Geometry of Isotropic Convex Bodies

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Contents

Preface ix

Chapter 1. Background from asymptotic convex geometry 1
  1.1. Convex bodies 1
  1.2. Brunn–Minkowski inequality 4
  1.3. Applications of the Brunn-Minkowski inequality 8
  1.4. Mixed volumes 12
  1.5. Classical positions of convex bodies 16
  1.6. Brascamp-Lieb inequality and its reverse form 22
  1.7. Concentration of measure 25
  1.8. Entropy estimates 34
  1.9. Gaussian and sub-Gaussian processes 38
  1.10. Dvoretzky type theorems 43
  1.11. The $\ell$-position and Pisier’s inequality 50
  1.12. Milman’s low $M^*$-estimate and the quotient of subspace theorem 52
  1.13. Bourgain-Milman inequality and the $M$-position 55
  1.14. Notes and references 58

Chapter 2. Isotropic log-concave measures 63
  2.1. Log-concave probability measures 63
  2.2. Inequalities for log-concave functions 66
  2.3. Isotropic log-concave measures 72
  2.4. $\psi_\alpha$-estimates 78
  2.5. Convex bodies associated with log-concave functions 84
  2.6. Further reading 94
  2.7. Notes and references 100

Chapter 3. Hyperplane conjecture and Bourgain’s upper bound 103
  3.1. Hyperplane conjecture 104
  3.2. Geometry of isotropic convex bodies 108
  3.3. Bourgain’s upper bound for the isotropic constant 116
  3.4. The $\psi_2$-case 123
  3.5. Further reading 128
  3.6. Notes and references 134

Chapter 4. Partial answers 139
  4.1. Unconditional convex bodies 139
  4.2. Classes with uniformly bounded isotropic constant 144
  4.3. The isotropic constant of Schatten classes 150
  4.4. Bodies with few vertices or few facets 155
## CONTENTS

4.5. Further reading 161  
4.6. Notes and references 170  

Chapter 5. $L_q$-centroid bodies and concentration of mass 173  
5.1. $L_q$-centroid bodies 174  
5.2. Paouris' inequality 182  
5.3. Small ball probability estimates 190  
5.4. A short proof of Paouris' deviation inequality 197  
5.5. Further reading 202  
5.6. Notes and references 209  

Chapter 6. Bodies with maximal isotropic constant 213  
6.1. Symmetrization of isotropic convex bodies 214  
6.2. Reduction to bounded volume ratio 223  
6.3. Regular isotropic convex bodies 227  
6.4. Reduction to negative moments 231  
6.5. Reduction to $I_1(K, Z_q(K))$ 234  
6.6. Further reading 239  
6.7. Notes and references 242  

Chapter 7. Logarithmic Laplace transform and the isomorphic slicing problem 243  
7.1. Klartag's first approach to the isomorphic slicing problem 244  
7.2. Logarithmic Laplace transform and convex perturbations 249  
7.3. Klartag's solution to the isomorphic slicing problem 251  
7.4. Isotropic position and the reverse Santaló inequality 254  
7.5. Volume radius of the centroid bodies 256  
7.6. Notes and references 270  

Chapter 8. Tail estimates for linear functionals 271  
8.1. Covering numbers of the centroid bodies 273  
8.2. Volume radius of the $\psi_2$-body 284  
8.3. Distribution of the $\psi_2$-norm 292  
8.4. Super-Gaussian directions 298  
8.5. $\psi_\alpha$-estimates for marginals of isotropic log-concave measures 301  
8.6. Further reading 304  
8.7. Notes and references 310  

Chapter 9. $M$ and $M^*$-estimates 313  
9.1. Mean width in the isotropic case 313  
9.2. Estimates for $M(K)$ in the isotropic case 322  
9.3. Further reading 330  
9.4. Notes and references 332  

Chapter 10. Approximating the covariance matrix 333  
10.1. Optimal estimate 334  
10.2. Further reading 349  
10.3. Notes and references 354  

Chapter 11. Random polytopes in isotropic convex bodies 357  
11.1. Lower bound for the expected volume radius 358
11.2. Linear number of points 363
11.3. Asymptotic shape 367
11.4. Isotropic constant 377
11.5. Further reading 381
11.6. Notes and references 387

Chapter 12. Central limit problem and the thin shell conjecture 389
12.1. From the thin shell estimate to Gaussian marginals 391
12.2. The log-concave case 397
12.3. The thin shell conjecture 402
12.4. The thin shell conjecture in the unconditional case 407
12.5. Thin shell conjecture and the hyperplane conjecture 415
12.6. Notes and references 422

Chapter 13. The thin shell estimate 425
13.1. The method of proof and Fleury’s estimate 427
13.2. The thin shell estimate of Guédon and E. Milman 436
13.3. Notes and references 458

Chapter 14. Kannan-Lovász-Simonovits conjecture 461
14.1. Isoperimetric constants for log-concave probability measures 462
14.2. Equivalence of the isoperimetric constants 473
14.3. Stability of the Cheeger constant 477
14.4. The conjecture and the first lower bounds 480
14.5. Poincaré constant in the unconditional case 485
14.6. KLS-conjecture and the thin shell conjecture 486
14.7. Further reading 505
14.8. Notes and references 509

Chapter 15. Infimum convolution inequalities and concentration 511
15.1. Property (τ) 512
15.2. Infimum convolution conjecture 523
15.3. Concentration inequalities 527
15.4. Comparison of weak and strong moments 533
15.5. Further reading 535
15.6. Notes and references 546

Chapter 16. Information theory and the hyperplane conjecture 549
16.1. Entropy gap and the isotropic constant 550
16.2. Entropy jumps for log-concave random vectors with spectral gap 552
16.3. Further reading 559
16.4. Notes and references 562

Bibliography 565
Subject Index 585
Author Index 591
Preface

Asymptotic convex geometry may be described as the study of convex bodies from a geometric and analytic point of view, with an emphasis on the dependence of various parameters on the dimension. This theory stands at the intersection of classical convex geometry and the local theory of Banach spaces, but it is also closely linked to many other fields, such as probability theory, partial differential equations, Riemannian geometry, harmonic analysis and combinatorics. The aim of this book is to introduce a number of basic questions regarding the distribution of volume in high-dimensional convex bodies and to provide an up to date account of the progress that has been made in the last fifteen years. It is now understood that the convexity assumption forces most of the volume of a body to be concentrated in some canonical way and the main question is whether, under some natural normalization, the answer to many fundamental questions should be independent of the dimension.

One such normalization, that in many cases facilitates the study of volume distribution, is the isotropic position. A convex body $K$ in $\mathbb{R}^n$ is called isotropic if it has volume 1, barycenter at the origin, and its inertia matrix is a multiple of the identity: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every $\theta$ in the Euclidean unit sphere $S^{n-1}$. It is easily verified that the affine class of any convex body $K$ contains a unique, up to orthogonal transformations, isotropic convex body; this is the isotropic position of $K$. A first example of the role and significance of the isotropic position may be given through the hyperplane conjecture (or slicing problem), which is one of the main problems in the asymptotic theory of convex bodies, and asks if there exists an absolute constant $c > 0$ such that $\max_{\theta \in S^{n-1}} |K \cap \theta^\perp| \geq c$ for every convex body $K$ of volume 1 in $\mathbb{R}^n$ that has barycenter at the origin. This question was posed by Bourgain [99], who was interested in finding $L_p$-bounds for maximal operators defined in terms of arbitrary convex bodies. It is not so hard to check that answering his question affirmatively is equivalent to the following statement:

**Isotropic constant conjecture.** There exists an absolute constant $C > 0$ such that

$$L_n := \max \{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leq C.$$ 

This problem became well-known due to an article of V. Milman and Pajor which remains a classical reference on the subject. Around the same time, K. Ball showed in his PhD Thesis that the notion of the isotropic constant and the conjecture can be reformulated in the language of logarithmically-concave (or log-concave
for short) measures; however, without the problem becoming essentially more general. Let us note here that a finite Borel measure $\mu$ on $\mathbb{R}^n$ is called log-concave if, for any $\lambda \in (0, 1)$ and any compact subsets $A, B$ of $\mathbb{R}^n$, we have
\[
\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda};
\]
ote also that the indicator function of a convex body is the density (with respect to the Lebesgue measure) of a compactly supported log-concave measure, but that not all log-concave measures are compactly supported. Isotropic convex bodies now form a genuine subclass of isotropic log-concave measures, but several properties and results that (may) hold for this subclass, including the boundedness or not of the isotropic constants, immediately translate in the setting of log-concave measures. Around 1990, Bourgain obtained the upper bound $L_n \leq c_4 \sqrt{n \log n}$ and, in 2006, this estimate was improved by Klartag to $L_n \leq c_4 \sqrt{n}$.

