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To the memory of our fathers,
Ozsváth István, Stipsicz István, and Dr. Szabó István

Contents

APPENDIX A

Homological algebra

For the sake of completeness, in this appendix we recall some basic notions and constructions of homological algebra. In fact, our discussion is slightly non-standard: the gradings natural in grid homology are somewhat different from the gradings that come up naturally in algebraic topology. We start with basics of chain complexes and their homologies in Sections A.1. In Section A.2 we describe exact triangles, and in Section A.3 we discuss mapping cones. In Section A.4 we describe the structure of the homology groups over the ring $\mathbb{F}[U]$. In Section A.5 we describe the relationship between a complex and its dual. In Section A.7 we discuss minimal models of filtered chain complexes; this result gives economical chain complexes which represent a fixed filtered quasi-isomorphism type. Finally, in Section A.8 we discuss the relation between chain homotopies and quasi-isomorphisms.

This appendix is intentionally brief; for a more detailed discussion of the topics the reader is advised to turn to [**70, 83, 200**].

A.1. Chain complexes and their homology

Let $\mathbb{K}$ denote either the finite field $\mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{N}$, or $\mathbb{Q}$ or the ring $\mathbb{Z}$. In the following, $\mathcal{R}$ will denote the polynomial ring $\mathbb{K}[V_1, \dots, V_n]$ of n variables. We include the $n = 0$ case with the understanding that in this case $\mathcal{R} = \mathbb{K}$. In particular, $\mathcal{R}$ is a field if $n = 0$ and $\mathbb{K}$ is not $\mathbb{Z}$; and $\mathcal{R}$ is a principal ideal domain (PID) if either $n = 0$, or $n = 1$ and $\mathbb{K}$ is not $\mathbb{Z}$.

DEFINITION A.1.1. A ***chain complex*** is an $\mathcal{R}$-module C, equipped with an $\mathcal{R}$-module homomorphism $\partial\colon C \to C$ with the property that $\partial \circ \partial = 0$. The map ∂ is called the ***boundary map*** or the ***differential*** for C. A ***cycle*** is an element $z \in C$ with $\partial z = 0$ and a ***boundary*** is an element b of the form $b = \partial a$ for some $a \in C$; i.e. the cycles are the elements in the kernel $\mathrm{Ker}\partial$, and the boundaries are the elements in the image $\mathrm{Im}\partial$.

An $\mathcal{R}$-submodule $C' \subset C$ of a chain complex (C, ∂) is a *subcomplex* if $\partial(C') \subset C'$. In this case, the pair $(C', \partial|_{C'})$ is a chain complex. Similarly, if $(C', \partial|_{C'})$ is a subcomplex, the quotient module C/C' inherits a boundary operator $\partial_{C/C'}$, induced from ∂. The pair $(C/C', \partial_{C/C'})$ is the *quotient complex* of C by C'.

The condition $\partial^2 = 0$ says $\mathrm{Im}\partial \subset \mathrm{Ker}\partial$, so we can make the following definition:

DEFINITION A.1.2. The ***homology*** $H(C, \partial)$ of the chain complex (C, ∂) is the quotient $\mathcal{R}$-module $\mathrm{Ker}\partial/\mathrm{Im}\partial$.

In standard homological algebra, it is customary to consider $\mathbb{Z}$-graded chain complexes, where the differential ∂ drops grading by 1 and where the action of the ring $\mathcal{R}$ preserves gradings. Such a complex admits a direct sum splitting $C = \bigoplus_{d\in\mathbb{Z}} C_d$, so that C_d is an $\mathcal{R}$-submodule for all d. In this case, the homology $H(C)$

inherits a grading, where $H_d(C)$ is the cycles modulo the boundaries in C_d. In the present volume, we consider a mild variation of this, where the action of the ring $\mathcal{R} = \mathbb{K}[V_1, \dots, V_n]$ changes the grading on the chain complex. In fact, quite often our complexes come naturally with two different gradings (see Theorem 4.6.3, for example). We formalize this in the following:

Definition A.1.3. Throughout this book, a ***bigraded chain complex over*** $\mathbb{K}[V_1, \dots, V_n]$ is a chain complex (C, ∂) over $\mathbb{K}$, equipped with endomorphisms $V_i \colon C \to C$ for $i = 1, \dots n$, called the ***algebra actions***, and a splitting $C = \bigoplus_{(d,s)\in\mathbb{Z}\oplus\mathbb{Z}} C_{d,s}$, satisfying the following compatibility conditions:

- for $i = 1, \dots, n$, $\partial \circ V_i = V_i \circ \partial$;
- for all $i, j \in \{1, \dots, n\}$, $V_i \circ V_j = V_j \circ V_i$;
- ∂ maps $C_{d,s}$ to $C_{d-1,s}$;
- V_i maps $C_{d,s}$ to $C_{d-2,s-1}$.

