of **C**, there exists an open neighborhood $W \subset V$ of r_0 such that $\lambda'_{\beta''}(r) \neq 0$ for all $r \in W$. Therefore, $\lambda_{\beta''}(r) \neq 0$ for all $r \in W$. Thus, for all $r \in W$, and for this index β'', $\zeta(r) = \lambda'_{\beta''}(r)/\lambda_{\beta''}(r)$. Hence, the function ζ is locally smooth. Therefore, ζ is smooth.

3) Let us define $dP(r)$ by its local expression $\sum_{\alpha \in A} d\lambda_\alpha(r) Z_\alpha$. Then, the form ϖ evaluated on the plot P, writes

$$\varpi(P)(r) = \frac{1}{2i}[P(r) \cdot dP(r) - dP(r) \cdot P(r)].$$

Evaluated on the plot $P' : r \mapsto \zeta(r)P(r)$, we have

$$\varpi(P')(r) = \frac{1}{2i}[(\zeta(r)P(r)) \cdot d(\zeta(r)P(r)) - d(\zeta(r)P(r)) \cdot (\zeta(r)P(r))].$$

After developing this expression, we get

$$\begin{aligned} \varpi(P')(r) &= \frac{1}{2i}[P(r) \cdot dP(r) - dP(r) \cdot P(r)] + \frac{1}{2i}[\zeta(r)^* d\zeta(r) - d(\zeta(r)^*)\zeta(r)] \\ &= \varpi(P) + \frac{1}{2i}[\zeta(r)^* d\zeta(r) - d(\zeta(r)^*)\zeta(r)]. \end{aligned}$$

But using $\zeta(r)^* = 1/\zeta(r)$, we get

$$\varpi(\zeta P)(r) = \varpi(P)(r) + \frac{d\zeta(r)}{i\zeta(r)}.$$

Then, since $d\zeta(r)/(i\zeta(r))$ is just $\zeta^*(\theta)(r)$, we get $\varpi(P') = \varpi(P) + \zeta^*(\theta)$.

4) Now, since $d\theta = 0$, we have

$$d[\varpi(P')] = d[\varpi(P) + \zeta^*(\theta)] = d[\varpi(P)] + \zeta^*(d\theta) = d[\varpi(P)].$$

Thus, by application of criterion (art. 6.38), there exists a 2-form ω on $\mathcal{P}_{\mathbf{C}} = \mathcal{S}_{\mathbf{C}}/U(1)$, such that $\pi^*(\omega) = d\varpi$. Since $d[\pi^*(\omega)] = dd\varpi = 0$, and thanks to (art. 6.39), $d\omega = 0$.

☞ **Exercise 101, p. 160** (Irrational tori are orientable). Let us consider two plots $P : U \to \mathbf{R}^n$ and $P' : U \to \mathbf{R}^n$ such that $\pi_\Gamma \circ P = \pi_\Gamma \circ P'$. Thus, the map $r \mapsto P'(r) - P(r)$ takes its values in Γ, but since Γ is discrete in $\mathbf{R}^n$, and $P' - P$ is smooth, the map $P' - P$ is locally constant. Then, there exist an open neighborhood V of r and an element $\gamma \in \Gamma$ such that $P' \upharpoonright V = \tau_\gamma \circ P \upharpoonright V$, where τ_γ is the translation by γ. Hence, $(P' \upharpoonright V)^*(\mathrm{vol}_n) = (\tau_\gamma \circ P \upharpoonright V)^*(\mathrm{vol}_n) = (P \upharpoonright V)^*(\tau_\gamma^*(\mathrm{vol}_n)) = (P \upharpoonright V)^*(\mathrm{vol}_n)$. Since $P^*(\mathrm{vol}_n)$ and $P'^*(\mathrm{vol}_n)$ coincide locally they coincide globally. Now, by application of (art. 6.38), there exists an n-form $\mathrm{vol}_\Gamma \in \Omega^n(T_\Gamma)$ such that $\pi_\Gamma^*(\mathrm{vol}_\Gamma) = \mathrm{vol}_n$. Since vol_n does not vanish anywhere, vol_Γ does not vanish anywhere (art. 6.39), and vol_Γ is a volume of the torus T_Γ. Hence, the torus T_Γ is orientable. Now, let vol be another volume on T_Γ. Its pullback $\pi_\Gamma^*(\mathrm{vol})$ is a volume of $\mathbf{R}^n$. Thus, there exists a smooth real function f on $\mathbf{R}^n$ such that $\pi_\Gamma^*(\mathrm{vol}) = f \times \mathrm{vol}_n$ (art. 6.44). But thanks to the invariance of $\pi_\Gamma^*(\mathrm{vol})$ and vol_n under the action of Γ, the function f also is invariant by Γ. Hence, since Γ is assumed to be dense, the function f is constant and $\mathrm{vol} = c \times \mathrm{vol}_\Gamma$, with $c \in \mathbf{R}$.

☞ **Exercise 102, p. 168** (The k-forms bundle on a real domain). For the first question, $P : V \to \Omega^k(U)$ is smooth if for all plots $Q : W \to U$, $(r, s) \mapsto [P(r)(Q)](s)$, defined on $V \times W$ is smooth. In particular, for $Q = \mathbf{1}_U$ that gives $(r, x) \mapsto [P(r)(\mathbf{1}_U)](x) = (r, x) \mapsto a_r(x)$ smooth, where $a_r(x) \in \Lambda^k(\mathbf{R}^n)$. Now, $(r, s) \mapsto [P(r)(Q)](s) = [P(r)(\mathbf{1}_U \circ Q)](s) = Q^*(P(r)(\mathbf{1}_U))(s) = Q^*(a_r)(s)$, hence, since pullback preserves smoothness, if $(r, x) \mapsto a_r(x)$ is smooth, then $(r, s) \mapsto Q^*(a_r)(s)$ also is smooth. Therefore, P is smooth if and only if $(r, x) \mapsto a_r(x)$ is smooth. For the second question, the condition $\alpha_x = \beta_x$ means that $\alpha(Q)(0) = \beta(Q)(0)$ for all plots $Q : V \to U$ centered at x, that is, $Q(0) = x$. In particular, for $Q = T_x$, where T_x is the translation $x' \mapsto x' + x$, $\alpha(T_x)(0) = \beta(T_x)(0)$, that is, $T_x^*(\alpha(\mathbf{1}_U))(0) = T_x^*(\beta(\mathbf{1}_U))(0)$, which gives $a(x) = b(x)$. Conversely, for every k-tuple of vectors of $\mathbf{R}^n$, denoted by $[v_i]$, $\alpha(Q)(0)[v_i] = Q^*(a)(0)[v_i] = a(x)[D(Q)(0)(v_i)]$. Thus, if $a(x) = b(x)$, then $\alpha(Q)(0) = \beta(Q)(0)$ and $\alpha_x = \beta_x$. Therefore, the value of a k-form α on U at the point x is defined by the value $a(x) = \alpha(\mathbf{1}_U)(x) \in \Lambda^k(\mathbf{R}^n)$. Since for every $a \in \Lambda^k(\mathbf{R}^n)$ we can choose the constant form $a(x) = a$, every $a \in \Lambda^k(\mathbf{R}^n)$ is the value of some smooth k-form α on U, and $\Lambda_x^k(U) \simeq \Lambda^k(\mathbf{R}^n)$. For the third question, note first that the map ϕ is well defined. We have seen that if $\alpha_x = \beta_x$, then $a(x) = b(x)$. Now, let $P : V \to \Lambda^k(U)$ be a plot, and let $Q : W \to U \times \Omega^k(U)$ be a smooth local lift of P along the projection $\pi : U \times \Omega^k(U) \to \Lambda^k(U)$, defined by $\pi(x, \alpha) = (x, \alpha_x)$. Let $Q(r) = (x_r, A(r))$ and $a_r = A(r)(\mathbf{1}_U)$, then $\phi \circ (P \upharpoonright W) = \phi \circ \pi \circ Q$ gives $\phi(P(r)) = (x_r, A(r)(\mathbf{1}_U)(x_r) = a_r(x_r))$. Since $A : W \to \Omega^k(U)$ is smooth, $r \mapsto a_r = A(r)(\mathbf{1}_U)$ is smooth, and since $r \mapsto x_r$ is smooth, $r \mapsto a_r(x_r)$ is smooth. Therefore $\phi \circ (P \upharpoonright W)$ is smooth, which implies that $\phi \circ P$ is smooth, and therefore ϕ. Conversely, let $P : r \mapsto (x_r, a_r)$ be a plot of $U \times \Lambda^k(\mathbf{R}^n)$, let $\mathbf{a}_r$ be the constant k-form on U with value a_r. Then, $\phi^{-1}(x_r, a_r) = \pi(x_r, \mathbf{a}_r)$, and since $r \mapsto x_r$, π and $r \mapsto \mathbf{a}_r$ are smooth, $\phi^{-1} \circ P$ is smooth. Therefore, ϕ is a diffeomorphism.

☞ **Exercise 103, p. 169** (The p-form bundle on a manifold). For the first question, let $x \in U$. We can choose $\varepsilon > 0$ such that the ball $B(x, \varepsilon)$ is strictly contained in U, that is, $U - B(x, \varepsilon) \neq \varnothing$. Then, there exists a smooth real function λ equal to 1 on the ball of radius $\varepsilon/2$ centered at x, and equal to zero, outside $B(x, \varepsilon)$. The smooth p-form $\bar{a} : x \mapsto \lambda(x)\, a$ satisfies $\bar{a}(x) = a$ and vanishes outside $B(x, \varepsilon)$. For the second question, Let $\mathcal{O} = F(U)$, since F is a local diffeomorphism, $\mathcal{O}$ is D-open and $F_*(\bar{a}) = (F^{-1})^*(\bar{a})$ is a p-form on $\mathcal{O}$, vanishing outside the D-open subset $F(B(x, \varepsilon))$. Since differential forms are local (art. 6.36), $F_*(\bar{a})$ can be extended by a differential p-form α on M such that $\alpha \upharpoonright (M - \mathcal{O}) = 0$. Now, since F is a local diffeomorphism, and thanks to Exercise 102, p. 168, the value of α at the point m is characterized by the value of $\bar{a}$ at x, that is, a. More precisely $\alpha_m = F_*(\bar{a}_x)$, that is, with our identification, $\alpha_m = F_*(a)$, where F_* is defined in (art. 6.51). Therefore, $\Lambda_m^p(M) \simeq \Lambda^p(\mathbf{R}^n)$. That answers the third question. For the fourth question, we just built the map $\mathcal{F}$, from $U \times \Lambda^p(\mathbf{R}^n)$

to $\Lambda^p(M)$, defined by $\mathcal{F}(x,a) = (F(x), F_*(a))$. By construction, $\mathcal{F}$ is bijective. Let us check rapidly that $\mathcal{F}$ is a local diffeomorphism. Let $x_0 \in U$, there exists an open ball B_0 centered at x_0 and $\varepsilon > 0$ such that, for all $x \in B_0$, $B(x,\varepsilon)$ is strictly included in U. Then, by the previous construction, we get for each $(x,a) \in B_0 \times \Lambda^p(\mathbf{R}^n)$ a smooth p-form $\bar{a}$ such that $\bar{a}_x = a$. Moreover, we can choose for each $x \in B_0$ the bump-function λ depending smoothly on x, thus the map $(x,a) \mapsto (x,\bar{a})$ is smooth. Hence, $\mathcal{F} \restriction B_0$ is smooth, therefore $\mathcal{F}$. Conversely, a smooth parametrization $r \mapsto (m_r, \alpha_r)$ in $\Lambda^p(M)$ can be locally lifted by a smooth parametrization $r \mapsto (m_r, A(r))$ in $M \times \Omega^p(M)$, that is, $\alpha_r = A(r)_{m_r}$. Then, by pullback, we get a smooth parametrization $r \mapsto (F^{-1}(m_r), F^*(A(r)))$ in $U \times \Omega^p(U)$ such that $\mathcal{F}^{-1}(m_r, \alpha_r) = (x_r = F^{-1}(m_r), [F^*(A(r))]_{x_r})$. Therefore, $\mathcal{F}^{-1}$ is smooth and $\mathcal{F}$ is a local diffeomorphism. When F runs over an atlas of M the charts $\mathcal{F}$ run obviously over an atlas of $\Lambda^p(M)$. Finally, $\Lambda^p(M)$ is a manifold, and $\dim(\Lambda^p(M)) = \dim(M) + \dim(\Lambda^p(\mathbf{R}^n)) = \dim(M) + C^p_n$. Note that also shows that $\Lambda^p(M)$ is a locally trivial bundle over M (art. 8.9).

☞ **Exercise 104, p. 169** (Smooth forms on diffeological vector spaces). 1) Let $\alpha \in \Omega^1(\mathcal{O})$, and let us show that $A(x) : u \mapsto \alpha(t \mapsto x + tu)_0(1)$ is linear. Let $u, v \in E$, $A(x)(u+v) = \alpha(t \mapsto x+tu+tv)_0(1)$. Let $\phi : (t,s) \mapsto x+tu+sv$ and $\Delta : t \mapsto (t,t)$. Then, $\alpha(t \mapsto x+tu+tv)_0(1) = \alpha(\phi \circ \Delta)_0(1) = \Delta^*(\alpha(\phi))_0(1) = \alpha(\phi)_{(0,0)}(1,1) = \alpha(\phi)_{(0,0)}(1,0) + \alpha(\phi)_{(0,0)}(0,1)$. Now, let $j_1 : t \mapsto (t,0)$ and $j_2 : s \mapsto (0,s)$, then $\alpha(\phi)_{(0,0)}(1,0) = j_1^*(\alpha(\phi))_0(1) = \alpha(\phi \circ j_1)_0(1) = \alpha(t \mapsto x+tu)_0(1) = A(x)(u)$ and $\alpha(\phi)_{(0,0)}(0,1) = j_2^*(\alpha(\phi))_0(1) = \alpha(\phi \circ j_2)_0(1) = \alpha(s \mapsto x+sv)_0(1) = A(x)(v)$. Thus, $A(x)(u+v) = A(x)(u) + A(x)(v)$. Next, let $\lambda \in \mathbf{R}$, $A(x)(\lambda u) = \alpha(t \mapsto x + t\lambda u)_0(1) = \alpha(t \mapsto s = t\lambda \mapsto x + su)_0(1) = \alpha(s \mapsto x+su)_0(\lambda) = \lambda\alpha(s \mapsto x+su)_0(1) = \lambda A(x)(u)$. Therefore, $A(x)$ is linear, that is, $A(x) \in E^*$. Let us show that $A(x) \in E^*_\infty$, that is, for every n-plot $Q : V \to E$, $A(x) \circ Q \in \mathcal{C}^\infty(V, \mathbf{R})$. Let $T_x : u \mapsto u + x$, T_x is a diffeomorphism of E, then $A(x)(Q(s)) = \alpha(t \mapsto x + tQ(s))_0(1) = T_x^*(\alpha)(t \mapsto tQ(s))_0(1)$. But $(t,s) \mapsto tQ(s)$ is a plot of E, and $T_x^*(\alpha)$ is a differential 1-form on E, thus $T_x^*(\alpha)((t,s) \mapsto tQ(s))$ is a smooth 1-form on $\mathcal{I} \times V$, where $\mathcal{I}$ is a small interval around $0 \in \mathbf{R}$. Hence, the map $(t,s) \mapsto T_x^*(\alpha)((t,s) \mapsto tQ(s))_{(t,s)}$ is a smooth parametrization in $\Lambda^1(\mathbf{R}^{1+n})$, and $s \mapsto T_x^*(\alpha)(t \mapsto tQ(s))_0(1) = T_x^*(\alpha)((t,s) \mapsto tQ(s))_{(0,s)}(1,0)$ is smooth. Let us show now that $A \in \mathcal{C}^\infty(\mathcal{O}, E^*_\infty)$. A is smooth if for every plot $P : U \to \mathcal{O}$ and $Q : V \to E$, the parametrization $(r,s) \mapsto A(P(r))(Q(s)) = \alpha(t \mapsto P(r) + tQ(s))_0(1)$ is smooth. But $(r,s,t) \mapsto P(r) + tQ(s)$ is a plot of E, we are in a situation analogous to the previous one and thus A is smooth.

2) Conversely, let $A \in \mathcal{C}^\infty(\mathcal{O}, E^*_\infty)$ and $\alpha(P)_r = d[A(x) \circ P]_r$, with $x = P(r)$, that is, $\alpha(P)_r = d[s \mapsto A(P(r)) \circ P(s)]_{s=r}$. Note first that, since $A(P(r)) \circ P$ is a real function defined on U, $d[s \mapsto A(P(r)) \circ P(s)]_{s=r}$ is a smooth 1-form on U. Next, let $F : V \to U$ be a smooth parametrization, let $t \in V$ and $r = f(t)$. On the one hand, $\alpha(P \circ F)_t = d[s \mapsto A(P \circ F(t)) \circ P \circ F(s)]_{s=t} = D(A(P(F(t))) \circ P \circ F)(s = t)$, and on the other hand, $F^*(\alpha(P))_t = \alpha(P)_{r=F(t)} \circ D(F)(t) = d[s \mapsto A(P(r)) \circ P(s)]_{s=r=F(t)} \circ D(F)(t) = D(A(P(F(t))) \circ P)(r = F(t)) \circ D(F)(t)$, but $D(A(P(F(t))) \circ P \circ F)(s = t) = D(A(P(F(t))) \circ P)(r = F(t)) \circ D(F)(t)$, by the chain rule. Thus, $\alpha(P \circ F)_t = F^*(\alpha(P))_t$. Therefore, α is a differential 1-form on $\mathcal{O}$.

3) Let us check now that $\sigma \circ \pi = \mathbf{1}$. Let $\sigma(A) = \alpha$ and $\pi(\alpha) = A'$, then $A'(x)(u) = \alpha(t \mapsto P(t) = x+tu)_{t=0}(1) = D(t \mapsto A(P(0)) \circ P(t))_{t=0}(1) = D[t \mapsto A(x)(x+tu) = A(x)(x) + tA(x)(u)]_{t=0}(1) = A(x)(u)$, thus $A' = A$.

4) We remark that π is linear, and let $\alpha \in \ker(\pi)$, that is, $\alpha(t \mapsto x + tu)_0(1) = 0$, for all $x \in \mathcal{O}$ and all $u \in E$. We know that the value of a 1-form is given by its values on the 1-plots (art. 6.37), that is, if $\alpha(c) = 0$, for all 1-plots c of E, then $\alpha = 0$. Now, let c be defined on an interval around $t_0 \in \mathbf{R}$, and let $x_0 = c(t_0)$. Decompose c into $t \mapsto s = t - t_0 \mapsto \bar{c}(s) = c(s + t_0) - x_0 \mapsto \bar{c}(s) + x_0$, that is, $c = T_{x_0} \circ \bar{c} \circ T_{-t_0}$. Then, $\alpha(c)_{t_0}(1) = \alpha(T_{x_0} \circ \bar{c} \circ T_{-t_0})_{t_0}(1) = T^*_{x_0}(\alpha)(\bar{c})_0(1)$. But, by hypothesis, there exists $u \in E$ such that $T^*_{x_0}(\alpha)(\bar{c})_0(1) = T^*_{x_0}(\alpha)(t \mapsto tu)_0(1)$, Then, since $\alpha \in \ker(\pi)$, $\alpha(c)_{t_0}(1) = \alpha(t \mapsto x_0 + tu)_0(1) = 0$, thus $\alpha = 0$. Therefore, π is injective and surjective, since $\pi \circ \sigma = \mathbf{1}$, that is, a linear isomorphism.