The problem remains open and has become the starting point for many other questions and challenging conjectures in high-dimensional geometry, one of those being the central limit problem. The latter in the asymptotic theory of convex bodies means the task to identify those high-dimensional distributions which have approximately Gaussian marginals. It is a question inspired by a general fact that has appeared more than once in the literature and states that, if $\mu$ is an isotropic probability measure on $\mathbb{R}^n$ which satisfies the thin shell condition
\[
\mu \left( \{ \|x\|_2 - \sqrt{n} \geq \varepsilon \} \right) \leq \varepsilon
\]
for some $\varepsilon \in (0, 1)$, then, for all directions $\theta$ in a subset $A$ of $S^{n-1}$ with $\sigma(A) \geq 1 - \exp(-c_1 \sqrt{n})$, one has
\[
|\mu \left( \{ x : \langle x, \theta \rangle \leq t \} \right) - \Phi(t) | \leq c_2 (\varepsilon + n^{-\alpha}) \quad \text{for all } t \in \mathbb{R},
\]
where $\Phi(t)$ is the standard Gaussian distribution function and $c_1, c_2, \alpha > 0$ are absolute constants. Thus, the central limit problem is reduced to the question of identifying those high-dimensional distributions that satisfy a thin shell condition. It was the work of Anttila, Ball and Perissinaki that made this type of statement widely known in the context of isotropic convex bodies or, more generally, log-concave distributions. One of the main results in this area, first proved by Klartag in a breakthrough work, states that the assumption of log-concavity guarantees a thin shell bound, and hence an affirmative answer to the central limit problem. In fact, the following quantitative conjecture has been proposed.

**Thin shell conjecture.** There exists an absolute constant $C > 0$ such that, for any $n \geq 1$ and any isotropic log-concave measure $\mu$ on $\mathbb{R}^n$, one has
\[
\sigma^2_\mu := \int_{\mathbb{R}^n} (\|x\|_2 - \sqrt{n})^2 d\mu(x) \leq C^2.
\]

A third conjecture concerns the Cheeger constant $I_{\mu}$ of an isotropic log-concave measure $\mu$ which is defined as the best constant $\kappa \geq 0$ such that
\[
\mu^+(A) \geq \kappa \min\{\mu(A), 1 - \mu(A)\}
\]
for every Borel subset $A$ of $\mathbb{R}^n$, and where
\[
\mu^+(A) := \liminf_{\varepsilon \to 0^+} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}
\]
is the Minkowski content of $A$ (also, $A_\varepsilon := \{ x : \text{dist}(x, A) < \varepsilon \}$ is the $\varepsilon$-extension of $A$).
**Kannan-Lovász-Simonovits conjecture.** There exists an absolute constant $c > 0$ such that

$$ \text{Is}_n := \min \{ \text{Is}_\mu : \mu \text{ is isotropic log-concave measure on } \mathbb{R}^n \} \geq c. $$

Another way to formulate this conjecture is to ask if there exists an absolute constant $c > 0$ such that, for every isotropic log-concave measure $\mu$ on $\mathbb{R}^n$ and for every smooth function $\varphi$ with $\int_{\mathbb{R}^n} \varphi \, d\mu = 0$, one has

$$ c \int_{\mathbb{R}^n} \varphi^2 \, d\mu \leq \int_{\mathbb{R}^n} \| \nabla \varphi \|_2^2 \, d\mu. $$

We then say that $\mu$ satisfies the **Poincaré inequality** with constant $c > 0$. The equivalence of the two formulations can be seen by checking that

$$ \text{Is}_n^2 \simeq \inf_{\mu} \inf_{\varphi} \frac{\int_{\mathbb{R}^n} \| \nabla \varphi \|_2^2 \, d\mu}{\int_{\mathbb{R}^n} \varphi^2 \, d\mu}. $$

In this book we discuss these three conjectures and what is currently known about them, as well as other problems that are related to and arise from them. We now give a brief account of the contents of every chapter; more details can be found in the introduction of each individual chapter. In Chapters 2–4, we present the hyperplane conjecture and the first attempts to an answer. This presentation is given in the more general setting of logarithmically concave probability measures, which are introduced in Chapter 2 along with their main concentration properties. Some of these properties follow immediately from the Brunn-Minkowski inequality (more precisely, from Borell’s lemma) and can be expressed in the form of reverse Hölder inequalities for seminorms: if $f : \mathbb{R}^n \to \mathbb{R}$ is a seminorm, then, for every log-concave probability measure $\mu$ on $\mathbb{R}^n$, one has $\| f \|_{\psi_1(\mu)} \leq c \| f \|_{L_1(\mu)}$, where

$$ \| f \|_{\psi_1(\mu)} = \inf \left\{ t > 0 : \int \exp\left(\frac{|f|}{t}\right)^\alpha \, d\mu \leq 2 \right\} $$

is the Orlicz $\psi_\alpha$-norm of $f$ with respect to $\mu$, $\alpha \in [1,2]$. Isotropic log-concave measures are the log-concave probability measures $\mu$ that have barycenter at the origin and satisfy the isotropic condition

$$ \int_{\mathbb{R}^n} \langle x, \theta \rangle^2 \, d\mu(x) = 1 $$

for every $\theta \in S^{n-1}$. The isotropic constant of a measure $\mu$ in this class is defined as

$$ L_\mu := \left( \sup_{x \in \mathbb{R}^n} f(x) \right)^{1/n} \simeq (f(0))^{1/n}, $$

where $f$ is the log-concave density of $\mu$. K. Ball introduced a family of convex bodies $K_p(\mu)$, $p \geq 1$, that can be associated with a given log-concave measure $\mu$ and showed that these bodies allow us to reduce the study of log-concave measures to that of convex bodies, but also enable us to use tools from the broader class of measures to tackle problems that have naturally, or merely initially, been formulated for bodies. A first example of their use, as mentioned above, is the fact that studying the magnitude of the isotropic constant of log-concave measures is completely equivalent to the respective task inside the more restricted class of convex bodies.
The isotropic constant conjecture is discussed in detail in Chapter 3; it reads that there exists an absolute constant $C > 0$ such that

$$L_{\mu} \leq C$$

for every $n \geq 1$ and every log-concave measure $\mu$ on $\mathbb{R}^n$. In order to understand its equivalence to the hyperplane conjecture we formulated above, we recall that

$$\max\{ L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n \} \simeq \sup\{ L_{\mu} : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n \},$$

and then we have to explain the relation of the moments of inertia of a centered convex body to the volume of its hyperplane sections passing through the origin. In particular, in Section 3.1.2 we show that, if $K$ is an isotropic convex body in $\mathbb{R}^n$, then for every $\theta \in S^{n-1}$ we have

$$\frac{c_1}{L_K} \leq |K \cap \theta^\perp| \leq \frac{c_2}{L_K},$$

where $c_1, c_2 > 0$ are absolute constants, and, thus, all hyperplane sections through the barycenter of $K$ have approximately the same volume, this volume being large enough if and only if $L_K$ is small enough. The hyperplane conjecture is also equivalent to the asymptotic versions of several classical problems in convex geometry. We discuss two of them: Sylvester’s problem on the expected volume of a random simplex contained in a convex body and the Busemann-Petty problem. In Sections 3.3 and 3.4 we discuss Bourgain’s upper bound $L_K \leq C \sqrt{n} \log n$ for the isotropic constant of convex bodies $K$ in $\mathbb{R}^n$. We describe two proofs of Bourgain’s result. A key observation is that, if $K$ is an isotropic convex body in $\mathbb{R}^n$, then, as we saw above for every log-concave probability measure $\mu$, one has $\|\langle \cdot, \theta \rangle_{\psi_1(K)}\| \leq C \|\langle \cdot, \theta \rangle\|_{L_1(K)} \leq CL_K$ for all $\theta \in S^{n-1}$, where $C > 0$ is an absolute constant. In fact, Alesker’s theorem shows that one has a stronger $\psi_2$-estimate for the function $f(x) = \|x\|_2$: one has $\|f\|_{\psi_2(K)} \leq C \|f\|_{L_2(K)} \leq C \sqrt{n}L_K$. Markov’s inequality then implies exponential concentration of the mass of $K$ in a strip of width $CL_K$ and normal concentration in a ball of radius $C \sqrt{n}L_K$.

Chapter 4 is devoted to some partial affirmative answers to the hyperplane conjecture that were obtained soon after the problem became known. In order to make this statement more precise, we say that a class $C$ of centered convex bodies satisfies the hyperplane conjecture uniformly if there exists a positive constant $C$ such that $L_K \leq C$ for all $K \in C$. The hyperplane conjecture has been verified for several important classes of convex bodies. A first example is the class of unconditional convex bodies; these are the centrally symmetric convex bodies $K$ in $\mathbb{R}^n$ that have a position that is symmetric with respect to the standard coordinate subspaces, namely they have a position $\tilde{K}$ such that, if $(x_1, \ldots, x_n)$ belongs to $\tilde{K}$, then $(\epsilon_1 x_1, \ldots, \epsilon_n x_n)$ also belongs to $\tilde{K}$ for every $(\epsilon_1, \ldots, \epsilon_n) \in \{ -1, 1 \}^n$. The class of unconditional convex bodies will appear often in this book, mainly as a model for results or conjectures regarding the general cases. In this chapter, we also describe uniform bounds for the isotropic constants of some other classes of convex bodies and we give simple geometric proofs of the best known estimates for the isotropic constants of polytopes with $N$ vertices or polyhedra with $N$ facets, estimates that are logarithmic in $N$.