The first grading is called the ***Maslov grading***, and the second the ***Alexander grading***. The actions can be viewed as endowing C with the structure of a module over $\mathbb{K}[V_1, \dots, V_n]$. The condition that ∂ commutes with V_i is equivalent to the condition that ∂ is a $\mathbb{K}[V_1, \dots, V_n]$-module homomorphism.

Remark A.1.4. The above definition of bigraded complexes fits naturally into a more general framework of bigraded complexes over a bigraded ring. For this purpose, consider the polynomial ring $\mathbb{K}[V_1, \dots, V_n]$ to be bigraded, so that V_i has bigrading $(-2, -1)$. Our bigraded complexes, then, are bigraded complexes over this bigraded ring, equipped with a differential which has bigrading $(-1, 0)$.

When $n = 0$, Definition A.1.3 specializes to a bigraded complex over $\mathbb{K}$. Of particular relevance to us is the case when $n = 1$, in which case we often denote the single variable by U. According to Theorem 4.6.3, the grid complex is a bigraded complex as above with $\mathbb{K}$ chosen to be $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. In Chapter 11 (see especially Theorem 11.1.7), the Alexander grading set is also enlarged. In Chapter 13, the Alexander grading is relaxed to an Alexander filtration. In Chapter 15, $\mathbb{K}$ is no longer the field $\mathbb{F}$, but the ring $\mathbb{Z}$ of integers.

For a bigraded chain complex, the homology $H(C)$ inherits both the structure of a $\mathbb{K}[V_1, \dots, V_n]$-module, and a bigrading compatible with that action. The portion of $H(C)$ in bigrading (d, s) is given by the quotient of the cycles in C by the boundaries in bigrading (d, s), and multiplication by V_i on C induces a corresponding action on $H(C)$.

Let (C, ∂) and (C', ∂') be two chain complexes over $\mathcal{R}$. An $\mathcal{R}$-module map $f \colon C \to C'$ is a *chain map* if it commutes with the boundary operators, that is, for every $c \in C$, $f(\partial c) = \partial' f(c)$. If C and C' are bigraded complexes over $\mathcal{R}$ as in Definition A.1.3, then a *bigraded chain map* is a chain map f that maps $C_{d,s}$ into $C'_{d,s} \subset C'$. More generally, if C and C' are as above, $f \colon C \to C'$ is a chain map, and there is a pair of integers (m, t) so that f maps $C_{d,s}$ into $C'_{d+m,s+t}$, then we say that f is a *homogeneous map with bidegree* (m, t).

Suppose that C is a bigraded chain complex over $\mathcal{R}$, and fix integers m, t. The (m, t)*-shift of* C is the bigraded chain complex $C[\![m, t]\!]$ defined by $C[\![m, t]\!]_{d,s} = C_{d+m,s+t}$. A chain map from C to C' that is homogeneous of degree (m, t) is the same as a bigraded chain map from C to $C'[\![m, t]\!]$.

A chain map $f \colon C \to C'$ maps cycles to cycles and boundaries to boundaries, and so it induces an $\mathcal{R}$-module map $H(f) \colon H(C, \partial) \to H(C', \partial')$ on the homology modules. When C and C' are bigraded complexes over $\mathcal{R}$ and $f\colon C \to C'$ is a bigraded chain map, the induced map $H(f)$ respects the induced bigrading on the homology modules.

Given two bigraded chain maps $f\colon (C,\partial) \to (C',\partial')$ and $g\colon (C',\partial') \to (C'',\partial'')$, their composite $g\circ f$ is another bigraded chain map, whose induced map on homology satisfies

$$H(g\circ f) = H(g)\circ H(f). \tag{A.1}$$

The identity map Id_C is a chain map that induces the identity map on $H(C,\partial)$. Consequently, an isomorphism between two chain complexes obviously induces an isomorphism between the corresponding homology modules. More generally:

Definition A.1.5. Let (C,∂) and (C',∂') be two chain complexes. A chain map $f\colon C \to C'$ is a ***quasi-isomorphism*** if it induces an isomorphism on homology. Two chain complexes (C,∂) and (C',∂') are said to be ***quasi-isomorphic*** if there is a chain complex (C'',∂'') and two quasi-isomorphisms $f\colon C'' \to C$ and $g\colon C'' \to C'$. In cases where (C,∂) and (C',∂') are bigraded complexes over $\mathcal{R}$, we require our quasi-isomorphisms to be bigraded maps over $\mathcal{R}$.

