Asymptotic Geometric Analysis, Part I

Shiri Artstein-Avidan
Apostolos Giannopoulos
Vitali D. Milman
Contents

Preface vii

Chapter 1. Convex bodies: classical geometric inequalities 1
  1.1. Basic convexity 1
  1.2. Brunn–Minkowski inequality 9
  1.3. Volume preserving transformations 16
  1.4. Functional forms 22
  1.5. Applications of the Brunn-Minkowski inequality 26
  1.6. Minkowski’s problem 36
  1.7. Notes and remarks 38

Chapter 2. Classical positions of convex bodies 47
  2.1. John’s theorem 49
  2.2. Minimal mean width position 62
  2.3. Minimal surface area position 69
  2.4. Reverse isoperimetric inequality 66
  2.5. Notes and remarks 73

Chapter 3. Isomorphic isoperimetric inequalities and concentration of measure 79
  3.1. An approach through extremal sets, and the basic terminology 80
  3.2. Deviation inequalities for Lipschitz functions on classical metric probability spaces 94
  3.3. Concentration on homogeneous spaces 97
  3.4. An approach through conditional expectation and martingales 100
  3.5. Khintchine type inequalities 107
  3.6. Raz’s Lemma 120
  3.7. Notes and remarks 123

Chapter 4. Metric entropy and covering numbers estimates 131
  4.1. Covering numbers 131
  4.2. Sudakov’s inequality and its dual 139
  4.3. Entropy numbers and approximation numbers 143
  4.4. Duality of entropy 148
  4.5. Notes and remarks 156

Chapter 5. Almost Euclidean subspaces of finite dimensional normed spaces 161
  5.1. Dvoretzky type theorems 163
  5.2. Milman’s proof 164
  5.3. The critical dimension $k(X)$ 172
  5.4. Euclidean subspaces of $\ell_p^n$ 177
5.5. Volume ratio and Kashin’s theorem 179
5.6. Global form of the Dvoretzky-Milman theorem 183
5.7. Isomorphic phase transitions and thresholds 187
5.8. Small ball estimates 193
5.9. Dependence on $\varepsilon$ 196
5.10. Notes and remarks 197

Chapter 6. The $\ell$-position and the Rademacher projection 203
6.1. Hermite polynomials 204
6.2. Pisier’s inequality 209
6.3. The Rademacher projection 211
6.4. The $\ell$-norm 218
6.5. The $M M^*$-estimate 222
6.6. Equivalence of the two projections 224
6.7. Bourgain’s example 227
6.8. Notes and remarks 229

Chapter 7. Proportional Theory 233
7.1. Introduction 233
7.2. First proofs of the $M^*$-estimate 234
7.3. Proofs with the optimal dependence 238
7.4. Milman’s quotient of subspace theorem 241
7.5. Asymptotic formulas for random sections 244
7.6. Linear duality relations 249
7.7. Notes and remarks 253

Chapter 8. $M$-position and the reverse Brunn–Minkowski inequality 257
8.1. Introduction 257
8.2. The Bourgain-Milman inequality 261
8.3. Isomorphic symmetrization 263
8.4. Milman’s reverse Brunn-Minkowski inequality 267
8.5. Extension to the non-symmetric case 271
8.6. Applications of the $M$-position 273
8.7. $\alpha$-regular $M$-position: Pisier’s approach 275
8.8. Notes and remarks 284

Chapter 9. Gaussian approach 287
9.1. Dudley, and another look at Sudakov 288
9.2. Gaussian proof of Dvoretzky theorem 296
9.3. Gaussian proof of the $M^*$-estimate 301
9.4. Random orthogonal factorizations 304
9.5. Comparison principles for Gaussian processes 307
9.6. Notes and remarks 312

Chapter 10. Volume distribution in convex bodies 315
10.1. Isotropic position 317
10.2. Isotropic log-concave measures 325
10.3. Bourgain’s upper bound for the isotropic constant 333
10.4. Paouris’ deviation inequality 336
10.5. The isomorphic slicing problem 346
Preface

In this book we present the theory of asymptotic geometric analysis, a theory which stands at the midpoint between geometry and functional analysis. The theory originated from functional analysis, where one studied Banach spaces, usually of infinite dimensions. In the first few decades of its development it was called “local theory of normed spaces”, which stood for investigating infinite dimensional Banach spaces via their finite dimensional features, for example subspaces or quotients. Soon, geometry started to become central. However, as we shall explain below in more detail, the study of “isometric” problems, a point of view typical for geometry, had to be substituted by an “isomorphic” point of view. This became possible with the introduction of an asymptotic approach to the study of high-dimensional spaces (asymptotic with respect to dimensions increasing to infinity). Finally, these finite but very high-dimensional questions and results became interesting in their own right, influential on other mathematical fields of mathematics, and independent of their original connection with infinite dimensional theory. Thus the name asymptotic geometric analysis nowadays describes an essentially new field.

Our primary object of study will be a finite dimensional normed space $X$; we may assume that $X$ is $\mathbb{R}^n$ equipped with a norm $\| \cdot \|$. Such a space is determined by its unit ball $K_X = \{ x \in \mathbb{R}^n : \| x \| \leq 1 \}$, which is a compact convex set with non-empty interior (we call this type of set “a convex body”). Conversely, if $K$ is a centrally symmetric convex body in $\mathbb{R}^n$, then it is the unit ball of a normed space $X_K = (\mathbb{R}^n, \| \cdot \|_K)$. Thus, the study of finite dimensional normed spaces is in fact a study of centrally symmetric convex bodies, but again, the low-dimensional type questions and the corresponding intuition are very different from what is needed when the emphasis is on high-dimensional asymptotic behaviour. An example that clarifies this difference is given by the following question: does there exist a universal constant $c > 0$ such that every convex body of volume one has a hyperplane section of volume more than $c$? In any fixed dimension $n$, simple compactness arguments show that the answer is affirmative (although the question to determine the optimal value of the corresponding constant $c_n$ may remain interesting and challenging). However, this is certainly not enough to conclude that a constant $c > 0$ exists which applies to any body of volume one in any dimension. This is already an asymptotic type question. In fact, it is unresolved to this day and will be discussed in Chapter [10].

Classical geometry (in a fixed dimension) is usually an isometric theory. In the field of asymptotic geometric analysis, one naturally studies isomorphic geometric objects and derives isomorphic geometric results. By an “isomorphic” geometric object we mean a family of objects in different spaces of increasing dimension and by an “isomorphic” geometric property of such an “isomorphic” object we mean a property shared by the high-dimensional elements of this family. One is interested
in the asymptotic behaviour with respect to some parameter (most often it is the dimension $n$) and in the control of how the geometric quantities involved depend on this parameter. The appearance of such an isomorphic geometric object is a new feature of asymptotic high-dimensional theory. Geometry and analysis meet here in a non-trivial way. We will encounter throughout the book many geometric inequalities in isomorphic form. Basic examples of such inequalities are the “isomorphic isoperimetric inequalities” that led to the discovery of the “concentration phenomenon”, one of the most powerful tools of the theory, responsible for many counterintuitive results. Let us briefly describe it here, through the primary example of the sphere. A detailed account is given in Chapter 3. Consider the Euclidean unit sphere in $\mathbb{R}^n$, denoted $S^{n-1}$, equipped with the Lebesgue measure, normalized to have total measure 1. Let $A$ be a subset of the sphere of measure $1/2$. Take an $\varepsilon$-extension of this set, with respect to Euclidean or geodesic distance, for some fixed but small $\varepsilon$; this is the set of all points which are at a distance of at most $\varepsilon$ from the original set (usually denoted by $A_\varepsilon$). It turns out that the remaining set (that is, the set $S^{n-1} \setminus A_\varepsilon$ of all points in the sphere which are at a distance more than $\varepsilon$ from $A$) has, in high dimensions, a very small measure, decreasing to zero exponentially fast as the dimension $n$ grows. This type of statement has meaning only in asymptotic language, since in fact we are considering a sequence of spheres of increasing dimensions, and a sequence of subsets of these spheres, each of measure one half of its corresponding sphere, and the sequence of the measures of the $\varepsilon$-extensions (where $\varepsilon$ is fixed for all $n$) is a sequence tending to 1 exponentially fast with dimension. We shall see how the above statement, which is proved very easily using the isoperimetric inequality on the sphere, plays a key role in some of the very basic theorems in this field.

We return to the question of changing intuition. The above paragraph shows that, for example, an $\varepsilon$-neighbourhood of the equator $x_1 = 0$ on $S^{n-1}$ already contains an exponentially close to 1 part of the total measure of the sphere (since the sets $x_1 \leq 0$ and $x_1 \geq 0$ are both of measure $1/2$, and this set is the intersection of their $\varepsilon$-neighbourhoods). While this is again easy to prove (say, by integration) once it is observed, it does not correspond to our three-dimensional intuition. In particular, the far reaching consequences of these observations are hard to anticipate in advance. So, we see that in high dimension some of the intuition which we built for ourselves from what we know about three-dimensional space fails, and this “break” in intuition is the source of what one may call “surprising phenomena” in high dimensions. Of course, the surprise is there until intuition corrects itself, and the next surprise occurs only with the next break of intuition.

Here is a very simple example: The volume of the Euclidean ball $B^n_2$ of radius one seems to be increasing with dimension. Indeed, denote this by $\kappa_n$ and compute:

$$
\kappa_1 = 2 < \kappa_2 = \pi < \kappa_3 = \frac{4\pi}{3} < \kappa_4 < \kappa_5 < \kappa_6.
$$

However, a simple computation which is usually performed in Calculus III classes shows that

$$
\text{Vol}_n(B^n_2) = \kappa_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} = (c_n/\sqrt{n})^n
$$

where $c_n \to \sqrt{2\pi e}$. We thus see that in fact the volume of the Euclidean unit ball decreases like $n^{-n/2}$ with dimension (and one has the recursion formula $\kappa_n = \frac{2\pi}{n} \kappa_{n-2}$). So, for example, if one throws a point into the cube circumscribing the
ball, at random, the chance that it will fall inside the ball, even in dimension 20, say, is practically zero. One cannot find this ball inside the cube.

Let us try to develop an intuition of high-dimensional spaces. We illustrate, with another example, how changing the intuition can help us understand, and anticipate, results. To begin, we should understand how to draw “high-dimensional” pictures, or, in other words, to try and imagine what do high-dimensional convex bodies “look like”. The first non-intuitive fact is that the volume of parallel hypersections of a convex body decays exponentially after passing the median level (this is a consequence of the Brunn-Minkowski inequality, see Section 3.5). If we want to capture this property, it means that our two- or three-dimensional pictures of a high-dimensional convex body should have a “hyperbolic” form! Thus, K is a convex set but, as the rate of volume decay has a crucial influence on the geometry, we should find a way to visualize it in our pictures. For example, one may draw the convex set K as follows:

The convexity is no longer seen in the picture, but the volumetric properties are apparent. Next, with such a picture in mind, we may intuitively understand the following fact (it is a special case of Theorem 5.5.4 in Section 5.5): Consider the convex body \( K = \sqrt{n}B_1^n := \text{conv}(\pm \sqrt{n}e_i) \) (also called the unit ball of \( L_1^n \)). Take a random rotation \( UK \) of \( K \) and intersect it with the original body.