In Chapters 5–7, we discuss more recent approaches to the slicing problem and some very useful tools that have been developed for these approaches as well as for
related problems in the theory. Bourgain’s approach exploited the $\psi_1$-information we have for the behavior of the linear functionals $x \mapsto \langle x, \theta \rangle$ on an isotropic convex body. The aim to understand the distribution of linear functionals in an isotropic convex body or, more precisely, the behavior of their $L_q$-norms with respect to the uniform measure on the body, has been furthered by the introduction of the family of $L_q$-centroid bodies of a convex body $K$ of volume 1 or, more generally, of a log-concave probability measure $\mu$. For every $q \geq 1$, the $L_q$-centroid body $Z_q(K)$ of $K$ or, respectively, the $L_q$-centroid body $Z_q(\mu)$ of $\mu$ is defined through its support function, which is given by

$$h_{Z_q(K)}(y) := \|\langle \cdot, y \rangle\|_{L_q(K)} = \left(\int_K |\langle x, y \rangle|^q dx\right)^{1/q},$$

or by

$$h_{Z_q(\mu)}(y) := \|\langle \cdot, y \rangle\|_{L_q(\mu)} = \left(\int |\langle x, y \rangle|^q d\mu(x)\right)^{1/q},$$

respectively, for every vector $y$. Note that, according to our normalization, a convex body $K$ of volume 1 in $\mathbb{R}^n$ is isotropic if and only if it is centered and $Z_2(K) = L_K B_n^2$ and, respectively, a log-concave probability measure $\mu$ on $\mathbb{R}^n$ is isotropic if and only if it is centered and $Z_2(\mu) = B_n^2$. The development of an asymptotic theory for this family of bodies, and for their behavior as $q$ increases from 2 up to the dimension $n$, was initiated by Paouris and has proved to be a very fruitful idea.

In Chapter 5 we present the basic properties of the family $\{L_q(\mu) : q \geq 2\}$ of the centroid bodies of a centered log-concave probability measure $\mu$ on $\mathbb{R}^n$ and prove some fundamental formulas. The first main application of this theory is a striking and very useful deviation inequality of Paouris: for every isotropic log-concave probability measure $\mu$ on $\mathbb{R}^n$ one has

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 \geq ct\sqrt{n}\}) \leq \exp\left(-t\sqrt{n}\right)$$

for every $t \geq 1$, where $c > 0$ is an absolute constant. This is a consequence of the following statement: there exists an absolute constant $C_1 > 0$ such that, if $\mu$ is an isotropic log-concave measure on $\mathbb{R}^n$, then

$$I_q(\mu) \leq C_1 I_2(\mu)$$

for every $q \leq \sqrt{n}$, where $I_q(\mu)$ is defined by

$$I_q(\mu) = \left(\int_{\mathbb{R}^n} \|x\|^q_2 d\mu(x)\right)^{1/q}$$

for all $0 \neq q > -n$. Paouris has, moreover, proved an extension to this theorem which we also present: there exists an absolute constant $c_2$ such that, if $\mu$ is an isotropic log-concave measure on $\mathbb{R}^n$, then for any $1 \leq q \leq c_2 \sqrt{n}$ one has

$$I_{-q}(\mu) \simeq I_q(\mu).$$

In particular, this shows that, for all $1 \leq q \leq c_2 \sqrt{n}$, one has $I_q(\mu) \leq CI_2(\mu)$, where $C > 0$ is an absolute constant. Using the extended result one can derive a small ball probability estimate: for every isotropic log-concave measure $\mu$ on $\mathbb{R}^n$ and for any $0 < \varepsilon < \varepsilon_0$, one has

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 < \varepsilon\sqrt{n}\}) \leq \varepsilon^{C_3 \sqrt{n}},$$
where $\varepsilon_0, c_3 > 0$ are absolute constants. In a few words, the main results of Paouris imply that for any isotropic log-concave measure one has
\[ \mu(\{x : c\sqrt{n} \leq \|x\|_2 \leq C\sqrt{n}\}) \geq 1 - \exp(-\sqrt{n}). \]
This is a rough version of the thin shell estimate, that is often enough for the applications. In fact, as we will explain in Chapter 13, a way to obtain a thin shell estimate is to prove a more precise version of (1), with the constant $C_1$ being, for example, of the form $1 + cq/\sqrt{n}$ for some absolute constant $c > 0$ and for as large $q \in [1, \sqrt{n}]$ as possible.

In Chapters 6 and 7 we discuss some recent approaches to and reductions of the hyperplane conjecture. Chapter 6 deals with properties that bodies with maximal isotropic constant have, namely bodies whose isotropic constant is equal to or very close to $L_n$. It turns out that the isotropic position of such bodies is closely related to their $M$-position and this enables one to establish several interesting facts: for example, a reduction of the hyperplane conjecture, due to Bourgain, Klartag and V. Milman, to the question of boundedness of the isotropic constant of a restricted class of convex bodies, those that have volume ratio bounded by an absolute constant. Next, we give two more reductions of the conjecture to the study of parameters that can be associated with any isotropic convex body. The proofs of these reductions rely heavily on the existence of convex bodies with maximal isotropic constant whose isotropic position is not only closely related to their $M$-position, but is also compatible with regular covering estimates. The first of these reductions is a continuation of the work of Paouris on the behavior of the negative moments of the Euclidean norm with respect to an isotropic measure $\mu$ on $\mathbb{R}^n$. As we mentioned above, we already know that $I_{-q}(\mu) \simeq I_2(\mu) = \sqrt{n}$ for $0 < q \leq \sqrt{n}$, however, the behavior of the negative moments $I_{-q}(\mu)$ for $q > \sqrt{n}$ is not known at all and, in fact, our current knowledge does not exclude the possibility that the moments stay constant for all positive $q$ up to $n - 1$. Dafnis and Paouris have actually proved that this question is equivalent to the hyperplane conjecture: they introduce a parameter that, for each $\delta \geq 1$, is given by
\[ q_{-c}(\mu, \delta) := \max\{1 \leq q \leq n - 1 : I_{-q}(\mu) \geq \delta^{-1}I_2(\mu) = \delta^{-1}\sqrt{n}\}, \]
and they establish that
\[ L_n \leq C\delta \sup_{\mu} \sqrt{\frac{n}{q_{-c}(\mu, \delta)}} \log^2\left(\frac{en}{q_{-c}(\mu, \delta)}\right) \]
for every $\delta \geq 1$; additionally, they show that, if the hyperplane conjecture is correct, then we must have $q_{-c}(\mu, \delta_0) = n - 1$ for some $\delta_0 \simeq 1$, for every isotropic log-concave measure $\mu$ on $\mathbb{R}^n$. The other reduction is a work of Giannopoulos, Paouris and Vritsiou, based on the study of the parameter
\[ I_1(K, Z_q^o(K)) = \int_K \mathcal{h}Z_q^o(K)(x)dx = \int_K \|\langle \cdot, x\rangle\|_{L_q(K)}dx, \]
and can be viewed as a continuation of Bourgain’s initial approach that led to the upper bound $L_K \leq c\sqrt{n}\log n$. Roughly speaking, this last reduction can be formulated as follows: given $q \geq 2$ and $\frac{1}{2} \leq s \leq 1$, an upper bound of the form
\[ I_1(K, Z_q^o(K)) \leq C_1q^s\sqrt{n}L_K^2 \]
for all bodies $K$ in isotropic position.
leads to the estimate

\[ L_n \leq \frac{C_2 \sqrt{n} \log^2 n}{q^{\frac{1+2}{4}}} \]

Bourgain’s estimate is (almost) recovered by choosing \( q = 2 \), however, the behavior of \( I_1(K, Z_q^0(K)) \) may allow one to use \( s < 1 \) along with large values of \( q \) to obtain improved bounds if possible.

In Chapter 7 we first discuss Klartag’s solution to the isomorphic slicing problem, an isomorphic variation of the hyperplane conjecture that asks whether, given any convex body, we can find another convex body, with absolutely bounded isotropic constant, that is geometrically close to the first body. Klartag’s method relies on properties of the logarithmic Laplace transform of the uniform measure on a convex body. In general, given a finite Borel measure \( \mu \) on \( \mathbb{R}^n \), the logarithmic Laplace transform of \( \mu \) is given by

\[ \Lambda_\mu(\xi) := \log \left( \frac{1}{\mu(\mathbb{R}^n)} \int_{\mathbb{R}^n} e^{\langle \xi, x \rangle} d\mu(x) \right) \]