☞ **Exercise 105, p. 170** (Forms bundles of irrational tori). Let α be a p-form on T_Γ. Let $a = \pi^*(\alpha)$. Thus, for each $\gamma \in \Gamma$, $\gamma^*(a) = \gamma^*(\pi^*(\alpha)) = (\pi \circ \gamma)^*(\alpha) = \pi^*(\alpha) = a$. Hence, a is invariant under the translations $\gamma \in \Gamma$. Now, let us decompose the form a of $\mathbf{R}^n$ in the canonical basis of $\Lambda^p(\mathbf{R}^n)$ (art. 6.16),

$$a(x) = \sum_{ij\cdots k} a_{ij\cdots k}(x)\ \boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \cdots \wedge \boldsymbol{e}^k,$$

where $x \in \mathbf{R}^n$ and the $a_{ij\cdots k}$ are smooth real functions. Since the monomials $\boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \cdots \wedge \boldsymbol{e}^k$ are invariant by translation, we get

$$\gamma^*(a) = a \quad \Rightarrow \quad a_{ij\cdots k}(x + \gamma) = a_{ij\cdots k}(x),$$

for all families of indices $ij \cdots k$. But since Γ is a dense subgroup of $\mathbf{R}^n$, and the $a_{ij\cdots k}$ are smooth, they are constant. Therefore, a is a constant p-form of $\mathbf{R}^n$. Now, let $a \in \Lambda^p(\mathbf{R}^n)$, and let us show that there exists α, p-form of T_Γ, such that $a = \pi^*(\alpha)$. We shall apply the criterion (art. 6.38). Let P and P' be two plots of $\mathbf{R}^n$, defined on the same domain U, such that $\pi \circ P = \pi \circ P'$. Thus, for every $r \in U$, $P'(r) - P(r) \in \Gamma$. But the parametrization $P' - P$ is smooth, hence locally constant, since Γ is discrete. Thus, restricting P and P' to connected parts of U, the difference $P' - P$ is constant, that is, $P'(r) = P(r) + \gamma$. Thus, $P'^*(a) = P^*(a)$ on the whole U. The criterion is satisfied and there exists $\alpha \in \Omega^p(T_\Gamma)$ such that $a = \pi^*(\alpha)$. Conversely, the map $a \mapsto \alpha$ is injective, indeed, thanks to (art. 6.39), if $\pi^*(\alpha) = 0$, then $\alpha = 0$. Therefore, the map $a \mapsto \alpha$, defined on $\Lambda^p(\mathbf{R}^n)$ to $\Omega^p(T_\Gamma)$ is an isomorphism. Next, let us consider $T_\Gamma \times \Omega^p(T_\Gamma) \simeq T_\Gamma \times \Lambda^p(\mathbf{R}^n)$. A pair (τ, α) is equivalent to (τ', α') if and only if $\tau = \tau'$ and $\alpha_\tau = \alpha'_{\tau'}$. But $\alpha_\tau = \alpha'_{\tau'}$ implies $a = a'$, where $a = \pi^*_\Gamma(\alpha)$ and $a' = \pi^*_\Gamma(\alpha')$. Hence,

$$\Lambda^p(T_\Gamma) \simeq T_\Gamma \times \Lambda^p(\mathbf{R}^n).$$

Note. We can check that a smooth section $\tau \mapsto (\tau, a)$ of $\Lambda^p(T_\Gamma)$ is just a smooth map $\tau \mapsto a$, and since $\mathcal{C}^\infty(T_\Gamma, \Lambda^p(\mathbf{R}^n)) \simeq \Lambda^p(\mathbf{R}^n)$ are just the constant maps, then every smooth p-form on T_Γ is *constant.* Now, we can look at the Liouville form on $\Lambda^p(T_\Gamma)$. But, rather than looking at Liouv, let us consider its pullback $\pi^*_\Gamma(\mathrm{Liouv})$ on $\mathbf{R}^n \times \Lambda^p(\mathbf{R}^n)$. Let $Q \times A$ be a plot of $\mathbf{R}^n \times \Lambda^p(\mathbf{R}^n)$, it is just a pair of smooth parametrizations defined on some domain U. Then, let $r \in U$ and $\delta r = (\delta_1 r) \cdots (\delta_p r)$ be a p-vector of $\mathbf{R}^n$,

$$\begin{aligned} \pi^*_\Gamma(\mathrm{Liouv})(Q \times A)(r)(\delta r) &= A(r)(Q)(r)(\delta r) \\ &= \sum_{i<j<\cdots<k} A(r)_{ij\cdots k}(\delta_i q)(\delta_j q) \cdots (\delta_k q), \end{aligned}$$

where the $\delta_i q = D(Q)(r)(\delta_i r)$ are again vectors of $\mathbf{R}^n$.

☞ **Exercise 106, p. 170** (Vector bundles of irrational tori). Let T_Γ be the irrational torus $\mathbf{R}^n/\Gamma$ where Γ is a dense generating subgroup of $\mathbf{R}^n$ and $\Lambda^p(T_\Gamma) \simeq T_\Gamma \times \Lambda^p(\mathbf{R}^n)$; see Exercise 105, p. 170. Let Q be a global p-plot of T_Γ. There exist a small ball B, centered at $0 \in \mathbf{R}^p$, and a lifting $\tilde{Q} : B \to \mathbf{R}^n$, such that $\pi \circ \tilde{Q} = Q \restriction B$. For the same reason developed in the proof of Exercise 105, p. 170, two such liftings differ from a constant element of Γ. Now, let $\alpha \in \Omega^p(T_\Gamma)$, and let $a \in \Lambda^p(\mathbf{R}^n)$ be the unique p-form such that $a = \pi^*(\alpha)$ (see Exercise 105, p. 170). We have, $\alpha(Q)(0) = \alpha(\pi \circ \tilde{Q})(0) = \pi^*(\alpha)(\tilde{Q})(0) = a(\tilde{Q})(0) = a(q_1, \ldots, q_p)$, where $q_i = D(\tilde{Q})(0)(\mathbf{e}_i) \in \mathbf{R}^n$, $i = 1 \cdots p$. A translation of $\tilde{Q}$ by a constant element $\gamma \in \Gamma$ does not change the q_i. Hence, $\alpha(Q)(0) = \alpha(Q')(0)$ if and only if $q_i = q'_i$ for all $i = 1 \cdots p$. On the other hand, for any p vectors $(v_1, \ldots, v_p) \in (\mathbf{R}^n)^p$, there exists a p-plot $Q = (t_1, \ldots, t_p) \mapsto \pi(\sum_{i=1}^p t_i v_i)$ such that $q_i = v_i$. Hence, the map j_p from $\mathrm{Paths}_p(T_\Gamma)$ to $T_\Gamma \times L^\infty(\Omega^p(T_\Gamma), \mathbf{R}) = T_\Gamma \times L(\Lambda^p(\mathbf{R}^n), \mathbf{R})$ is given by

$$j_p : Q \mapsto (Q(0), [a \mapsto a(q_1, \ldots, q_p)]).$$

But the map $[a \mapsto a(q_1, \ldots, q_p)]$ is just an element of the dual of $\Lambda^p(\mathbf{R}^n)$. Then, since each element of the dual $\Lambda^p(\mathbf{R}^n)^*$ can be associated with a global p-plot,

$$T^p(T_\Gamma) \simeq T_\Gamma \times [\Lambda^p(\mathbf{R}^n)]^* \simeq T_\Gamma \times \mathbf{R}^{n!/p!(n-p)!}.$$

In particular,

$$T_x(T_\Gamma) \simeq \mathbf{R}^n \quad \text{and} \quad T(T_\Gamma) \simeq T_\Gamma \times \mathbf{R}^n.$$

NOTE. If we had tested the paths on differential of function, which is one of the usual ways in classical differential geometry of manifolds, we should have get $T_x(T_\Gamma) = \{0\}$, since the only real functions defined on T_Γ are constant. And this is clearly unsatisfactory. Thus, the definition suggested in (art. 6.53) is, for diffeological spaces at least, better. However, note that a section of the tangent bundle is necessarily constant $x \mapsto (x, v)$, with $v \in \mathbf{R}^n$, which is not really surprising.

☞ **Exercise 107, p. 170** (Differential 1-forms on $\mathbf{R}/\{\pm 1\}$). Let us check that the 1-form $d[t^2] = 2t \times dt$ passes to the quotient Δ. We apply the criterion (art. 6.38). Let P and P' be two plots such that $\mathrm{sq} \circ P = \mathrm{sq} \circ P'$, that is, $P(r)^2 = P'(r)^2$ for all $r \in \mathrm{def}(P) = \mathrm{def}(P')$, then $d[t^2](P)_r = D(r \mapsto P(r)^2)_r = D(r \mapsto P'(r)^2)_r = d[t^2](P')_r$. Thus, there exists a 1-form θ on Δ such that $\mathrm{sq}^*(\theta) = d[t^2]$. Now, let α be a differential 1-form on Δ, its pullback $\mathrm{sq}^*(\alpha)$ is a differential 1-form on $\mathbf{R}$, that is $\mathrm{sq}^*(\alpha) = F(t)dt$. But $\mathrm{sq}^*(\alpha)$ is invariant by $\{\pm 1\}$, hence $F(-t) = -F(t)$, then F vanishes at $t = 0$. Since F is $\mathcal{C}^\infty$, there exists a smooth function φ such that $F(t) = 2t\varphi(t)$. Thus, $\mathrm{sq}^*(\alpha) = 2t\varphi(t)dt = \varphi(t)d[t^2]$. Now, by invariance, there exists a smooth function $f \in \mathcal{C}^\infty(\Delta, \mathbf{R})$ such that $\mathrm{sq} \circ \varphi = f$. Therefore, $\mathrm{sq}^*(\alpha) = \mathrm{sq}^*(f \times \theta)$. By (art. 6.39), $\alpha = f \times \theta$. Next, every 1-plot γ of Δ such that $\gamma(0) = 0$ has a local lift $\bar{\gamma}$ in $\mathbf{R}$ such that $\bar{\gamma}(0) = 0$, and then $\alpha(\gamma)_0(1) = \mathrm{sq}^*(\alpha)(\bar{\gamma})_0(1) = \varphi(\bar{\gamma}(0)) \times 2\bar{\gamma}(0)\bar{\gamma}'(0) = 0$, where the prime $'$ denotes the derivative with respect to t. Therefore, $\alpha_0 = 0$ and by consequence $T_0(\Delta) = \{0\}$.

☞ **Exercise 108, p. 175** (Anti-Lie derivative). Let $P : U \to X$ be an n-plot, and let us shortly denote p vectors by $[v] = (v_1) \cdots (v_p)$, we have on the one hand

$$\frac{\partial}{\partial t}\left\{ F(t)^*(F(t)_*(\alpha))(P)_r[v] \right\}_{t=0} = \frac{\partial}{\partial t}\left\{ \alpha(P)_r[v] \right\}_{t=0} = 0,$$

and on the other hand,

$$
\begin{aligned}
\frac{\partial}{\partial t}\left\{F(t)^*(F(t)_*(\alpha))(P)_r[\nu]\right\}_{t=0} &= D\left(\begin{pmatrix} t \\ s \end{pmatrix} \mapsto F(t)^*(F(s)_*(\alpha))(P)_r[\nu]\right)_{\binom{0}{0}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= D\left(\begin{pmatrix} t \\ s \end{pmatrix} \mapsto F(t)^*(F(s)_*(\alpha))(P)_r[\nu]\right)_{\binom{0}{0}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&+ D\left(\begin{pmatrix} t \\ s \end{pmatrix} \mapsto F(t)^*(F(s)_*(\alpha))(P)_r[\nu]\right)_{\binom{0}{0}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= D(t \mapsto F(t)^*(\alpha)(P)_r[\nu])_0(1) \\
&+ D(s \mapsto F(s)_*(\alpha)(P)_r[\nu])_0(1) \\
&= \frac{\partial}{\partial t}\left\{F(t)^*(\alpha)(P)_r[\nu]\right\}_{t=0} \\
&+ \frac{\partial}{\partial s}\left\{F(s)_*(\alpha)(P)_r[\nu]\right\}_{s=0}.
\end{aligned}
$$

Therefore,

$$\frac{\partial}{\partial t}\left\{F(t)_*(\alpha)(P)_r[\nu]\right\}_{t=0} = -\frac{\partial}{\partial t}\left\{F(t)^*(\alpha)(P)_r[\nu]\right\}_{t=0},$$

that is,

$$\frac{\partial}{\partial t}\left\{F(t)_*(\alpha)\right\}_{t=0} = -\pounds_F(\alpha).$$

Exercise 109, p. 175 (Multi-Lie derivative). First of all, the proof that this generalization of the Lie derivative is well defined is a slight adaptation of the first proposition of (art. 6.54). For the second question, let $P : U \to X$ be a plot. The Lie derivative decomposes in the canonical basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q)$ of $\mathbf{R}^q$,

$$
\begin{aligned}
\pounds_h(\alpha)(\nu)(P) &= D[s \mapsto h(s)^*(\alpha(P))](0)(\nu) \\
&= \sum_{i=1}^{q} \nu^i \times D[s \mapsto h(s)^*(\alpha(P))](0)(\boldsymbol{e}_i) \\
&= \sum_{i=1}^{q} \nu^i \times D[s \mapsto h(s\boldsymbol{e}_i)^*(\alpha(P))](0)(1) \\
&= \sum_{i=1}^{q} \nu^i \times D[s \mapsto h_i(s)^*(\alpha(P))](0)(1).
\end{aligned}
$$

But

$$D[s \mapsto h_i(s)^*(\alpha(P))](0)(1) = \frac{\partial}{\partial s}\left(h_i(s)^*(\alpha(P))\right)\Big|_{s=0} = \pounds_{h_i}(\alpha)(P).$$

Therefore,

$$\pounds_h(\alpha)(P) = \sum_{i=1}^{q} \nu^i \pounds_{h_i}(\alpha)(P).$$

Exercise 110, p. 175 (Variations of points of domains). Let F be an arc of the plot x, defined on $]-\varepsilon, +\varepsilon[$. So, $F(0) = x$, $[(s,0) \mapsto F(s)(0)] \in \mathcal{C}^\infty(]-\varepsilon, +\varepsilon[\times \{0\}, U)$.

Thus, F (or $\bar{F}$) is just equivalent to a path $f : s \mapsto F(s)(0)$ of U, such that $f(0) = x$. Now, let $\hat{x}^i = dx^i$ be the i-th coordinate 1-form of U (art. 6.23), and let

$$\nu^i = \hat{x}^i(\bar{F})_{\binom{0}{0}}\begin{pmatrix}1\\0\end{pmatrix} = \left.\frac{df^i(s)}{ds}\right|_{s=0}.$$

Now, since every smooth 1-form α of U is a combination $\alpha = \sum_{i=1}^{n} \alpha_i(x)dx^i$, if two paths f and f', pointed at x, are equivalent, then their derivatives ν^i at 0 are equal. Conversely, let v be the vector of $\mathbf{R}^n$ with coordinates ν^i. The arc $f_\nu : t \mapsto t\nu$ can be chosen as a representative of f. And, since $\mathbf{x}$ is a 0-plot, we do not have to check the equivalence on p-forms with $p > 1$. Therefore, a variation of a point x of U is just a vector ν of $\mathbf{R}^n$, which we could summarize by $\delta\mathbf{x} = (x, \nu) \in U \times \mathbf{R}^n$.

☞ **Exercise 111, p. 176** (Liouville rays). 1) Since the p-form ω is not the zero form, there exist a plot $P : U \to X$, a point $r \in U$, and p vectors $\nu_1, \ldots, \nu_p$ of $\mathbf{R}^n$, $n = \dim(U)$, such that $\omega(P)(r)(\nu_1)\cdots(\nu_p) \neq 0$. So, $h(t)^*(\omega) = \lambda(t)\omega$ implies

$$\lambda(t) = \frac{h(t)^*(\omega)(P)(r)(\nu_1)\cdots(\nu_p)}{\omega(P)(r)(\nu_1)\cdots(\nu_p)}.$$

Since h is the smooth homomorphism from $\mathbf{R}$ to $\mathrm{Diff}(X)$, by definition of the functional diffeology, $t \mapsto h(t)^*(\omega)(P)(r)(\nu_1)\cdots(\nu_p)$ is smooth. Hence, λ is smooth.

2) Thanks to (art. 6.55), we have for all $t \in \mathbf{R}$,

$$\frac{\partial h(t)^*(\omega)}{\partial t} = h(t)^*(\pounds_h(\omega)), \quad \text{thus} \quad \lambda'(t) \times \omega = \lambda(t) \times \omega,$$

where the prime denotes the derivative. Hence, $(\lambda'(t) - \lambda(t)) \times \omega = 0$, and since $\omega \neq 0$, $\lambda'(t) = \lambda(t)$, that is, $\lambda(t) = ce^t$. But $h(0) = 1_X$, thus $\omega = h(0)^*(\omega) = \lambda(0) \times \omega = c \times \omega$. Therefore, $c = 1$ and $\lambda(t) = e^t$.

☞ **Exercise 112, p. 186** (The boundary of a 3-cube). Using the notation $\{t_1t_2t_3\}$ for $\sigma(t_1)(t_2)(t_3)$ and formulas in (art. 6.59), we have

$$\partial\sigma(t_1)(t_2) = \{1t_1t_2\} - \{0t_1t_2\} - \{t_11t_2\} + \{t_10t_2\} + \{t_1t_21\} - \{t_1t_20\},$$

and therefore

$$\begin{aligned}
\partial(\partial\sigma)(t) &= [\partial\sigma(1)(t) - \partial\sigma(0)(t)] - [\partial\sigma(t)(1) - \partial\sigma(t)(0)] \\
&= \{11t\} - \{01t\} - \{11t\} + \{10t\} + \{1t1\} - \{1t0\} \\
&- \{10t\} + \{00t\} + \{01t\} - \{00t\} - \{0t1\} + \{0t0\} \\
&- \{1t1\} + \{0t1\} + \{t11\} - \{t01\} - \{t11\} + \{t10\} \\
&+ \{1t0\} - \{0t0\} - \{t10\} + \{t00\} + \{t01\} - \{t00\} \\
&= 0.
\end{aligned}$$

☞ **Exercise 113, p. 186** (Cubic homology of a point). First of all let us note that $\mathrm{Cub}_p(\star) = \{\hat{0} = [t \mapsto 0]\}$, thus $\mathrm{C}_p(\star) = \{n\,\hat{0} \mid n \in \mathbf{Z}\} \simeq \mathbf{Z}$, for all $p \in \mathbf{N}$. Since $\partial\,\hat{0} = 0$ (the zero chain), $\partial[\mathrm{C}_p(\star)] = \{0\}$, that is, $\mathrm{Z}_p(\star) = \mathrm{C}_p(\star) \simeq \mathbf{Z}$ and $\mathrm{B}_p(\star) = \{0\}$, for all $p \in \mathbf{N}$. Therefore, for all $p \in \mathbf{N}$, $\mathrm{H}_p(\star) \simeq \mathbf{Z}$. Now, let us consider the reduced cubic chains. For $p = 0$, there is no degenerate 0-chain, $\mathrm{C}^\bullet_0(X) = \{0\}$, then $\mathbf{C}_0(\star) = \mathrm{C}_0(X)/\mathrm{C}^\bullet_0(X) = \mathrm{C}_0(X) \simeq \mathbf{Z}$. For $p \geq 1$, since every cubic chain is constant, every cubic chain is degenerate, that is, $\mathrm{C}_p(X) = \mathrm{C}^\bullet_p(X)$, and then $\mathbf{C}_p(\star) = \mathrm{C}_p(X)/\mathrm{C}^\bullet_p(X) = \{0\}$. Therefore, $\mathbf{H}_0(\star) = \mathbf{Z}$ and $\mathbf{H}_p(\star) = \{0\}$, for all $p \geq 1$.

☞ **Exercise 114, p. 198** (Liouville rays and closed forms). By application of the Cartan formula (art. 6.72), $\pounds_h(\omega) = i_h(d\omega) + d(i_h(\omega))$. Since $d\omega = 0$, and $\pounds_h(\omega) = \omega$, we get $\omega = d[i_h(\omega)]$. The p-form ω is exact and $i_h(\omega)$ is one of its primitives.

☞ **Exercise 115, p. 198** (Integrals on homotopic cubes). In our case, $d\alpha = 0$ and $\delta\alpha = 0$, the variation of the integral on a p-cube σ (art. 6.70) is just

$$\delta \int_\sigma \alpha = \int_{\partial I^p} \alpha(\delta\sigma),$$

with

$$\alpha(\delta\sigma)_r(v_2)\cdots(v_p) = \alpha(\boldsymbol{\sigma})_{\binom{0}{r}} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\v_2\end{pmatrix} \cdots \begin{pmatrix}0\\v_p\end{pmatrix},$$

$\boldsymbol{\sigma} : (s,r) \mapsto \sigma_s(r)$, and $v_i \in \mathbf{R}^p$. But the variation involves the restriction of $\boldsymbol{\sigma}$ to the boundary ∂I^p, and by hypothesis, $\boldsymbol{\sigma}(s,r) \upharpoonright \partial I^p = \sigma(r) \upharpoonright \partial I^p$, for all s. Thus, restricted to the boundary,

$$\begin{aligned}
\alpha(\boldsymbol{\sigma})_{\binom{0}{r}} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\v_2\end{pmatrix} \cdots \begin{pmatrix}0\\v_p\end{pmatrix} &= \alpha(\sigma \circ \mathrm{pr}_2)_{\binom{0}{r}} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\v_2\end{pmatrix} \cdots \begin{pmatrix}0\\v_p\end{pmatrix} \\
&= \mathrm{pr}_2^*[\alpha(\sigma)]_{\binom{0}{r}} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\v_2\end{pmatrix} \cdots \begin{pmatrix}0\\v_p\end{pmatrix} \\
&= \alpha(\sigma)_r(0)(v_2)\cdots(v_p) \\
&= 0.
\end{aligned}$$

Therefore, the variation vanishes, and the integral of a closed p-form is constant along a fixed-boundary homotopy of p-cubes.