The resulting body, \( K \cap UK \) is, with high probability over the choice of the random rotation \( U \), contained in a Euclidean ball of radius \( C \), where \( C \) is a universal constant independent of the dimension. Note that the original body, which contains a Euclidean ball of radius 1 (as does the intersection), has points in distance \( \sqrt{n} \) from the origin. That is, the smallest Euclidean ball containing \( K \) is \( \sqrt{n}B_2^n \). However, the simple (random) procedure of rotation and intersection, with high probability cuts out all these “remote regions” and regularizes the body completely so that it becomes an isomorphic Euclidean ball.
This was an example of a very concrete body, but it turns out the same property holds for a large class of bodies (called “finite volume ratio” bodies, see Section 5.5). Actually, if one allows more rotations, $\frac{n}{\log n}$ of them in dimension $n$, one may always regularize any body by the same process to become an isomorphic Euclidean ball. This last claim needs a small correction to be completely true: we have not explained how one chooses a random rotation. To this end one considers the Haar probability measure on the space of orthogonal rotations. To consider orthogonal rotations one must first fix a Euclidean structure, and the above statement is true after fixing the “right” structure corresponding to the body in question. The story of choosing a Euclidean structure, which is the same as choosing a “position” for the body, is an important topic, and for different goals different structures should be chosen. This topic is covered in Chapter 2.

Let us emphasize that while the geometric picture is what helps us understand which phenomena may occur, the picture is of course not a proof, and in each case a proof should be developed and is usually non-trivial.

This last example brings us to another important point which will be a central theme in this book, and this is the way in which, in this theory, randomness and patterns appear together. A perceived random nature of high dimensions is at the root of the reasons for the patterns produced and the unusual phenomena observed in high dimensions. In the dictionary, “randomness” is the exact opposite of “pattern”. Randomness means “no pattern”. But, in fact, objects created by independent identically distributed random processes, while being different from one another, are many times indistinguishable and similar in the statistical sense. Consider for example the unit cube, $[0, 1]^n$. Choosing a random point inside it with respect to the uniform distribution means simply picking the $n$ coordinates independently and uniformly at random in $[0, 1]$. We know that such a point has some very special statistical properties (the simplest of which is the law of large numbers and the central limit theorem regarding the behaviour of the sum of these coordinates). It turns out that similar phenomena occur when the unit cube is replaced by a general convex body (again, a position should be specified). It is a challenge to uncover these similarities, a pattern, in very different looking objects. When we discover very similar patterns in arbitrary, and apparently very diverse convex bodies or normed spaces, we interpret them as a manifestation of the randomness principle mentioned above.

On the one hand, high dimension means many variables and many “possibilities”, so one may expect an increase in the diversity and complexity as dimension increases. However, the concentration of measure and similar effects caused by the convexity assumption imply in fact a reduction of the diversity with increasing dimension, and the collapse of many different possibilities into one, or, in some cases, a few possibilities only. We quote yet another simple example which is a version of the “global Dvoretzky-type theorem”. For details see Section 5.6. (The Minkowski sum of two sets is defined by $A + B = \{a + b : a \in A, b \in B\}$.)

Let $n \in \mathbb{N}$ and let $K \subset \mathbb{R}^n$ be a convex body such that the Euclidean ball $B_2^n$ is the ellipsoid of maximal volume inside $K$. Then, for $N = Cn/\log n$ random orthogonal transformations $U_i \in O(n)$, with probability at least $1 - e^{-cn}$ we have that

$$B_2^n \subset \frac{1}{N}(U_1K + U_2K + \cdots + U_NK) \subset C'B_2^n,$$
where $0 < c, C, C' < \infty$ are universal constants (independent of $K$ and of $n$).

One way in which diversity is compensated and order is created in the mixture caused by high dimensionality, is the concentration of measure phenomenon. As the dimension $n$ increases, the covering numbers of a generic body of the same volume as the unit Euclidean ball, say, by the Euclidean ball itself (this means the number of translates of the ball needed to cover the body, see Sections 4.1 and 4.2) become large, usually exponentially so, meaning $e^{cn}$ for some constant $c > 0$, and so seem impossible to handle. The concentration of measure is, however, of exponential order too (this time $e^{-c'n}$ for some constant $c' > 0$), so that in the end proofs become a matter of comparison of different constants in the various exponents (this is, of course, a very simplistic description of what is going on).

Let us quote from the preface of P. Lévy to the second edition of his book of 1951 [380]:

“It is quite paradoxical, that an increase in the number of variables might cause simplifications. In reality, any law of large numbers presupposes the existence of some rule governing the influence of sequential variables; starting with such a rule, we often obtain simple asymptotic results. Without such a rule, complete chaos ensues, and since we are unable to describe, for instance, an infinite sequence of numbers, without resorting to an exact rule, we are unable to find order in the chaos, where, as we know, one can find mysterious non-measurable sets, which we can never truly comprehend, but which nevertheless will not cease to exist.”

As we shall see below, the above facts reflect the probabilistic nature of high dimensions. We mean by this more than just the fact that we are using probabilistic techniques in many steps of the proofs. Let us mention one more very concrete example to illustrate this “probabilistic nature”: Assume you are given a body $K \subset \mathbb{R}^n$, and you know that there exist 3 orthogonal transformations $U_1, U_2, U_3 \in O(n)$ such that the intersection of $U_1K, U_2K$ and $U_3K$ is, up to constant 2, say, a Euclidean ball. Then, for a random choice of 10 rotations, $\{V_i\}_{i=1}^{10} \subset O(n)$, with high probability on their choice, one has that $\bigcap_{i=1}^{10} V_iK$ is up to constant $C$ (which depends on the numbers 2, 3 and 10, not on the dimension $n$, and may be computed) a Euclidean ball. This is a manifestation of a principle which is sometimes called “random is the best”, namely that in various situations the results obtained by a random method cannot be substantially improved if the random choice is replaced by the best choice for the specific goal.

There are a number of reasons for this observed ordered behaviour. One may mention “repetition”, which creates order, as statistics demonstrates. What we explain here and shall see throughout the book is that very high dimensions, or more generally, high parametric families, are another source of order.

We mention at this point that historically we observe the study of finite, but very high-dimensional spaces and their asymptotic properties as dimension increases already in Minkowski’s work, who for the purposes of analytic number theory considered $n$-dimensional space from a geometric point of view. Before him, as well as long after him, geometry had to be two- or three-dimensional, see, e.g., the works of Blaschke. A paper of von Neumann from 1942 also portrays the same asymptotic
point of view. We quote below from Sections 4 and 5 of the introduction of [601]. Here $E_n$ denotes $n$-dimensional Euclidean space and $M_n$ denotes the space of all $n \times n$ matrices. Whatever is in brackets is the present authors’ addition.

“Our interest will be concentrated in this note on the conditions in $E_n$ and $M_n$ - mainly $M_n$ - when $n$ is finite, but very great. This is an approach to the study of the infinite dimensional, which differs essentially from the usual one. The usual approach consists in studying an actually infinite dimensional unitary space, i.e. the Hilbert space $E_\infty$. We wish to investigate instead the asymptotic behaviour of $E_n$ and $M_n$ for finite $n$, when $n \to \infty$.

We think that the latter approach has been unjustifiably neglected, as compared with the former one. It is certainly not contained in it, since it permits the use of the notions $\|A\|$ and $t(A)$ (normalized Hilbert Schmidt norm, and trace) which, owing to the factors $1/n$ appearing in (their definitions) possess no analogues in $E_\infty$. Since Hilbert space $E_\infty$ was conceived as a limiting case of the $E_n$ for $n \to \infty$, we feel that such a study is necessary in order to clarify to what extent $E_\infty$ is or is not the only possible limiting case. Indeed we think that it is not, and that investigations on operator rings by F. J. Murray and the author show that other limiting cases exist, which under many aspects are more natural ones.

Our present investigations originated in fact mainly from the desire to solve certain questions... We hope, however, that the reader will find that they also have an interest of their own, mainly in the sense indicated above: as a study of the asymptotic behaviour of $E_n$ and $M_n$ for finite $n$, when $n \to \infty$.

From the point of view described (above) it seems natural to ask this question: How much does the character of $E_n$ and $M_n$ change when $n$ increases - especially if $n$ has already assumed very great values?”

Let us turn to a short description of the various chapters of the book; this will give us the opportunity to comment on additional fundamental ideas of the theory.

In Chapter 1 we recall basic notions from classical convexity. In fact, a relatively large portion of this book is dedicated to convexity theory, since a large part of the development of asymptotic geometric analysis is connected strongly with the classical theory. We present several proofs of the Brunn-Minkowski inequality and some of its fundamental applications. We have chosen to discuss in detail those proofs as they allow us to introduce fruitful ideas which we shall revisit throughout the book. In the appendices we provide a more detailed exposition of basic facts from elementary convexity, convex analysis and the theory of mixed volumes. In particular, we describe the proof of Minkowski’s theorem on the polynomiality of the volume of the sum of compact convex sets, and of the Alexandrov-Fenchel inequality, one of the most beautiful, non-trivial and profound theorems in convexity, which is linked with algebraic geometry and number theory. We emphasize the functional analytic point of view into classical convexity. This point of view
opened a new field which is sometimes called “functionalization of geometry” or “geometrization of probability”: It turns out that almost any notion or inequality connected with convex bodies has an analogous notion or inequality in the world of convex functions. This analogy between bodies and functions is fruitful in many different ways. On the one hand, it allows to predict functional inequalities which then are interesting in their own right. On the other hand, the generalization into the larger world of convex functions enables one to see the bigger picture and better understand what is going on. Finally, the results for functions may sometimes have implications back in the convex bodies world. This general idea is considered in parallel with the classical theory throughout the book.

In Chapter 2 we introduce the most basic and classical positions of convex bodies: Given a convex body $K$ in $\mathbb{R}^n$, the family of its positions is the family of its affine images $\{x_0 + T(K)\}$ where $x_0 \in \mathbb{R}^n$ and $T \in GL_n$. In the context of functional analysis, one is given a norm (whose unit ball is $K$) and the choice of a position reflects a choice of a Euclidean structure for the linear space $\mathbb{R}^n$. Note that the choice of a Euclidean structure specifies a unit ball of the Euclidean norm, which is an ellipsoid. Thus, we may equivalently see a “position” as a choice of a special ellipsoid. The different ellipsoids connected with a convex body (or the different positions, corresponding to different choices of a Euclidean structure) that we consider in this chapter reflect different traces of symmetries which the convex body has. We introduce John position (also called maximal volume ellipsoid position), minimal surface area position and minimal mean width position. It turns out that when a position is extremal then some differential must vanish, and its vanishing is connected with isotropicity of some connected measure.

We also discuss some applications, mainly of John position, and introduce a main tool, which is useful in many other results in the theory, called the Brascamp-Lieb inequality. We state and prove one of its most useful forms, which is the so-called “normalized form” put forward by K. Ball, together with its reverse form, using F. Barthe’s transportation of measure argument. In the second volume of this book we shall discuss the general form of the Brascamp-Lieb inequality, its various versions, proofs, and reverse form, as well as further applications to convex geometric analysis.

In Chapter 3 we discuss the concentration of measure phenomenon, first put forward in V. Milman’s version of Dvoretzky theorem. Concentration is the central phenomenon that is responsible for the main results in this book. We present a number of approaches, all leading to the same type of behaviour: in high parametric families, under very weak assumptions of various types, a function tends to concentrate around its mean or median. Classical isoperimetric inequalities for metric probability spaces, such as the sphere, Gauss space and the discrete cube, are at the origin of measure concentration, and we start our exposition with these examples. Once the extremal sets (the solutions of the isoperimetric problem) are known, concentration inequalities come as a consequence of a simple computation. However, in very few examples are the extremal sets known. We therefore do not focus on extremal sets but mainly on different ways to get concentration inequalities. We explore various such ways, and determine the different sources for concentration. In the second volume of this book we shall come back to this subject and study its functional aspects: Sobolev and logarithmic Sobolev inequalities, tensorization...
techniques, semi-group approaches, Laplace transform and infimum convolutions, and investigate in more detail the subject of transportation of measure.