Klartag proved that, if \( K \) is a convex body in \( \mathbb{R}^n \), then, for every \( \varepsilon \in (0, 1) \), we can find a centered convex body \( T \subset \mathbb{R}^n \) and a point \( x \in \mathbb{R}^n \) such that \( \frac{1}{1+\varepsilon} T \subset K + x \subset (1+\varepsilon)T \) and

\[ L_T \leq C/\sqrt{\varepsilon}, \]

where \( C > 0 \) is an absolute constant. Most remarkably, by combining this fact with the deviation inequality of Paouris, one may also deduce the currently best known upper bound for the isotropic constant, which is that

\[ L_\mu \leq C' n^{1/4} \]

for every isotropic log-concave measure \( \mu \) on \( \mathbb{R}^n \). The logarithmic Laplace transform is another important tool of the theory that, since it was first employed in the setting of isotropic convex bodies and log-concave measures, has proved to be extremely useful given its various and interesting applications; these include Klartag’s solution to the isomorphic slicing problem, that we already mentioned, as well as an alternative approach of Klartag and E. Milman that combines the advantages of both the logarithmic Laplace transform and the extensive theory of the \( L_q \)-centroid bodies, and occupies the second part of Chapter 7. Klartag and E. Milman looked for lower bounds for the volume radius of the \( L_q \)-centroid bodies of an isotropic log-concave measure \( \mu \). Through a delicate analysis of the logarithmic Laplace transform of \( \mu \), they showed that

\[ (2) \quad \left| Z_q(\mu) \right|^{1/n} \geq c_1 \sqrt{q/n} \]

for all \( q \leq \sqrt{n} \), where \( c_1 > 0 \) is an absolute constant. Apart from being interesting on its own, this result leads again to the estimate \( L_\mu \leq c_2 \sqrt{n} \). It is also plausible that (2) can hold for larger values of \( q \in [1, n] \) as well; this is the content of a recent work of Vritsiou that is also discussed in the chapter. She showed that (2) holds for every \( q \) up to a variant of the parameter \( q_{-c}(\mu, \delta) \) of Dafnis and Paouris, which, as we previously mentioned, could be of the order of \( n \) (in fact, recall that the hyperplane conjecture is correct if and only if \( q_{-c}(\mu, \delta_0) \) is of the order of \( n \) for some \( \delta_0 \simeq 1 \) and every isotropic log-concave measure \( \mu \) on \( \mathbb{R}^n \)). However, even a small improvement to the estimates we currently have for \( q_{-c}(\mu, \delta) \) and its variant could permit one to extend the range of \( q \) with which the method of Klartag and Milman can be applied, and also improve on the currently known bounds for the isotropic
constant problem. Other applications of the logarithmic Laplace transform are discussed in some of the following chapters, the most important of these appearing in Chapters 12 and 15.

In Chapters 8–11, we deviate a little from those lines of results that are directly related to the hyperplane conjecture and the other two main conjectures of the theory so as to look at different applications of the tools that were developed in the previous part. Chapters 8 and 9 are devoted to some open questions, whose study so far has already shed more light on various geometric properties of convex bodies and log-concave measures. The first question was originally posed by V. Milman in the framework of convex bodies: it asks if there exists an absolute constant $C > 0$ such that every centered convex body $K$ of volume 1 has at least one sub-Gaussian direction with constant $C$. Following some positive results for special classes of convex bodies, Klartag was the first to prove the existence of “almost sub-Gaussian” directions for any isotropic convex body. More precisely, using again properties of the logarithmic Laplace transform, he proved that for every log-concave probability measure $\mu$ on $\mathbb{R}^n$, there exists a sub-Gaussian direction with constant $b = O(\sqrt{\log n})$. The main tool in the proof of this result is estimates for the covering numbers $N(Z_q(K), sB_n^2)$.

In Chapter 9 we discuss the questions of obtaining an upper bound for the mean width $w(K) := \int_{S^{n-1}} h_K(x) d\sigma(x)$, that is, the $L_1$-norm of the support function of $K$ with respect to the Haar measure on the sphere, as well as the respective $L_1$-norm of the Minkowski functional of $K$.

Chapters 10 and 11 contain applications of the theory of $L_q$-centroid bodies and of the main inequalities of Paouris to random matrices and random polytopes.
In Chapter 10 we discuss a question of Kannan, Lovász and Simonovits on the approximation of the covariance matrix of a log-concave measure. If \( K \) is an isotropic convex body in \( \mathbb{R}^n \), then one has

\[
I = \frac{1}{L_K^2} \int_K x \otimes x \, dx,
\]

where \( I \) is the identity operator. Given \( \varepsilon \in (0, 1) \), the question is to find \( N_0 \), as small as possible, for which the following holds true: if \( N \geq N_0 \), then \( N \) independent random points \( x_1, \ldots, x_N \) that are uniformly distributed in \( K \) must have, with probability greater than \( 1 - \varepsilon \), the property that

\[
(1 - \varepsilon)L_K^2 \leq \frac{1}{N} \sum_{i=1}^{N} \langle x_i, \theta \rangle^2 \leq (1 + \varepsilon)L_K^2
\]

for every \( \theta \in S^{n-1} \). The question had its origin in the problem of finding a fast algorithm for the computation of the volume of a given convex body, and Kannan, Lovász and Simonovits proved that one can take \( N_0 = C(\varepsilon)n^2 \) for some constant \( C(\varepsilon) > 0 \) depending only on \( \varepsilon \). This was improved to \( N_0 = C(\varepsilon)n(\log n)^3 \) by Bourgain and to \( N_0 = C(\varepsilon)n(\log n)^2 \) by Rudelson. It was finally proved by Adamczak, Litvak, Pajor and Tomczak-Jaegermann that the best estimate for \( N_0 \) is \( C(\varepsilon)n \). We describe the history and the solution of the problem.

In Chapter 11 we discuss the asymptotic shape of the random polytope \( K_N := \text{conv}\{\pm x_1, \ldots, \pm x_N\} \) that is spanned by \( N \) independent random points \( x_1, \ldots, x_N \) uniformly distributed in an isotropic convex body \( K \) in \( \mathbb{R}^n \). The literature on the approximation of convex bodies by random polytopes is very rich, but the main point here is that \( N \) is fixed in the range \([n, e^n]\) and we are interested in estimates which do not depend on the affine class of a convex body \( K \). Some basic tasks in this spirit are: to determine the asymptotic behavior of the volume radius \(|K|^{1/n}\), to understand the typical “asymptotic shape” of \( K_N \) and to estimate the isotropic constant of \( K_N \). The same questions can be formulated and studied more generally if we assume that we have \( N \) independent copies \( X_1, \ldots, X_N \) of an isotropic log-concave random vector \( X \). A general, and rather precise, description was obtained by Dafnis, Giannopoulos and Tsolomitis: given any isotropic log-concave measure \( \mu \) on \( \mathbb{R}^n \) and any \( n \leq N \leq \exp(n) \), the random polytope \( K_N \) defined by \( N \) i.i.d. random points \( X_1, \ldots, X_N \) which are distributed according to \( \mu \) satisfies, with high probability, the next two conditions: (i) \( K_N \supseteq cZ_{\log(n/n)}(\mu) \) and (ii) for every \( \alpha > 1 \) and \( q \geq 1 \),

\[
\mathbb{E} \left[ \sigma(\theta : h_{K_N}(\theta) \geq \alpha h_{Z_{\log(n/n)}(\mu)}(\theta)) \right] \leq N\alpha^{-q}.
\]

Using this description of the shape of \( K_N \) and the theory of centroid bodies which was developed in the previous chapters, one can determine the volume radius and the quermassintegrals of a random \( K_N \), at least in the range \( n \leq N \leq \exp(\sqrt{n}) \). A question concerning the isotropic constant of \( K_N \) can be made precise in the following way: one would like to show that, with probability tending to 1 as \( n \to \infty \), the isotropic constant of the random polytope \( K_N := \text{conv}\{\pm x_1, \ldots, \pm x_N\} \) is bounded by \( CL_K \) where \( C > 0 \) is a constant independent of \( K \), \( n \) and \( N \). We describe a method that was initiated by Klartag and Kozma when dealing with the class of Gaussian random polytopes. Variants of the method also work in the cases that the vertices \( x_i \) of \( K_N \) are distributed according to the uniform measure on an
isotropic convex body which is either $\psi_2$ (with constant $b$) or unconditional. The general case remains open.

Chapters 12–14 provide an exposition of our state of knowledge on the thin shell and Kannan-Lovász-Simonovits (or KLS for short) conjectures. Historical and other information about the thin shell conjecture and its connections with the central limit problem is given in Chapter 12. We present the work of Anttila, Ball and Perissinaki and various central limit theorems for isotropic convex bodies which would follow from thin shell estimates. This question has been studied by many authors and has been verified in some special cases. Klartag was the first to give a positive answer in full generality. In fact, aside from the immediate consequence of a general thin shell estimate that, as we mentioned again earlier in the Introduction, is that most one-dimensional marginals are close to Gaussian distributions, Klartag also established normal approximation for multidimensional marginal distributions. In Section 12.4 we give an account of Klartag’s positive answer to the thin shell conjecture for the class of unconditional isotropic log-concave random vectors, which is one of the special cases for which this question was fully verified. Klartag proved that if $K$ is an unconditional isotropic convex body in $\mathbb{R}^n$, then

$$\sigma_K^2 := \mathbb{E}_{\mu_K} (\|x\|_2 - \sqrt{n})^2 \leq C^2,$$

where $C \leq 4$ is an absolute positive constant. We also describe a result of Eldan and Klartag which shows that the thin shell conjecture is stronger than the hyperplane conjecture and implies it; more precisely, they prove that $L_n \leq C\sigma_n$ where

$$\sigma_n := \max\{\sigma_\mu : \mu \text{ is isotropic log-concave measure on } \mathbb{R}^n\},$$

and, hence, any estimate one establishes for the former conjecture immediately holds for the latter too. Chapter 13 is then devoted to a complete proof of the currently best known estimate for the thin shell conjecture, $\sigma_n \leq Cn^{1/3}$, which is due to Guédon and E. Milman.