Special kinds of quasi-isomorphisms are supplied by the following:

Definition A.1.6. Suppose that $f,g\colon (C,\partial) \to (C',\partial')$ are two chain maps between two chain complexes over $\mathcal{R}$. The maps f and g are said to be ***chain homotopic*** if there is an $\mathcal{R}$-module map $h\colon C \to C'$ satisfying

$$f - g = \partial'\circ h + h\circ\partial.$$

A map $f\colon (C,\partial) \to (C',\partial')$ is a ***chain homotopy equivalence*** if there is a map $g\colon (C',\partial') \to (C,\partial)$ so that $f\circ g$ and $g\circ f$ are both chain homotopic to the respective identity maps $\mathrm{Id}_{C'}$ and Id_C.

Lemma A.1.7. *Let (C,∂) and (C',∂') be chain complexes. Then chain homotopic maps from C to C' induce the same map in homology. Consequently, a chain homotopy equivalence is a quasi-isomorphism.*

Proof. Suppose that $z\in C$ is a cycle. By definition, $H(f)([z]) = [f(z)]$, hence

$$H(f)([z]) = [g(z) + \partial'(h(z)) + h(\partial(z))] = [g(z)] = H(g)([z]),$$

since $\partial(z) = 0$ and $g(z)$ and $g(z)+\partial'(h(z))$ are homologous. Thus, a chain homotopy equivalence is a quasi-isomorphism in view of Equation (A.1). □

Note that not every quasi-isomorphism is a chain homotopy equivalence; compare Example 13.1.8, and see Proposition A.8.1. The definition of chain homotopy equivalence extends to the bigraded setting as follows.

Definition A.1.8. Suppose that (C,∂) and (C',∂') are two bigraded chain complexes over $\mathcal{R}$. Suppose that $f,g\colon C \to C'$ are two chain maps of degree (m,t). We say that f and g are ***chain homotopic as homogeneous maps of degree*** (m,t) if there is an $\mathcal{R}$-module map $h\colon C \to C'$ that sends $C_{d,s}$ to $C'_{d+m+1,s+t}$ with

$$\partial'\circ h + h\circ\partial = f - g. \tag{A.2}$$

A.2. Exact sequences

A sequence $\{C^i\}_{i\in\mathbb{Z}}$ of $\mathcal{R}$-modules equipped with $\mathcal{R}$-module maps $f^i\colon C^i \to C^{i+1}$ is called an *exact sequence of $\mathcal{R}$-modules* if $\mathrm{Im} f^i = \mathrm{Ker} f^{i+1}$. A special case is when $\mathcal{C}$ is a *short exact sequence*, that is, $C^i = 0$ unless $i = 1, 2, 3$. In this case the maps $f^1\colon C^1 \to C^2$ and $f^2\colon C^2 \to C^3$ satisfy that

- f^1 is injective,
- f^2 is surjective, and
- $\mathrm{Im} f^1 = \mathrm{Ker} f^2$.

An *exact triangle* is a 3-periodic exact sequence; i.e. in which there are three $\mathcal{R}$-modules C^1, C^2, and C^3, and maps $f^1\colon C^1 \to C^2$, $f^2\colon C^2 \to C^3$ and $f^3\colon C^3 \to C^1$, with $\mathrm{Ker}(f^i) = \mathrm{Im}(f^{i-1})$, where $C^i = C^{i+3}$ and $f^i = f^{i+3}$.

An *exact sequence of chain complexes* (C^i, ∂^i) *over* $\mathcal{R}$ is an exact sequence of $\mathcal{R}$-modules, where the maps $f^i\colon C^i \to C^{i+1}$ are also chain maps.

A short exact sequence of chain complexes induces a long exact sequence in homology, according to the following standard result. (See also [**83**, Theorem 2.16].)

Lemma A.2.1. *To each short exact sequence of chain complexes of $\mathcal{R}$-modules*

$$0 \to (C, \partial) \xrightarrow{f} (C', \partial') \xrightarrow{g} (C'', \partial'') \to 0,$$

there is an associated $\mathcal{R}$-module homomorphism $\delta\colon H(C'', \partial'') \to H(C, \partial)$, called the connecting homomorphism, such that

$$\begin{tikzcd}
H(C,\partial) \arrow[rr, "H(f)"] & & H(C',\partial') \arrow[dl, "H(g)"] \\
& H(C'',\partial'') \arrow[ul, "\delta"] &
\end{tikzcd}$$

is an exact triangle. Moreover, if the three complexes are bigraded complexes of $\mathcal{R}$-modules, and f and g are homogeneous $\mathcal{R}$-module homomorphisms of degree (m_1, t_1) and (m_2, t_2) respectively, then δ is a homogeneous map with degree $(-m_1 - m_2 - 1, -t_1 - t_2)$.