In Chapter 4 we introduce the covering numbers $N(A, B)$ and the entropy numbers $e_k(A, B)$ as a way of measuring the “size” of a set $A$ in terms of another set $B$. As we will see in the next chapters, they are a very useful tool and play an important role in the theory. Here, we explain some of their properties, derive relations and duality between these numbers, and estimate them in terms of other parameters of the sets involved — estimates which shall be useful in the sequel.

Chapter 5 is the starting point for our exposition of the asymptotic theory of convex bodies. It is devoted to the Dvoretzky-Milman theorem and to the main developments around it. In geometric language the theorem states that every high-dimensional centrally symmetric convex body has central sections of high dimension which are almost ellipsoidal. The dependence of the dimension $k$ of these sections on the dimension $n$ of the body is as follows: for every $n$-dimensional normed space $X = (\mathbb{R}^n, \| \cdot \|)$ and every $\varepsilon \in (0, 1)$ there exist an integer $k \geq c\varepsilon^2 \log n$ and a $k$-dimensional subspace $F$ of $X$ which satisfies $d_{BM}(F, \ell_k^n) \leq 1 + \varepsilon$, where $d_{BM}$ denotes Banach-Mazur distance, a natural geometric distance between two normed spaces, and $c$ is some absolute constant. The proof of the Dvoretzky-Milman theorem exploits the concentration of measure phenomenon for the Euclidean sphere $S^{n-1}$, in the form of a deviation inequality for Lipschitz functions $f : S^{n-1} \to \mathbb{R}$, which implies that the values of $\| \cdot \|$ on $S^{n-1}$ concentrate near their average

$$M = \int_{S^{n-1}} \|x\| d\sigma(x).$$

A remarkable fact is that in Milman’s proof, a formula for such a $k$ is given in terms of $n$, $M$ and the Lipschitz constant (usually called $b$) of the norm, and that this formula turns out to be sharp (up to a universal constant) in full generality. This gives us the opportunity to introduce one more new idea of the theory, which is universality. In different fields, and also in the origins of asymptotic geometric analysis, for a long time one knew how to write very precise estimates reflecting different asymptotic behaviour of certain specific high-dimensional (or high parametric) objects (say, for the spaces $\ell_p^n$). Usually, one could show that these estimates are sharp, in an isomorphic sense at least. However, an accumulation of results indicates that, in fact, available estimates are exact for every sequence of spaces in increasing dimension (and thus one is tempted to say “for every space”). These kinds of estimates are called “asymptotic formulae”. Let us demonstrate another such formula, concerning the diameter of a random projection of a convex body. All constants appearing in the statement below $(C, c_1, C_2, c')$ are universal and do not depend on the body or on the dimension. Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. One denotes by $h_K(u)$ the support function of $K$ in direction $u$, that is,

$$h_K(u) = \max \{ \langle x, u \rangle : x \in K \}.$$ 

Denote by $d = d(K)$ the smallest constant such that $K \subset dB^n_d$, that is, half of the diameter of $K$, and actually $d = \max_{u \in S^{n-1}} h_K(u)$. Denote by $M^* = M^*(K)$ the average of $h_K$ over $S^{n-1}$, that is,

$$M^*(K) = \int_{S^{n-1}} h_K(u) d\sigma(u).$$
where $\sigma$ is the Haar probability measure on $S^{n-1}$. It turns out that for dimensions larger than $k^* = C(M^*/d)^2n$, the diameter of the projection of $K$ onto a random $k$-dimensional subspace is, with high probability, approximately $d\sqrt{k/n}$. That is, between $c_1d\sqrt{k/n}$ and $C_2d\sqrt{k/n}$. Around the critical dimension $k^* = k^*(K)$, the projection becomes already (with high probability on the choice of a subspace) a Euclidean ball of radius approximately $M^*(K)$, and this will be, again up to constants, the diameter (and the inner-radius) of a random projection onto dimension $c/k^*$ and less. In this result the isomorphic nature of the result is very apparent. Indeed, the diameter need not be $\varepsilon$-isometrically close to $d\sqrt{k/n}$ for $k$ in the range between $k^*$ and $n$, but only isomorphically. Isometric results are known in the regime $k \leq c/k^*$ when the projection is already with high probability a Euclidean ball (this is actually the Dvoretzky-Milman theorem). We describe this result in detail in Section 5.7.1. Another property of this last example is a threshold behaviour of the function $f(t)$ giving the average diameter of a projection into dimension $tn$. The function, which is monotone, attains its maximum, $d$, at $t = 1$, behaves like $d\sqrt{t}$ in the range $[C(M^*/d)^2, 1]$, and like a constant, close to $M^*$, in the range $[0, c'(M^*/d)^2]$. Threshold phenomena have been known for a long time in many areas of mathematics, for example, in mathematical physics. Here we see that these occur in complete generality (for any convex body, the same type of threshold). More examples of threshold behaviour in asymptotic geometric analysis shall be demonstrated in the book.

Before moving to the description of Chapters 6–10 we mention another point of view one should keep in mind when reading the book: the comparison between local and global type results. The careful readers may have already noted the similarity of two of the statements given so far in this preface: a part of the statement about decrease of diameter in fact said that after some critical dimension, a random projection of a convex body is with high probability close to a Euclidean ball (this also follows from the Dvoretzky-Milman theorem by duality of projections and sections). This is called a “local” statement. Two other theorems quoted above regarded what happens when one intersects random rotations of a convex body (for example, $B_2^n$), or when one takes the Minkowski sum (average) of random rotations of a convex body (for example, the cube). Again the results were that after a suitable (and not very large) number of such rotations, the resulting body is an isomorphic Euclidean ball. These types of results, pertaining to the body as a whole and not its sections or projections, are called “global” results. At the heart of the global results presented in this book, which have convex geometric flavor, stand methods which come from functional analysis (considering norms, their averages, etc). Again, by global properties we refer to properties of the original body or norm in question, while the local properties pertain to the structure of lower dimensional sections and projections of the body or normed space. From the beginning of the 1970’s the need for geometric functional analysis led to a deep investigation of the linear structure of finite dimensional normed spaces (starting with Dvoretzky theorem). However, it had to develop a long way before this structure was understood well enough to be used for the study of the global properties of a space. The culmination of this study was an understanding of the fact that subspaces (and quotient spaces) of proportional dimension behave very predictably. An example is the theorem quoted above regarding the decay of diameter. This understanding formed a bridge between the problems of functional analysis and the global asymptotic properties of
convex sets, and is the reason the two fields of convexity and of functional analysis work together nowadays.

In Chapter 6 we discuss upper bounds for the parameter $M(K)M^*(K)$, or equivalently, the product of the mean width of $K$ and the mean width of its polar, the main goal being to minimize this parameter over all positions of the convex body. (The polar of a convex body $K$ is the closed convex set generating the norm given by $h_K$, and is denoted $K^\circ$.) We will see that the quantity $MM^*$ can be bounded from above by a parameter of the space $(X, \|\cdot\|_K)$ which is called its $K$-convexity constant, and which in turn can be bounded from above, for $X$ of dimension $n$, by $c\log(d_{BM}(X, \ell_2^n)) + 1 \leq c' \log n$ for universal $c, c'$. This estimate for the $K$-convexity constant is due to G. Pisier and as we will see it is one of the fundamental facts in the asymptotic theory. The estimate for $M(K)M^*(K)$ brings us to one more main point, which concerns duality, or polarity. In many situations two dual operations performed one after the other already imply complete regularization. That is, one operation cancels a certain type of “bad behaviour”, and the dual operation cancels the “opposite” bad behaviour. Other examples include the quotient of a subspace theorem (see Chapter 7) or its corresponding global theorem: if one takes the sum of a body and a random (in the right coordinate system) rotation of it, then considers the polar of this set, to which again one applies a random rotation and takes the sum, the resulting body will be with high probability on the choice of rotations, an isomorphic Euclidean ball. If one uses just one of these two operations, there may be a need for $n/\log n$ such operations.

Chapter 7 is devoted to results about proportional subspaces and quotients of an $n$-dimensional normed space, i.e., of dimension $\lambda n$, where the “proportion” $\lambda \in (0, 1)$ can sometimes be very close to 1. The first step in this direction is Milman’s $M^*$-estimate. In a geometric language, it says that there exists a function $f : (0, 1) \to \mathbb{R}^+$ such that, for every centrally symmetric convex body $K$ in $\mathbb{R}^n$ and every $\lambda \in (0, 1)$, a random $\lfloor \lambda n \rfloor$-dimensional section $K \cap F$ of $K$ satisfies the inclusion

$$K \cap F \subseteq \frac{M^*(K)}{f(\lambda)}B_2^n \cap F.$$ 

In other words, the diameter of a random “proportional section” of a high-dimensional centrally symmetric convex body $K$ is controlled by the mean width $M^*(K)$ of the body. We present several proofs of the $M^*$-estimate; based on these, we will be able to say more about the best possible function $f$ for which the theorem holds true and about the corresponding estimate for the probability of subspaces in which this occurs. As an application of the $M^*$ estimate we obtain Milman’s quotient of a subspace theorem. We also complement the $M^*$ estimate by a lower bound for the outer-radius of sections of $K$, which holds for all subspaces, we compare “best” sections with “random” ones of slightly lower dimension, and we provide a linear relation between the outer-radius of a section of $K$ and the outer-radius of a section of $K^\circ$.

In Chapter 8 we present one of the deepest results in asymptotic geometric analysis: the existence of an $M$-position for every convex body $K$. This position can be described “isometrically” (if, say, $K$ has volume 1) as minimizing the volume of $T(K) + B_2^n$ over all $T \in SL_n$. However, such a characterization hides its main properties and advantages that are in fact of an “isomorphic” nature. The isomorphic formulation of the result states that there exists an ellipsoid of the same
volume as the body $K$, which can replace $K$, in many computations, up to universal constants. This result, which was discovered by V. Milman, leads to the reverse Santaló inequality and the reverse Brunn-Minkowski inequality. The reverse Santaló inequality concerns the volume product, sometimes called the Mahler product, of $K$ which is defined by

$$s(K) := \text{Vol}_n(K)\text{Vol}_n(K^o).$$

The classical Blaschke-Santaló inequality states that, given a centrally symmetric convex body $K$ in $\mathbb{R}^n$, the volume product $s(K)$ is less than or equal to the volume product $s(B_2^n) = \kappa_n^2$, and that equality holds if and only if $K$ is an ellipsoid. In the opposite direction, a well-known conjecture of Mahler states that $s(K) \geq 4^n/n!$ for every centrally symmetric convex body $K$ (i.e., the cube is a minimizer for $s(K)$ among centrally symmetric convex bodies) and that $s(K) \geq (n + 1)^n/(n!)^2$ in the not necessarily symmetric case, meaning that in this case the simplex is a minimizer. The reverse Santaló inequality of Bourgain and Milman verifies this conjecture in the asymptotic sense: there exists an absolute constant $c > 0$ such that

$$\langle s(K) \rangle^{1/n} \geq c$$

for every centrally symmetric convex body $K$ in $\mathbb{R}^n$. Milman’s reverse Brunn-Minkowski inequality states that for any pair of convex bodies $K$ and $T$ that are in $M$-position, one has

$$\text{Vol}_n(K + T)^{1/n} \leq C \left[ \text{Vol}_n(K)^{1/n} + \text{Vol}_n(T)^{1/n} \right].$$

(The reverse inequality, with constant 1, is simply the Brunn-Minkowski inequality of Chapter 1.)