Chapter 14 is devoted to the Kannan-Lovász-Simonovits conjecture. We first introduce various isoperimetric constants which provide information on the interplay between a log-concave probability measure $\mu$ and the underlying Euclidean metric (the Cheeger constant $I_{\mu}$, the Poincaré constant $\text{Poin}_\mu$, the exponential concentration constant $\text{Exp}_\mu$ and the first moment concentration constant $\text{FM}_\mu$) and we discuss their relation. Complementing classical results of Maz’ya, Cheeger, Gromov, V. Milman, Buser, Ledoux and others, E. Milman established the equivalence of all four constants in the log-concave setting: one has

$$I_{\mu} \simeq \sqrt{\text{Poin}_\mu} \simeq \text{Exp}_\mu \simeq \text{FM}_\mu$$

for every log-concave probability measure, where $a \simeq b$ means that $c_1 a \leq b \leq c_2 b$ for some absolute constants $c_1, c_2 > 0$. As an application, E. Milman obtained stability results for the Cheeger constant of convex bodies. Loosely speaking, if $K$ and $T$ are two convex bodies in $\mathbb{R}^n$ and if $|K| \simeq |T| \simeq |K \cap T|$, then $I_{K} \simeq I_{T}$. We introduce the KLS-conjecture in Section 14.4 and we present the first general lower bounds for $I_{\mu}$ in the isotropic log-concave case. From the work of Kannan, Lovász and Simonovits and Bobkov one has that $\sqrt{n}I_{\mu} \geq c$, where $c > 0$ is an absolute constant. Actually, Bobkov proved that

$$\sqrt{n} \sigma_{I_{\mu}} \geq c;$$
this provides a direct link between the KLS-conjecture and the thin shell conjecture: combined with the thin shell estimate of Guédon and E. Milman his result leads to the bound $n^{5/12} \mu \geq c$. In Section 14.5 we describe Klartag’s logarithmic in the dimension lower bound for the Poincaré constant $\text{Poin}_K$ of an unconditional isotropic convex body $K$ in $\mathbb{R}^n$; one has $\text{Is}_K \simeq \sqrt{\text{Poin}_K} \geq c \log n$, where $c > 0$ is an absolute positive constant. We close this discussion with a result of Eldan which, again, connects the thin shell conjecture with the KLS-conjecture: there exists an absolute constant $C > 0$ such that

$$\frac{1}{\text{Is}_n^2} \leq C \log n \sum_{k=1}^{n} \frac{\sigma_k^2}{k}.$$ 

Taking into account the result of Guédon and E. Milman, one gets the currently best known bound for $\text{Is}_n$: $\text{Is}_n^{-1} \leq C n^{1/3} \log n$.

In the last two chapters of the book we are concerned with two more approaches to the main questions in this theory. Chapter 15 is devoted to a probabilistic approach and related conjectures of Latała and Wojtaszczyk on the geometry of log-concave measures. The starting point is an infimum convolution inequality which was first introduced by Maurey when he gave a simple proof of Talagrand’s two level concentration inequality for the product exponential measure. In general, if $\mu$ is a probability measure and $\varphi$ is a non-negative measurable function on $\mathbb{R}^n$, one says that the pair $(\mu, \varphi)$ has property $(\tau)$ if, for every bounded measurable function $f$ on $\mathbb{R}^n$,

$$\left( \int_{\mathbb{R}^n} e^{f \Box \varphi} d\mu \right) \left( \int_{\mathbb{R}^n} e^{-f} d\mu \right) \leq 1,$$

where $f \Box \varphi$ is the infimum convolution of $f$ and $\varphi$, defined by

$$(f \Box \varphi)(x) = \inf \{ f(x - y) + \varphi(y) : y \in \mathbb{R}^n \}.$$ 

That the property $(\tau)$ is satisfied by a pair $(\mu, \varphi)$ is directly related to concentration properties of the measure $\mu$ since the former property implies that, for every measurable $A \subseteq \mathbb{R}^n$ and every $t > 0$, we have

$$\mu(x \notin A + B_\varphi(t)) \leq (\mu(A))^{-1} e^{-t},$$

where $B_\varphi(t) = \{ \varphi \leq t \}$. Therefore, given a measure $\mu$ it makes sense to ask for the optimal cost function $\varphi$ for which we have that $(\mu, \varphi)$ has property $(\tau)$. The first main observation is that, if we restrict ourselves to even probability measures $\mu$ and convex cost functions $\varphi$, then the (pointwise) largest candidate for a cost function is the Cramer transform $\Lambda^*_\mu$ of $\mu$; this is the Legendre transform of the logarithmic Laplace transform of $\mu$. In the setting of log-concave probability measures, the conjecture Latała and Wojtaszczyk formulate is that the pair $(\mu, \Lambda^*_\mu)$ always has property $(\tau)$. A detailed analysis shows that this conjecture would imply an affirmative answer to most of the conjectures addressed in this book: among them, the thin shell conjecture as well as the hyperplane conjecture. The problems that are raised through this approach are very interesting and challenging. An affirmative answer has been given for some rather restricted classes of measures: even log-concave product measures, uniform distributions on $\ell_p^n$-balls and rotationally invariant log-concave measures.

In the last chapter we give an account of K. Ball’s information theoretic approach, which is based on the study of the Shannon entropy $\text{Ent}(X) = - \int_{\mathbb{R}^n} f \log f$ of an isotropic random vector $X$ with density $f$. It is known that, among all
isotropic random vectors, the standard Gaussian random vector $G$ has the largest entropy, and the main observation is that comparing the entropy gap $\text{Ent} \left( \frac{X + Y}{\sqrt{2}} \right) - \text{Ent}(X)$ (with $Y$ being an independent copy of $X$) to $\text{Ent}(G) - \text{Ent}(X)$ provides a link between the KLS-conjecture and the hyperplane conjecture. A first result of this type was obtained by Ball, Barthe and Naor for one-dimensional distributions. The main result of this chapter is a recent high-dimensional analogue for isotropic log-concave random vectors, which is due to Ball and Nguyen: if $X$ is an isotropic log-concave random vector in $\mathbb{R}^n$ and its density $f$ satisfies the Poincaré inequality with constant $\kappa > 0$, then

$$\text{Ent} \left( \frac{X + Y}{\sqrt{2}} \right) - \text{Ent}(X) \geq \frac{\kappa}{4(1 + \kappa)} (\text{Ent}(G) - \text{Ent}(X)), $$

where $G$ is a standard Gaussian random vector in $\mathbb{R}^n$. In addition, Ball and Nguyen show that this implies $L_X \leq e^{17/\kappa}$. Thus, for each individual isotropic log-concave distribution $X$, a lower bound for the Poincaré constant implies a bound for the isotropic constant.

The book is primarily addressed to readers who are familiar with the basic theory of convex bodies and the asymptotic theory of finite dimensional normed spaces as these are developed in the books of Milman and Schechtman and of Pisier. Nevertheless, we have included an introductory chapter where all the prerequisites are described; short proofs are also provided for the most important results that are used in the sequel.

This book grew out of our working seminar in the last fifteen years. Among the main topics that were discussed in our meetings were the developments on the basic questions addressed in the text. A large part of the material forms the basis of PhD and MSc theses that were written at the University of Athens and the University of Crete. We are grateful to Nikos Dafnis, Dimitris Gatzouras, Marianna Hartzoulaki, Labrini Hioni, Lefteris Markessinis, Nikos Markoulakis, Grigoris Paouris, Eirini Perissinaki, Pantelis Stavrakakis and Antonis Tsolomitis for their active participation in our seminar, for collaborating with us at various stages, for numerous discussions around the subject of this book and for their friendship over the years.

