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Preface

Polytope exchange transformations are higher dimensional generalizations of interval exchange transformations, one dimensional maps which have been extensively and very fruitfully studied for the past 40 years or so. Polytope exchange transformations have the added appeal that they produce intricate fractal-like tilings. At this point, the higher dimensional versions are not nearly as well understood as their 1-dimensional counterparts, and it seems natural to focus on such questions as finding a robust renormalization theory for a large class of examples.

In this monograph, we introduce a general method of constructing polytope exchange transformations (PETs) in all dimensions. Our construction is functorial in nature. One starts with a multigraph such that the vertices are labeled by convex polytopes and the edges are labeled by Euclidean lattices in such a way that each vertex label is a fundamental domain for all the lattices labeling incident edges. There is a functor from the fundamental groupoid of this multigraph into the category of PETs, and the image of this functor contains many interesting examples. For instance, one can produce huge multi-parameter families based on finite reflection groups.

Most of this monograph is devoted to the study of the simplest examples of our construction. These examples are based on the order 8 dihedral reflection group D_4. The corresponding multigraph is a digon (two vertices connected by two edges) decorated by 2-dimensional parallelograms and lattices. This input produces a 1-parameter family of polygon exchange transformations which we call the Octagonal PETs. One particular parameter is closely related to a system studied by Adler-Kitchens-Tresser.

We show that the family of octagonal PETs has a renormalization scheme in which the $(2, 4, \infty)$ hyperbolic reflection triangle group acts on the parameter space (by linear fractional transformations) as a renormalization symmetry group. The underlying hyperbolic geometry symmetry of the system allows for a complete classification of the shapes of the periodic tiles and also a complete classification of the topology of the limit sets.

We also establish a local equivalence between outer billiards on semi-regular octagons and the octagonal PETs, and this gives a similarly complete description of outer billiards on semi-regular octagons. Finally, we show how the octagonal PETs arise naturally as invariant slices of certain 4-dimensional PETs based on deformations of the E_4 lattice.

I discovered almost all the material in this monograph by computer experimentation, and then later on found rigorous proofs. Most of the proofs here are traditional, but they do rely on 12 computer calculations. These calculations are described in detail in the last part of the monograph.

I wrote two interactive Java programs, OctaPET and BonePET, which illustrate essentially all the mathematics in this monograph. The reader can download these programs from my website (as explained at the end of the introduction) and can use them while reading the manuscript. I wrote the monograph with the intention that a serious reader would also use the programs.

I would like to thank Nicolas Bedaride, Pat Hooper, Injee Jeong, John Smillie, and Sergei Tabachnikov for interesting conversations about topics related to this work. Some of this work was carried out at ICERM in Summer 2012, and most of it was carried out during my sabbatical at Oxford in 2012–2013. I would especially like to thank All Souls College, Oxford, for providing a wonderful research environment.

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Oxford, November 2012

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A polytope exchange transformation is a (discontinuous) map from a polytope to itself that is a translation wherever it is defined. The 1-dimensional examples, interval exchange transformations, have been studied fruitfully for many years and have deep connections to other areas of mathematics, such as Teichmüller theory. This book introduces a general method for constructing polytope exchange transformations in higher dimensions and then studies the simplest example of the construction in detail. The simplest case is a 1-parameter family of polygon exchange transformations that turns out to be closely related to outer billiards on semi-regular octagons. The 1-parameter family admits a complete renormalization scheme, and this structure allows for a fairly complete analysis both of the system and of outer billiards on semi-regular octagons. The material in this book was discovered through computer experimentation. On the other hand, the proofs are traditional, except for a few rigorous computer-assisted calculations.

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