Another way to define the $M$-position of a convex body is through covering numbers, as was presented in Milman’s proof. Pisier has proposed a different approach to these results, which allows one to find a whole family of special $M$-ellipsoids satisfying stronger entropy estimates. We describe his approach in the last part of Chapter 8.

In Chapter 9 we introduce a “Gaussian approach” to some of the main results which were presented in previous chapters, including sharp versions of the Dvoretzky-Milman theorem and of the $M^*$-estimate. The proof of these results is based on comparison principles for Gaussian processes, due to Gordon, which extend a theorem of Slepian. The geometric study of random processes, and especially of Gaussian processes, has strong connections with asymptotic geometric analysis. The tools presented in this chapter will appear again in the second volume of the book.

In the last chapter of this volume, Chapter 10, we discuss more recent discoveries on the distribution of volume in high-dimensional convex bodies, together with the unresolved “slicing problem” which was mentioned briefly at the beginning of this preface, with some of its equivalent formulations. A natural framework for this study is the isotropic position of a convex body: a convex body $K \subset \mathbb{R}^n$ is called isotropic if $\text{Vol}_n(K) = 1$, its barycenter (center of mass) is at the origin and its inertia matrix is a multiple of the identity; that is, there exists a constant $L_K > 0$
such that
\[ \int_K \langle x, \theta \rangle^2 dx = L_K^2 \]
for every \( \theta \) in the Euclidean unit sphere \( S^{n-1} \). The number \( L_K \) is then called the isotropic constant of \( K \). The isotropic position arose from classical mechanics back in the 19th century. It has a useful characterization as a solution of an extremal problem: the isotropic position \( \tilde{K} = T(K) \) of \( K \) minimizes the quantity
\[ \int_{\tilde{K}} |x|^2 dx \]
over all \( T \in GL_n \) such that \( \text{Vol}_n(\tilde{K}) = 1 \) and \( \int_K x dx = 0 \).

The central theme in Chapter 10 is the hyperplane conjecture (or slicing problem): it asks whether there exists an absolute constant \( c > 0 \) such that \( \max_{\theta \in S^{n-1}} \text{Vol}_{n-1}(K \cap \theta^\perp) \geq c \) for every \( n \) and every convex body \( K \) of volume 1 in \( \mathbb{R}^n \) with barycenter at the origin. We will see that an affirmative answer to this question is equivalent to the fact that there exists an absolute constant \( C > 0 \) such that
\[ L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leq C. \]

We shall work in the more general setting of a finite log-concave measure \( \mu \), where a corresponding notion of isotropicity is defined via the covariance matrix \( \text{Cov}(\mu) \) of \( \mu \). We present the best known upper bounds for \( L_n \). Around 1985-86, Bourgain obtained the upper bound \( L_n \leq c \sqrt{n} \log n \) and, in 2006, this estimate was improved by Klartag to \( L_n \leq c \sqrt{n} \). In fact, Klartag obtained a solution to an isomorphic version of the hyperplane conjecture, the “isomorphic slicing problem”, by showing that, for every convex body \( K \) in \( \mathbb{R}^n \) and any \( \varepsilon \in (0, 1) \), one can find a centered convex body \( T \subset \mathbb{R}^n \) and a point \( x \in \mathbb{R}^n \) such that \( (1 + \varepsilon)^{-1}T \subseteq K + x \subseteq (1 + \varepsilon)T \) and \( L_T \leq C/\sqrt{\varepsilon} \) for some absolute constant \( C > 0 \). An additional essential ingredient in Klartag’s proof of the bound \( L_n \leq c \sqrt{n} \), which is a beautiful and important result in its own right, is the following very useful deviation inequality of Paouris: if \( \mu \) is an isotropic log-concave probability measure on \( \mathbb{R}^n \), then
\[ \mu\{x \in \mathbb{R}^n : |x| \geq ct \sqrt{n}\} \leq \exp\left(-t^2 \sqrt{n}\right) \]
for every \( t \geq 1 \), where \( c > 0 \) is an absolute constant. The proof is presented in Section 10.4 along with the basic theory of the \( L_q \)-centroid bodies of an isotropic log-concave measure. Another important result regarding isotropic log-concave measures is the central limit theorem of Klartag, which states that the 1-dimensional marginals of high-dimensional isotropic log-concave measures \( \mu \) are approximately Gaussian with high probability. We will come back to this result and related ones in the second volume of the book and we will see that precise quantitative relations exist between the hyperplane conjecture, the optimal answer to the central limit problem, and other conjectures regarding volume distribution in high dimensions.

Acknowledgements. This book is based on material gathered over a long period of time with the aid of many people. We would like to mention two ongoing working seminars in which many of the ideas and results were presented and discussed: these are the Asymptotic Geometric Analysis seminars at the University of Athens and at Tel Aviv University. The active participation of faculty members, students and visitors in these seminars, including many discussions and collaborations, have made a large contribution to the possibility of this book. We would like to mention the names of some people whose contribution was especially important, whether
in offering us mathematical and technical advice, in reading specific chapters of
the book, in allowing us to make use of their research notes and material, and for
sending us to correct and less known references and sources. We thank S. Alesker,
S. Bobkov, D. Faifman, B. Klartag, H. König, A. Litvak, G. Pisier, R. Schneider,
B. Slomka, S. Sodin and B. Vritsiou. Finally, we would like to thank S. Gelfand and
the AMS team for offering their publishing house as a home for this manuscript,
and for encouraging us to complete this project.

The second named author would like to acknowledge partial support from the
ARISTEIA II programme of the General Secretariat of Research and Technology
of Greece during the final stage of this project. The first and third named authors
would like to acknowledge partial support from the Israel Science Foundation.

November 2014
Bibliography


[34] K. M. Ball, Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$, Studia Math. 88 (1988), 69-84.


[74] B. J. Birch, *Homogeneous forms of odd degree in a large number of variables*, Mathematika **4** (1957), 102-105.


[488] O. Palmon, The only convex body with extremal distance from the ball is the simplex, Israel J. Math. 80 (1992), 337-349.


## Subject Index

<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-symmetric basis</td>
<td>163</td>
</tr>
<tr>
<td>$K^\circ$</td>
<td>360</td>
</tr>
<tr>
<td>$K_p(f)$</td>
<td>328</td>
</tr>
<tr>
<td>$K_r(X)$</td>
<td>213</td>
</tr>
<tr>
<td>$L_2(\Omega, \mathcal{P}) \otimes X$</td>
<td>207</td>
</tr>
<tr>
<td>$L_2(\Omega, \mathcal{P}; X)$</td>
<td>207</td>
</tr>
<tr>
<td>$\mathcal{L}(f)$</td>
<td>376</td>
</tr>
<tr>
<td>$M_0(X)$</td>
<td>390</td>
</tr>
<tr>
<td>$P_K(x)$</td>
<td>372</td>
</tr>
<tr>
<td>$QS(X)$</td>
<td>241</td>
</tr>
<tr>
<td>$R(K)$</td>
<td>189</td>
</tr>
<tr>
<td>$R_k(K)$</td>
<td>189</td>
</tr>
<tr>
<td>$S_u, \alpha^*(u)$</td>
<td>218</td>
</tr>
<tr>
<td>$\delta H$</td>
<td>382</td>
</tr>
<tr>
<td>$\ell$-norm, $\ell(u)$</td>
<td>219</td>
</tr>
<tr>
<td>$\ell_p^n$</td>
<td>177</td>
</tr>
<tr>
<td>$\ell_q^n$</td>
<td>221</td>
</tr>
<tr>
<td>$\varepsilon(y_1, \ldots, y_m)$</td>
<td>297</td>
</tr>
<tr>
<td>$\kappa_m$</td>
<td>6</td>
</tr>
<tr>
<td>$\nabla^2(f)$</td>
<td>378</td>
</tr>
<tr>
<td>$\partial(\varphi)$</td>
<td>2</td>
</tr>
<tr>
<td>$\psi_1$ measure</td>
<td>115</td>
</tr>
<tr>
<td>$\psi_2$-norm</td>
<td>114</td>
</tr>
<tr>
<td>$\psi_\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>direction</td>
<td>112</td>
</tr>
<tr>
<td>estimate</td>
<td>112</td>
</tr>
<tr>
<td>norm</td>
<td>113</td>
</tr>
<tr>
<td>random variable</td>
<td>113</td>
</tr>
<tr>
<td>$a_k$</td>
<td>297</td>
</tr>
<tr>
<td>conv</td>
<td>370</td>
</tr>
<tr>
<td>$\text{conv}(f)$</td>
<td>375</td>
</tr>
<tr>
<td>$d$-parameter</td>
<td>198</td>
</tr>
<tr>
<td>$d(K, T)$</td>
<td>19</td>
</tr>
<tr>
<td>$d_{BM}(K, T)$</td>
<td>19</td>
</tr>
<tr>
<td>$\text{dom}(f)$</td>
<td>375</td>
</tr>
<tr>
<td>$\text{exp}(K)$</td>
<td>375</td>
</tr>
<tr>
<td>$\text{ext}(K)$</td>
<td>373</td>
</tr>
<tr>
<td>$f^*$</td>
<td>374</td>
</tr>
<tr>
<td>$h_K$</td>
<td>3</td>
</tr>
<tr>
<td>$k(X)$</td>
<td>172</td>
</tr>
<tr>
<td>$\tilde{k}(X)$</td>
<td>172</td>
</tr>
<tr>
<td>$k_0(X)$</td>
<td>172</td>
</tr>
<tr>
<td>$\text{relint}$</td>
<td>371</td>
</tr>
<tr>
<td>$t$-extension</td>
<td>80</td>
</tr>
<tr>
<td>$t(X, r)$</td>
<td>249</td>
</tr>
<tr>
<td>$t_q(X)$</td>
<td>192</td>
</tr>
<tr>
<td>vr($K$)</td>
<td>179</td>
</tr>
<tr>
<td>$\alpha$-concave function</td>
<td>14</td>
</tr>
<tr>
<td>$\alpha$-regular $M$-position</td>
<td>201</td>
</tr>
<tr>
<td>affine hull</td>
<td>360</td>
</tr>
<tr>
<td>Alesker theorem</td>
<td>56</td>
</tr>
<tr>
<td>Alesker-Dar-Milman map</td>
<td>211</td>
</tr>
<tr>
<td>Alexandrov inequalities for quermassintegrals</td>
<td>67</td>
</tr>
<tr>
<td>inequality for mixed discriminants</td>
<td>7</td>
</tr>
<tr>
<td>Alexandrov-Fenchel inequality</td>
<td>159</td>
</tr>
<tr>
<td>angular radius</td>
<td>173</td>
</tr>
<tr>
<td>approximation numbers</td>
<td>141</td>
</tr>
<tr>
<td>area measure</td>
<td>36</td>
</tr>
<tr>
<td>Artstein-Milman-Szarek theorem</td>
<td>149</td>
</tr>
<tr>
<td>Artstein-Milman-Szarek-Tomczak duality theorem</td>
<td>164</td>
</tr>
<tr>
<td>Azuma’s inequality</td>
<td>102</td>
</tr>
<tr>
<td>B-theorem</td>
<td>193</td>
</tr>
<tr>
<td>Ball</td>
<td>527</td>
</tr>
<tr>
<td>$K_p(\mu)$-bodies</td>
<td>527</td>
</tr>
<tr>
<td>Proposition on contact points</td>
<td>56</td>
</tr>
<tr>
<td>convexity of $K_p(f)$</td>
<td>328</td>
</tr>
<tr>
<td>normalized form of the Brascamp-Lieb inequality</td>
<td>67</td>
</tr>
<tr>
<td>proposition</td>
<td>330</td>
</tr>
<tr>
<td>reverse isoperimetric inequality</td>
<td>66</td>
</tr>
<tr>
<td>theorem on maximal volume ratio</td>
<td>74</td>
</tr>
<tr>
<td>Banach-Mazur distance</td>
<td>49</td>
</tr>
<tr>
<td>Barthe reverse Brascamp-Lieb inequality</td>
<td>69</td>
</tr>
<tr>
<td>barycenter</td>
<td>14</td>
</tr>
<tr>
<td>basis</td>
<td>163</td>
</tr>
<tr>
<td>1-symmetric</td>
<td>163</td>
</tr>
<tr>
<td>symmetric</td>
<td>17</td>
</tr>
<tr>
<td>Bergstrom inequality</td>
<td>106</td>
</tr>
<tr>
<td>Bernstein inequality</td>
<td>211</td>
</tr>
<tr>
<td>Beta distribution</td>
<td>245</td>
</tr>
<tr>
<td>Blaschke selection theorem</td>
<td>154</td>
</tr>
<tr>
<td>Blaschke-Santaló inequality</td>
<td>92</td>
</tr>
<tr>
<td>body</td>
<td>70</td>
</tr>
<tr>
<td>projection</td>
<td>70</td>
</tr>
<tr>
<td>star-shaped</td>
<td>104</td>
</tr>
</tbody>
</table>
Borell and Sudakov-Tsirelson theorem, 85
Borell’s lemma, 30
mentioned, 109, 338
Borell’s lemma for log-concave measures, 115
Bourgain-Milman inequality, mentioned, 265, 266, 310, 311, 351
Brenier map, 17
Brenier-McCann theorem, 20
Brunn
concentration, 114
mentioned, 27, 39, 122, 359
Brunn-Minkowski inequality, 9
Caffarelli regularity result, 407
Carathéodory theorem, 55
Carathéodory theorem, 371
Carl’s theorem, 145
Cauchy formula, 6, 109
centered
log-concave measure, 114
centrally symmetric convex body, 11
Chevet inequality, 288, 305, 312
Christoffel problem, 16
concentration
exponential, 91
function, 91
Giannopoulos-Milman type, 124
Lévy-Milman type, 124
normal, 91
of measure, 79, 91
of volume, 80
property, 128
without measure, 128
conditional expectation, 101
conjecture
Dar, 43
duality, 154
Hadwiger, 135
Hyperplane, 315
hyperplane for log-concave functions, 326
isotropic constant, 320
KLS, 317
Mahler, 257
thin-shell, 317
conjugate exponent, 158
constant
$K$-convexity, 209
isotropic, 48
contact point, 59, 61, 80, 71
convex body, 1
area measure, 36
barycenter, 15
centrally symmetric, 1
circumradius, 8
diameter, 8
inradius, 8
mean width, 33
polar, 8
width, 8
convex function, 2, 374
closed, 2
domain of, 2
epigraph of, 2
lower semi continuous, 2
proper, 2
subdifferential of, 2, 379
subgradient of, 2
convex set, 1
convexified duality theorem, 154
dual Sudakov inequality of Pajor-Tomczak, 139
duality of entropy theorem, 149
duality relations, 249
Dudley-Sudakov theorem, 290
Dvoretzky theorem, 165
algebraic form, 203
Dvoretzky-Rogers lemma, 59
elementary set, 13
ellipsoid, 47, 171
$M$, 48
maximal volume, 10, 51, 52, 57, 58, 60
minimal volume, 62
elliptic inequalities, 6
entropy
extension, 158
number, 143
entropy extension theorem, 158
epigraph, 2
extremal point, 373
faces, 87, 9
facet, 175
facets, 573
Fernique theorem, 292
Figiel-Tomczak-Jaegermann theorem, 220
SUBJECT INDEX