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Subject Index

(T, E)-symmetrization, 214
(k, τ)-regular body, 234
(τ)-property, 512
0 - 1-polytope, 383
B-theorem, 190
I_q(K, C), 234
I_q(μ), 182
K_p(f)
  Ball’s bodies, 84
  convexity of, 87
  volume, 90
L_n
  monotonicity, 220
L_q-Rogers-Shephard inequality, 179
L_q-affine isoperimetric inequality, 175
L_q-centroid body, 174
  volume, 181
M-ellipsoid, 56
M-position, 57
  of order α, 57
M M*-estimate, 52
Q S(X), 54
Z_p^+(f), 437
Z_q-projection formula, 179
Λ_p(μ), 257
α-regular measure, 529
β-center, 384
ℓ-norm, 50
γ-concave
  function, 66
ϕ(K)
  functional, 133
ψ_2-body, 284
ψ_α
  body, 114
  estimate, 79
  measure, 79
ψ_α-norm, 78
ε-concentration hypothesis, 394
d_α-parameter, 193
j-th area measure, 14
p-median, 273
q-c-parameter, 231
q_τ-parameter, 186
τ^H-parameter, 264
s-concave
  measure, 66
Alexandrov
  inequalities for quermassintegrals, 15
  uniqueness theorem, 16
Alexandrov-Fenchel inequality, 15
asymptotic shape, 357
axis
  of inertia, 215
Ball
  normalized version of the Brascamp-Lieb
  inequality, 24
  reverse isoperimetric inequality, 24
Barthe
  reverse Brascamp-Lieb inequality, 22
  barycenter, 2
  of a function, 64
  of a measure, 64
basis
  of inertia axes, 215
Binet ellipsoid, 107
Blaschke
  formula, 129
  selection theorem, 3
Blaschke-Santaló inequality, 11
Borell lemma, 10, 80
Bourgain
  upper bound for ψ_2-bodies, 123
  upper bound for the isotropic constant, 117
Bourgain-Milman inequality, 55
Brascamp-Lieb inequality, 22
Brunn concavity principle, 4
Brunn-Minkowski inequality, 4
Busemann formula, 131
Busemann-Petty problem, 132
Cauchy formula, 16
Cauchy-Binet formula, 157
central limit problem, 389
centroïd bodies
  covering numbers, 273
  centroïd body, 174
    normalized, 279
  Cheeger constant, 462
  Cheeger inequality, 462
  comparison
    weak and strong moments, 533
concentration
  exponential, 27
  first moment, 473
  normal, 26
  of measure, 25
concentration function, 26
concentration inequality, 531
conjecture
  Gaussian correlation, 159
  hyperplane, 104
  infimum convolution, 527
  isotropic constant, 104
  Kannan-Lovász-Simonovits, 480
  Mahler, 55
  random simplex, 128
  thin shell, 403
  weak and strong moments, 534
constant
  Cheeger, 462
  exponential concentration, 471
  isoperimetric, 462
  isotropic, 75
  log-Lipschitz, 443
  Poïncaré, 465
  super-Gaussian, 272
contact point, 17
convex
  body, 2
  set, 2
convex body, 2
  2-convex, 147
  almost isotropic, 92
  barycenter, 2
  centered, 2
  isotropic, 72
  mean width, 12
  mixed width, 185
  polar, 2
  position, 16
  small diameter, 117
  symmetric, 2
  unconditional, 139
  width, 2
convolution, 275
covariance matrix, 76
covering number, 34
Cramér transform, 523
difference body, 8
direction
  super-Gaussian, 272
discrete cube, 26
distance
  Banach-Mazur, 3
  geometric, 3
  Hausdorff, 3
  Wasserstein, 410
dual
  norm, 3
  space, 3
duality of entropy
  theorem, 37
Dudley-Fernique decomposition, 121
Dvoretzky theorem, 43
Dvoretzky-Rogers lemma, 20
elipsoid, 17
  M-ellipsoid, 56
  Binet, 107
  maximal volume, 17
  minimal volume, 17
empirical process
  boundedness, 544
entropy
  duality conjecture, 37
  duality theorem, 37
  monotonicity, 559
Euclidean
  norm, 1
  unit ball, 1
  unit sphere, 1
exponential
  inequality, 471
exponential concentration
  constant, 471
first moment concentration, 473
Fisher information, 552
formula
  $Z_q$-projection, 179
  Blaschke, 129
  Busemann, 131
  Cauchy, 16
  Cauchy-Binet, 157
  Fourier inversion, 170
  Holmstedt, 368
  Kubota, 13
  Saint-Raymond, 151
  Steiner, 13
Fourier
  inversion formula, 170
  transform, 163
Fradelizi inequality, 67
function
  $\gamma$-concave, 66
  $\psi_K$, 292
  barycenter, 64
  centered, 64
  concentration, 26
indicator, 7
isotropic, 76
Lipschitz, 27
logarithmically concave, 64
radial, 2
support, 2
Walsh, 51
functional
\(\phi(K)\), 133

Gauss space, 26
Gaussian
correlation conjecture, 159
isoperimetric inequality, 31
measure, 26
Gr"unbaum lemma, 71
Grassmann manifold, 2

Haar measure, 2
Hausdorff
limit, 16
Hausdorff metric, 3
hyperplane conjecture, 104
hyperplane sections, 105
hypothesis
\(\varepsilon\)-concentration, 394
variance, 402

indicator function, 7
inequality
\(L_q\)-Rogers-Shephard, 179
Alexandrov-Fenchel, 15
Barthe, 22
Blaschke-Santaló, 11
Bobkov-Nazarov, 202
Borell, 69
Brascamp-Lieb, 22
Brunn-Minkowski, 4
Busemann, 398
Cheeger, 462, 522
concentration, 531
dual Sudakov, 35
Dudley, 40
exponential, 471
Fradelizi, 67
Hadamard, 145
Kahane-Khintchine, 34
Khintchine, 33
log-Sobolev, 431
Loomis-Whitney, 140
Lyapunov, 69
MacLaurin, 157
Minkowski, 13
non-commutative Khintchine, 352
Poincaré, 192, 465
Prékopa-Leindler, 5
Rogers-Shephard, 8
Sudakov, 35
Talagrand, 516

Urysohn, 12
Vaaler, 161
Young, 22

inertia
axis, 215
moments, 105
inertia matrix, 76
infimum convolution, 512
property, 524
infimum convolution conjecture, 527
information
Fisher, 552
isomorphic slicing problem, 244
isoperimetric
coefficient, 480
constant, 462
problem, 29
isoperimetric inequality
discrete cube, 32
for convex bodies, 10
in Gauss space, 31
spherical, 29
isotropic
condition, 73
constant of a convex body, 75
constant of a measure, 76
convex body, 72
function, 76
measure, 17, 75
position, 73
random vector, 78
isoperimetric constant
monotonicity, 220
stability, 117
isotropic convex body
\((\kappa, \tau)\)-regular, 234
circumradius, 108
covering numbers, 109
inradius, 108
with small diameter, 284