☞ **Exercise 116, p. 198** (Closed 1-forms on connected spaces). We have seen in Exercise 115, p. 198, that the integral of a closed p-form on a p-cube does not depend on the fixed-boundary homotopy class of the p-cube. Applied to 1-forms it just says that the integral $\int_\ell \alpha$, where $\alpha \in \Omega^1(X)$, $d\alpha = 0$ and $\ell \in \mathrm{Loops}(X,x)$, does not depend on the fixed-ends homotopy class of ℓ. Now let $\ell' \in \mathrm{Loops}(X,x)$, since ℓ and ℓ' are always fixed-ends homotopic to two stationary loops (which do not change the integrals), we can assume that ℓ and ℓ' are stationary. Then,

$$\begin{aligned}
\int_{\ell\vee\ell'} \alpha &= \int_0^1 \alpha(\ell\vee\ell')_t(1)\,dt \\
&= \int_0^{1/2} \alpha(\ell\vee\ell')_t(1)\,dt + \int_{1/2}^1 \alpha(\ell\vee\ell')_t(1)\,dt \\
&= \int_0^{1/2} \alpha(t\mapsto \ell(2t))_t(1)\,dt + \int_{1/2}^1 \alpha(t\mapsto \ell'(2t-1))_t(1)\,dt \\
&= \int_0^1 \alpha(t\mapsto \ell(t))_t(1)\,dt + \int_0^1 \alpha(t\mapsto \ell'(t))_t(1)\,dt \\
&= \int_\ell \alpha + \int_{\ell'} \alpha.
\end{aligned}$$

Thus, $\mathrm{class}(\ell) \mapsto \int_\ell \alpha$ is a homomorphism from $\pi_1(X,x)$ to $\mathbf{R}$ and P_α is the image of this homomorphism. Now, let $x' \in X$, since X is connected, there exists a stationary path c connecting x to x', and clearly $\int_{c\vee\ell\vee\bar{c}} \alpha = \int_c \alpha + \int_\ell \alpha - \int_c \alpha = \int_\ell \alpha$, where

$\ell \in \mathrm{Loops}(X, x')$ and $\bar{c}(t) = 1 - t$ is the reverse of c. Therefore, the group P_α does not depend on the point where it is computed.

☞ **Exercise 117, p. 198** (Closed 1-forms on simply connected spaces). Let us assume that the connected component of x is simply connected. Let $s \mapsto \ell_s$ be a smooth path in X connecting $\ell = \ell_0$ to the constant path $\ell_1 : t \mapsto x$. We have

$$\frac{\partial}{\partial s}\left\{\int_\ell \alpha\right\}_s = \int_0^1 d\alpha(\delta\ell_s) + \int_0^1 d[\alpha(\delta\ell_s)] = 0 + \Big[\alpha(\delta\ell_s)\Big]_{t=0}^{t=1}.$$

Now, let $\bar{\ell} : (s,t) \mapsto \ell_s(t)$ and $j_t : s \mapsto (s,t)$, we have

$$\begin{aligned}
\alpha(\delta\ell_s)(0) &= \alpha(\bar{\ell})_{\binom{s}{0}}\begin{pmatrix}1\\0\end{pmatrix} = j_0^*[\alpha(\bar{\ell})](s)(1)\\
&= \alpha(\bar{\ell}\circ j_0)(s)(1) = \alpha(s \mapsto \ell_s(0) = x)(s)(1)\\
&= 0,
\end{aligned}$$

and the same holds for $t = 1$, $\alpha(\delta\ell_s)(1) = 0$. Thus,

$$\frac{\partial}{\partial s}\left\{\int_\ell \alpha\right\}_s = 0 \quad\Rightarrow\quad \int_{\ell_s}\alpha = \mathrm{cst} \quad\Rightarrow\quad \int_\ell \alpha = \int_{[t\mapsto x]}\alpha = 0.$$

So, if the integral of a closed 1-form on a loop is nonzero, then the space cannot be simply connected.

☞ **Exercise 118, p. 202** (1-forms vanishing on loops). Let $\sigma : \mathbf{R}^2 \to X$ be a 2-cube, and let its boundary $\partial\sigma$ be a loop. It is the concatenation ℓ of the four paths $\gamma_1 : t \mapsto \sigma(t,0)$, $\gamma_2 : t \mapsto \sigma(1,t)$, $\gamma_3 : t \mapsto \sigma(1-t,1)$ and $\gamma_4 : t \mapsto \sigma(0,1-t)$. Then, because the integral of α vanishes on loops, and thanks to Stokes' theorem,

$$\int_\sigma d\alpha = \int_{\partial\sigma}\alpha = \int_{\gamma_1}\alpha + \int_{\gamma_2}\alpha + \int_{\gamma_3}\alpha + \int_{\gamma_4}\alpha = \int_\ell \alpha = 0.$$

Thus, since the integral of $d\alpha$ vanishes on every 2-cube, $d\alpha = 0$ (art. 6.66). Next, since α is closed and vanishes on every loop, α is exact (art. 6.89). If this proof is essentially correct, we could have been more careful, indeed the concatenation of the four paths may be not smooth. We should have smashed the paths before concatenation, that is to say, exchanged γ_i into $\gamma_i^\star = \gamma_i \circ \lambda$, where λ is the smashing function described in Figure 5.1. Since this operation leads to a change of variable under the integral, it does not change the result.

☞ **Exercise 119, p. 202** (Forms on irrational tori are closed). Let α be a p-form on T_Γ. Let $a = \pi_\Gamma^*(\alpha)$ be the pullback of α by the projection $\pi_\Gamma : \mathbf{R}^n \to T_\Gamma$. By construction, the p-form a is invariant by the action of Γ, that is, for all $\gamma \in \Gamma$, $\gamma^*(a) = a$, where $\gamma(x) = x + \gamma$, $x \in \mathbf{R}^n$. But the invariance of a under Γ and the density of Γ in $\mathbf{R}^n$ imply that every component $a_{i\ldots k}$ of a is constant. Hence $a \in \Lambda^p(\mathbf{R}^n)$ and therefore, $\Omega^p(T_\Gamma) \simeq \Lambda^p(\mathbf{R}^n)$. Now, since the components of any $a = \pi_\Gamma^*(\alpha)$ are constant, the form a is closed, $da = 0$. But, since π_Γ is a subduction (art. 6.38), $da = 0$ and $da = \pi_\Gamma^*(d\alpha)$ imply $d\alpha = 0$. Hence, all the differential forms of T_Γ are closed, $Z^p_{\mathrm{dR}}(T_\Gamma) = \Omega^p(T_\Gamma) \simeq \Lambda^p(\mathbf{R}^n)$. Now, if $\alpha = d\beta$, $\beta \in \Omega^{p-1}(T_\Gamma, \mathbf{R})$, then $\alpha = 0$ since $d\beta = 0$ for any form on T_Γ. Then, $B^p_{\mathrm{dR}}(T_\Gamma) = \{0\}$ and $H^p_{\mathrm{dR}}(T_\Gamma) \simeq \Lambda^p(\mathbf{R}^n)$.

☞ **Exercise 120, p. 202** (Is the group $\mathrm{Diff}(S^1)$ simply connected?) Clearly, $\alpha(P)$ is a 1-form on U. Let $F : V \to U$ be a smooth m-parametrization. We have

$$\alpha(P \circ F)_s(\delta s) = \int_0^{2\pi} \left\langle J\{[P(F(s))](X(\theta))\}, \frac{\partial [P(F(s))](X(\theta))}{\partial s}(\delta s) \right\rangle d\theta.$$

Let $r = F(s)$, then

$$\alpha(P \circ F)_s(\delta s) = \int_0^{2\pi} \left\langle J[P(r)(X(\theta))], \frac{\partial P(r)(X(\theta))}{\partial r} \frac{\partial r}{\partial s}(\delta s) \right\rangle d\theta,$$

where $\partial r/\partial s = D(F)(s)$. Thus,

$$\alpha(P \circ F)_s(\delta s) = \alpha(P)_{F(s)}(D(F)(s))(\delta s) = F^*(\alpha(P))_s(\delta s).$$

Therefore, α is a 1-form on $\mathrm{Diff}(S^1)$. Now, let us consider the canonical 1-form Θ on S^1 defined by

$$\Theta(Q)_s(\delta s) = \left\langle J[Q(s)], \frac{\partial Q(s)}{\partial s}(\delta s) \right\rangle,$$

where Q is a plot of S^1, with the same kind of notation as above. Next, let $z \in S^1$ and $R(z) : \mathrm{Diff}(S^1) \to S^1$ be the orbit map, $R(z)(\varphi) = \varphi(z)$. The pullback of Θ by $R(z)$ is then given, on the plot P, by

$$[R(z)^*(\Theta)](P)_r(\delta r) = \left\langle J[P(r)(z)], \frac{\partial P(r)(z)}{\partial r}(\delta r) \right\rangle,$$

and then

$$\alpha(P)_r(\delta r) = \int_0^{2\pi} [R(X(\theta))^*(\Theta)](P)_r(\delta r)\, d\theta.$$

Thus, by additivity of the integral and since $d\Theta = 0$, $d\alpha = 0$. Note that this last expression of α proves directly that α is a 1-form on $\mathrm{Diff}(S^1)$. Next, to compute $\int_\sigma \alpha$, we need

$$\sigma(t)(X(\theta)) = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \cos(2\pi t + \theta) \\ \sin(2\pi t + \theta) \end{pmatrix}.$$

Then,

$$\begin{aligned}
\int_\sigma \alpha &= \int_0^1 \alpha(\sigma)_t(1)\, dt \\
&= \int_0^1 dt \int_0^{2\pi} d\theta \left\langle J \begin{pmatrix} \cos(2\pi t + \theta) \\ \sin(2\pi t + \theta) \end{pmatrix}, \frac{\partial}{\partial t} \begin{pmatrix} \cos(2\pi t + \theta) \\ \sin(2\pi t + \theta) \end{pmatrix} \right\rangle \\
&= 2\pi \int_0^1 dt \int_0^{2\pi} d\theta \\
&= 4\pi^2.
\end{aligned}$$

Therefore, the identity component of $\mathrm{Diff}(S^1)$ is not simply connected; see Exercise 117, p. 198. Actually, it was not necessary to integrate the pullback $R(z)^*(\Theta)$ on the loop X, we could have considered just the pullback $R(\mathbf{e}_1)^*(\Theta)$.

☞ **Exercise 121, p. 212** (The Fubini-Study form is locally exact). We know that the infinite projective space $\mathcal{P}_\mathbf{C}$ is a diffeological manifold modeled on the Hilbert space $\mathcal{H}_\mathbf{C}$ (art. 4.11). Every point $p \in \mathcal{P}_\mathbf{C}$ is in the range of some chart $F_k : \mathcal{H}_\mathbf{C} \to \mathcal{P}_\mathbf{C}$, for some $k \in \mathbf{N}$, (art. 4.11, item 2). Since $F_k(\mathcal{H}_\mathbf{C})$ is D-open and contractible (because $\mathcal{H}_{\mathbf{C}}$ is contractible), every closed differential form on $\mathcal{P}_{\mathbf{C}}$ is locally exact, in particular the Fubini-Study form ω.

☞ **Exercise 122, p. 213** (Closed but not locally exact). From Exercise 119, p. 202, we know that every differential form on an irrational torus $T_\Gamma = \mathbf{R}^n/\Gamma$, where Γ is a dense discrete generating subgroup of $\mathbf{R}^n$, $n \geq 1$, is closed. But since the D-topology of T_Γ is the coarse topology, and since T_Γ is not simply connected, $\pi_1(T_\Gamma) = \Gamma$, T_Γ is also not locally simply connected. Hence, none of the closed forms, except the form 0, is locally exact.

☞ **Exercise 123, p. 213** (A morphism from $H^\star_{\mathrm{dR}}(X)$ to $H^\star_{\mathrm{dR}}(\mathrm{Diff}(X))$). Let x_0 and x_1 be two points of X connected by a path $t \mapsto x_t$, the map $\hat{x}_t : \varphi \mapsto \varphi(x_t)$ is a homotopy from $\hat{x}_0$ to $\hat{x}_1$. Then, thanks to (art. 6.88), they induce the same map in cohomology, $\hat{x}^*_{0\mathrm{dR}} = \hat{x}^*_{1\mathrm{dR}}$. Next, let us consider $X = S^1 \subset \mathbf{R}^2$. Note first that S^1 is connected and $H^p_{\mathrm{dR}}(S^1) = \{0\}$ for all $p \geq 1$, then the morphism from $H^p_{\mathrm{dR}}(S^1)$ to $H^p_{\mathrm{dR}}(\mathrm{Diff}(S^1))$ is unique and obviously injective for $p = 0$ and $p \geq 1$. Let $\hat{\mathbf{e}}_1 : \mathrm{Diff}(S^1) \to S^1$ be the orbit map of the point $\mathbf{e}_1 = (1, 0)$. Let α be a closed 1-form on S^1 such that $\hat{\mathbf{e}}^*_{1\mathrm{dR}}(\mathrm{class}(\alpha)) = 0$, that is, $\hat{\mathbf{e}}^*_1(\alpha) = dF$, where $F \in \mathcal{C}^\infty(\mathrm{Diff}(S^1), \mathbf{R})$. Now, since α is closed, the integral $\int_\ell \alpha$, where $\ell \in \mathrm{Loops}(S^1, \mathbf{e}_1)$, depends only on the homotopy class of ℓ (see Exercise 116, p. 198), and thanks to Exercise 133, p. 266, we know that there exists $k \in \mathbf{Z}$ such that $\ell \sim [t \mapsto \mathcal{R}(2\pi kt)(\mathbf{e}_1)]$, where $\mathcal{R}(\theta)$ is the rotation of angle θ. Let $\sigma_k(t) = \mathcal{R}(2\pi kt)$, then σ_k belongs to $\mathrm{Loops}(\mathrm{Diff}(S^1), \mathbf{1}_{S^1})$, and $\ell \sim [t \mapsto \sigma_k(t)(\mathbf{e}_1)]$. Thus, on the one hand, $\int_{\sigma_k} dF = 0$, and on the other hand, $\int_{\sigma_k} dF = \int_{\sigma_k} \hat{\mathbf{e}}^*_1(\alpha) = \int_{\hat{\mathbf{e}}_1 \circ \sigma_k} \alpha = \int_{[t \mapsto \sigma_k(t)(\mathbf{e}_1)]} \alpha = \int_\ell \alpha$. Hence, $\int_\ell(\alpha) = 0$ for all loops in S^1, and thanks to (art. 6.89), α is exact. Therefore, $\hat{\mathbf{e}}^*_{1\mathrm{dR}}$ is injective. Compared with Exercise 120, p. 202, this also shows that the identity component of $\mathrm{Diff}(S^1)$ is not simply connected.

This example is a particular case in which a smooth map $f : X \to X'$ induces a surjection from $\pi_1(X, x)$ onto $\pi_1(X', x')$, where $x' = f(x)$. We assume X and X' connected. In this situation, for all $\alpha \in \Omega^1(X)$ such that $d\alpha = 0$, if $f^*(\alpha)$ is exact, that is, if $\int_{\ell'} f^*(\alpha) = 0$ for all $\ell' \in \mathrm{Loops}(X', x')$, then $\int_\ell \alpha = 0$ for all $\ell \in \mathrm{Loops}(X, x)$, and thus α is exact. Therefore, the associated homomorphism $f^*_{\mathrm{dR}} : H^1_{\mathrm{dR}}(X') \to H^1_{\mathrm{dR}}(X)$ is injective.

☞ **Exercise 124, p. 221** (Subgroups of $\mathbf{R}$). For the first question, there are two possibilities:

a) There exists $\varepsilon > 0$ such that $]-\varepsilon, +\varepsilon[\subset K$.

b) For all $\varepsilon > 0$ there exists $t \in \mathbf{R}$ such that $0 < t < \varepsilon$ and $t \notin K$.

If we are in the first situation, then $K = \mathbf{R}$. Indeed, for every $t \in \mathbf{R}$, there exists $N \in \mathbf{N}$ such that $N\varepsilon \leq t < (N+1)\varepsilon$. So, $x = t/(N+1) \in K$. And, since K is a group for the addition, $t = (N+1)x$ belongs to K. Therefore, in this case K is not a strict subgroup of $\mathbf{R}$. Thus, we are in the second case. Let $P : U \to K$ be a plot, that is $P \in \mathcal{C}^\infty(U, \mathbf{R})$ and $P(U) \subset K$. Let us assume that $0 \in U$ and that $P(0) = 0$. If it is not the case, we can compose P at the source and the target such that it will be the case. Since U is open, there exists $R > 0$ such that for every $\rho < R$ the open balls $B(\rho)$ of radius ρ, centered at $0 \in U$, are contained in U. Then, for every $0 < \rho < R$ let us choose $r \in B(\rho)$ and let $k = P(r)$. Thus, $k \in K$, but also all the $P(sr)$, $0 \leq s \leq 1$, are elements of K. Since $p : s \mapsto P(sr)$ is smooth, therefore continuous, p takes all the values between $0 = p(0)$ and $k = p(1)$. But by hypothesis a) this is not possible except if p is constant and $k = 0$. Therefore, P is locally constant and the group K is discrete.

Next, let K be a strict subgroup of $\mathbf{R}$, K is discrete and therefore its D-topology also is discrete (art. 2.11). There are two cases: either K is generated by one number a, $K = a\mathbf{Z}$; or there exist two numbers $a \neq 0$ and $b \neq 0$, independent over $\mathbf{Q}$, such that $\{na\}_{n\in\mathbf{Z}} \subset K$, $\{mb\}_{m\in\mathbf{Z}} \subset K$, and $\{na\}_{n\in\mathbf{Z}} \cap \{mb\}_{m\in\mathbf{Z}} = \{0\}$. In the first case, the group is embedded in $\mathbf{R}$. Indeed, any subset A of a discrete space is D-open, if $K = a\mathbf{Z}$, then one can find an open interval I_x centered around each point x of A such that $I_x \cap K = \{x\}$, and the intersection of this union of intervals with A is just A. In the second case, K contains the dense subgroup $a \times [\mathbf{Z} + \alpha\mathbf{Z}]$, with $\alpha = b/a \in \mathbf{R} - \mathbf{Q}$. Then, the intersection of every open interval of $\mathbf{R}$ with $\mathbf{K}$ contains an infinite number of points of $\mathbf{K}$. Therefore, $\mathbf{K}$ is not embedded in $\mathbf{R}$.

Exercise 125, p. 221 (Diagonal diffeomorphisms). A plot of $\Delta(\mathrm{Diff}(X)) \subset \mathrm{Diff}(X^N)$ writes, in a unique way, $r \mapsto \Delta(\varphi_r)$, where $r \mapsto \varphi_r$ is some parametrization in $\mathrm{Diff}(X)$. Now, $r \mapsto \Delta(\varphi_r)$ is smooth only if $(r, x_1, \ldots, x_N) \mapsto (\varphi_r(x_1), \ldots, \varphi_r(x_N))$ is smooth, which means that, for every $k = 1, \ldots, N$, the map $(r, x_1, \ldots, x_N) \mapsto \varphi_r(x_k)$ is smooth. Since the projection $\mathrm{pr}_k : (x_1, \ldots, x_N) \mapsto x_k$ is a subduction, that is equivalent to $(r, x_k) \mapsto \varphi_r(x_k)$ being smooth, which means then that $r \mapsto \varphi_r$ is a plot of $\mathrm{Diff}(X)$. Therefore, Δ is an induction.

Exercise 126, p. 221 (The Hilbert sphere is homogeneous). Let us give the proof in two steps.

1. *The map π is surjective.* Let Z and Z' be two elements of $\mathcal{S}_{\mathbf{C}}$. If Z and Z' are collinear, then there exists $\tau \in S^1 \simeq \mathbf{U}(\mathbf{C})$ such that $Z' = \tau Z$, and the map $Z \mapsto \tau Z$ belongs to $\mathbf{U}(\mathcal{H})$. Otherwise, let E be the plane spanned by these two vectors, and let F be its orthogonal for the Hermitian product. According to Bourbaki [**Bou55**], E and F are supplementary $\mathcal{H} = E \oplus F$. The vectors Z and Z' are vectors of the unit sphere $S^3 \subset E \simeq \mathbf{C}^2$, now the group $\mathbf{U}(\mathbf{C}^2)$ acts transitively on S^3, there exists $A \in \mathbf{U}(\mathbf{C}^3)$ such that $Z' = AZ$. This map, extended to $\mathcal{H}$ by the identity on F, belongs to $\mathbf{U}(\mathcal{H})$ and maps Z to Z'. Therefore, the action of $\mathbf{U}(\mathcal{H})$ is transitive on $\mathcal{S}_{\mathbf{C}}$, which is equivalent to the assertion that π is surjective.