- Cauchy, 6, 109
- Cauchy-Binet, 68
- Green’s, 61
- Kubota, 6, 108
- Steiner, 5

- formula
  - Cauchy, 6, 409
  - Cauchy-Binet, 68
  - Green’s, 64
  - Kubota, 6, 408
  - Steiner, 5

- function
  - α-concave, 11
  - concave, 375
  - concentration, 91
  - convex, 2
  - Lipschitz, 92
  - log-concave, 3, 114
  - Rademacher, 107, 211
  - radial, 27
  - radial extension, 64
  - functional inequality, 22

- Gauss curvature, 65
- Gauss space, 80
- Gaussian
  - isoperimetric inequality, 65
  - measure, 80
  - process, 288
  - projection, 209
  - random variable, 165
- Gelfand numbers, 144
- geodesic metric, 50
- Giannopoulos-Milman theorem, 65
- Giannopoulos-Milman-Tsolomitis theorem, 245, 248
- Gordon theorem, 231
- Gordon’s min-max principle, 301
- Grünbaum’s lemma, 66
- Grassmann manifold, 98
- Gromov
  - isoperimetry of waists theorem, 245
  - Gromov-Milman theorem, 100

- Haar measure, 97
- Hadwiger conjecture, 133, 137
- Hadwiger’s theorem, 413
- Hamming metric, 81, 86
- Harper’s theorem, 56
- Hausdorff metric, 41, 582
- Helly theorem, 276
- Hermite polynomials, 239
- homogeneous space, 97
- hyperbolic inequalities, 6

- identity
  - operator, 53
  - representation, 53, 57

- Inequality
  - Rogers-Shephard, 45

- inequality
  - Alexandrov, 6
  - mentioned, 40, 405
  - Alexandrov, for mixed discriminants, 7
  - Alexandrov-Fenchel, 6, 598
  - for discriminant, 407
  - mentioned, 7, 39, 41, 299, 104, 105
  - 107, 408, 111, 112
  - Azuma, 102
  - Ball’s normalized form of the
    Brascamp-Lieb inequality, 67
  - Bergström, 106
  - Bernstein, 116, 119, 210
  - Blaschke-Santaló, 32
  - mentioned, 45, 210, 217, 260, 207, 272
  - Brascamp-Lieb
    mentioned, 415, 377, 177
    reverse, 69
  - Brunn-Minkowski, 9
    functional form, 22
    mentioned, 11, 10, 11, 13, 16, 17, 21, 23, 26, 27, 29, 30, 53, 131, 19, 22, 48, 53
    83, 84, 115, 121, 138, 156, 257, 259
    355, 365, 399
  - Chevet, 228, 478, 482
    deviation for Lipschitz functions, 94
    dual Sudakov, 130
    Dudley-Sudakov, 290
    elliptic, 9
    hyperbolic, 6
    isoperimetric in \( \mathbb{R}^n \), 65
    Jensen, 373
    Kahane, 62, 109
    mentioned, 110, 110, 124, 126, 105, 357
    Khintchine, 108
    Marcus-Pisier, 184, 306, 313
    Minkowski, 32
    Minkowski’s first, 399
    Minkowski’s first and second, 398
    Minkowski’s second, 399
    Minkowski, inequality for determinants, 7
    Prékopa-Leindler, 22
    mentioned, 11, 22, 23, 18, 70, 86, 86, 85
    114, 125
    reverse Brascamp-Lieb, 69
    reverse isoperimetric, 66, 67
    reverse Santaló, 261
    Rogers-Shephard, 26
    mentioned, 45, 137, 221, 257, 271, 276
    540, 394, 388
    Sudakov, 139
    Tomczak-Jaegermann, 137
    Urysohn, 83, 89
  - inertia
    matrix, 48
    infimum convolution, 95
    invariant measure, 97
    inverse
      Hölder inequality, 107
      isoperimetric
      inequality, 31
isoperimetric inequality
for the discrete cube, 86
Gaussian, 85
reverse, 66, 67
spherical, 82
isoperimetry of waists, 245
isotropic
Borel measure, 55
constant, 48
position, 48
subgroup, 97
Jensen inequality, 375
John
time, 49
orem, 50, 52
mentioned, 58, 60, 72, 74, 174, 176,
216, 223, 265, 320
K-convexity constant, 209
Kadets-Snobar theorem, 58
Kahane-Khintchine inequality, 109
Kashin decomposition, 182
Khintchine inequality, 108
Klartag
bound for $L_K$, 130
isomorphic slicing theorem, 346
Klartag-Vershynin theorem, 203
Knaster’s problem, 200
Knothe map, 16
Kolmogorov numbers, 126
König-Milman theorem, 262
Kubota integral formula, 6, 408
Lévy
family, 91
normal, 91
mean, 92
Lévy-Gromov theorem, 60
Lévy-Schmidt theorem, 82
Laplace-Beltrami operator, 64
Latala theorem, 356
Legendre-Fenchel transform, 376
lemma
Bernstein, 210
Borell’s, 50
Borell’s log-concave measures, 115
distance, 230
Dvoretzky-Rogers, 50, 60
Grünbaum’s, 55
Johnson-Lindenstrauss, 237
Lewis, 218
Maurey, 115
Maurey-Pisier, 203, 229
Raz, 120
Slepian, 291
trace duality, 218
length
of a metric space, 105
linear operator
projection, 58
Litvak-Milman-Schechtman theorem, 190
localization technique, 39
log-concave
function, 3, 114
function convolution, 24
measure, 114
Ludwig-Reitzner theorem, 413
Lutwak’s containment theorem, 387
\(M\)-ellipsoid, 108
\(M\)-position, 48, 258
of order \(\alpha\), 261
Maurey
difference, 257
product, 32, 257
manifold
Grassmann, 98
Riemannian, 99
Stiefel, 98
map
Alesker-Dar-Milman, 21
Brenier, 17
mentioned, 11, 16, 20, 21, 33, 77, 407
Gauss, 60
Knothe, 16
mentioned, 11, 16, 17, 43
Marcus-Pisier inequality, 306
martingale, 102
matrix
linear symmetric, 53
positive semi-definite, 55
Maurey
Maurey’s lemma, 155
theorem on gaussian property \(\tau\), 95
theorem on permutations, 103
Maurey-Pisier Lemma, 202, 229
maximal
volume ellipsoid, 39, 51, 52, 57, 58, 60
volume ratio, 71
maximal volume
position, 71
McCann
Brenier-McCann theorem, 20
mean width, 63, 259
minimal, 63
measure
\(\psi_1\), 115
area measure, 36
berycenter of, 114
centered, 114
concentration, 79, 91
Gaussian, 80
Haar, 91
invariant, 97
isotropic, 55
log-concave, 114
marginal, 18
normalized invariant, 98
push forward, 20
transportation, 70
median, 92
metric
geodesic, 80
Hamming, 81, 86
probability space, 80
Milman
$M$-ellipsoid, 268, 267
$M$-position, 268
$M^*$-estimate, 233
duality relation theorem, 249
isomorphic symmetrization, 268
$M^*$-estimate, 248
quotient of subspace theorem, 241
reverse Brunn-Minkowski inequality, 259
version of Dvoretzky theorem, 169
Milman-Pajor theorem, 137
minimal
mean width position, 92
surface area position, 65
surface invariant, 66
volume ellipsoid, 51, 52
Minkowski
content, 81
existence theorem, 56
first and second inequalities, 398
first inequality for mixed volumes, 32
functional, 164
inequality for determinants, 7
polarization theorem, 5
problem, 38
sum, 1
symmetrization, 45
theorem on determinants, 51, 105
theorem on extremal points, 373
mixed discriminant, 4, 105
polarization formula, 7
mixed volumes, 5, 359
polarization formula, 5
$MM^*$-estimate, 234
$M^*$-estimate, 233
Gordon’s formulation, 240, 284
optimal dependence, 248
proof, 234, 235, 237
net, 167
Newton inequalities, 494
Newton polyhedron, 49
norm
$\psi_\alpha$, 111
Orlicz, 111
trace dual, 218
number
convexified separation, 151
covering, 131
entropy, 144
packing, 134
separation, 134
operator norm, 218
optimal transportation, 139
Orlicz norm, 111
orthogonal
transformation, 63
orthogonal group, 97
oscillation stability, 124
outer normal, 65
packing number, 134
Pajor-Tomczak inequality, 140
Pajor-Tomczak theorem, 238
Paouris theorem, 336, 338
parameter
$d$, 193
permutation group, 103
phenomenon
concentration of measure, 79, 91
Pisier
$\alpha$-regular $M$-position, 261
estimate for the $K$-convexity constant, 240
estimate for the Rademacher constant, 240
theorem on $\alpha$-regular position, 276
polar body, 8, 369
polarization formula
mixed discriminant, 7
mixed volumes, 5
polyhedral set, 370
polytope, 4, 370
centrally symmetric, 170
density of, 383
strongly isomorphic, 383
position, 17
isotropic, 317
mentioned, 48, 73, 315, 318, 321
350, 353, 360, 361
John, 19
mentioned, 47, 50, 52, 55, 58, 74
23, 27, 103, 231
$L_p$, 283
mentioned, 62, 212, 238
Löwner, 52, 74$M$, 228
mentioned, 48, 73, 76, 258, 264, 270
272, 273, 275
maximal volume ellipsoid, 47
minimal mean width, 62
mentioned, 17, 62, 76, 231
minimal surface area, 65
mentioned, 47, 66, 76, 231
of maximal volume, 74
Prékopa-Leindler inequality, 22
problem
Christoffel, 46
isoperimetric, 82
Knaster, 200
projection
Gaussian, 209
Rademacher, 213
projection body, 76
property (τ), 95
proposition
Ball, 56, 330
Grunbaum’s lemma, 35
reverse isoperimetric inequality, 67
Rockafellar, 19
push forward, 20
quermassintegrals, 6, 108
quotient of subspace theorem, 241
Rademacher
constant, 213
functions, 107, 211
projection, 213
radial function, 27
radius
angular, 175
Radon theorem, 370
random rotations, 255
Raz’s lemma, 120
reverse
Brascamp-Lieb inequality, 69
Brunn-Minkowski inequality, 259, 267
isoperimetric inequality, 66, 67
Santaló inequality, 237, 201
Urysohn inequality, 223
Ricci
curvature, 91
Rockafellar proposition, 19
Rogers-Shephard inequality, 26
Santaló
Blaschke-Santaló inequality, 32
Schechtman theorem, 195
Schneider’s steiner point theorem, 388
Schwarz symmetrization, 39
separated set, 134
separation number, 134
set
elementary, 13
separated, 134
simplex, 266, 310
slab, 194
Slepian lemma, 291
spectrum, 124
sphere, 80
spherical isoperimetric inequality, 82
Steiner
formula, 6
point, 388
symmetral, 7
symmetrization, 7
Stiefel manifold, 98
Straszewicz theorem, 374
strongly isomorphic polytopes, 383
subdifferential, 2
subgradient, 2
successive approximation, 168, 188
Sudakov inequality, 139, 292
mentioned, 238, 240, 263, 264, 268, 293
Sudakov’s comparison theorem, 292
support function, 3
supporting hyperplane, 12
surface area, 5
symmetric
basis, 47
symmetric basic sequences, 126
symmetrization
dimension descending, 261
isomorphic, 263
Minkowski, 35
Schwarz, 39
Steiner, 7
system of moving shadows, 14
$t$-extension, 32
Talagrand
Concentration on the discrete cube, 87
isoperimetric inequality for the discrete cube, 87
(τ)-property, 95
theorem, 291
Alesker, 336, 413
Alexandrov inequality for mixed discriminants, 7
Artstein-Milman-Szarek, 149
Artstein-Milman-Szarek-Tomczak, 153
Azuma’s inequality, 192
Ball, 66
Ball’s normalized form of the Brascamp-Lieb inequality, 67
Ball’s theorem on maximal volume ratio, 71
Bartle’s reverse Brascamp-Lieb inequality, 69
Bernstein inequality, 117, 119
Blaschke selection, 15
Blaschke-Santaló inequality, 32
Borell, Sudakov-Tsirelson, 83
Bourgain-Lindenstrauss-Milman, 184
Bourgain-Milman, 261
Brenier-McCann, 20
Carathéodory, 58, 374
Carl, 146
SUBJECT INDEX 445