John
representation of the identity, 18
theorem, 19

\(k(X)\)
critical dimension, 45
Kahane-Khintchine inequality, 34
Kannan-Lovasz-Simonovits conjecture, 480
Kashin decomposition, 50
Khintchine inequality, 33
Klartag
bound for the isotropic constant, 252
isomorphic slicing problem, 251
KLS-conjecture, 480
Knothe map, 7
Kubota formula, 13
Löwner position, 20
Lévy family, 26
Lévy mean, 27
Legendre transform, 523
lemma
Arias de Reyna-Ball-Villa, 30
Borell, 10, 80
Dvoretzky-Rogers, 20
Grübaum, 71
Lewis, 50
Lovász-Simonovits, 94, 481
Lozanovskii, 146
Sidak, 158
Slepian, 40
Lipschitz function, 27
localization lemma, 94, 481
log-concave
density, 64
function, 64
measure, 64
log-Lipschitz constant, 443
logarithmic Laplace transform, 249
Loomis-Whitney inequality, 140
Mahler conjecture, 55
majorizing measure theorem, 42
manifold
Grassmann, 2
map
Knothe, 7
Minkowski, 16
marginal, 178
matrix
covariance, 76
inertia, 76
mean width, 12
measure
α-regular, 529
s-concave, 66
barycenter, 64
centered, 64
concentration, 25
convolution, 275, 513
even, 64
Gaussian, 26
Haar, 2
isotropic, 17, 75
logarithmically concave, 64
majorizing, 42
marginal, 178
mixed area, 14
peakedness, 161
product exponential, 516
surface area, 14
symmetric, 64
symmetric exponential, 515
metric
Haussdorff, 3
metric probability space, 25
Milman
$M$-ellipsoid, 56
isomorphic symmetrization, 55
low $M^*$-estimate, 52
quotient of subspace theorem, 54
reverse Brunn-Minkowski inequality, 57
version of Dvoretzky theorem, 44
minimal
mean width, 21
surface, 21
surface invariant, 21
Minkowski
content, 10, 29, 462
existence theorem, 15
inequality for mixed volumes, 13
map, 16
sum, 2
theorem on mixed volumes, 12
mixed
area measure, 14
volume, 13
mixed width, 185
monotonicity
of $L_\alpha$, 220
movement
type-$i$, 443
Neumann Laplacian, 408
norm
ψα-norm, 78
Orlicz, 78
Rademacher projection, 52
trace dual, 50
unconditional, 139
unitarily invariant, 151
order statistics, 536
Orlicz norm, 78
Ornstein-Uhlenbeck semigroup, 553
packing number, 34
Pisier
deviation inequality, 182
small ball estimate, 190
parameter
d+, 193
$q_{-c}$, 231
$q^{-\cdot}$, 268
$q_*$, 186
$q_{t, c}$, 267
$q^H_{t, c}$, 267
$r^H_{t, c}$, 264
Pisier
α-regular $M$-position, 57
norm of the Rademacher projection, 52
Poincaré constant, 465
Poincaré inequality, 192, 465
polar body, 2
polynomials
Khintchine type inequalities, 97
polytope, 15
0 – 1, 383
random, 357
polytopes
with few facets, 160
with few vertices, 156
position
$M$-position, 57
isotropic, 19, 73
John, 17
Löwner, 20
Lewis, 149
minimal mean width, 21
minimal surface, 21
of a convex body, 16
Prékopa-Leindler inequality, 5
principle
Brunn, 4
problem
Busemann-Petty, 132
Sylvester, 128
process
empirical, 348
Gaussian, 38
sub-Gaussian, 38
projection body, 16
property ($\tau$), 512
quermassintegrals, 13
Alexandrov inequalities, 15
normalized, 14
Rademacher
functions, 33
projection, 51
radial function, 2
Radon transform, 163
spherical, 163
random
polytope, 357
random polytopes
asymptotic shape, 368
isotropic constant, 377
volume radius, 375
random vector
isotropic, 78
reverse
Brascamp-Lieb inequality, 22
Brunn-Minkowski inequality, 57
isoperimetric inequality, 24
Santaló inequality, 55, 254
Urysohn inequality, 52
Riemannian
manifold, 418
metric, 417
package, 417
package, isomorphism, 417
package, log-concave, 417
Rogers-Shephard inequality, 8
Saint-Raymond
formula, 151
Schatten class, 150
semigroup
Ornstein-Uhlenbeck, 553
separated set, 34
set
convex, 2
separated, 34
star-shaped, 2
Sidák lemma, 158
Slepian lemma, 40
slicing problem, 104
reduction to $I_1 (K, Z^n_0 (K))$, 234
reduction to $q_{-c}(K, \delta)$, 231
reduction to bounded volume ratio, 223
space
normed, 3
spectral gap, 465
spherical cone, 46
isoperimetric inequality, 29
symmetrization, 30
star body, 2
Steiner
formula, 13
symmetrization, 4
sub-Gaussian
direction, 271
subindependence of coordinate slabs, 405
Sudakov inequality, 35
super-Gaussian
constant, 272
direction, 272
support function, 2
surface area, 10
measure, 14
Sylvester’s problem, 128
symmetrization
$(T, E)$, 214
isomorphic, 55
spherical, 30
Steiner, 4, 358
Talagrand
comparison theorem, 42
isoperimetric inequality for the discrete cube, 32
majorizing measure theorem, 42
theorem
Adamczak-Litvak-Pajor-Tomczak, 334, 335, 347
Anttila-Ball-Perissinaki, 397
Artstein-Ball-Barthe-Naor, 559
Artstein-Milman-Szarek, 37
Bakry-Ledoux, 469
Ball, 24, 87, 91, 166, 169, 550
Ball-Nguyen, 552
Ball-Perissinaki, 405
Blaschke, 3
Bobkov, 391, 395, 506
Bobkov-Nazarov, 142, 202, 203, 306, 307
Borell, 64, 66
Bourgain, 97, 118, 121, 123, 350
Bourgain-Klartag-Milman, 215, 220, 223
Bourgain-Milman, 55
Busemann, 398
Carl-Pajor, 158
Cordero-Fradelizi-Maurey, 190
Dafnis-Giannopoulos-Guédon, 381
Dafnis-Giannopoulos-Tsolomitis, 368, 375
Dafnis-Paouris, 227, 232, 233
Dudley-Sudakov, 39
Dvoretzky, 43
Eldan, 487
Eldan-Klartag, 415
Figiel-Tomczak, 51
Fleury, 433
Fradelizi-Guédon, 94
Gatgouras-Giannopoulos, 386
Gatgouras-Giannopoulos-Markoulakis, 384, 386
Giannopoulos-Milman, 21
Giannopoulos-Paouris-Valettas, 273, 284
Giannopoulos-Paouris-Vritsiou, 234
Gluskin, 158
Gromov-V. Milman, 471
Guédon-E. Milman, 436, 443
Kaner, 161
Kashin, 48
Klartag, 91, 207, 244, 251, 407, 485
Klartag-E. Milman, 260, 262, 453
Klartag-Kozma, 380
Klartag-Vershynin, 194, 205
Koldobsky, 164, 165
Latała, 82
Latała-Oleszkiewicz, 204
Litvak-Milman-Schechtman, 183
Lutwak-Yang-Zhang, 175, 181
Meyer-Pajor, 161
Milman, E., 474–477, 488
Milman, V. D., 44, 52, 54, 56
Milman-Pajor, 9
Milman-Schechtman, 45
Minkowski, 12, 15
Pajor-Tomczak, 35
Paouris, 179, 181–183, 187, 188, 190
Petty, 21
Pisier, 52, 57, 227
Rudelson, 352
Szarek-Tomczak, 48
Talagrand, 32, 42
thin shell conjecture, 403
transform
Cramer, 523
Fourier, 163
Legendre, 523
Radon, 163
type-\(i\) movement, 443
unconditional convex body, 139
unitarily invariant norm, 151
Urysohn inequality, 12
variance hypothesis, 402
volume, 1
mixed, 13
of \(L_q\)-centroid bodies, 181
radius, 9
ratio, 17
sections of the cube, 166
volume product, 11
volume ratio, 47
outer, 144
theorem, 48
theorem, global form, 49
uniformly bounded, 47
Walsh functions, 51
Wasserstein distance, 410
weak and strong moments comparison, 533
Wiener process, 489
zonoïd, 16
Löwner position, 149
Lewis position, 149
minimal mean width position, 150
Author Index

Adamczak, R., 197, 334, 335, 347, 355, 547, 565
Aldaz, J. M., 134, 565
Alesker, S., 115, 135, 183, 354, 565
Alexandrov, A. D., 14–16, 58, 565
Alonso-Gutiérrez, D., 134, 137, 155, 156, 158, 171, 377, 387, 388, 565
Anderson, T. W., 171, 565
Anttila, M., 397, 422, 423, 565
Aomoto, A., 171, 565
Arias de Reyna, J., 30, 60, 171, 565
Artstein-Avidan, S., 37, 60, 273, 559, 563, 565
Aubrun, G., 134, 355, 566
Bárány, I., 171, 371, 384, 385, 566
Badrikian, A., 61, 566
Bakry, D., 469, 509, 566
Ball, K. M., 18, 24, 30, 58–60, 84, 100, 101, 134, 136, 137, 149, 155, 166, 169, 171, 172, 397, 405, 422, 423, 550, 552, 559, 562, 563, 565, 566
Banaszczyk, W., 210
Barlow, R. E., 100, 566
Barron, A. R., 566
Barthe, F., 22, 23, 59, 60, 101, 137, 172, 312, 368, 423, 510, 559, 563, 565–567
Bastero, J., 156, 332, 422, 565, 567
Batson, J., 355, 567
Bayle, V., 509, 567
Beckenbach, E. F., 567
Bellman, R., 567
Bermúdez, J., 156, 422, 565, 567
Berwald, L., 567
Blachman, N. M., 567
Blaschke, W., 3, 11, 58, 135, 387, 567
Bogachev, V., 1, 568
Bolker, E. D., 149, 171, 568
Bonnees, T., 58, 568
Borell, C., 10, 31, 59, 60, 65, 66, 69, 100, 568
Brascamp, H. J., 22, 59, 569
Brazitikos, S., 456, 569
Brehm, U., 422, 569
Brenier, Y., 410, 569
Brezis, H., 1, 569
Brunn, H., 4, 58, 569
Buchta, C., 384, 569
Burago, Y. D., 58, 569
Busemann, H., 131, 132, 136, 398, 422, 569
Buser, P., 467, 509, 569
Campi, S., 209, 569
Carbery, A., 101, 134, 569
Carl, B., 158, 171, 569
Carlen, E. A., 569
Cauchy, A. L., 16
Chakerian, G. D., 59, 569
Chavel, I., 509, 569
Cheeger, J., 462, 466, 509, 570
Chevet, S., 61, 566
Cordero-Erausquin, D., 171, 190, 210, 273, 567, 570
Costa, M., 562, 570
Cover, T., 562, 570
Csizsár, I., 570
Dalla, L., 135, 570
Dar, S., 117, 120, 121, 135, 155, 156, 171, 570
Das Gupta, S., 59, 570
Dembo, A., 562, 570
Diaconis, P., 390, 422, 570
Dudley, R. M., 39, 40, 61, 570
Durrett, R., 570
Dvoretzky, A., 20, 43, 59, 61, 570
Dyer, M. E., 384, 385, 387, 570