2. *The map π is a subduction.* Let $Q : U \to \mathcal{S}_{\mathbf{C}}$ be a plot. We want to lift Q locally along the projection π, that is, for any $r_0 \in U$, to find a plot $P : V \to \mathbf{U}(\mathcal{H})$, defined on some open neighborhood V of r_0, such that $P(r)(\boldsymbol{e}_1) = Q(r)$, for all $r \in V$. So, let $r_0 \in U$, let V be an open neighborhood of r_0, let $j : \mathbf{C}^m \to \mathcal{H}$ be an injection, and let $\phi : V \to \mathbf{C}^m$ be a smooth parametrization such that $Q \restriction V = j \circ \phi$. Let us denote $E = j(\mathbf{C}^m)$. The plot Q of $\mathcal{S}_{\mathbf{C}}$ takes its values in E, and hence in the unit sphere of E: $S(E) = E \cap \mathcal{S}_{\mathbf{C}}$. The diffeology induced on $S(E)$ is the standard diffeology: $S(E) \simeq S^{2m-1}$. Thus, $Q \restriction V$ is an ordinary smooth map from V into $S(E)$. But we know that the projection from $\mathbf{U}(m)$ onto $S(\mathbf{C}^m)$ is a submersion, *a fortiori* a subduction. Thus, for any $r_0 \in V$ there exist a domain $W \subset V$ and a smooth lifting $\varphi : W \to \mathbf{U}(m)$ such that $Q(r) = \varphi(r)(\boldsymbol{e}_1^m)$, for all $r \in W$, where $\boldsymbol{e}_1^m$ is the vector $(1, 0, \ldots, 0) \in \mathbf{C}^m$. Let us assume that $\boldsymbol{e}_1 = j(\boldsymbol{e}_1^m)$, if it is not the case we conjugate everything with some suitable linear map. Now, let F be the orthogonal of E. The space $\mathcal{H}$ is the direct sum of E and F, *i.e.*, $\mathcal{H} = E \oplus F$. Every vector $Z \in \mathcal{H}$ has a unique decomposition $Z = Z_E + Z_F$ such that $Z_E \in E$ and $Z_F \in F$. Let then

$$P(r)(Z) = \varphi(r)(Z_E) + Z_F,$$

for all $r \in W$ and all $Z \in \mathcal{H}$. For all $r \in W$, the map $P(r)$ is smooth because the decomposition $Z \mapsto (Z_E, Z_F)$ is linear, and then smooth for the fine diffeology. Moreover $P(r)$ clearly preserves the Hermitian product, and is obviously invertible. The map P lifts Q locally,

$$P(r)(\boldsymbol{e}_1) = \varphi(r)(\boldsymbol{e}_1^m) + 0 = Q(r),$$

for all $r \in W$. It remains then to check that P is a plot of the functional diffeology of $\mathbf{U}(\mathcal{H})$. But this is quite clear—a finite family of vectors decomposes into components belonging to E and to F, and because the family is finite, one has only a finite intersection of open sets which is open, we get the property we are looking for. The inverse of $P(r)$ does not give more problems. Therefore, we get that $\mathcal{S}_{\mathbf{C}}$ is homogeneous under $U(\mathcal{H}_{\mathbf{C}})$.

Now, $\mathcal{P}_{\mathbf{C}}$ is the quotient of $\mathcal{S}_{\mathbf{C}}$ by $\mathbf{U}(1)$. Since the composite of subductions $\mathbf{U}(\mathcal{H}_{\mathbf{C}}) \to \mathcal{S}_{\mathbf{C}} \to \mathcal{P}_{\mathbf{C}}$ is a subduction (art. 1.47), and since the action of $\mathbf{U}(\mathcal{H}_{\mathbf{C}})$ on $\mathcal{S}_{\mathbf{C}}$ passes to $\mathcal{P}_{\mathbf{C}}$, $\mathcal{P}_{\mathbf{C}}$ is a homogeneous space of $\mathbf{U}(\mathcal{H}_{\mathbf{C}})$.

Exercise 127, p. 227 (Pullback of 1-forms by multiplication). Let us develop the form $m^*(\alpha)(P \times Q)$,

$$\begin{aligned} m^*(\alpha)(P\times Q)_{\binom{r}{s}}\begin{pmatrix}\delta r\\ \delta s\end{pmatrix} &= \alpha[(r,s)\mapsto (P(r),Q(s))\mapsto P(r)\cdot Q(s)]_{\binom{r}{s}}\begin{pmatrix}\delta r\\ \delta s\end{pmatrix}\\ &= [\alpha_{U,s}(r) \quad \alpha_{V,r}(s)]\begin{pmatrix}\delta r\\ \delta s\end{pmatrix},\end{aligned}$$

because any 1-form on $U \times V$, at a point (r, s), writes $[\alpha_{U,s}(r)\ \alpha_{V,r}(s)]$, where $\alpha_{U,s}$ is a 1-form of U depending on s, and $\alpha_{V,r}$ is a 1-form of V depending on r. Thus,

$$\begin{aligned} m^*(\alpha)(P\times Q)_{\binom{r}{s}}\begin{pmatrix}\delta r\\ \delta s\end{pmatrix} &= \alpha_{U,s}(r)(\delta r) + \alpha_{V,r}(s)(\delta s)\\ &= \alpha[r\mapsto P(r)\cdot Q(s)]_r(\delta r)\\ &+ \alpha[s\mapsto P(r)\cdot Q(s)]_s(\delta s)\\ &= (R(Q(s))^*\alpha)(P)_r(\delta r)\\ &+ (L(P(r))^*\alpha)(Q)_s(\delta s).\end{aligned}$$

Each term of the right sum above is computed by considering successively, in $m^*(\alpha)(P \times Q)_{(r,s)}(\delta r, \delta s)$, s constant and $\delta s = 0$, then r constant and $\delta r = 0$. We get finally, considering the diagonal map $\Delta : r \mapsto (r, r)$,

$$\begin{aligned} \alpha[r\mapsto P(r)\cdot Q(r)]_r(\delta r) &= \Delta^*(m^*(\alpha)(P\times Q))_r(\delta r)\\ &= (R(Q(r))^*\alpha)(P)_r(\delta r)\\ &+ (L(P(r))^*\alpha)(Q)_r(\delta r).\end{aligned}$$

Exercise 128, p. 227 (Liouville form on groups). Let $F : V \to U$ be a smooth parametrization, then $Q \circ F = (P \circ F, A \circ F)$. On the one hand,

$$\begin{aligned} \lambda(Q\circ F)_s(\delta s) &= A(F(s))(P\circ F)_s(\delta s)\\ &= F^*[A(F(s))(P)]_s(\delta s)\\ &= A(F(s))(P)_{F(s)}(D(F)(s)(\delta s)),\end{aligned}$$

and on the other hand,

$$\begin{aligned} F^*[\lambda(Q)]_s(\delta s) &= \lambda(Q)_{F(s)}(D(F)(s)(\delta s)) \\ &= A(F(s))(P)_{F(s)}(D(F)(s)(\delta s)). \end{aligned}$$

Therefore, $\lambda(Q \circ F) = F^*(\lambda(Q))$, and λ defines a differential 1-form on $G \times \mathcal{G}^*$. Now, $j^*_\alpha(\lambda)(P)_r(\delta r) = \lambda(j_\alpha \circ P)_r(\delta r) = \lambda(r \mapsto (P(r), \alpha))_r(\delta r) = \alpha(P)_r(\delta r)$. Thus, $j^*_\alpha(\lambda) = \alpha$. Next, let $g' \in G$, recalling that $\mathrm{Ad}_*(g') = \mathrm{Ad}(g'^{-1})^*$, we have

$$\begin{aligned} g^*_{G\times\mathcal{G}^*}(\lambda)(Q)_r(\delta r) &= \lambda(g_{G\times\mathcal{G}^*} \circ Q)_r(\delta r) \\ &= \lambda(r \mapsto (\mathrm{Ad}(g')(P(r)), \mathrm{Ad}_*(g')[A(r)]))_r(\delta r) \\ &= [\mathrm{Ad}_*(g')(A(r))](\mathrm{Ad}(g') \circ P)_r(\delta r) \\ &= \mathrm{Ad}(g')^*[\mathrm{Ad}(g'^{-1})^*(A(r))](P)_r(\delta r) \\ &= A(r)(P)_r(\delta r). \end{aligned}$$

Therefore, λ is invariant under this action of G on $G \times \mathcal{G}^*$.

Exercise 129, p. 242 (Groupoid associated with $x \mapsto x^3$). Let $X = \mathbf{R}$ and $Q = \mathbf{R}$, equipped with the standard diffeology, and let $\pi : X \to Q$, $\pi(x) = x^3$. Let $\mathbf{K}$ be the groupoid associated with π. Since π is injective, $X_q = \pi^{-1}(q)$ is equal to the singleton $\{x = \sqrt[3]{q}\}$. Hence, $\mathrm{Mor}(q, q')$ is itself reduced to the singleton $\{[\sqrt[3]{q} \mapsto \sqrt[3]{q'}]\}$. Then,

$$\mathrm{Mor}(\mathbf{K}) = \{[x \mapsto x'] \mid x, x' \in \mathbf{R}\},$$

and set theoretically $\mathrm{Mor}(\mathbf{K}) \simeq \mathbf{R} \times \mathbf{R}$. The question is about the diffeology, on $\mathbf{R} \times \mathbf{R}$, induced by the diffeology of $\mathrm{Mor}(\mathbf{K})$. Let $P : U \to \mathrm{Mor}(\mathbf{K})$ be a plot, and let $P(r) = \{[x_r \mapsto x'_r]\}$. Let us make explicit the spaces $X_{\mathrm{src}\circ P}$ and $X_{\mathrm{trg}\circ P}$,

$$X_{\mathrm{src}\circ P} = \{(r, x_r) \in U \times \mathbf{R} \mid r \in U\} \quad \text{and} \quad X_{\mathrm{trg}\circ P} = \{(r, x'_r) \in U \times \mathbf{R} \mid r \in U\}.$$

The maps P_{src} and P_{trg} are then given by

$$P_{\mathrm{src}}(r, x_r) = P(r)(x_r) = x'_r \quad \text{and} \quad P_{\mathrm{trg}}(r, x'_r) = P(r)^{-1}(x'_r) = x_r.$$

These maps are smooth if and only if the parametrizations $r \mapsto x'_r$ and $r \mapsto x_r$ are smooth, which implies in particular that $\chi \circ P : r \mapsto (x_r^3, x_r'^3)$ is smooth. Therefore, the diffeology induced on $\mathbf{R} \times \mathbf{R}$ by the groupoid diffeology of $\mathbf{K}$ is the standard diffeology. Note that the injection $i_Q : Q \to \mathrm{Mor}(\mathbf{K})$ is indeed smooth, even if the presence of the cubic root is disturbing. Let $r \mapsto q_r$ be a plot of $Q = \mathbf{R}$, defined on U, and let $P : r \mapsto \mathbf{1}_{X_{q_r}}$ be the composite with i_Q. We have to check that P is a plot of $\mathrm{Mor}(\mathbf{K}) \simeq \mathbf{R} \times \mathbf{R}$. In this case $X_{\mathrm{src}\circ P} = X_{\mathrm{trg}\circ P} = \{(r, \sqrt[3]{q_r}) \mid r \in U\}$ and $P_{\mathrm{src}}(r, \sqrt[3]{q_r}) = P_{\mathrm{trg}}(r, \sqrt[3]{q_r}) = \sqrt[3]{q_r}$. A parametrization $s \mapsto (x_{r_s}, \sqrt[3]{q_{r_s}})$ of $X_{\mathrm{src}\circ P}$ is a plot if and only if $s \mapsto x_{r_s}$ and $s \mapsto \sqrt[3]{q_{r_s}}$ are smooth, and thus P_{src} and P_{trg} are smooth. Eventually, the real question is, Is this groupoid fibrating? The answer is No! Because the map $\chi : (x, x') \mapsto (x^3, x'^3)$ is not a subduction, and now the reason is exactly because $q \mapsto \sqrt[3]{q}$ is not smooth.

Exercise 130, p. 253 (Polarized smooth functions). A line $\mathbf{D}$ passing through the origin has two unit direction vectors $\{\pm u\}$. So, we can equip $P^1(\mathbf{R})$ with the quotient diffeology of $S^1 \subset \mathbf{R}^2$ by $\{\pm 1\}$. This diffeology is equivalent to the powerset diffeology; see Exercise 63, p. 61. Therefore, we can regard T as

$$T = \{(\pm u, f) \in P^1(\mathbf{R}) \times \mathcal{C}^\infty(\mathbf{R}^2, \mathbf{C}) \mid f(x + su) = f(x),\ \forall x \in \mathbf{R}^2,\ \forall s \in \mathbf{R}\}.$$

Now, let us consider the pullback of the projection $\pi : T \to P^1(\mathbf{R})$ by the projection $p : S^1 \to P^1(\mathbf{R})$, that is,

$$p^*(T) = \{(u, f) \in S^1 \times \mathcal{C}^\infty(\mathbf{R}^2, \mathbf{C}) \mid f(x + su) = f(x)\ \forall x \in \mathbf{R}^2,\ \forall s \in \mathbf{R}\}.$$

Thus, T is now equivalent to the quotient of $p^*(T)$ by the equivalence relation $(u, f) \sim (\pm u, f)$. Then, let us introduce the $\pi/2$ rotation J in the plane $\mathbf{R}^2$, and let us consider the map

$$\phi : (u, F) \mapsto (u, f = [x \mapsto F(u \cdot Jx)]), \quad \text{for all} \quad (u, F) \in S^1 \times \mathcal{C}^\infty(\mathbf{R}, \mathbf{C}),$$

where the dot $\cdot$ denotes the usual scalar product. Next, since $u \cdot Ju = 0$, $f(x + su) = F(u \cdot J(x + su)) = F((u \cdot Jx) + (su \cdot Ju)) = F(u \cdot Jx) = f(x)$, and $f = [x \mapsto F(u \cdot Jx)]$ is constant on all the lines parallel to $\mathbf{R}u$. Moreover, ϕ is bijective, for all $u \in S^1$ and $f \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{C})$ such that $f(x + su) = f(x)$,

$$\phi^{-1}(u, f) = (u, F = [t \mapsto f(t \times J^{-1}u)]).$$

The maps ϕ and ϕ^{-1} are clearly smooth, thus $p^*(T)$ is trivial, equivalent to $\mathrm{pr}_1 : S^1 \times \mathcal{C}^\infty(\mathbf{R}, \mathbf{C}) \to S^1$. This is sufficient to prove that $\pi : T \to P^1$ is a diffeological fiber bundle.

Then, the pullback of the action of $\varepsilon \in \{\pm 1\}$, by ϕ, writes $\varepsilon(u, F) = (\varepsilon u, F \circ \hat{\varepsilon})$, where $\hat{\varepsilon} : t \mapsto \varepsilon \times t$. Hence, T is equivalent to the quotient of $S^1 \times \mathcal{C}^\infty(\mathbf{R}, \mathbf{C})$ by this action. Finally, let us consider the induction of $\mathbf{R}$ into $\mathcal{C}^\infty(\mathbf{R}, \mathbf{C})$ by $t \mapsto [x \mapsto tx]$. The subbundle $\phi(S^1 \times \mathbf{R})/\{\pm 1\} \subset T$ is the quotient of the product $S^1 \times \mathbf{R}$ by the action $\varepsilon(u, t) = (\varepsilon u, \varepsilon t)$, $\varepsilon \in \{\pm 1\}$. But this is exactly the Möbius strip, and the Möbius strip is not trivial over $P^1(\mathbf{R})$, thus the fiber bundle T also is not trivial.

Exercise 131, p. 253 (Playing with SO(3)). Since $SO(2, \boldsymbol{e}_1)$ is a subgroup of SO(3), the projection $\pi : SO(3) \to SO(3)/SO(2, \boldsymbol{e}_1)$ is a principal fibration, (art. 8.15). Now, let us prove that the map $p : SO(3) \to S^2$, defined by $A \mapsto A\boldsymbol{e}_1$, is a subduction and then identifies $SO(3)/SO(2, \boldsymbol{e}_1)$ with S^2. First of all, the map p is surjective. Indeed, let $u \in S^2$, there exists a vector $v \in S^2$ orthogonal to u. Then, the matrix $A = [u\ v\ u \times v]$, where $\times$ denotes the vector product, belongs to SO(3), and $A\boldsymbol{e}_1 = u$. Now, let $r \mapsto u_r$ be a smooth parametrization of S^2, let r_0 be a point in its domain, and let $u_0 = u_{r_0}$. Let $w \in \mathbf{R}^3$ such that $[1 - u_0\bar{u}_0]w \neq 0$, where $[1 - u_0\bar{u}_0]$ is the orthogonal projector along u_0. Since the map $r \mapsto [1 - u_r\bar{u}_r]w$ is smooth, there exists a small open ball B centered at r_0 such that, for all $r \in B$, the vector $w_r = [1 - u_r\bar{u}_r]w$ is not zero. Now, let $v_r = w_r/\|w_r\|$. Since w_r is not zero, $r \mapsto v_r$ is smooth, it belongs to S^2 and it is orthogonal to u_r. Thus, $r \mapsto A_r = [u_r\ v_r\ u_r \times v_r]$ is a plot of SO(3) such that $A_r(\boldsymbol{e}_1) = u_r$. Therefore p is a subduction. Now we just observe that $p(A) = p(A')$ if and only if there exists $k \in SO(2, \boldsymbol{e}_1)$ such that $A' = Ak^{-1}$. Therefore, thanks to the uniqueness of quotients (art. 1.52), we conclude that p is a fibration, and moreover a principal fibration. Finally, the map $(A, v) \mapsto (A\boldsymbol{e}_1, Av)$ from $SO(3) \times \boldsymbol{e}_1^\perp$ to $S^2 \times \mathbf{R}^3$ takes its values in TS^2 and represents the associated fiber bundle $SO(3) \times_{SO(2, \boldsymbol{e}_1)} \boldsymbol{e}_1^\perp$.

Exercise 132, p. 253 (Homogeneity of manifolds). Let ε_n be a smooth bump-function defined on $\mathbf{R}^n$, equal to 1 on the ball B′ of radius $r/2$, centered at 0_n, and equal to 0 outside the ball B of radius r. Let ε be a smooth bump-function defined on $\mathbf{R}$, equal to 1 on the interval $[-\delta/2, 1 + \delta/2]$ and equal to 0 outside the interval $]-\delta, 1 + \delta[$. Let f be the vector field defined on $\mathbf{R}^n \times \mathbf{R}$ by $f(x, t) = (0_n, \varepsilon_n(x) \times \varepsilon(t))$. The vector field f is equal to zero outside the cylinder

C and equal to $(0_n, 1)$ into the cylinder $C' = B' \times [-\delta/2, 1+\delta/2]$. Since the support of f is contained in C, f is integrable, and the time 1 of its exponential is a compactly supported diffeomorphism, whose support is contained in C. Moreover, since inside C′ the vector field f is constant equal to $(0_n, 1)$, its exponential is the translation $\exp(sf)(x,t) = (x, t+s)$ as soon as (x,t) and $(x, t+s)$ belong to C′. Thus $\exp(f)(0_n, 0) = (0_n, 1)$, and $\exp(f)$ satisfies the conditions of the exercise. Now, let $F : \mathcal{B} \to M$ be some chart of M, where $\mathcal{B}$ is an open ball. Let $r, r' \in \mathcal{B}$, let $x = F(r)$ and $x' = F(r')$. There exists a small open cylinder $\mathcal{C}$ containing the segment $\{r + s(r'-r)\}_{s=0}^{1}$ and contained in $\mathcal{B}$. According to the first part of the exercise (modulo smooth equivalence) there exists a diffeomorphism of $\mathcal{B}$ with compact support contained in $\mathcal{C}$, and mapping r to r'. Thus, the image of this diffeomorphism by the chart F defines a local diffeomorphism of M, defined on the open subset $F(\mathcal{B})$, mapping x to x' and which is the identity outside a closed subset. It can thus be extended, by the identity, into a global compactly supported diffeomorphism of M, mapping x to x'. Next, let us choose x and x', any two points in M. Let us note first that there exists an atlas of M made of charts whose domains are open balls, let us call these charts *round charts*. Since M is connected, there exists a path γ connecting x to x', let $\{F_t\}_{t=0}^{1}$ be a family of round charts such that $\gamma(t) \in \mathrm{val}(F_t)$. The set $\{\gamma^{-1}(\mathrm{val}(F_t))\}_{t=0}^{1}$ is an open covering of $[0,1]$, by compacity, after re-indexation, there exists a finite family $\{F_i\}_{i=1}^{N}$ such that $\{\gamma^{-1}(\mathrm{val}(F_i))\}_{i=1}^{N}$ is an open covering of $[0,1]$. We can even assume that for every index $i = 1, \ldots, N$, $J_i = \gamma^{-1}(\mathrm{val}(F_i))$ is an open interval of $\mathbf{R}$ such that only two successive intervals intersect. Choosing a point $t_i \in J_i \cap J_{i+1}$, $i = 1, \ldots, N-1$, we get a family of points $x_i = \gamma(t_i)$ such that two consecutive points (x_i, x_{i+1}) belong to the values of one round chart F_i. Now, thanks to the previous result we get a finite family of compactly supported diffeomorphisms of M mapping every point of this family to its successor, the first point being x and the last x'. Thus, after composition we get a compactly supported diffeomorphism mapping x to x'. And we conclude that the map $\hat{x}_0 : \mathrm{Diff}_K(M) \to M$ is surjective. Now, to prove that the projection $\hat{x}_0$ is a principal fibration, with structure group the stabilizer $\mathrm{Diff}_K(M, x_0)$, we need to lift locally any plot of M. But since M is a manifold, it is sufficient to lift locally any chart $F : U \to M$. Without loss of generality, we can assume that $0 \in U$, and for simplicity that U is connected. Let us prove first that there exists a smooth map $r \mapsto \Psi_r$, where r belongs to an open ball centered at 0, contained in U, and Ψ_r is a compactly supported diffeomorphism of U, mapping 0 to r. For that, let us chose two balls B and B′ of radii $R < R'$, contained in U and centered at 0. Let λ be a smooth bump-function defined on U, equal to 1 in B and equal to 0 outside B′. Let us define $f_r(r') = \lambda(r') \times r$, for $r' \in U$ and $|r| < R$. The map $r \mapsto f_r$ is a smooth family of vector fields on U, with supports contained in B′. For all t, r and r' such that r' and $r' + tr$ belong to B, the exponential of f_r coincides with the translation $\exp(tf_r)(r') = r' + tr$. Thus, for $t = 1$, $r' = 0$, and $|r| < R$, we get $\exp(f_r)(0) = r$. Thanks to the differentiability of $r \mapsto f_r$ and to the differentiability of the solutions of an ordinary differential equation with respect to the parameters, the map $r \mapsto \exp(f_r)$ is smooth for the functional diffeology. Now, since the support of $\exp(f_r)$ is contained in U, $F \circ \exp(f_r) \circ F^{-1}$ is a local compactly supported diffeomorphism of M mapping $F(U)$ into itself, therefore it can be extended, by the identity, into a global compactly supported diffeomorphism Φ_r of M, mapping $x_0' = F(0)$ to $x = F(r)$, that is, $\Phi_r(x_0') = F(r)$. Next, we know that there exists a diffeomorphism

φ mapping x_0 to x'_0, thus the diffeomorphism $\phi_r = \Phi_r \circ \varphi$ maps x_0 to $F(r)$, that is, $\hat{x}_0(\phi_r) = \phi_r(x_0) = F(r)$. Finally, since $r \mapsto \exp(f_r)$ is smooth and the composition with smooth maps is a smooth operation, the local lift $r \mapsto \phi_r$ of F along $\hat{x}_0$, in $\mathrm{Diff}_K(M)$, is smooth. Therefore, $\hat{x}_0$ is a diffeological principal fibration. The homogeneity of manifolds has first been proved by Donato in his dissertation [**Don84**].