Chevet inequality, 305
convexity of $K_p(f)$, 928
Cordero-Fradelizi-Maurey, 193
dol’nikov and Karasev, 200
Dudley-Sudakov, 290
Dvoretzky, 163
dvoretzky-milman, 163, 173
dvoretzky-Rogers, 163
dvoretzky-Rogers lemma, 59
Fernique, 292
figiel-lindenstrauss-milman, 174, 176
figiel-tomczak-jaegermann, 220
Giannopoulos-Milman, 64
Giannopoulos-Milman-Tsolomitis, 245
Gordon, 241
Gromov, 240
Gromov-Milman, 100
Hadwiger, 413
Harper, 86
Helly, 376
John, 50, 52
Kadets-Snobar, 58
Kahane inequality, 106
Kashin, 180
Khintchine inequality, 108
Klartag, 340, 350
Klartag-Vershynin, 193
König-Milman, 202
Levy-Gromov, 99
Levy-Schmidt, 52
Latała, 356
Litvak-Milman-Pajor-Tomczak, 158
Litvak-Milman-Schechtman, 190, 192
Ludwig-Reitzner, 113
Lutwak, 387
Marcus-Pisier, 306
Maurey, on gaussian property $\tau$, 95
Maurey, on permutations, 101
Milman, 169, 283, 291, 219
Milman-Pajor, 157
Milman-Schechtman, 173, 186
Minkowski on determinants, 84
Minkowski existence theorem, 36
Minkowski on extremal points, 373
Minkowski polarization, 5
Minkowski’s theorem on determinants, 405
Pajor-Tomczak, 238
Paouris, 340, 388
Pisier, 211, 210, 215, 276
Radon, 370
Radon-Nikodym, 101
reverse Brunn-Minkowski, 48
Rogers-Shephard-Minkowski, 48
Schechtman, 190
Schneider, 588
Slepian, 291
Straszewicz, 374
Sudakov comparison, 292
Szarek-Tomczak, 180
Talagrand, concentration on the discrete cube, 57
Tomczak-Jaegermann, 147
Urysohn, 34
Tomczak-Jaegermann theorem on entropy, 184
trace duality, 218
transformation
volume preserving, 16
transportation optimal, 43
unit sphere, 80
Urysohn inequality, 34
vertex, 175
volume
concentration, 30
product, 32
ratio, 179
volume preserving
transformation, 16
volume product, 257
volume radius, 28
volume ratio, 71
maximal, 73
theorem, 180
theorem, global form, 181
Walsh functions, 211
width
in direction $u$, 3
zonoid, 26
Author Index