591
Eldan, R., 415, 423, 459, 487, 510, 570
Emery, M., 566
Erhard, A., 60, 571
Füredi, Z., 171, 384, 385, 387, 566, 570
Feller, W., 1, 571
Fenchel, W., 15, 58, 568, 571
Figiel, T., 51, 60, 61, 571
Fleiner, T., 384, 571
Fleury, B., 433, 571
Folland, G. B., 571
Fradelizi, M., 67, 94, 100, 101, 137, 190, 210, 216, 219, 273, 567, 570, 571
Freedman, D., 390, 422, 570
Frieze, A., 571
Fukuda, K., 384, 571
Gardner, R. J., 58, 136, 137, 571
Gatzouras, D., xx, 384–386, 571
Giannopoulos, A., 17, 21, 58, 59, 100, 135–137, 171, 234, 242, 254, 270, 273, 279, 280, 284, 301, 310, 322, 332, 354, 359, 367, 368, 374, 375, 381, 384–388, 570–572
Giertz, M., 136, 572
Gilbarg, D., 572
Gluskin, E. D., 158, 171, 572
Goodey, P. R., 171, 572
Gordon, Y., 53, 61, 572
Gozlan, N., 546, 573
Gröchenig, K., 58, 136, 573
Gromov, M., 60, 100, 171, 471, 573
Gronchi, P., 209, 569
Gross, L., 458, 573
Grothendieck, A., 42, 59
Gruber, P. M., 1, 58, 573
Guédon, O., 94, 101, 210, 312, 354, 368, 381, 388, 436, 443, 567, 570, 571, 573
Hörmander, L., 408, 574
Haagerup, U., 34, 60, 573
Hadwiger, H., 58, 136, 573
Hardy, G. H., 69, 573
Hargé, G., 171, 573
Harper, L. H., 32, 60, 573
Hartzoulaki, M., xx, 109, 135, 314, 332, 354, 359, 367, 387, 572, 573
Helffer, B., 408, 574
Hensley, D., 100, 106, 574
Henstock, R., 58, 574
Hernandez Cifre, M. A., 387, 565
Hinow, P., 422, 569
Hioni, L., xx
Holmstedt, T., 368, 574
Houdré, C., 509, 568
Huet, N., 510, 574
Jessen, B., 58, 571
John, F., 18, 19, 59, 574
Johnson, W. B., 574
Junge, M., 155, 171, 574
König, H., 37, 60, 150, 155, 171, 575
Kadlec, J., 408, 574
Kahane, J.-P., 34, 60, 574
Kaibel, V., 384, 571
Kannan, R., 101, 109, 135, 334, 354, 510, 571, 574
Kanter, M., 161, 171, 574
Kashin, B., 48, 50, 61, 574
Khintchine, A., 34, 60, 574
Knothe, H., 7, 59, 575
Koldobsky, A., 137, 163, 172, 402, 422, 423, 567, 571, 575
Koszma, G., 377, 380, 388, 575
Krasnosel’skii, M. A., 100, 575
Kubota, T., 13
Kullback, S., 575
Kuperberg, G., 61, 137, 575
Kuwart, E., 509, 575
Kwapień, S., 60, 575
Lévy, P., 30, 60, 576
Larman, D. G., 135, 136, 371, 566, 570, 575
Latala, R., 82, 101, 197, 204, 210, 510, 533, 536, 544, 547, 565, 576
Ledoux, M., 60, 171, 254, 347, 348, 467, 469, 509, 566, 576
Lehec, J., 546, 576
Leindler, L., 5, 59, 576
Lewis, D. R., 61, 149, 171, 576
Li, P., 509, 510, 576
Lichnerowicz, A., 408, 576
Lieb, E. H., 22, 59, 563, 569, 576
Limshits, M. A., 199, 422, 575, 576
Lindenschauss, J., 58, 60, 100, 171, 379, 568, 571, 574, 576
Littlewood, J. E., 60, 69, 573, 576
Litvak, A. E., 183, 197, 210, 332, 334, 335, 347, 355, 367, 369, 387, 547, 565, 576
Loomis, L. H., 140, 170, 577
Lovász, L., 101, 109, 135, 334, 354, 510, 574, 577
Lozanovskii, G. J., 146, 577
Lust-Piquard, F., 352, 577
Lusternik, L., 58, 577
Luttinger, J. M., 59, 569
Lutwak, E., 136, 175, 181, 209, 241, 577
Müller, C., 579
Müller, D., 134, 579
Müller, J., 384, 569
Macbeath, A. M., 58, 359, 387, 574, 577
Madiman, M., 562, 568
Mahl, K., 59, 61, 577
Markessinis, E., xx, 59, 577
Markoualakis, N., xx, 384–386, 572
Marshall, A. W., 100, 566, 577
Maurey, B., 32, 60, 137, 190, 210, 273, 516, 546, 567, 570, 577
Maz’ya, V. G., 462, 466, 509, 577
McDiarmid, C., 384, 385, 387, 570
Meckes, E., 422, 577
Meckes, M. W., 422, 577
Mehta, M. L., 171, 577
Mendelson, S., 312, 355, 368, 567, 577
Meyer, M., 11, 59, 61, 150, 155, 161, 162, 171, 569, 573, 575, 577
Milman, E., 476
Minkowski, H., 4, 12, 13, 15, 58, 578
Montgomery-Smith, S. J., 579
Naor, A., 134, 172, 312, 368, 422, 559, 563, 565–567, 579
Nguyen, V. H., 423, 552, 563, 566
Ohmann, D., 58, 573
Oksendal, B., 579
Oleszkiewicz, K., 172, 197, 204, 210, 565, 576, 579
Olkin, I., 100, 577
Pólya, G., 69, 573
Pór, A., 384, 385, 566
Paley, R. E. A. C., 274, 579
Papadimitriakis, M., 59, 136, 171, 572, 579
Payne, L. E., 510, 580
Pelczynski, A., 172, 579
Perissinaki, E., xx, 397, 405, 422, 423, 565, 573, 576, 579
Pett, C. M., 22, 59, 132, 136, 569, 570
Pichorides, S. K., 136, 570
Pilipczuk, M., 423, 570
Pisier, M. S., 580
Pisier, G., 1, 52, 57–59, 61, 227, 352, 577, 580
Pitt, L., 159, 171, 580
Pivovarov, P., 311, 315, 332, 364, 387, 388, 579
Podkorytov, A. N., 167, 172, 579
Prékopa, A., 5, 59, 580
Prochno, J., 387, 565
Proschon, F., 100, 566, 577
Rao, M. M., 100, 580
Reisner, S., 61, 573, 580
Ren, Z. D., 100, 580
Rinott, Y., 171, 580
Rockafellar, R. T., 1, 580
Rogers, C. A., 8, 20, 43, 59, 136, 570, 575, 580
Romik, D., 422, 579
Rosales, C., 509, 567
Rote, G., 384, 571
Rotem, L., 100, 580
Rothaus, O. S., 462, 509
Rudelson, M., 332, 333, 352, 354, 367, 369, 387, 573, 577, 580
Rutickii, J. B., 100, 575
Ryabogin, D., 172, 575
Saint-Raymond, J., 61, 151, 152, 171, 581
Santaloš, L. A., 11, 59, 571
Saroglou, C., 59, 135, 577, 581
Schechtman, G., 1, 45, 58, 60, 61, 101, 171, 183, 185, 210, 576, 578, 581
Schlumprecht, T., 171, 571, 581
Schmidt, E., 60, 581
Schmuckenschlager, M., 170, 171, 210, 581
Schneider, R., 1, 58, 250, 581
Shannon, C. E., 581
Shephard, G. C., 8, 59, 580
Sherman, S. A., 171, 581
Sidák, Z., 158, 151, 581
Simonovits, M., 101, 135, 334, 354, 510, 574, 577
Sjöstrand, J., 408, 574
Slepian, D., 40, 61, 581
Sodin, M., 101, 579
Sodin, S., 422, 459, 510, 578, 581
Soffer, A., 569
Spielman, D., 355, 567
Spingarn, J., 228, 581
Srivastava, N., 355, 567, 581
Stam, A., 581
Stavralakis, P., xx, 322, 332, 456, 569, 572, 581
Stein, E., 134, 581
Steiner, J., 13, 58, 582
Sternberg, P., 509, 582
Strömberg, J. O., 134, 582
Stroock, D. W., 60, 582
Sudakov, V. N., 31, 35, 39, 40, 60, 61, 390, 422, 582
Szarek, S. J., 34, 37, 48, 60, 61, 273, 281, 383, 566, 578, 582
Talagrand, M., 32, 35, 42, 43, 60, 61, 117, 135, 254, 347, 348, 516, 544, 576, 582
Tao, T., 134, 579
Thomas, J., 562, 570
Tichy, R. F., 384, 569
Tomczak-Jaegermann, N., 1, 35, 36, 48, 51, 53, 58, 60, 61, 197, 254, 332, 334, 335, 347, 355, 367, 369, 387, 547, 565, 571, 576, 579, 582
Trudinger, N. S., 572
Tserpes, B. S., 31, 60, 582
Tsolomitis, A., xx, 135, 322, 332, 354, 359, 367, 368, 374, 375, 387, 570, 572
Tzafriri, L., 100, 171, 576
Urysohn, P. S., 12, 59, 61, 582
Vaaler, J. D., 161, 171, 582
Valettas, P., 59, 273, 279, 284, 301, 310, 322, 332, 572, 581
Vempala, S., 577
Vershynin, R., 193–195, 205, 210, 332, 333, 355, 575, 581, 582
Villa, R., 30, 60, 171, 565
Villani, C., 410, 582
Vogt, H., 90, 422, 569, 582
Voigt, J. A., 90, 422, 569, 582
Volberg, A., 101, 579
von Weizsäcker, H., 390, 582
Vritsiou, B.-H., 234, 242, 254, 266, 270, 322, 332, 572, 582
Weaver, W., 581
Weil, W., 171, 572
Weinberger, H. F., 510, 580
Werner, E., 371, 579
Whitney, H., 140, 170, 577
Wojtaszczyk, J., 422, 423, 510, 533, 547, 576, 582
Wolff, P., 156, 565
Wright, J., 101, 569
Yang, D., 175, 181, 209, 241, 577
Yaskin, V., 172, 575
Yau, S. T., 509, 510, 576
Zalgaller, V. A., 58, 569
Zhang, G., 136, 175, 181, 209, 241, 577, 583
Ziegler, G., 384, 583
Zinn, J., 171, 210, 581
Zong, C., 583
Zumbrun, K., 509, 575
Zvavitch, A., 172, 575
Zygmund, A., 274, 579
Zymonopoulou, M., 172, 575
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