Let $T(M)$ be the tangent bundle to M, that is, the space of 1-jets of paths in M. Precisely, $T(M)$ is the quotient of Paths(M) by the equivalence relation $\gamma \sim \gamma'$ if $\gamma(0) = \gamma'(0)$, and for all differential 1-forms α on M, $\alpha(\gamma)(0) = \alpha(\gamma')(0)$. Then, $T(M)$ is the associated fiber bundle $\mathrm{Diff}_K(M) \times_{\mathrm{Diff}_K(M,x_0)} E$, where $E = T_{x_0}(M) = \mathrm{Paths}(M, x_0, \star)/\sim$.

☞ **Exercise 133, p. 266** (Covering tori). First of all, since $(\mathbf{R}^n, +)$ is a group and $\Gamma \subset \mathbf{R}^n$ is a subgroup, the projection $p : \mathbf{R}^n \to T_\Gamma$ is a fibration (art. 8.15), a principal fibration. Then, thanks to the unicity, up to equivalence, of universal coverings (art. 8.26), since $\mathbf{R}^n$ is simply connected and Γ is discrete, $p : \mathbf{R}^n \to T_\Gamma$ is the universal covering of T_Γ, and Γ identifies with $\pi_1(T_\Gamma)$. Now every path $t \mapsto t\gamma$, where $\gamma \in \Gamma$, projects into a loop in T_Γ. This family of loops, one for each element of Γ, gives a favorite representative for each element of $\pi_1(T_\Gamma, \mathbf{1}_{T_\Gamma})$, that is, each class of homotopy of loop based at $\mathbf{1}_{T_\Gamma} = p(0)$. Next, considering the circle, we know that the map $p : t \mapsto (\cos(t), \sin(t))$ is a subduction, making the circle $S^1 \subset \mathbf{R}^2$ diffeomorphic to $\mathbf{R}/2\pi\mathbf{Z}$; see Exercise 27, p. 27. Each loop of S^1 is then homotopic to some $t \mapsto p(2\pi k t)$, for $k \in \mathbf{Z}$. But $p(2\pi t k) = (\cos(2\pi k t), \sin(2\pi k t))$ is also equal to $\mathcal{R}(2\pi k t)(\mathbf{1}_{S^1})$. By translation with an element of S^1, we get the general statement.

☞ **Exercise 134, p. 272** (De Rham homomorphism and irrational tori). There are two different cases, $s = \alpha r$ and $s \neq \alpha r$.

1. Case $s = \alpha r$. The action of $\mathbf{Z} \times \mathbf{Z}$ on $\mathbf{R} \times \mathbf{R}$ is given by $(x, y) \mapsto (x + n + \alpha m, y + r(n + \alpha m))$, with $(n, m) \in \mathbf{Z} \times \mathbf{Z}$. The following map

$$\Phi : (x, y) \mapsto ([x], y - ax),$$

defined from $\mathbf{R} \times \mathbf{R}$ to $T_\alpha \times \mathbf{R}$, where $[x] \in T_\alpha$, is a realization of the quotient, $\Phi(x, y) = \Phi(x', y') \Leftrightarrow (x', y') = (x + n + \alpha m, y + r(n + \alpha m))$. We find again the situation with $\rho = \rho_c$, and the quotient is trivial.

2. Case $s \neq \alpha r$. Let us consider the following linear map,

$$M : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{s - \alpha r} \begin{pmatrix} sx - \alpha y \\ -rx + y \end{pmatrix}.$$

The map M is a linear isomorphism of $\mathbf{R} \times \mathbf{R}$ with determinant 1. The image by M of the action of $\mathbf{Z} + \alpha\mathbf{Z}$ is the standard action of $\mathbf{Z} \times \mathbf{Z}$, $(n, m) : (u, v) \mapsto (u + n, v + m)$. Thus, the quotient $\mathbf{R} \times_\rho \mathbf{R}$ is diffeomorphic to the 2-torus $T^2 = T \times T$. The projection from $\mathbf{R} \times_\rho \mathbf{R} \simeq T \times T$ onto the irrational torus T_α is given in terms of (u, v), above, by

$$p : T^2 \to T_\alpha \quad \text{with} \quad p([u, v]) = [u + \alpha v].$$

Then, the action of $(\mathbf{R}, +)$ on the variables (u, v) is generated by the translation along the vector

$$\xi = \frac{1}{s - \alpha r} \begin{pmatrix} -\alpha \\ 1 \end{pmatrix}.$$

Note that the various choices of homomorphisms ρ change only the speed of the action of $\mathbf{R}$ on the 2-torus.

☞ **Exercise 135, p. 272** (The fiber of the integration bundle). Let $r \mapsto \hat{x}_r$ be a plot of X_α with values in the fiber $\mathrm{pr}_\alpha^{-1}(x_0)$. By definition of the quotient diffeology, there exists, at least locally, a smooth lift $r \mapsto \ell_r$ such that $\hat{x}_r = \mathrm{class}(\ell_r)$. Let us represent the quotient by the values of the integrals $\int_\ell \alpha$, $\ell \in \mathrm{Loops}(X, x_0)$, and let us prove that the map $r \mapsto \int_{\ell_r} \alpha$ is locally constant. Let us thus compute the variation of the integral, according to (art. 6.70). We can reduce the question to a 1-parameter variation $s \mapsto \ell_s$, and because $d\alpha = 0$, the variation is reduced to

$$\delta \int_{\ell_s} \alpha = \int_{\partial I} \alpha(\delta\ell) = \Big[\alpha(\delta\ell)\Big]_0^1 = \alpha(\delta\ell)(1) - \alpha(\delta\ell)(0).$$

But $\alpha(\delta\ell)$ is given (art. 6.56) by

$$\begin{aligned}
\alpha(\delta\ell)(t) &= \alpha\left(\begin{pmatrix} s \\ t \end{pmatrix} \mapsto \ell_s(t)\right)_{\binom{s=0}{t}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \alpha\left(\begin{pmatrix} s \\ t \end{pmatrix} \mapsto \ell_s(t)\right)_{\binom{s=0}{t}} \left[D\left(s \mapsto \begin{pmatrix} s \\ t \end{pmatrix}\right)(s=0)(1)\right] \\
&= \alpha\left(s \mapsto \begin{pmatrix} s \\ t \end{pmatrix} \mapsto \ell_s(t)\right)_{s=0} (1) \\
&= \alpha\left(s \mapsto \ell_s(t)\right)_{s=0} (1).
\end{aligned}$$

Now, since $\ell_s(0) = \ell_s(1)$ for all s, we get $\alpha(\delta\ell)(1) = \alpha(\delta\ell)(0)$, and hence $\delta \int_{\ell_s} \alpha = 0$. Therefore, the map $r \mapsto \hat{x}_r$ is locally constant and the fiber in X_α over x_0, equipped with the subset diffeology, is discrete.

☞ **Exercise 136, p. 287** (Spheric periods on toric bundles). Since σ belongs to Loops(Loops$(X, x), \mathbf{x}$), we have $\sigma(0)(t) = \sigma(1)(t) = \sigma(s)(0) = \sigma(s)(1) = x$ for all s, t, that is, $\sigma \circ j_1(0) = \sigma \circ j_1(1) = \sigma \circ j_2(0) = \sigma \circ j_2(1) = \mathbf{x}$, where j_k is defined in (art. 6.59). We know that the pullback $\mathrm{pr}_1 : \sigma^*(Y) \to \mathbf{R}^2$ is trivial (art. 8.9), so there exists a global lifting $\tilde{\sigma}$ of σ, $\pi_*(\tilde{\sigma}) = \pi \circ \tilde{\sigma} = \sigma$. Now,

$$\int_\sigma \omega = \int_{\pi_*(\tilde{\sigma})} \omega = \int_{\tilde{\sigma}} \pi^*(\omega) = \int_{\tilde{\sigma}} d\lambda = \int_{\partial\tilde{\sigma}} \lambda = \sum_{k=1}^{2} (-1)^k \left[\int_{\tilde{\sigma}\circ j_k(0)} \lambda - \int_{\tilde{\sigma}\circ j_k(1)} \lambda\right].$$

Let $\gamma_{k,a} = \tilde{\sigma} \circ j_k(a)$, with $k = 1, 2$ and $a = 0, 1$, and let $\bar{\gamma}_{k,a}(t) = \gamma_{k,a}(1-t)$. We have

$$\sum_{k=1}^{2} (-1)^k \left[\int_{\tilde{\sigma}\circ j_k(0)} \lambda - \int_{\tilde{\sigma}\circ j_k(1)} \lambda\right] = \int_{\gamma_1=\gamma_{2,0}} \lambda + \int_{\gamma_2=\gamma_{1,1}} \lambda + \int_{\gamma_3=\bar{\gamma}_{2,1}} \lambda + \int_{\gamma_4=\bar{\gamma}_{1,0}} \lambda.$$

Next, since $\sigma \circ j_k(a) = \mathbf{x}$, the paths γ_i are paths in $Y_x = \pi^{-1}(x)$, and they describe a closed circuit: $\tilde{\sigma}(1)(0) = \gamma_1(1) = \gamma_2(0)$, $\tilde{\sigma}(1)(1) = \gamma_2(1) = \gamma_3(0)$, $\tilde{\sigma}(0)(1) = \gamma_3(1) = \gamma_4(0)$, $\tilde{\sigma}(0)(0) = \gamma_4(1) = \gamma_1(0)$. Then, choosing a point $y_0 \in Y_x$ and identifying Y_x with T, thanks to the orbit map $\tau \mapsto \tau_Y(y_0)$, we can regard the γ_i as paths in T, describing a close circuit starting and ending at $0 \in T$. But since $\mathbf{R}$ is a covering of T (actually, the universal covering), thanks to the monodromy theorem, we can lift each path γ_i by a path x_i in $\mathbf{R}$, $\gamma_i(t) = \mathrm{class}(x_i(t))$. Moreover, we can choose these liftings such that $x_1(0) = 0$, $x_1(1) = x_2(0)$, $x_2(1) = x_3(0)$, $x_3(1) = x_4(0)$, and since $\gamma_4(1) = 0$, $x_4(1) \in \Gamma$. Then,

$$\int_{\gamma_i} \lambda = \int_0^1 \lambda(\gamma_i)_t(1)\, dt = \int_0^1 \theta[t \mapsto \mathrm{class}(x_i(t))]_t(1)\, dt.$$

Let us recall that $\mathrm{class}^*(\theta) = dt$ means $\theta[t \mapsto \mathrm{class}(t)] = dt$, that is, $\theta[r \mapsto \mathrm{class}(x(r))] = x^*(dt) = dx$, for all smooth real parametrizations x, thus

$$\int_{\gamma_i} \lambda = \int_0^1 \frac{dx_i(t)}{dt}\, dt = x_i(1) - x_i(0),$$

and, finally

$$\int_\sigma \omega = \sum_{i=1}^4 \int_{\gamma_i} \lambda = \gamma_4(1) - \gamma_1(0) = \gamma_4(1) \in \Gamma.$$

Exercise 137, p. 287 (Fiber bundles over tori). Let us recall that every fiber bundle is associated with a principal fiber bundle (art. 8.16). Thus, it is sufficient, for this exercise, to assume the fiber bundle to be principal. Let $\pi : Y \to T_\Gamma$ be a principal fiber bundle with structure group G. Let $p : \mathbf{R}^n \to T_\Gamma$ be the universal covering. The pullback $\mathrm{pr}_1 : p^*(Y) \to \mathbf{R}^n$ is a G-principal fiber bundle, with the action $g(x, y) = (x, g_Y(y))$, where g_Y denotes the action of $g \in G$ on Y. Since the base of the pullback is $\mathbf{R}^n$, the fibration is trivial (art. 8.19). Therefore, it admits a smooth section, that is, there exists $\varphi \in \mathcal{C}^\infty(\mathbf{R}^n, Y)$ such that $\pi \circ \varphi = p$. Then, for each $x \in \mathbf{R}^n$ and each $\gamma \in \Gamma$, there exists $h(\gamma)(x) \in G$ such that $\varphi(x) = h(\gamma)(x)_Y(\varphi(x+\gamma))$. Thus, $h \in \mathrm{Maps}(\Gamma, \mathcal{C}^\infty(\mathbf{R}^n, G))$ and $h(\gamma+\gamma')(x) = h(\gamma)(x) \cdot h(\gamma')(x+\gamma)$. Note also that, since Γ is Abelian, $h(\gamma)(x) \cdot h(\gamma')(x+\gamma) = h(\gamma')(x) \cdot h(\gamma)(x+\gamma')$. Next, $\mathrm{pr}_2 : p^*(Y) \to Y$ is a subduction, indeed if $r \mapsto y_r$ is a plot of Y, then locally $y_r = \mathrm{pr}_2(x_r, y_r)$, where $r \mapsto x_r$ is a local lift in $\mathbf{R}^n$ of $r \mapsto \pi(y_r)$. Then, Y is equivalent to the quotient of $\mathbf{R}^n \times G$ by the action of Γ induced by h, that is, $\gamma(x, g) = (x+\gamma, g \cdot h(\gamma)(x))$. Conversely, given such a map h, we get a G-principal fiber bundle by quotient. These maps h are kinds of cocycles, the above construction gives a trivial bundle if and only if h is trivial, that is, if and only if there exists $f \in \mathcal{C}^\infty(\mathbf{R}^n, G)$ such that $h(\gamma)(x) = f(x)^{-1} \cdot f(x+\gamma)$, such h can be regarded as coboundaries.

Exercise 138, p. 287 (Flat connections on toric bundles). Let $\pi : Y \to X$ be a T-principal bundle, with T a 1-dimensional torus. Let λ be a connection 1-form and let us assume that Θ, the associated connection, is flat, which means, by definition, that the holonomy group is discrete (art. 8.35). Thus, the holonomy bundle $p : Y_\Theta(y) \to X$ is a covering, (art. 8.35, (♠)) and (art. 8.35, Note 1). Let us recall that $Y_\Theta(y) \subset Y$ is made of the ends of horizontal paths starting at y, and let us denote by $j : Y_\Theta(y) \to Y$ the inclusion. Let $\gamma \in \mathrm{Paths}(Y_\Theta(y), y)$, the horizontal path $\Theta(\gamma, 0)$ takes necessarily its values in $Y_\Theta(y)$ and projects on the same path $\pi \circ \gamma$ as γ in X. But since $Y_\Theta(y)$ is a covering, there is one and only one lift of $\pi \circ \gamma$ in $Y_\Theta(y)$ starting at y, thus $\Theta(\gamma, 0) = \gamma$. Hence, all the 1-plots in $Y_\Theta(y)$ are horizontal and $\lambda(\gamma) = 0$ for all $\gamma \in \mathrm{Paths}(Y_\Theta(y))$. Therefore, the connection form λ vanishes on $Y_\Theta(y)$ (art. 6.37), that is, $j^*(\lambda) = 0$. Now $\pi \circ j = p$ implies, on the one hand, $j^*(d\lambda) = d(j^*(\lambda)) = 0$, and on the other hand $j^*(d\lambda) = j^*(\pi^*(\omega)) = (\pi \circ j)^*(\omega) = p^*(\omega)$, where ω is the curvature of λ. Thus, $p^*(\omega) = 0$, and since p is a subduction, $\omega = 0$ (art. 6.39).

☞ **Exercise 139, p. 287** (Connection forms over tori). Let λ be a connection 1-form on Y. Let $\mathrm{pr} : (x, t) \mapsto [x, t]$ be the projection from $\mathbf{R} \times \mathbf{R}$ to Y, the pullback $\Lambda = \mathrm{pr}^*(\lambda)$ is a connection 1-form on $\mathbf{R} \times \mathbf{R}$ and thus writes, with our usual notation for points and vectors,

$$\Lambda_{\binom{x}{t}} \begin{pmatrix} \delta x \\ \delta t \end{pmatrix} = a(x)\delta x + \delta t,$$

where a is a smooth real function. Now, Λ is invariant by Γ. Thus, for all $k \in \Gamma$,

$$\begin{aligned} \left[k^*_{\mathbf{R}\times\mathbf{R}}(\Lambda)\right]_{\binom{x}{t}} \begin{pmatrix} \delta x \\ \delta t \end{pmatrix} &= \Lambda_{\binom{x}{t}} \begin{pmatrix} \delta x \\ \delta t \end{pmatrix}, \\ \Lambda_{\binom{x+k}{t+\tau(k)(x)}} \begin{pmatrix} \delta(x+k) \\ \delta(t+\tau(k)(x)) \end{pmatrix} &= \Lambda_{\binom{x}{t}} \begin{pmatrix} \delta x \\ \delta t \end{pmatrix}, \\ a(x+k)\delta x + \delta t + \tau(k)'(x)\delta x &= a(x)\delta x + \delta t, \\ a(x+k)\delta x + \tau(k)'(x)\delta x &= a(x)\delta x. \end{aligned}$$

Hence, $\tau(k)'(x) = a(x) - a(x+k)$. Let A be a primitive of a. We get by integration, $\tau(k)(x) - \tau(k)(0) = A(x) - A(0) - A(x+k) + A(k)$. Then, $\tau(k)(x) = B(k) + \sigma(x+k) - \sigma(x)$, for some $B : \Gamma \to \mathbf{R}$ and $\sigma \in \mathcal{C}^\infty(\mathbf{R})$. Thus, τ is equivalent to B (art. 8.39, (♡)). The condition of cocycle writes then $B(k+k') = B(k) + B(k')$, and B is a homomorphism from Γ to $\mathbf{R}$. Therefore, if there exists a connection 1-form on Y, then the cocycle τ defining π is equivalent to a homomorphism. Conversely, if τ is equivalent to a homomorphism B, we just consider $\tau = B$. Then, any 1-form $\Lambda_{(x,t)} : (\delta x, \delta t) \mapsto a(x)\delta x + \delta t$, where a is a Γ-invariant real function on $\mathbf{R}$, is invariant by the action of Γ on $\mathbf{R} \times \mathbf{R}$, and since $\mathbf{R} \times \mathbf{R}$ is a covering of the quotient $Y = \mathbf{R} \times_\Gamma \mathbf{R}$, there exists a 1-form λ on Y such that $\mathrm{pr}^*(\lambda) = \Lambda$ (art. 6.38). Next, since Λ is clearly a connection 1-form, so is λ.

In the special case $\Gamma = \mathbf{Z} + \alpha\mathbf{Z}$, a homomorphism writes $B(n + \alpha m) = an + bm$, but considering the function $\sigma(x) = ax$, B is equivalent to $\beta(n + \alpha m) = cm$ for some constant c. If $c \neq 0$, then we can choose $c = 1$, for simplicity. Thus, the quotient of $\mathbf{R} \times \mathbf{R}$ by the action $(n, (x, t)) \mapsto (x + n + \alpha m, t + m)$ of $\mathbf{Z}$ is equivalent to the 2-torus T^2, with $T = \mathbf{R}/\mathbf{Z}$, for the projection $(x, t) \mapsto ([t], [x - \alpha t])$.