Alesker, S., 21, 43, 334, 411, 413, 415
Alexandrova, A. D., 6, 7, 40, 41, 415
Alon, N., 125, 115
Alonso-Gutiérrez, D., 362, 367, 115
Amir, D., 125, 115
Anderson, G. W., 198, 115
Andrews, G. E., 230, 115
Anttila, M., 364, 115
Archimedes
Arias de Reyna, J., 83, 124, 115
Artstein-Avidan, S., 45, 149, 154, 158, 198, 253, 388, 414–416
Askey, R., 231, 115
Azuma, K., 102, 126, 416
Badrikian, A., 83, 116
Banaszczyk, W., 199
Bárány, I., 367, 116
Barthe, F., 39, 43, 48, 69, 70, 77, 364, 416, 417
Bastero, J., 83, 116
Beckenbach, E. F., 413, 417
Behrend, F., 77, 116
Bellman, R., 114, 116
Ben-Tal, A., 102
Bennett, G., 128, 107, 117
Benyamini, Y., 313, 117
Berg, C., 40, 117
Bergh, J., 231, 117
Bernstein, D. M., 117
Bernstein, S., 110, 126, 127, 117
Bernués, J., 364, 117
Bezdek, K., 117
Bieberbach, L., 25, 117
Birch, B. J., 417
Blaschke, W., 5, 32, 33, 33, 117
Blichfeldt, H. F., 128, 117
Bobkov, S. G., 121, 126, 385, 386, 417, 418
Bogachev, V., 418
Bolker, E. D., 418
Bonnesen, T., 364, 115
Borel, E., 123
Borell, C., 30, 35, 115, 116, 123, 126, 118
Böröczky, K., 358
Bourgain, J., 44, 145, 176, 107, 126, 157, 158, 163, 184, 193, 199, 230, 249, 256, 261, 284, 313, 361, 362, 418, 419
Brascamp, H. J., 18
Brass, P., 117
Brass, P., 419
Brazitikos, S., 317, 366, 419
Brehm, U., 303, 119
Brenier, Y., 20, 143, 119
Brönnstein, E. M., 159, 119
Brunn, H., 8, 110, 119
Burago, Y. D., 128, 387, 119
Busseman, H., 419
Buser, P., 419
Caffarelli, L., 20, 143, 407, 419
Campi, S., 45, 362, 119, 120
Carbery, A., 126, 361, 120
Carl, B., 115, 117, 120
Carlier, G., 15, 120
Cauty, L. A., 40, 109
Chakerian, G. D., 45, 120
Chavel, I., 128, 120
Chen, W., 209, 120
Cheng, S. Y., 46, 120
Chevet, S., 472, 116
Choquet, G., 45, 120
Christoffel, E. B., 46, 120
Colesanti, A., 49, 120
Colesanti, A., 49
Cordero-Erausquin, D., 164, 193, 199, 364, 417, 420
Craig, C. C., 128, 120
Cramér, H., 120
Dafris, N., 303, 367, 420
Danzer, L., 358
Danzer, L., 124
Dar, S., 211, 32, 33, 362, 411, 115, 421
Das Gupta, S., 43, 421
Davidović, J. S., 43, 421
Davis, W., 316, 421
Diaconis, F., 369, 421
<table>
<thead>
<tr>
<th>Author</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dilworth, S. J.</td>
<td>74, 255</td>
</tr>
<tr>
<td>Diskant, V. I.</td>
<td>421</td>
</tr>
<tr>
<td>Dolnikov, V. L.</td>
<td>200, 421</td>
</tr>
<tr>
<td>Dor, L. E.</td>
<td>197, 417</td>
</tr>
<tr>
<td>Dudley, R. M.</td>
<td>257, 290, 292, 294, 295, 312, 421</td>
</tr>
<tr>
<td>Dvoretzky, A.</td>
<td>59, 60, 195, 196, 197, 198, 199, 219, 220, 241, 421</td>
</tr>
<tr>
<td>Ebin</td>
<td>126, 128</td>
</tr>
<tr>
<td>Eggleston, H. G.</td>
<td>45</td>
</tr>
<tr>
<td>Eidelman, Y.</td>
<td>387, 421</td>
</tr>
<tr>
<td>Eldan</td>
<td>364, 421</td>
</tr>
<tr>
<td>Elton</td>
<td>256, 421</td>
</tr>
<tr>
<td>Enflo</td>
<td>421</td>
</tr>
<tr>
<td>Erdal</td>
<td>124, 125</td>
</tr>
<tr>
<td>Falconer, K. J.</td>
<td>141, 122</td>
</tr>
<tr>
<td>Fan, K.</td>
<td>113, 122</td>
</tr>
<tr>
<td>Fejes-Tóth, G.</td>
<td>157, 122</td>
</tr>
<tr>
<td>Fenchel, W.</td>
<td>63, 101, 102, 103, 118, 122</td>
</tr>
<tr>
<td>Fernique, X.</td>
<td>172, 112, 122</td>
</tr>
<tr>
<td>Figalli, A.</td>
<td>421</td>
</tr>
<tr>
<td>Figiel, T.</td>
<td>121, 174, 176, 180, 181, 189, 192, 219</td>
</tr>
<tr>
<td>Fimiani, M.</td>
<td>46, 120</td>
</tr>
<tr>
<td>Firey, W.</td>
<td>122</td>
</tr>
<tr>
<td>Firey, W. J.</td>
<td>126, 101, 122</td>
</tr>
<tr>
<td>Fleuryst, B.</td>
<td>122</td>
</tr>
<tr>
<td>Florentin, D.</td>
<td>111, 113, 158, 114, 116, 122</td>
</tr>
<tr>
<td>Floyd, E. E.</td>
<td>200, 122</td>
</tr>
<tr>
<td>Fradelizi, M.</td>
<td>102</td>
</tr>
<tr>
<td>Fradelizi, M.</td>
<td>125, 126, 135, 136, 141, 150, 152</td>
</tr>
<tr>
<td>Freedman, D.</td>
<td>363, 421</td>
</tr>
<tr>
<td>Galichon, A.</td>
<td>13, 120</td>
</tr>
<tr>
<td>Gallot, S.</td>
<td>125, 122</td>
</tr>
<tr>
<td>Gardner, R. J.</td>
<td>39, 111, 157, 122</td>
</tr>
<tr>
<td>Garling, D. J.</td>
<td>313</td>
</tr>
<tr>
<td>Garnaev, A.</td>
<td>426</td>
</tr>
<tr>
<td>Giannopoulos, A.</td>
<td>255</td>
</tr>
<tr>
<td>Giannopoulos, A.</td>
<td>131, 157, 165, 155, 145, 146, 148, 151, 152</td>
</tr>
<tr>
<td>Gluck, H.</td>
<td>116, 126</td>
</tr>
<tr>
<td>Gluskin, E. D.</td>
<td>7, 157, 255, 123, 124</td>
</tr>
<tr>
<td>Godwin, H. J.</td>
<td>198, 124</td>
</tr>
<tr>
<td>Gohberg, L.</td>
<td>157, 124</td>
</tr>
<tr>
<td>Goodey, P.</td>
<td>14, 124</td>
</tr>
<tr>
<td>Goodman, V.</td>
<td>117, 124</td>
</tr>
<tr>
<td>Gordon, Y.</td>
<td>71, 157, 158, 162, 163, 238, 241</td>
</tr>
<tr>
<td>Götze, F.</td>
<td>118</td>
</tr>
<tr>
<td>Green, J. W.</td>
<td>211, 124</td>
</tr>
<tr>
<td>Groemer, H.</td>
<td>125, 101, 187, 124</td>
</tr>
<tr>
<td>Gronov, M.</td>
<td>131, 121, 103, 109, 100, 125, 200, 125, 253, 255, 111, 121</td>
</tr>
<tr>
<td>Gronchi, P.</td>
<td>43, 163, 119, 120</td>
</tr>
<tr>
<td>Gross, W.</td>
<td>148, 124</td>
</tr>
<tr>
<td>Grothendieck, A.</td>
<td>75, 197, 125</td>
</tr>
<tr>
<td>Gruber, P. M.</td>
<td>355, 357, 357, 358, 358, 358, 358, 358, 358, 358</td>
</tr>
<tr>
<td>Grünbaum, B.</td>
<td>425</td>
</tr>
<tr>
<td>Guan, B.</td>
<td>16, 125</td>
</tr>
<tr>
<td>Guan, P.</td>
<td>425</td>
</tr>
<tr>
<td>Guédon, O.</td>
<td>362, 367, 120, 121, 121, 121</td>
</tr>
<tr>
<td>Guruswami, V.</td>
<td>195, 124</td>
</tr>
<tr>
<td>Gutman, K.</td>
<td>115</td>
</tr>
<tr>
<td>Haagerup, U.</td>
<td>108, 126, 126</td>
</tr>
<tr>
<td>Hacet, B. L.</td>
<td>23, 121</td>
</tr>
<tr>
<td>Hadwiger, H.</td>
<td>159, 157, 157, 113, 126</td>
</tr>
<tr>
<td>Hanner, O.</td>
<td>425</td>
</tr>
<tr>
<td>Hardy, G. H.</td>
<td>116, 126</td>
</tr>
<tr>
<td>Harper, L. H.</td>
<td>86, 126, 126</td>
</tr>
<tr>
<td>Hartzoulaki, M.</td>
<td>73, 366, 114, 126, 126</td>
</tr>
<tr>
<td>Henstock, R.</td>
<td>39, 123, 125</td>
</tr>
<tr>
<td>Hilbert, D.</td>
<td>39, 201, 126</td>
</tr>
<tr>
<td>Hinow, P.</td>
<td>363, 110</td>
</tr>
<tr>
<td>Hioni, L.</td>
<td>366, 367, 111, 128</td>
</tr>
<tr>
<td>Hoefding, W.</td>
<td>128, 126</td>
</tr>
<tr>
<td>Hörmander, L.</td>
<td>126</td>
</tr>
<tr>
<td>Houdré, C.</td>
<td>118</td>
</tr>
<tr>
<td>Hug, D.</td>
<td>11, 367, 122, 126</td>
</tr>
<tr>
<td>Hulin, D.</td>
<td>129, 122</td>
</tr>
<tr>
<td>Hurwitz, A.</td>
<td>201, 126</td>
</tr>
<tr>
<td>Ibragimov, I. A.</td>
<td>185, 126</td>
</tr>
<tr>
<td>Indyk, P.</td>
<td>198, 126</td>
</tr>
<tr>
<td>Jerison, D.</td>
<td>111, 106, 122, 126</td>
</tr>
<tr>
<td>Jessen, B.</td>
<td>106, 122</td>
</tr>
<tr>
<td>Jiménez, C. H.</td>
<td>75, 126</td>
</tr>
<tr>
<td>John, F.</td>
<td>17, 50, 52, 114, 126</td>
</tr>
<tr>
<td>Johnson, W. B.</td>
<td>75, 197, 257, 117, 192, 126, 126</td>
</tr>
<tr>
<td>Junge, M.</td>
<td>302, 126</td>
</tr>
<tr>
<td>Kabatianski, G. A.</td>
<td>135, 156, 126</td>
</tr>
<tr>
<td>Kadets, M. I.</td>
<td>53, 126</td>
</tr>
<tr>
<td>Kahane, J.-P.</td>
<td>107, 109, 146, 122, 126</td>
</tr>
<tr>
<td>Kakutani, S.</td>
<td>124</td>
</tr>
<tr>
<td>Kannan, R.</td>
<td>39, 126, 365, 126</td>
</tr>
<tr>
<td>Kantorovich, L.</td>
<td>15</td>
</tr>
<tr>
<td>Karasev, R. N.</td>
<td>200, 121</td>
</tr>
<tr>
<td>Kashin, B. S.</td>
<td>130, 197, 198, 200, 126</td>
</tr>
<tr>
<td>Khintchine, A.</td>
<td>107, 108, 126, 127</td>
</tr>
<tr>
<td>Khovanski, A. G.</td>
<td>127</td>
</tr>
<tr>
<td>Klain, D.</td>
<td>413, 123</td>
</tr>
<tr>
<td>Klartag, B.</td>
<td>23, 123, 129, 165, 168, 172, 121</td>
</tr>
<tr>
<td>Klee, V.</td>
<td>15, 387, 121</td>
</tr>
<tr>
<td>Knaster, B.</td>
<td>200, 127</td>
</tr>
<tr>
<td>Kneser, H.</td>
<td>14, 124, 127</td>
</tr>
<tr>
<td>Knothe, H.</td>
<td>10, 123, 124</td>
</tr>
<tr>
<td>Koldobsky, A.</td>
<td>39, 363, 364, 113, 127</td>
</tr>
<tr>
<td>Kolmogorov, A. N.</td>
<td>131, 157, 127</td>
</tr>
<tr>
<td>König, H.</td>
<td>77, 157, 158, 262, 283, 362, 124</td>
</tr>
<tr>
<td>Krasnikov, V. A.</td>
<td>127, 123</td>
</tr>
<tr>
<td>Author</td>
<td>Pages</td>
</tr>
<tr>
<td>---------------------</td>
<td>-------</td>
</tr>
<tr>
<td>Korenbljum, B. I.</td>
<td>449</td>
</tr>
<tr>
<td>Kozma, G.</td>
<td>365</td>
</tr>
<tr>
<td>Kubota, T.</td>
<td>6, 141, 195</td>
</tr>
<tr>
<td>Kuperberg, G.</td>
<td>284, 128</td>
</tr>
<tr>
<td>Kuperberg, W.</td>
<td>157, 122</td>
</tr>
<tr>
<td>Kushnirenko, A. G.</td>
<td>128</td>
</tr>
<tr>
<td>Kwapien, S.</td>
<td>126, 220, 128</td>
</tr>
<tr>
<td>Lőfstrom, J.</td>
<td>284</td>
</tr>
<tr>
<td>Lévy, P.</td>
<td>50, 99, 128, 124</td>
</tr>
<tr>
<td>Lóvasz, L.</td>
<td>39</td>
</tr>
<tr>
<td>Lafontaine, J.</td>
<td>126, 122</td>
</tr>
<tr>
<td>Lorman, D. G.</td>
<td>397, 128</td>
</tr>
<tr>
<td>Latała, R.</td>
<td>193, 356, 365, 357, 128</td>
</tr>
<tr>
<td>Ledoux, M.</td>
<td>121, 126, 157, 128</td>
</tr>
<tr>
<td>Lee, J.</td>
<td>138, 125</td>
</tr>
<tr>
<td>Leichtweiss, K.</td>
<td>20, 74, 75, 128</td>
</tr>
<tr>
<td>Leindler, L.</td>
<td>22, 148</td>
</tr>
<tr>
<td>Lettl, G.</td>
<td>388, 125</td>
</tr>
<tr>
<td>Levenshtein, V. I.</td>
<td>135, 156, 157, 128</td>
</tr>
<tr>
<td>Leaky, P.</td>
<td>39</td>
</tr>
<tr>
<td>Leichtweiss, K.</td>
<td>22, 148</td>
</tr>
<tr>
<td>Lieb, E. H.</td>
<td>126, 126, 126, 128</td>
</tr>
<tr>
<td>Lévy, P.</td>
<td>125, 126, 126, 128</td>
</tr>
<tr>
<td>Lewis, D. R.</td>
<td>74, 218, 234, 125</td>
</tr>
<tr>
<td>Li, P.</td>
<td>365, 128</td>
</tr>
<tr>
<td>Li, W. V.</td>
<td>125, 128</td>
</tr>
<tr>
<td>Lütz, L.</td>
<td>4, 125</td>
</tr>
<tr>
<td>Lifshits, M. A.</td>
<td>841, 127</td>
</tr>
<tr>
<td>Lin, C.</td>
<td>102, 125</td>
</tr>
<tr>
<td>Linde, W.</td>
<td>157, 128</td>
</tr>
<tr>
<td>Lindenstrauss, J.</td>
<td>171, 176, 181, 197, 198, 237, 199, 118</td>
</tr>
<tr>
<td>Littlewood, J. E.</td>
<td>126, 111, 125, 129</td>
</tr>
<tr>
<td>Litvak, A. E.</td>
<td>125, 125, 125, 125</td>
</tr>
<tr>
<td>Löring, J.</td>
<td>117</td>
</tr>
<tr>
<td>Lorentz, G. G.</td>
<td>157, 129</td>
</tr>
<tr>
<td>Lovász, L.</td>
<td>126, 165, 128, 129</td>
</tr>
<tr>
<td>Lovett, S.</td>
<td>128, 120</td>
</tr>
<tr>
<td>Lucas, E.</td>
<td>131</td>
</tr>
<tr>
<td>Ludwig, M.</td>
<td>168, 129</td>
</tr>
<tr>
<td>Lusternik, L.</td>
<td>129, 129</td>
</tr>
<tr>
<td>Luttik, E.</td>
<td>127, 129</td>
</tr>
<tr>
<td>Lyubarskii, Y.</td>
<td>136, 129</td>
</tr>
<tr>
<td>Ma, X.</td>
<td>146, 125</td>
</tr>
<tr>
<td>Macbeath, A. M.</td>
<td>391, 125</td>
</tr>
<tr>
<td>Maggi, M.</td>
<td>122, 122</td>
</tr>
<tr>
<td>Mahler, K.</td>
<td>129, 129</td>
</tr>
<tr>
<td>Makovec, V. V.</td>
<td>129, 129</td>
</tr>
<tr>
<td>Makovoz, Y.</td>
<td>129, 129</td>
</tr>
<tr>
<td>Mani, P.</td>
<td>129, 142, 129</td>
</tr>
<tr>
<td>Marcus, M. B.</td>
<td>134, 138, 128, 128, 128, 128, 128</td>
</tr>
<tr>
<td>Markessinis, E.</td>
<td>126, 139</td>
</tr>
<tr>
<td>Markus, A. S.</td>
<td>127, 120</td>
</tr>
<tr>
<td>Marshall, A. W.</td>
<td>120, 120</td>
</tr>
<tr>
<td>Maurey, B.</td>
<td>129, 130, 130, 130, 130, 130, 129, 129, 129, 129, 129</td>
</tr>
<tr>
<td>Maz'ja, V. G.</td>
<td>120, 120</td>
</tr>
<tr>
<td>McCann, R.</td>
<td>120, 129</td>
</tr>
<tr>
<td>McDermid, C.</td>
<td>125, 130</td>
</tr>
<tr>
<td>McMillen, P.</td>
<td>125, 130</td>
</tr>
<tr>
<td>Meckes, E.</td>
<td>130, 130</td>
</tr>
<tr>
<td>Meckes, M. W.</td>
<td>130, 130</td>
</tr>
<tr>
<td>Meyer, M.</td>
<td>125, 125, 125, 125, 125, 125, 125, 125, 125, 125</td>
</tr>
<tr>
<td>Milman, E.</td>
<td>125, 130</td>
</tr>
<tr>
<td>Minding, F.</td>
<td>132</td>
</tr>
<tr>
<td>Minkowski, H.</td>
<td>132</td>
</tr>
<tr>
<td>Mitra, A. I.</td>
<td>132</td>
</tr>
<tr>
<td>Molchanov, G.</td>
<td>132</td>
</tr>
<tr>
<td>Monge, G.</td>
<td>132</td>
</tr>
<tr>
<td>Moser, W.</td>
<td>125, 111</td>
</tr>
<tr>
<td>Müller, D.</td>
<td>131, 132</td>
</tr>
<tr>
<td>Naor, A.</td>
<td>364, 132</td>
</tr>
<tr>
<td>Naszódi, M.</td>
<td>132</td>
</tr>
<tr>
<td>Nazarov, F.</td>
<td>132</td>
</tr>
<tr>
<td>Nemirovskii, A.</td>
<td>132</td>
</tr>
<tr>
<td>Newman, C. M.</td>
<td>132</td>
</tr>
<tr>
<td>Newton, I.</td>
<td>132</td>
</tr>
<tr>
<td>Nguyen, V. H.</td>
<td>360, 116</td>
</tr>
<tr>
<td>Ohmann, D.</td>
<td>132</td>
</tr>
<tr>
<td>Oleksikewicz, K.</td>
<td>132</td>
</tr>
<tr>
<td>Olkin, I.</td>
<td>132</td>
</tr>
<tr>
<td>Ostrovey, Y.</td>
<td>132</td>
</tr>
<tr>
<td>Pach, J.</td>
<td>132</td>
</tr>
<tr>
<td>Pajor, A.</td>
<td>132</td>
</tr>
<tr>
<td>Palmon, O.</td>
<td>132</td>
</tr>
<tr>
<td>Paouris, G.</td>
<td>132</td>
</tr>
<tr>
<td>Papadimitriakis, M.</td>
<td>132</td>
</tr>
<tr>
<td>Payne, L. E.</td>
<td>132</td>
</tr>
<tr>
<td>Pelczynski, A.</td>
<td>132</td>
</tr>
<tr>
<td>Perissinaki, I.</td>
<td>132</td>
</tr>
<tr>
<td>Pestov, V. G.</td>
<td>132</td>
</tr>
<tr>
<td>Petty, C. M.</td>
<td>132</td>
</tr>
<tr>
<td>Pietsch, A.</td>
<td>132</td>
</tr>
<tr>
<td>Pinelis, I.</td>
<td>132</td>
</tr>
<tr>
<td>Pinkus, A.</td>
<td>132</td>
</tr>
<tr>
<td>Pisier, G.</td>
<td>132</td>
</tr>
<tr>
<td>Pivovarov, P.</td>
<td>132</td>
</tr>
<tr>
<td>Papadimitriakis, M.</td>
<td>132</td>
</tr>
<tr>
<td>Payne, L. E.</td>
<td>132</td>
</tr>
<tr>
<td>Pelczynski, A.</td>
<td>132</td>
</tr>
<tr>
<td>Perissinaki, I.</td>
<td>132</td>
</tr>
<tr>
<td>Pestov, V. G.</td>
<td>132</td>
</tr>
<tr>
<td>Petty, C. M.</td>
<td>132</td>
</tr>
<tr>
<td>Pietsch, A.</td>
<td>132</td>
</tr>
<tr>
<td>Pinelis, I.</td>
<td>132</td>
</tr>
<tr>
<td>Pinkus, A.</td>
<td>132</td>
</tr>
<tr>
<td>Pisier, G.</td>
<td>132</td>
</tr>
<tr>
<td>Pivovarov, P.</td>
<td>132</td>
</tr>
</tbody>
</table>
Pogorelov, A. V., 46, 433
Pólya, G., 411, 416
Pontrjagin, L., 131, 433
Prékopa, A., 22, 43, 433
Pratelli, A., 42, 422
Proskor, F., 43, 433
Radström, H., 126
Raz, O., 120, 129, 433
Raz, R., 120, 129, 433
Razborov, A., 198, 425
Regev, O., 123, 129, 427
Reisner, S., 284, 424, 433
Reitzner, M., 367, 413, 429, 433
Rockafellar, R. T., 18, 19, 21, 38, 43, 387, 433
Rogers, C. A., 27, 44, 45, 59, 60, 75, 163, 421, 433, 434
Romance, M., 74, 76, 417
Romik, D., 364, 432
Rotem, L., 41, 431
Roy, R., 230, 415
Rudelson, M., 75, 76, 158, 198, 256, 423, 429, 434
Rudin, W., 198, 433
Saint-Raymond, J., 45, 284, 362, 434
Santaló, L., 32, 45, 434
Santambrogio, F., 43, 420
Saroglou, Ch., 76, 430, 434
Schür, C., 424, 435
Schwarz, H. A., 39, 435
Segal, A., 43, 435
Shepherd, G. C., 27, 41, 45, 433
Shiffman, B., 388, 435
Sidelnikov, V. M., 156, 435
Simonovits, M., 29, 126, 305, 126, 129
Slepian, D., 288, 291, 312, 435
Slomka, B., 416
Snobar, M. G., 58, 75, 436
Sodin, M., 120, 432
Sodin, S., 126, 126, 305, 305, 129, 135
Spencer, J. H., 126, 435
Spiegelman, D., 174, 171, 435
Srivastava, N., 75, 117, 435
Stavrakakis, P., 145, 366, 423
Steele, J. M., 129, 435
Stein, E., 401, 436
Steiner, J., 5, 7, 11, 435
Stephani, I., 137, 435
Stroock, D. W., 124, 435
Sukdakov, V. N., 85, 121, 125, 159, 167, 287
Talagrand, M., 4, 7, 9, 10, 12, 426
Teissier, B., 136
Temlyakov, V. N., 198, 426
Thompson, A. C., 435
Tikhomirov, V. M., 181, 427
Tomczak-Jaegermann, N., 74, 139, 433
Triebel, H., 157, 420, 436
Tsirelson, B. S., 85, 124, 125, 156, 158
Tsalomitis, A., 74, 158, 215, 218, 251, 255
Uhrin, B., 43, 436
Uryson, P. S., 43, 436
Valettas, P., 317, 366, 428
Vempala, S., 420
Vershynin, R., 158, 192, 193, 198, 199, 218
Villa, R., 83, 124, 420
Villani, C., 43, 436
Vogt, H., 363, 419
Voigt, J. A., 463, 419
Volčič, A., 430
Volberg, A., 126, 132
von Neumann, J., 131, 436
von Weizsäcker, H. C., 436
Vritsiou, B. H., 317, 366, 398, 398, 110, 426
Wagner, R., 122, 129, 436
Wang, D. L., 125, 436
Wang, P., 125, 436
Weil, W., 365, 418, 428, 122, 424
Weinberger, H. F., 365, 432
Weyl, H., 436
Weyl, W., 411
Wigderson, A., 198, 425
Wojtaszczyk, J., 363, 365, 437
Wolfson, H., 174, 230, 119, 132
Wright, J., 126, 420
Yamabe, H., 420, 437
Yang, D., 429
Yaskin, V., 46, 424
Yaskina, M., 46, 424
Yau, S. T., 46, 305, 420, 428
Yujobo, Z., 200, 437
Yurinskii, V. V., 126, 437
Zalgaller, V. A., 125, 387, 419
Zhang, G., 429
Zhou, F., 16, 425
Zinn, J., 336, 362, 434
Zong, C., 431, 437
The authors present the theory of asymptotic geometric analysis, a field which lies on the border between geometry and functional analysis. In this field, isometric problems that are typical for geometry in low dimensions are substituted by an “isomorphic” point of view, and an asymptotic approach (as dimension tends to infinity) is introduced. Geometry and analysis meet here in a non-trivial way. Basic examples of geometric inequalities in isomorphic form which are encountered in the book are the “isomorphic isoperimetric inequalities” which led to the discovery of the “concentration phenomenon”, one of the most powerful tools of the theory, responsible for many counterintuitive results.

A central theme in this book is the interaction of randomness and pattern. At first glance, life in high dimension seems to mean the existence of multiple “possibilities”, so one may expect an increase in the diversity and complexity as dimension increases. However, the concentration of measure and effects caused by convexity show that this diversity is compensated and order and patterns are created for arbitrary convex bodies in the mixture caused by high dimensionality.

The book is intended for graduate students and researchers who want to learn about this exciting subject. Among the topics covered in the book are convexity, concentration phenomena, covering numbers, Dvoretzky-type theorems, volume distribution in convex bodies, and more.

For additional information and updates on this book, visit www.ams.org/bookpages/surv-202