☞ **Exercise 140, p. 297** (Loops in the torus). The 1-form $\mathcal{K}\omega$ restricted to $\mathrm{Loops}(T^2)$ is closed, thanks to the fundamental property of the Chain-Homotopy operator. In particular, the restriction of $\mathcal{K}\omega$ to each component of $\mathrm{Loops}(T^2)$ is closed. Let $\mathrm{comp}(\ell)$ be the connected component of ℓ in $\mathrm{Loops}(T^2)$. The path $\sigma : s \mapsto \sigma_s$, with $\sigma_s(t) = \mathrm{class}(t, s)$, is a loop in $\mathrm{Loops}(T^2)$, based at ℓ, thus a loop in $\mathrm{comp}(\ell)$. Let us compute the integral of $\mathcal{K}\omega$ on σ, that is,

$$\int_\sigma \mathcal{K}\omega = \int_0^1 \mathcal{K}\omega(\sigma)_s(1)\, ds.$$

Let $\bar{\sigma}(t,s) = \sigma_s(t)$, and let $\tilde{\sigma}$ be the lift of $\bar{\sigma}$ in $\mathbf{R}^2$, $\mathrm{class}\circ\tilde{\sigma} = \bar{\sigma}$, such that $\tilde{\sigma}(0,0) = (0,0)$, by definition,

$$\begin{aligned}
\mathcal{K}\omega(\sigma)_s(1) &= \int_0^1 \omega\left(\begin{pmatrix} t \\ s\end{pmatrix} \mapsto \sigma_s(t)\right)_{\binom{t}{s}} \begin{pmatrix} 1 \\ 0\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} dt \\
&= \int_0^1 \omega(\bar{\sigma})_{\binom{t}{s}} \begin{pmatrix} 1 \\ 0\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} dt \\
&= \int_0^1 \bar{\sigma}^*(\omega)_{\binom{t}{s}} \begin{pmatrix} 1 \\ 0\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} dt \\
&= \int_0^1 (\mathrm{class}\circ\tilde{\sigma})^*(\omega)_{\binom{t}{s}} \begin{pmatrix} 1 \\ 0\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} dt \\
&= \int_0^1 (\tilde{\sigma})^*(\mathbf{e}^1 \wedge \mathbf{e}^2)_{\binom{t}{s}} \begin{pmatrix} 1 \\ 0\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} dt \\
&= \int_0^1 \det\left[D(\tilde{\sigma})_{\binom{t}{s}}\right] dt.
\end{aligned}$$

Hence,

$$\int_\sigma \mathcal{K}\omega = \int_0^1 ds \int_0^1 dt \det\left[D(\tilde{\sigma})_{\binom{t}{s}}\right]. \qquad (\diamondsuit)$$

In our case $\tilde{\sigma}$ is the identity and $\int_\sigma \mathcal{K}\omega = 1$. Then, since $d[\mathcal{K}\omega \restriction \mathrm{comp}(\ell)] = 0$ and $\int_\sigma \mathcal{K}\omega \neq 0$, $\mathrm{comp}(\ell)$ is not simply connected; see Exercise 117, p. 198.

☞ **Exercise 141, p. 297** (Periods of a surface). Since T^2 is a group, its fundamental group $\pi_1(T^2, \mathrm{class}(0,0))$ is Abelian, and the set of connected components $\pi_0(\mathrm{Loops}(T^2))$ is in a one-to-one correspondence with $\pi_1(T^2, \mathrm{class}(0,0))$; see Exercise 87, p. 123. Precisely, $\pi_1(T^2, \mathrm{class}(0,0)) = \pi_1(T, [0])^2$, where we simply denote $\mathrm{class}(x)$ by $[x]$, for $x \in \mathbf{R}$ and $\mathrm{class}(x) \in T = \mathbf{R}/\mathbf{Z}$, and then $\mathrm{class}(x,y) = ([x],[y])$. Now, $\pi_1(T,[0]) \simeq \mathbf{Z}$, each class is represented by a loop $t \mapsto [nt]$, with $n \in \mathbf{Z}$. Therefore, each class of loop in T^2 is represented by a loop $\ell_{n,m} : t \mapsto ([nt],[mt])$. Now, let $s \mapsto \sigma_s$ be a loop in $\mathrm{Loops}(T^2)$ based in $\ell_{n,m}$, the map $(t,s) \mapsto \sigma_s(t)$, from $\mathbf{R}^2$ to T^2 has a unique lift $t \mapsto (x_s(t), y_s(t))$ in $\mathbf{R}^2$ — that is, $\sigma_s(t) = ([x_s(t)],[y_s(t)])$ — such that $(x_0(0), y_0(0)) = (0,0)$, (art. 8.25). Since $\sigma_0 = \ell_{n,m}$, $x_0(t) = nt + k$ and $y_0(t) = mt + k'$, but $(x_0(0), y_0(0)) = (0,0)$ implies $k = k' = 0$. Then, $x_0(t) = nt$ and $y_0(t) = mt$. Next, since σ_s is a loop, $\sigma_s(1) = \sigma_s(0)$, that is, $x_s(1) = x_s(0) + k$ and $y_s(1) = y_s(0) + k'$, with $k, k' \in \mathbf{Z}$. Computed for $s = 0$, that gives $k = n$ and $k' = m$, thus $x_s(1) = x_s(0) + n$ and $y_s(1) = y_s(0) + m$. The last condition, $\sigma_0 = \sigma_1 = \ell_{n,m}$, gives $x_1(t) = x_0(t) + k$ and $y_1(t) = y_0(t) + k'$, that is, $x_1(t) = nt + k$ and $y_1(t) = mt + k'$, with $k, k' \in \mathbf{Z}$. Summarized, these conditions write

$$\begin{array}{lll}
x_0(t) = nt & x_1(t) = nt + k & x_s(1) - x_s(0) = n, \\
y_0(t) = mt & y_1(t) = mt + k' & y_s(1) - y_s(0) = m.
\end{array}$$

Now, let us define

$$\xi(u)_s(t) = \begin{pmatrix} [u(nt + sk) + (1-u)x_s(t)] \\ [u(mt + sk') + (1-u)y_s(t)] \end{pmatrix}.$$

With the conditions summarized above, we can check that $\xi(u)_s(0) = \xi(u)_s(1)$ and $\xi(u)_0(t) = \xi(u)_1(t) = \ell_{n,m}$. Thus, ξ is a fixed-ends homotopy of loops connecting $\xi(0) = [s \mapsto \sigma_s]$ to $\xi(1) = [s \mapsto \sigma'_s]$, with $\sigma'_s(t) = ([nt + sk], [mt + sk'])$. Next, since $\mathcal{K}\omega \restriction \mathrm{comp}(\ell_{n,m})$ is closed, the integrals of $\mathcal{K}\omega$ on the loops $[s \mapsto \sigma_s]$ and $[s \mapsto \sigma'_s]$ coincide; see Exercise 116, p. 198. Hence, thanks to Exercise 140, p. 297 ($\diamondsuit$), that gives

$$\begin{aligned}
\int_{[s\mapsto\sigma_s]} \mathcal{K}\omega &= \int_0^1 dt \int_0^1 ds \det \left[D\left(\begin{pmatrix} t \\ s \end{pmatrix} \mapsto \begin{pmatrix} nt+sk \\ mt+sk' \end{pmatrix}\right)_{\binom{t}{s}} \right] \\
&= \int_0^1 dt \int_0^1 ds \det \begin{pmatrix} n & k \\ m & k' \end{pmatrix} \\
&= nk' - mk.
\end{aligned}$$

Therefore,

$$\mathrm{Periods}(\mathcal{K}\omega \restriction \mathrm{comp}(\ell_{n,m})) = \{nk' - mk \mid k, k' \in \mathbf{Z}\}.$$

Note that on the component of the constant loop $\ell_{0,0}$, the periods of $\mathcal{K}\omega$ vanish, which means that $\mathcal{K} \restriction \mathrm{comp}(\ell_{0,0})$ is exact. Also note that, for $(n,m) \neq (0,0)$,

$$\mathrm{Periods}(\mathcal{K}\omega \restriction \mathrm{comp}(\ell_{n,m})) = \gcd(n,m)\,\mathbf{Z}.$$

Remark finally that, since T^2 is a group, the connected components of $\mathrm{Loops}(T^2)$ form a group for the pointwise addition. For all $(n,m) \in \mathbf{Z}^2$, the map $\phi_{n,m} : \ell \mapsto \ell + \ell_{n,m}$, defined on the connected component of the constant loop $\ell_{0,0}$, is a diffeomorphism from $\mathrm{comp}(\ell_{0,0})$ to $\mathrm{comp}(\ell_{n,m})$, mapping $\ell_{0,0}$ to $\ell_{n,m}$. The computation of the periods shows that $\phi_{n,m}$ is not an automorphism of $\mathcal{K}\omega$.

Exercise 142, p. 307 (Compact supported real functions, I). Let us first remark that the definition of $\omega(P)_r(\delta r, \delta' r)$ makes sense. Since, by definition of the compact diffeology, for any r_0, there exist an open neighborhood $V \subset U$ and a compact K of $\mathbf{R}$ such that $P(r)$ and $P(r_0)$ coincide outside K, the derivatives

$$t \mapsto \frac{\partial}{\partial r}\frac{\partial P(r)(t)}{\partial t}(\delta r) \quad \text{and} \quad t \mapsto \frac{\partial P(r)(t)}{\partial r}(\delta' r)$$

are compact supported real functions, with their supports in K.

1) Note that $\omega(P)_r$ is antisymmetric. Indeed,

$$\begin{aligned}
\omega(P)_r(\delta r, \delta' r) &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial r}\frac{\partial P(r)(t)}{\partial t}(\delta r)\frac{\partial P(r)(t)}{\partial r}(\delta' r)\ dt \\
&= \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}\left(\frac{\partial P(r)(t)}{\partial r}(\delta r)\right)\frac{\partial P(r)(t)}{\partial r}(\delta' r)\ dt \\
&= \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}\left(\frac{\partial P(r)(t)}{\partial r}(\delta r)\frac{\partial P(r)(t)}{\partial r}(\delta' r)\right)\ dt \\
&- \int_{-\infty}^{+\infty} \frac{\partial P(r)(t)}{\partial r}(\delta r)\frac{\partial}{\partial t}\left(\frac{\partial P(r)(t)}{\partial r}(\delta' r)\right)\ dt \\
&= 0 - \int_{-\infty}^{+\infty} \frac{\partial}{\partial r}\frac{\partial P(r)(t)}{\partial t}(\delta' r)\frac{\partial P(r)(t)}{\partial r}(\delta r)\ dt \\
&= -\omega(P)_r(\delta' r, \delta r).
\end{aligned}$$

Now, let $F : V \to U$ be a smooth m-parametrization. Let $s \in V$, $\delta s, \delta' s \in \mathbf{R}^m$, let us denote $r = F(s)$, $\delta r = D(F)_s(\delta s)$ and $\delta' r = D(F)_s(\delta' s)$. We have

$$\begin{aligned}
\omega(P \circ F)_s(\delta s, \delta' s) &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial s}\left[\frac{\partial P(F(s))(t)}{\partial t}\right](\delta s)\frac{\partial P(F(s))(t)}{\partial s}(\delta' s)\,dt \\
&= \int_{-\infty}^{+\infty} \frac{\partial}{\partial r}\left[\frac{\partial P(r)(t)}{\partial t}\right]\left(\frac{\partial r}{\partial s}(\delta s)\right)\frac{\partial P(r)(t)}{\partial r}\left(\frac{\partial r}{\partial s}(\delta' s)\right)\,dt \\
&= \int_{-\infty}^{+\infty} \frac{\partial}{\partial r}\left[\frac{\partial P(r)(t)}{\partial t}\right](\delta r)\frac{\partial P(r)(t)}{\partial r}(\delta' r)\,dt \\
&= F^*(\omega(P))_s(\delta s, \delta' s).
\end{aligned}$$

Thus, ω satisfies the conditions to be a differential 2-form on X.

2) Denoting

$$r = \begin{pmatrix} s \\ s' \end{pmatrix}, \quad \delta r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \delta' r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad P(r) = sf + s'g,$$

we have immediately

$$\begin{aligned}
\frac{\partial}{\partial r}\left[\frac{\partial P(r)(t)}{\partial t}\right](\delta r) &= \frac{\partial}{\partial s}\frac{\partial}{\partial t}\Big(sf(t) + s'g(t)\Big) &= \dot{f}(t), \\
\frac{\partial P(r)(t)}{\partial r}(\delta' r) &= \frac{\partial}{\partial s'}\Big(sf(t) + s'g(t)\Big) &= g(t).
\end{aligned}$$

Then,

$$\omega\left(\begin{pmatrix} s \\ s' \end{pmatrix} \mapsto sf + s'g\right)_{\binom{s}{s'}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \int_{-\infty}^{+\infty} \dot{f}(t)g(t)\ dt = \bar{\omega}(f, g).$$

3) Let $u \in X$ and P be a plot of X. Using the notation above, we have

$$\begin{aligned}
T_u^*(\omega)(P)_r(\delta r, \delta' r) &= \omega(T_u \circ P)_r(\delta r, \delta' r) \\
&= \omega(r \mapsto P(r) + u)_r(\delta r, \delta' r),
\end{aligned}$$

and since

$$\frac{\partial}{\partial r}\Big(P(r) + u\Big) = \frac{\partial P(r)}{\partial r},$$

we get $\omega(r \mapsto P(r) + u)_r(\delta r, \delta' r) = \omega(r \mapsto P(r))_r(\delta r, \delta' r)$, that is, $T_u^*(\omega) = \omega$.

4) The space X is contractible. Indeed, the map $s \mapsto sf$, $s \in \mathbf{R}$, is a (smooth) deformation retraction connecting the constant map $f \mapsto 0$ to $\mathbf{1}_X$. Thus, X is null-homotopic. Then, since the holonomy group is a homomorphic image of $\pi_1(X)$ (art. 9.7, item 2), we get $\Gamma = \{0\}$.

5) Since Γ vanishes, the path moment map $\Psi(p)$, for a path p connecting f to g, depends only on the ends f and g, we can chose the path $p : s \mapsto sg + (1 - s)f$. Let F be a plot of G, that is, a plot of X. The definition (art. 9.2, (♡)) gives then

$$\begin{aligned}
\Psi(p)(F)_r(\delta r) &= \int_0^1 \omega\left[\begin{pmatrix} s \\ u \end{pmatrix} \mapsto T_{F(u)}(p(s+t))\right]_{\binom{s=0}{u=r}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \delta r \end{pmatrix} dt \\
&= \int_0^1 \omega\left[\begin{pmatrix} s \\ u \end{pmatrix} \mapsto p(s+t) + F(u)\right]_{\binom{s=0}{u=r}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \delta r \end{pmatrix} dt.
\end{aligned}$$

Let us introduce

$$x = \begin{pmatrix} s \\ r \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ r \end{pmatrix}, \quad \delta x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \delta' x_0 = \begin{pmatrix} 0 \\ \delta r \end{pmatrix}.$$

Then, the integrand $\mathcal{I}$ in the above formula rewrites

$$\begin{aligned}
\mathcal{I} &= \omega \left[\begin{pmatrix} s \\ u \end{pmatrix} \mapsto p(s+t) + F(u) \right]_{\binom{s=0}{u=r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \delta r \end{pmatrix} \\
&= \omega \left[x \mapsto p(s+t) + F(u) \right]_{x_0} (\delta x_0, \delta' x_0) \\
&= \int_{-\infty}^{+\infty} \left\{ \frac{\partial}{\partial x} \Big((s+t)\dot{g}(\tau) + (1-s-t)\dot{f}(\tau) + \dot{F}(u)(\tau) \Big) \Big|_{x=x_0} \delta x_0 \right. \\
&\times \left. \frac{\partial}{\partial x} \Big((s+t)g(\tau) + (1-s-t)f(\tau) + F(u)(\tau) \Big) \Big|_{x=x_0} \delta' x_0 \right\} d\tau \\
&= \int_{-\infty}^{+\infty} \left\{ \frac{\partial}{\partial s} \Big((s+t)\dot{g} + (1-s-t)\dot{f} + \dot{F}(r) \Big) \Big|_{s=0} \right. \\
&\times \left. \frac{\partial}{\partial r} \Big(tg(\tau) + (1-t)f(\tau) + F(r)(\tau) \Big) \delta r \right\} d\tau \\
&= \int_{-\infty}^{+\infty} \left\{ \Big(\dot{g}(\tau) - \dot{f}(\tau) \Big) \frac{\partial F(r)(\tau)}{\partial r} \delta r \right\} d\tau.
\end{aligned}$$

We get finally

$$\begin{aligned}
\Psi(p)(F)_r(\delta r) &= \int_0^1 \left[\int_{-\infty}^{+\infty} \left\{ \Big(\dot{g}(\tau) - \dot{f}(\tau) \Big) \frac{\partial F(r)(\tau)}{\partial r} \delta r \right\} d\tau \right] dt \\
&= \int_{-\infty}^{+\infty} \left\{ \Big(\dot{g}(\tau) - \dot{f}(\tau) \Big) \frac{\partial F(r)(\tau)}{\partial r} \delta r \right\} d\tau.
\end{aligned}$$

Exercise 143, p. 310 (Compact supported real functions, II). Let ψ be the 2-points moment map associated with the action of G.

1) Thanks to Exercise 142, p. 307, it is clear that

$$\psi(f, g) = \mu(g) - \mu(f) \quad \text{with} \quad \mu(f)(F)_r(\delta r) = \int_{-\infty}^{+\infty} \dot{f}(t) \frac{\partial F(r)(t)}{\partial r} \delta r \; dt,$$

where F is a plot of G, that is, a plot of X. We have just to check that $\mu(f)$ is an element of $\mathcal{G}^*$, that is, invariant by G. Let $u \in X$, we have

$$\begin{aligned}
T_u^*(\mu(f))(F)_r(\delta r) &= \mu(T_u \circ F)_r(\delta r) \\
&= \mu(r \mapsto F(r) + u)_r(\delta r) \\
&= \int_{-\infty}^{+\infty} \dot{f}(t) \frac{\partial [F(r)(t) + u(t)]}{\partial r} \delta r \; dt \\
&= \int_{-\infty}^{+\infty} \dot{f}(t) \frac{\partial F(r)(t)}{\partial t} \delta r \; dt \\
&= \mu(f)(F)_r(\delta r).
\end{aligned}$$

Thus, μ is indeed a primitive of ψ.

2) Thanks to (art. 9.10, item 3), modulo a coboundary, a Souriau cocycle is given by $\theta(u) = \psi(f_0, T_u(f_0))$, where f_0 is element of X. We can choose $f_0 = 0$, which gives $\theta(u) = \psi(0, T_u(0)) = \psi(0, u) = \mu(u) - \mu(0) = \mu(u)$. Thus, $\theta = \mu$. Now,

a coboundary is defined by $\Delta c = [u \mapsto \mathrm{Ad}_*(u)(c) - c]$. But G is Abelian, thus $\mathrm{Ad} = \mathbf{1}_G$, hence there is no nontrivial coboundary except 0. Therefore, since $\mu \neq 0$, θ is not trivial.

☞ **Exercise 144, p. 320** (Compact supported real functions, III). First of all the moment map μ computed in Exercise 143, p. 310, is injective. Indeed, since μ is linear we have just to solve the equation $\mu(f) = 0$, that is,

$$0 = \int_{-\infty}^{+\infty} \dot{f}(t) \frac{\partial F(r)(t)}{\partial r} \delta r \; dt$$

for all n-plots $F : U \to X$, $n \in \mathbf{N}$, for all $r \in U$ and all $\delta r \in \mathbf{R}^n$. Let us choose $F(r) = rg$, with $g \in X$, $r \in \mathbf{R}$, and $\delta r = 1$. Then,

$$0 = \int_{-\infty}^{+\infty} \dot{f}(t) g(t) \, dt \quad \text{for all} \quad g \in X.$$

Thanks to the fundamental lemma of variational calculus, since f is smooth, $\dot{f} = 0$, that is, $f = \mathrm{cst}$. But f is compact supported, then $f = 0$. Hence, μ is injective. Now, X is obviously the quotient of itself by $u \mapsto T_u(0) = u$. Therefore, (X, ω) is a symplectic homogeneous space.

☞ **Exercise 145, p. 327** (The classical moment map). Let Ψ be the paths moment map of G on (M, ω). For all $p \in \mathrm{Paths}(M)$, $\Psi(p)$ is a 1-form on G. Thus, $\Psi(p)$ is characterized by its values on the 1-plots $F : t \mapsto g_t$ (art. 6.37). But since $\Psi(p)$ is a left-invariant 1-form, $\Psi(p)$ is characterized by its values on the 1-plots pointed at $\mathbf{1}_G$, that is, $F(0) = \mathbf{1}_G$. Next, since G is a Lie group, every path F, pointed at $\mathbf{1}_G$, is tangent to a homomorphism, thus we can assume that $F \in \mathrm{Hom}^\infty(\mathbf{R}, G)$, and then $F(t) = \exp(tZ)$, with $Z \in T_{\mathbf{1}_G}(G)$. According to (art. 9.20), since for $t = 0$ and $\delta t = 1$,

$$\delta p(s) = [D(F(0))(p(s))]^{-1} \left. \frac{\partial F(t)(p(s))}{\partial t} \right|_{t=0} (1) = Z_M(p(s)),$$

we get

$$\Psi(p)(F)_0(1) = \int_0^1 \omega_{p(s)}(\dot{p}(s), Z_M(p(s))) \, ds.$$

Now, let us assume that the action of G is Hamiltonian, that is, $\Psi(p)(F)_0(1) = \mu(m')(F)_0(1) - \mu(m)(F)_0(1)$, with $m' = p(1)$ and $m = p(0)$. With our notation, that gives $\Psi(p)(F)_0(1) = \mu_Z(m') - \mu_Z(m)$. Let us apply this computation to the path $p_s(t) = p(st)$. After a change of variable, we get

$$\mu_Z(m_s) = \int_0^s \omega_{p(t)}(\dot{p}(t), Z_M(p(t))) \, dt + \mu_Z(m)$$

with $m_s = p(s)$. The derivative of this identity, with respect to s, for $s = 0$, gives then

$$\left. \frac{\partial \mu_Z(m_s)}{\partial s} \right|_{s=0} = \omega(\delta m, Z_M(m)), \quad \text{with} \quad \delta m = \dot{p}(m).$$

Therefore, with obvious notation, μ_Z is the solution of the differential equation $\omega(Z_M, \cdot) = -d[\mu_Z](\cdot)$, that is, $i_{Z_M}(\omega) = -d\mu_Z$. Finally, decomposing Z in a basis $\{\xi^i\}_{i=1}^N$ of the tangent space $T_{\mathbf{1}_G}(G)$, $Z = \sum_{i=1}^N Z_i \, \xi^i$, gives $\mu_Z(m) = \sum_{i=1}^N Z_i \, \mu_i(m)$, where $\mu_i = \mu_{\xi^i}$. Then, $\bar{\mu}(m) : Z \mapsto \mu_Z(m)$ belongs to the dual

$[T1_G(G)]^*$, which is identified with the space of momenta $\mathcal{G}^*$. We find again, that way, the classical definition of the moment map $\bar{\mu} : M \to [T1_G(G)]^*$, [**Sou70**].

☞ **Exercise 146, p. 328** (The cylinder and $\mathrm{SL}(2,\mathbf{R})$). First of all, let X and X' be two vectors of $\mathbf{R}^2$ and $M \in \mathrm{SL}(2,\mathbf{R})$, $\omega(MX, MX') = \det(M)\omega(X,X') = \omega(X,X')$, thus $\mathrm{SL}(2,\mathbf{R})$ preserves ω. Let us next check that $\mathrm{SL}(2,\mathbf{R})$ is transitive on $\mathbf{R}^2 - \{0\}$. Let $X = (x,y) \neq (0,0)$. If $x \neq 0$ or $y \neq 0$, then

$$M = \begin{pmatrix} x & 1 \\ y & \frac{1+y}{x} \end{pmatrix} \text{ or } \begin{pmatrix} x & \frac{x-1}{y} \\ y & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbf{R}) \quad \text{and} \quad M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, $\mathrm{SL}(2,\mathbf{R})$ is transitive. Now, since $\mathbf{R}^2$ is simply connected, the action of $\mathrm{SL}(2,\mathbf{R})$ is Hamiltonian (art. 9.7), and since this action has a fixed point, it is exact (art. 9.10, Note 2). Consider now the path γ_X, connecting 0 to X, and the 1-plot $F_\sigma : s \mapsto e^{s\sigma}$, with $\sigma \in \mathfrak{sl}(2,\mathbf{R})$. The application of (art. 9.20, ($\diamondsuit$)) gives

$$\Psi(\gamma_X)(F_\sigma)_0(1) = \int_0^1 \omega(\dot{\gamma}_X(t), \delta\gamma_X(t))\, dt = \int_0^1 \omega(X, t\sigma X)\, dt = \frac{1}{2}\omega(X, \sigma X),$$

thanks to (art. 9.20, ($\heartsuit$)), and to

$$\delta\gamma_X(t) = \left.\frac{\partial e^{s\sigma}(\gamma_X(t))}{\partial s}\right|_{s=0} (1) = t\sigma X.$$

And we deduce the expression of the moment map μ, the constant is fixed with $\mu(0) = 0$. It is not difficult then to check that the preimages of μ_M are the pairs $\pm X$. Actually, the image of μ_M is a 1-sheet hyperboloid in $\mathbf{R}^3 \simeq \mathfrak{sl}(2,\mathbf{R})$.

☞ **Exercise 147, p. 350** (The moment of imprimitivity). First of all let us check the variance of Taut by the action of $\mathcal{C}^\infty(X,\mathbf{R})$. Let f be a smooth real function defined on X, and let us denote by $Q \times P$ a plot of $X \times \Omega^1(X)$. We have $\bar{f}^*(\text{Taut})(P \times Q)_r = \text{Taut}(\bar{f} \circ (Q \times P))_r = (P(r) - df)(Q)_r = P(r)(Q)_r - df(Q)_r = \text{Taut}(Q \times P)_r - df(\text{pr}_1 \circ (Q \times P))_r = \text{Taut}(Q \times P)_r - \text{pr}_1^*(df)(Q \times P)_r$. Thus, $\bar{f}^*(\text{Taut}) = \text{Taut} - \text{pr}_1^*(df)$. Now let us check that this action is compatible with the value relation. Let (x, α) and (x', α') be two elements of $X \times \Omega^1(X)$ such that $\text{value}(\alpha)(x) = \text{value}(\alpha')(x')$, that is, $x = x'$ and, for every plot Q of X centered at x, $\alpha(Q)_0 = \alpha'(Q)_0$. Then, $(\alpha - df)(Q)_0 = (\alpha' - df)(Q)_0$ and $\text{value}(\alpha - df)(x) = \text{value}(\alpha)(x) - \text{value}(df)(x)$, or $(\alpha - df)(x) = \alpha(x) - df(x)$. Thus, the action of $\mathcal{C}^\infty(X, \mathbf{R})$ projects on T^*X as the action $\bar{f} : (x, a) \mapsto a - df(x)$. Now, since $\bar{f}^*(\text{Taut}) = \text{Taut} - \text{pr}_1^*(df)$, clearly $\bar{f}^*(\text{Liouv}) = \text{Liouv} - \pi^*(df)$. Put differently, $\bar{f}^*(\text{Liouv}) = \text{Liouv} - dF(f)$ with $F \in \mathcal{C}^\infty(\mathcal{C}^\infty(X, \mathbf{R}), \mathcal{C}^\infty(T^*X, \mathbf{R}))$ and $F(f) = \pi^*(f) = f \circ \pi$.

Let us denote by $R(x, a)$ the orbit map $f \mapsto a - df(x)$. Let p be a path in T^*X such that $p(0) = (x_0, a_0)$ and $p(1) = (x_1, a_1)$. By definition, for $\omega = d\text{Liouv}$, $\Psi(p) = \hat{p}^*(\mathcal{K}\omega) = \hat{p}^*(\mathcal{K}d\text{Liouv})$. Now, applying the property of the Chain-Homotopy operator $\mathcal{K} \circ d + d \circ \mathcal{K} = \hat{1}^* - \hat{0}^*$, we get

$$\begin{aligned}
\Psi(p) &= \hat{p}^*(\mathcal{K}d\text{Liouv}) \\
&= \hat{p}^*(\hat{1}^*(\text{Liouv}) - \hat{0}^*(\text{Liouv}) - d\mathcal{K}\text{Liouv}) \\
&= (\hat{1} \circ \hat{p})^*(\text{Liouv}) - (\hat{0} \circ \hat{p})^*(\text{Liouv}) - d[(\mathcal{K}\text{Liouv}) \circ \hat{p}] \\
&= R(x_1, a_1)^*(\text{Liouv}) - R(x_0, a_0)^*(\text{Liouv}) - d[f \mapsto \mathcal{K}\text{Liouv}(\hat{p}(f))].
\end{aligned}$$

Let us consider first the term $[f \mapsto \mathcal{K}\text{Liouv}(\hat{p}(f))]$. Let $p(t) = (x_t, a_t)$, then $\hat{p}(f) = [t \mapsto (x_t, a_t - df(x_t))]$. Thus,

$$\begin{aligned}
\mathcal{K}\text{Liouv}(\hat{p}(f))) &= \int_0^1 a_t[s \mapsto x_s]_{s=t}\ dt - \int_0^1 df[t \mapsto x_t]\ dt \\
&= \int_0^1 a_t[s \mapsto x_s]_{s=t}\ dt - [f(x_1) - f(x_0)].
\end{aligned}$$

Hence,

$$\begin{aligned}
d[f \mapsto \mathcal{K}\text{Liouv}(\hat{p}(f))] &= d\left\{ f \mapsto \int_0^1 a_t[s \mapsto x_s]_{s=t}\ dt - [f(x_1) - f(x_0)] \right\} \\
&= -d[f \mapsto f(x_1) - f(x_0)].
\end{aligned}$$

Let us compute $R(x, a)^*(\text{Liouv})$, for (x, a) in T^*X. Let $P : U \to \mathcal{C}^\infty(X, \mathbf{R})$ be a plot, we have

$$\begin{aligned}
R(x, a)^*(\text{Liouv})(P) &= \text{Liouv}(R(x, a) \circ P) \\
&= \text{Liouv}(r \mapsto \overline{P(r)}(x, a)) \\
&= \text{Liouv}(r \mapsto a + d[P(r)](x)) \\
&= (a + d[P(r)](x))(r \mapsto x). \\
&= 0.
\end{aligned}$$

The last equality happens because the 1-form $a + d[P(r)](x)$ is evaluated on the constant plot $r \mapsto x$, and every form evaluated on a constant plot vanishes. We get finally

$$\Psi(p) = d[f \mapsto f(x_1)] - d[f \mapsto f(x_0)].$$

Now, clearly, $\Psi(p) = \psi(p(0), p(1))$, with $p(0) = (x_0, a_0)$ and $p(1) = (x_1, a_1)$. Then, the action of $\mathcal{C}^\infty(X, \mathbf{R})$ is Hamiltonian, $\Gamma = \{0\}$, and for the 2-points moment map, we have $\psi((x_0, a_0), (x_1, a_1)) = \mu(x_1, a_1) - \mu(x_0, a_0)$, with

$$\mu : (x, a) \mapsto d[f \mapsto f(x)].$$

Noting δ_x the real function $f \mapsto f(x)$, $\mu(x, a) = d\delta_x$. Let us now check the invariance of μ. Note that, for every $h \in \mathcal{C}^\infty(X, \mathbf{R})$, $\delta_x \circ L(h) = [f \mapsto f(x) + h(x)]$. Thus, for all $h \in \mathcal{C}^\infty(X, \mathbf{R})$, $\hat{h}^*(\mu)(x, a) = \hat{h}^*(d\delta_x) = d(\delta_x \circ L(h)) = d[f \mapsto f(x) + h(x)] = d[f \mapsto f(x)] = d\delta_x = \mu(x, a)$. Hence, μ is invariant, and it is a primitive of the 2-points moment map ψ. Therefore, the Souriau class of the action of $\mathcal{C}^\infty(X, \mathbf{R})$ on T^*X vanishes.

Afterword

I was a student of Jean-Marie Souriau, working on my doctoral dissertation, when he introduced « diffeology ». I remember well, we used to gather for a seminar at that time — the beginning of the 1980s — every Tuesday, at the Center for Theoretical Physics, at Marseille's Luminy campus. Jean-Marie was trying to generalize his quantization procedure to a certain kind of coadjoint orbits of infinite dimensional groups of diffeomorphisms. He wanted to regard these groups of diffeomorphisms as Lie groups, like everybody, but he also wanted to avoid topological finessing, feeling that that was not essential for this goal. He invented then a lighter « differentiable » structure on groups of diffeomorphisms. These groups quickly became autonomous objects. I mean, he gave up groups of diffeomorphisms for abstract groups, equipped with an abstract differential structure. He called them « *groupes différentiels* », this was the first name for the future diffeological groups.

Differential spaces **are born**. Listening to Jean-Marie talking about his differential groups, I had the feeling that these structures, the axiomatics of differential groups, could be easily extended to any set, not necessarily groups, and I remember a particularly hot discussion about this question in the Luminy campus cafeteria. It was during a break in our seminar. We were there, the whole group: JMS (as we call him), Jimmy Elhadad, Christian Duval, Paul Donato, Henry-Hugues Fliche, Roland Triay, and myself. Souriau denied the interest of considering anything other than orbits of differential groups (Souriau was really, but really, « group-oriented »), and I decided when I had the time — I was working on the classification of SO(3)-symplectic manifolds which has nothing to do with diffeology — to generalize his axiomatics for any sets. But I never got the opportunity to do it. Sometime later, days or weeks, I don't remember exactly, he outlined the general theory of « *espaces différentiels* » as he called them. I would have liked to do it, anyway... I must say that, at that time, these constructions appeared to us, his students, as a fine construction, but so general that it could not turn out into great results, it could give at most some intellectual satisfaction. We were dubious. I decided to forget differential spaces and stay focused on « real maths », the classification of SO(3)-symplectic manifolds. I went to Moscow, spent a year there, and came back with a complete classification in dimension 4 and some general results in any dimension. This work represented for me a probable doctoral thesis. It was the first global classification theorem in symplectic geometry after the homogeneous case, the famous Kirillov-Kostant-Souriau theorem, which states that any homogeneous symplectic manifold is a covering of some coadjoint orbit. But Jean-Marie didn't pay any attention to my work, looking away from it, as he was completely absorbed by his « differential spaces ». I was really disappointed, I thought that this work deserved to become my doctorate. At the same time, Paul Donato gave a general construction of the universal covering for any quotient of « differential groups », that is, the universal covering of any homogeneous « differential space ». This construction became his doctoral thesis. I decided then to give up, for a moment, symplectic geometry and to get into the world of differential spaces, since it was the only subject about which JMS was able, or willing, to talk at that time.

The coming of the irrational torus. It was the year 1984, we were taking part in a conference about symplectic geometry, in Lyon, when we decided, together with Paul, to test diffeology on the *irrational torus*, the quotient of the 2-torus by an irrational line. This quotient is not a manifold but remains a diffeological space, moreover a diffeological group. We decided to call it T_α, where α is the slope of the line. The interest for this example came, of course, from the Denjoy-Poincaré flow about which we heard so much during this conference. What had diffeology to say about this group, for which topology is completely dry? We used the techniques worked out by Paul and computed its homotopy groups, we found $\mathbf{Z} + \alpha\mathbf{Z} \subset \mathbf{R}$ for the fundamental group and zero for the higher ones. The real line $\mathbf{R}$ itself appeared as the universal covering of T_α. I remember how we were excited by this computation, as we didn't believe really in the capabilities of diffeology for saying anything serious about such « singular » spaces or groups. Don't forget that differential spaces had been introduced for studying infinite dimensional groups and not singular quotients. We continued to explore this group and found that, as diffeological space, T_α is characterized by α, up to a conjugation by $\mathrm{GL}(2, \mathbf{Z})$, and we found that the components of the group of diffeomorphisms of T_α distinguish the cases where α is quadratic or not. It became clear that diffeology was not such a trivial theory and deserved to be more developed. At the same time, Alain Connes introduced the first elements of noncommutative geometry and applied them to the irrational flow on the torus — our favorite example — and his techniques didn't give anything more (in fact less) than the diffeological approach, which we considered more in the spirit of ordinary geometry. We were in a good position to know the application of Connes' theory on irrational flows as he had many fans, in the Center for Theoretical Physics at that time, developing his ideas.

All in all, this example convinced me that diffeology was a good tool, not as weak as it seemed to be. And I decided to continue to explore this path. The result of the computation of the homotopy group of T_α made me think that everything was as if the irrational flow was a true fibration of the 2-torus: the fiber $\mathbf{R}$ being contractible the homotopy of the quotient T_α had to be the same as the total space T^2, and one should avoid Paul's group specific techniques to get it. But, of course, T_α being topologically trivial it could not be an ordinary locally trivial fibration. I decided to investigate this question and, finally, gave a definition of diffeological fiber bundles, which are not locally trivial, but locally trivial along the plots — the smooth parametrizations defining the diffeology. It showed two important things for me: The first one was that the quotient of a diffeological group by any subgroup is a diffeological fibration, and thus $T^2 \to T_\alpha$. The second point was that diffeological fibrations satisfy the exact homotopy sequence. I was done, I understood why the homotopy of T_α, computed with the techniques elaborated by Paul, gave the homotopy of T^2, because of the exact homotopy sequence. I spent one year on this job, and I returned to Jean-Marie with that and some examples. He agreed to listen to me and decided that it could be my dissertation. I defended it in November 1985, and became since then completely involved in the diffeology adventure.

Differential, differentiable, or diffeological spaces? The choice of the wording « differential spaces » or « differential groups » was not very happy, because « differential » is already used in maths and has some kind of usage, especially « differential groups » which are groups with an operation of derivation. This was quoted often to us. I remember Daniel Kastler insisting that JMS change this name. From time to time we tried to find something else, without success. Finally, it was during the defense of Paul's thesis, if memory serves me right, when Van Est suggested the word « *difféologie* » like « *topologie* » as a replacement for « *différentiel* ». We found the word accurate and we decided to use it, and « *espaces différentiels* » became « *espaces difféologiques* ». There was a damper, however, « *différentiel* » as well as « *topologique* » have four spoken syllables when « *difféologique* » has five. Anyway, I used and abused this new denomination, many friends laughed at me, and one of them once told me, Your « *dix fées au logis* » — which means "ten fairies at home" — since then, there is no time when I say diffeology without thinking of these ten fairies waiting at home... Later, Daniel Bennequin pointed out to me that Kuo-Tsai Chen, in his work, *Iterated path integrals* [**Che77**] in the 1970s, defined « differentiable spaces » which looked a lot like « diffeological spaces ». I got to the library, read Chen's paper and drew a rapid (but unfounded) conclusion that our « diffeological spaces » were just equivalent to Chen's « differentiable spaces », with a slight difference in the definition. I was very disappointed, I was working on a subject I thought really new and it appeared to be known and already worked out. I decided to drop « diffeology » for « differentiable » and to give honor to Chen, but my attempt to use Chen's vocabulary was aborted — the word « diffeology » had already moved into practice, having myself helped to popularize it. However, it is good to notice that, although Chen's and Souriau's axiomatics look alike, Souriau's choice is better adapted to the geometrical point of view. Defining plots on open domains, rather than on standard simplices or convex subsets, changes dramatically the scope of the theory.

Last word? I would add some words about the use or misuse of diffeology. Some friends have expressed their skepticism about diffeology, and told me that they are waiting for diffeology to prove something great. Well, I don't know any theory proving anything, but I know mathematicians proving theorems. Let me put it differently: number theory doesn't prove any theorem, mathematicians solve problems raised by number theory. A theory is just a framework to express questions and pose problems, it is a playground. The solutions of these problems depend on the skill of the mathematicians who are interested in them. As a framework for formulating questions in differential geometry, I think diffeology is a very good one, it offers good tools, simple axioms, simple vocabulary, simple but rich objects, it is a stable category, and it opens a wide field of research. Now, I understand my friends, there are so many attempts to extend the usual category of differential geometry, and so many expectations, that it is legitimate to be doubtful. Nevertheless, I think that we now have enough convincing examples, simple or more elaborate, for which diffeology brings concrete and formal results. And this is an encouragement to persist on this path, to develop new diffeological tools, and perhaps to prove some day, some great theorem :).

At the time I began this book, Jean-Marie Souriau was alive and well. He asked me frequently about my progress. He was eager to know if people were buying his theory, and he was happy when I could say sometimes that, yes, some people in Tel Aviv or *in Texas mentioned it in some paper or discussed it on some web forum. Now, as I'm finishing this book and writing the last sentences, Jean-Marie is no longer with us. He will not see the book published and complete. It is sad,* diffeology *was his last program, in which he had strong expectations regarding geometric quantization. I am not sure if diffeology will fulfill his expectations, but I am sure that it is now a mature theory, and I dedicate this work to his memory. Whether it is the right framework to achieve Souriau's quantization program is still an open question.*

Notation and Vocabulary

Diffeology and diffeological spaces

$\mathbf{R}$	The *real numbers*.
$\mathbf{N}$	The *natural integers*.
$\times$	The product of sets or the product of a number by something.
$\mathrm{def}(f)$	The *set of definition* of the map f.
$\mathrm{val}(f)$	The values of the map f.
$\mathrm{Maps}(X, Y)$	The maps from X to Y.
1_X	The identity map of X.
n-domain	A (nonempty) open set of the vector space $\mathbf{R}^n$.
domain	An arbitrary n-domain, for some n.
$\mathrm{Domains}(\mathbf{R}^n)$	The domains of $\mathbf{R}^n$.
Domains	The domains of the $\mathbf{R}^n$, n running over $\mathbf{N}$.
$\mathfrak{P}(E)$	The *powerset* of the set E, that is, the set of all the subsets of the set E.
$\dim(U)$	The *dimension* of the domain U, that is, if $U \subset \mathrm{R}^n$, $\dim(U) = n$.
Parametrization	Any map defined from a domain to some set.
$\mathrm{Param}(X)$	The parametrizations of a set X.
$\mathrm{Param}(U, X)$	The parametrizations of a set X defined on a domain U.
$\mathbf{x}$	A bold lower case letter, a constant map (here with value x).
$D(F)(s)$ or $D(F)_s$ or $\frac{\partial r}{\partial s}$	For any smooth map $F : s \mapsto r$ between real domains, the tangent linear map of F at the point s.
pr_k	From a product, the projection on the k-th factor.
$\mathcal{D}$	Refers to a diffeology.
$D1, D2, D3$	Refers to the three axioms of diffeology.
$\mathcal{D}(U, X)$	The plots, defined on U, of the space X.
$\mathcal{D}(X, X')$ or $\mathcal{C}^\infty(X, X')$	The smooth maps from X to X'.
$\mathrm{Diff}(X, X')$	The diffeomorphisms from X to X'.
{Set}, {Diffeology} etc.	Names of categories.
$\mathcal{D}_\circ(X)$	The discrete diffeology of X.
$X_\circ$	X equipped with the discrete diffeology.
$\mathcal{D}_\bullet(X)$	The coarse diffeology of X.
$X_\bullet$	X equipped with the coarse diffeology.
$f^*(\mathcal{D}')$	The pullback of the diffeology $\mathcal{D}'$ by f.
$\coprod_{i \in \mathcal{J}} X_i$	The sum (coproduct) of diffeological spaces.

class(x)	The equivalence class of x.
$f_*(\mathcal{D})$	The pushforward of the diffeology $\mathcal{D}$ by f.
$\prod_{i\in\mathcal{I}} X_i$	The product of diffeological spaces.
Paths(X)	The space of paths in X, that is, $\mathcal{C}^\infty(\mathbf{R}, X)$.
Paths$_k$(X)	The space of iterated k-paths in X, that is, $\mathcal{C}^\infty(\mathbf{R}, \mathrm{Paths}_{k-1}(X))$.
δx	Any vector of $\mathbf{R}^n$, when x is a generic point in an n-domain.
$\langle\,\mathcal{F}\,\rangle$	The diffeology generated by a family $\mathcal{F}$ of parametrizations.
Nebula($\mathcal{F}$)	The nebula of the family of parametrizations $\mathcal{F}$.
dim(X)	The global dimension of X, for a diffeological space.
Gen(X)	The generating families of the diffeology of X.

Locality and diffeologies

$\mathcal{C}^\infty_{\mathrm{loc}}(X, X')$	The locally smooth maps from X to X'.
$\mathrm{Diff}_{\mathrm{loc}}(X, X')$	The local diffeomorphisms from X to X'.
$=_{\mathrm{loc}}$	Is equal/coincides locally.
germ	The germ of a map.
D-open	Open for the D-topology.
$\dim_x(X)$	The pointwise dimension of X at x.

Diffeological vector spaces

$L^\infty(E, E')$	The smooth linear maps from E to E'.
$\mathcal{H}$, $\mathcal{H}_{\mathbf{R}}$, $\mathcal{H}_{\mathbf{C}}$	The standard Hilbert space, on $\mathbf{R}$, on $\mathbf{C}$.
$X \cdot Y$	The scalar or Hermitian product.

Modeling spaces, manifolds, etc.

$\mathcal{S}_{\mathbf{R}}$, $\mathcal{S}_{\mathbf{C}}$	The infinite Hilbert sphere, in $\mathcal{H}_{\mathbf{R}}$ or $\mathcal{H}_{\mathbf{C}}$.
$\mathcal{P}$, $\mathcal{P}_{\mathbf{C}}$	The infinite projective space.
$\mathbf{H}_n$, $\mathbf{K}_n$	The half n-space or the n-corner.

Homotopy of diffeological spaces

$\hat{0}(\gamma)$, $\hat{1}(\gamma)$, ends(γ)	Starting point, ending point, and ends of a path γ.
Paths(X, A, B)	The paths in X, starting in A and ending in B.
Loops(X)	Loops in X.
Loops(X, x), Loops$_n$(X, x)	Loops and iterated loops in X, based at x.
$\gamma \vee \gamma'$	Concatenation of paths.
rev(γ)	The reverse path of γ.
stPaths, stLoops	Stationary paths, loops.
$\gamma^\star$	Smashed path, making it stationary.
$\pi_0(X)$, $\pi_k(X, x)$	The components of X and k-th-homotopy groups based at x.
$\pi_k(X, A, x)$	Relative homotopy pointed-sets/groups.
$\Pi(X)$	The Poincaré groupoid of X.

Cartan-De Rham calculus

$E * F$ or $L(E, F)$	The space of linear maps from E to F.
E^*	Dual vector space E, that is, $E * \mathbf{R}$.
Vector — Covector	An element of a vector space — An element of its dual.
$\otimes$, $\wedge$	Tensor product, exterior product.
$\mathrm{sgn}(\sigma)$	Signature of the permutation σ.
$L^\infty(E, F)$, E^*_∞	Smooth linear maps, smooth dual.
$M^*(A)$	Pullback of a covariant tensor A by a linear map M.
vol, $\mathrm{vol}_{\mathcal{B}}$	Volume, canonical volume of a basis $\mathcal{B}$.
$\Lambda^k(E)$	The space of all k-linear forms on a vector space E.
$\Omega^k(X)$	The space of differentiable k-forms on a diffeological space X.
$\Lambda^k_x(X)$	The space of the values of the k-forms of X at the point x.
$\alpha(P)$	The value of the differential form α on the plot P.
$f^*(\alpha)$	The pullback of the differential form α by the smooth map f.
$d\alpha$	The *exterior derivative* of the differential form α.
θ	The canonical 1-form on $\mathbf{R}$ or on an irrational torus $\mathbf{R}/\Gamma$.
δP, $\delta\sigma$, δc	The *variation* of a plot P, a cube σ, a chain c.
$\alpha \rfloor \delta\sigma$	The *contraction* of a k-form α by a variation of a cube σ, a differential $(k-1)$-form on $\mathrm{def}(\sigma)$.
i_F	The *contraction* by an arc of diffeomorphisms F.
$\pounds_F$	The Lie derivative by an arc of diffeomorphisms F.
$\mathrm{Cub}_p(X)$, $C_p(X)$	The smooth p-cubes, p-chains, in X.
$Z_\star(X, \cdot)$, $Z^\star(X, \cdot)$	The cycles/cocycles groups with coefficients.
$B_\star(X, \cdot)$, $B^\star(X, \cdot)$	The boundary/coboundary groups with coefficients.
$H_\star(X, \cdot)$, $H^\star(X, \cdot)$	The homology/cohomology groups with coefficients.
$\int_\sigma \alpha$, $\int_c \alpha$	The integral of a differential form on a cube/cubic-chain.
Magma	A set A equipped with an internal operation $\star$.
$\mathcal{K}$	The Chain-Homotopy operator.
$L(\gamma)$, $R(\gamma)$	Pre- or post-concatenation by γ.

Diffeological groups

$\mathrm{Hom}^\infty(G, G')$	The smooth homomorphisms from G to G'.
G^{Ab}	The Abelianized group $G/[G, G]$.
$R(x)$ or $\hat{x}$	The orbit map from G to X.
$L(g)$, $R(g)$	The left/right multiplication by g in G.
$\mathrm{Ad}(g)$	The adjoint action of g on G, $\mathrm{Ad}(g) = L(g) \circ R(g^{-1})$.
Coset	The left or right orbits of a subgroup in a group.
$\mathcal{G}^*$	The *space of momenta* of a diffeological group G.
Ad_*, $\mathrm{Ad}_*^{\Gamma,\theta}$	The coadjoint action of g on $\mathcal{G}^*$, on $\mathcal{G}^*/\Gamma$ with cocycle θ.

Diffeological fiber bundles

src, trg, χ	The source, target and characteristic maps on a groupoid.
$f^*(T)$	The pullback by f of the total space T of a projection.
$T \times_G E$ or $T \times_\rho E$	An associate bundle.
Θ	A connection on a principal bundle.
HorPaths(Y, Θ)	The horizontal paths in Y, for the connection Θ.
T_Γ	A *torus* $\mathbf{R}^n/\Gamma$, Γ discrete (diffeologically) and generator of $\mathbf{R}^n$.
$\mathcal{R}(X)$	The diffeological group of $(\mathbf{R}, +)$-principal bundles over X.
T_α	The *torus of periods* of a closed form α.

Symplectic diffeology

Ψ, Ψ_ω	The *moment of paths*, the index ω is for *universal.*
ψ, ψ_ω	The 2-*points moment map.*
μ, μ_ω	A 1-*point moment map.*
θ, θ_ω	The *lack of equivariance* of a moment map, a group cocycle in $\mathcal{G}^*/\Gamma$.
σ, σ_ω	The *Souriau class*, the class of the cocycles θ, θ_ω.

Bibliography

[Arn65] Vladimir I. Arnold, *Small denominators, mapping of the circumference onto itself*, Isvestja A.N. SSSR series math., vol. 25, 1965.

[Arn80] Vladimir I. Arnold, *Chapitres supplémentaires à la théorie des équations différentielles*, MIR, Moscou, 1980.

[BaHo09] John C. Baez and Alexander E. Hoffnung, *Convenient Categories of Smooth Spaces*, 2009. `http://arXiv.org/abs/0807.1704`.

[Ban78] Augustin Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv., vol. 53, pp. 174–227, 1978.

[BaWe97] Sean Bates and Alan Weinstein, *Lectures on the geometry of quantization*, Berkeley Mathematics Lecture Notes, American Mathematical Society, Providence, RI, 1997.

[BFBF09] Daniel Bennequin, Ronit Fuchs, Alain Berthoz, Tamar Flash, *Movement Timing and Invariance Arise from Several Geometries*, PLoS Comput. Biol., vol. 5, **7**, 2009.

[BeGo72] Marcel Berger and Bernard Gostiaux, *Cours de géométrie*, PUF, Armand Colin, Paris, 1972.

[BFW10] Christian Blohmann, Marco C. B. Fernandes, and Alan Weinstein, *Groupoid Symmetry And Constraints In General Relativity. 1: Kinematics*, 2010. `http://arxiv.org/abs/1003.2857v1`.

[Bom67] Jan Boman, *Differentiability of a function and of its compositions with functions of one variable*, Mathematica Scandinavica, vol. 20, pp. 249–268, 1967.

[Boo69] William M. Boothby, *Transitivity of the automorphisms of certain geometric structures*, Trans. Amer. Math. Soc., vol. 137, pp. 93–100, 1969.

[Bou55] Nicolas Bourbaki, *Espaces vectoriels topologiques*, Eléments de mathématiques, Hermann, Paris, 1955.

[Bou61] Nicolas Bourbaki, *Topologie générale*, Eléments de mathématiques, Hermann, Paris, 1961.

[Bou72] Nicolas Bourbaki, *Théorie des ensembles*, Eléments de mathématiques, Hermann, Paris, 1972.

[Bou82] Nicolas Bourbaki, *Variétés différentielles et analytiques*, Eléments de mathématiques, Diffusion CCLS, Paris, 1982.

[BoCh67] Zenon I. Borevitch and Igor R. Chafarevitch, *Théorie des nombres*, Monographies Internationales de mathématiques Modernes, Gauthier-Villars, Paris, 1967.

[Bot78] Raoul Bott, *On some formulas for the characteristic classes of group actions*, Differential Topology, Foliations and Gelfand-Fuchs Cohomology, Lect. Notes in Math., vol. 652, Springer Verlag, 1978.

[BoGa07] Charles P. Boyer and Krzysztof Galicki, *Sasakian Geometry*, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131, 2007.

[Bry93] Jean-Luc Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Birkäuser, 1993.

[Car87] Pierre Cartier, *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki – 40ème année, vol. 687, 1987–88.

[Che77] Kuo-Tsai Chen, *Iterated path integrals*, Bull. Amer. Math. Soc., vol. 83, **5**, pp. 831–879, 1977.

[Die70a] Jean Dieudonné, *Eléments d'analyse*, vol. I, Gauthier-Villars, Paris, 1970.

[Die70c] Jean Dieudonné, *Eléments d'analyse*, vol. III, Gauthier-Villars, Paris, 1970.

[Doc76] Manfred P. Do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall Inc. Englewood Cliffs, New Jersey, 1976.

[Dnl99] Simon K. Donaldson, *Moment maps and diffeomorphisms*, Asian Journal of Math., vol. 3, pp. 1–16, 1999.

[Don84] Paul Donato, *Revêtement et groupe fondamental des espaces différentiels homogènes*, Thèse de doctorat d'état, Université de Provence, Marseille, 1984.

[Don88] Paul Donato, *Géométrie des orbites coadjointes des groupes de difféomorphismes*, in Proc. Lect. Notes in Math. vol. 1416 pp. 84–104 1988

[Don94] Paul Donato, $\mathrm{Diff}(\mathrm{S}^1)$ *as coadjoint orbit in the Virasoro space of moments*, Journal of Geometry and Physics, vol. 13, pp. 299–305, 1994.

[DoIg85] Paul Donato and Patrick Iglesias, *Exemple de groupes difféologiques : flots irrationnels sur le tore*, C.R. Acad. Sci. Paris, vol. 301, **4**, 1985.

[DNF82] Boris Doubrovine, Serguei Novikov and Anatoli Fomenko, *Géométrie contemporaine*, MIR, Moscou, 1982.

[EOM02] *Encyclopedia of Mathematics*, Springer Verlag, New York-Heidelberg-Berlin, 2002.

[FuRo81] Dimitry B. Fuchs and Vladimir A. Rohlin, *Premier cours de topologie, Chapitres géométriques*, MIR, Moscou, 1981.

[God40] Kurt Gödel, *The consistency of the axiom of choice and of the generalised continuum-hypothesis with the axioms of set theory*, Princeton University Press, Princeton–New Jersey, 1940.

[God64] Roger Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1964.

[GuPo74] Victor Guillemin and Alan Pollack, *Differential topology*, Prentice Hall, New Jersey, 1974.

[Her79] Michael R. Hermann *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publication Mathématiques, I.H.E.S. Bures sur Yvette, 1979.

[HoYo61] John G. Hocking and Gail S. Young, *Topology*, Dover Publication, 1961.

[Hu59] Sze-Tsen Hu, *Homotopy Theory*, Academic Press, New York, 1959.

[Igl85] Patrick Iglesias, *Fibrations difféologiques et homotopie*, Thèse de doctorat d'État, Université de Provence, Marseille, 1985.

[Igl86] Patrick Iglesias, *Difféologie d'espace singulier et petits diviseurs*, C.R. Acad. Sci. Paris, vol. 302, pp. 519–522, 1986.

[Igl87] Patrick Iglesias, *Connexions et difféologie*, Aspects dynamiques et topologiques des groupes infinis de transformation de la mécanique, Hermann, Paris, vol. 25, pp. 61–78, 1987.

[Igl87b] Patrick Iglesias, *Bicomplexe cohomologique des espaces différentiables*, Preprint CPT-87/P.2052, Marseille, France, 1987.

[Igl91] Patrick Iglesias, *Les SO(3)-variétés symplectiques et leur classification en dimension 4*, Bull. Soc. Math. France, vol. 119, **4**, pp. 371–396, 1991.

[Igl95] Patrick Iglesias, *La trilogie du Moment*, Ann. Inst. Fourier, vol. 45, 1995.

[Igl98] Patrick Iglesias, *Les origines du calcul symplectique chez Lagrange*, L'Enseignement Mathématique, vol. 44, pp. 257–277, 1998.

[IgLa90] Patrick Iglesias and Gilles Lachaud, *Espaces différentiables singuliers et corps de nombres algébriques*, Ann. Inst. Fourier, Grenoble, vol. 40, **1**, pp. 723–737, 1990.

[IKZ05] Patrick Iglesias, Yael Karshon and Moshe Zadka, *Orbifolds as diffeologies*, Trans. Am. Math. Soc., vol. 362, **6**, pp. 2811–2831, 2010.

[IZK10] Patrick Iglesias-Zemmour and Yael Karshon, *Smooth Lie groups actions are diffeological subgroups*, Proc. Am. Math. Soc., vol. 140, pp. 731–739, 2012.

[Kak43] Shizuo Kakutani, *Topological properties of the unit sphere of a Hilbert space*, Proc. Imp. Acad. Tokyo, **19**, pp. 269–271, 1943.

[KaMe85] Mikhail I. Kargapolov and Iuri I. Merzliakov, *Elements de la théorie des groupes*, MIR, Moscou, 1985.

[KaZo12] Yael Karshon and Masrour Zoghi, *Orbifold groupoids and their underlying diffeology*, In preparation, 2012.

[Kir76] Alexandre A. Kirillov, *Elements of the theory of representations*, Berlin, Springer Verlag, 1976.

[Kle74] Felix Klein, *Le programme d'Erlangen*, Discours de la méthode, Gauthier-Villars, Paris, 1974.

[KoNo63] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry*, Interscience Publishers, 1963.

[YKS10] Yvette Kosmann-Schwarzbach, *The Noether Theorems*, Sources and Studies in the History of Mathematics and Physical Sciences, Springer Verlag, 2010.

[Kos70] Bertram Kostant, *Orbits and quantization theory*, in Proc. International Congress of Mathematicians, 1970-1971.

[Lee06] John M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics, Springer Verlag, New York, 2006.

[Les03] Joshua Leslie, *On a Diffeological Group Realization of Certain Generalized Symmetrizable Kac-Moody Lie Algebras*, Journal of Lie Theory, vol. 13, **3**, 2003.

[McK87] Kirill Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Notes Series, **124**, Cambridge University Press,Cambridge, UK, 1987.

[McL71] Saunders Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Springer Verlag, New York-Heidelberg-Berlin, 1971.

[McL75] Saunders Mac Lane, *Homology*, Graduate Texts in Mathematics Springer Verlag, New York-Heidelberg-Berlin, 1975.

[Man70] Youri I. Manine, *Cours de géométrie algébrique*, Moscow State University Ed., 1970.

[MaWe74] Jerrold Marsden and Alan Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys., vol. 5, pp. 121–130, 1974.

[Mil74] John Milnor, *Caracteristic classes*, Annals of Math. Studies, Princeton University Press, vol. 76, 1974.

[Mos66] Jurgen Moser, *A rapidly convergent iteration method and non-linear differential equations II*, Ann. Scuola Norm. di Pisa, vol. III, **20**, 1966.

[Omo86] Stephen Malvern Omohundro, *Geometric Perturbation Theory in Physics*, World Scientific, 1986.

[Sat56] Ichiro Satake, *On a generalization of the notion of manifold*, Proceedings of the National Academy of Sciences, vol. 42, pp. 359–363, 1956.

[Sat57] Ichiro Satake, *The Gauss-Bonnet theorem for V-manifolds*, Journal of the Mathematical Society of Japan, vol. 9, **4**, pp. 464–492, 1957.

[Sch11] Urs Schreiber, *Differential cohomology in a cohesive ∞-topos*, Habilitation thesis, Universität Hamburg, Germany, 2011.

[Sch75] Gerald Schwarz, *Smooth Functions Invariant under the Action of a Compact Lie Group*, Topology, vol. 14, pp. 63–68, 1975.

[Sik67] Roman Sikorski, *Abstract covariant derivative*, Colloquium Mathematicum, vol. 18, pp. 151–172., 1967.

[Smi66] Wolfgang J. Smith, *The De Rham theorem for generalized spaces*, Tohôku Math. Journ., vol. 18, **2**, 1966.

[Sou64] Jean-Marie Souriau, *Calcul linéaire*, P.U.F. Paris, 1964.

[Sou70] Jean-Marie Souriau, *Structure des systèmes dynamiques*, Dunod, Paris, 1970.

[Sou83] Jean-Marie Souriau, *Géométrie globale du problème à deux corps*, Modern Developments in Analytical Mechanics, Accademia della Scienza di Torino, pp. 369–418, 1983.

[Sou80] Jean-Marie Souriau, *Groupes différentiels*, Lect. Notes in Math., Springer Verlag, New York, vol. 836, pp. 91–128, 1980.

[Sou84] Jean-Marie Souriau, *Groupes différentiels et physique mathématique*, Collection travaux en cours, Hermann, Paris, pp. 73–119, 1984.

[Sta10] Andrew Stacey, *Comparative Smootheology*, Theory and Applications of Categories, vol. 25, **4**, pp. 64–117, 2011.

[Tyc35] Andrei Tychonoff, *Ein Fixpunktsatz*, Math. Ann., vol. 111, pp. 767–776, 1935.

[Wey39] Hermann Weyl, *Classical Groups*, Princeton University Press, New Jersey, 1939–1946.

[Wht78] Georges W. Whitehead, *Elements of Homotopy Theory*, vol. 61, Graduate Texts in Math, Springer Verlag, New York, 1978.

[Whi43] Hassler Whitney, *Differentiable even functions*, Duke Mathematics Journal, vol. 10, pp. 159–160, 1943.

[Zie96] François Ziegler, *Théorie de Mackey symplectique, in Méthode des orbites et représentations quantiques*, Thèse de doctorat d'Université, Université de Provence, Marseille, 1996. `http://arxiv.org/pdf/1011.5056v1.pdf`.

Diffeology is an extension of differential geometry. With a minimal set of axioms, diffeology allows us to deal simply but rigorously with objects which do not fall within the usual field of differential geometry: quotients of manifolds (even non-Hausdorff), spaces of functions, groups of diffeomorphisms, etc. The category of diffeology objects is stable under standard set-theoretic operations, such as quotients, products, coproducts, subsets, limits, and colimits. With its right balance between rigor and simplicity, diffeology can be a good framework for many problems that appear in various areas of physics.

Actually, the book lays the foundations of the main fields of differential geometry used in theoretical physics: differentiability, Cartan differential calculus, homology and cohomology, diffeological groups, fiber bundles, and connections. The book ends with an open program on symplectic diffeology, a rich field of application of the theory. Many exercises with solutions make this book appropriate for learning the subject.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-185

AMS *on the* Web
www.ams.org

ISBN 978-0-8218-9131-5

9 780821 891315

SURV/185