Proof. Given a homology class $x \in H(C'', \partial'')$, define $\delta(x)$ as follows. Pick a cycle $c'' \in C''$ representing x, and find $c \in C$ and $c' \in C'$ so that

$$g(c') = c'' \qquad \text{and} \qquad f(c) = \partial' c'. \tag{A.3}$$

The element c' can be found since g is surjective; the element c can be found since

$$g(\partial' c') = \partial''(g(c')) = \partial''(c'') = 0;$$

i.e. $\partial c' \in \mathrm{Ker}(g) = \mathrm{Im}(f)$. Moreover, the element c is a cycle since

$$f(\partial c) = \partial'(f(c)) = \partial'(\partial' c') = 0,$$

and f is injective. We then define $\delta(x)$ to be the homology class represented by c. We made three choices above: the cycle c'' representing x, the choice of c' with $g(c') = c''$, and the choice of c with $f(c) = \partial c'$. It is straightforward to verify that different choices result in homologous cycles $c \in C$; i.e. δ is a well defined map in homology. Furthermore, the map δ is an $\mathcal{R}$-module map. For example, for fixed cycle c'', if the elements $c \in C$ and $c' \in C'$ solve Equations (A.3), then for the cycle $r \cdot c''$, the elements $r \cdot c$ and $r \cdot c'$ solve the corresponding version of Equation (A.3).

In the bigraded case, if c'' is supported in bigrading (d, s), then we can find c' in bigrading $(d - m_2, s - t_2)$, and hence c in bigrading $(d - m_1 - m_2 - 1, s - t_1 - t_2)$.

Next, we verify exactness of the triangle at $H(C')$. The exactness of the short exact sequence ensures that $g \circ f = 0$, and hence $H(g) \circ H(f) = 0$; i.e.

$$\mathrm{Im} H(f) \subseteq \mathrm{Ker} H(g). \tag{A.4}$$

We must check that this inclusion is an equality. An element in $\mathrm{Ker}H(g)$ is represented by an element $c' \in C'$ with $\partial' c' = 0$ and $g(c') = \partial'' c''$ for some $c'' \in C''$. Since g is surjective, there is some c_2' with $g(c_2') = c''$. Thus,

$$c' - \partial' c_2' \in \mathrm{Ker}(g) = \mathrm{Im}(f)$$

so we can find $c \in C$ with $f(c) = c' - \partial' c_2'$. Since f is injective, it follows that c is a cycle; and so $[c'] = H(f)([c])$. Since $[c'] \in \mathrm{Ker}H(g)$ is arbitrary, we have verified that $\mathrm{Ker}H(g) \subseteq \mathrm{Im}H(f)$ which, together with Equation (A.4), implies that $\mathrm{Ker}H(g) = \mathrm{Im}H(f)$. Exactness at the other two terms can be verified by a similar diagram chase. ☐

A further elaboration on the proof of Lemma A.2.1 is the following:

LEMMA A.2.2. *The connecting homomorphism is natural, in the sense that for a map of short exact sequences of chain complexes, i.e. for a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "f"] \arrow[d, "\phi"'] & C' \arrow[r, "g"] \arrow[d, "\phi'"'] & C'' \arrow[r] \arrow[d, "\phi''"'] & 0 \\ 0 \arrow[r] & B \arrow[r, "f'"] & B' \arrow[r, "g'"] & B'' \arrow[r] & 0 \end{tikzcd}$$

(where the rows are exact sequences and the squares commute), the following diagram commutes

$$\begin{tikzcd} H(C'') \arrow[r, "\delta"] \arrow[d, "H(\phi'')"'] & H(C) \arrow[d, "H(\phi)"] \\ H(B'') \arrow[r, "\delta'"] & H(B) \end{tikzcd}$$

where the maps δ and δ' are the connecting homomorphisms for the two short exact sequences.

Proof. The proof is a straightforward application of the definition of the connecting homomorphism (Equation (A.3)). ☐

The above naturality gives a useful method for establishing isomorphisms between modules (see for example Lemma A.3.9 below), when combined with the following *five lemma*:

LEMMA A.2.3 (Five lemma). *Suppose that the diagram*

$$\begin{tikzcd} C^1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & C^2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"'] & C^3 \arrow[r, "f_3"] \arrow[d, "\alpha_3"'] & C^4 \arrow[r, "f_4"] \arrow[d, "\alpha_4"'] & C^5 \arrow[d, "\alpha_5"'] \\ B^1 \arrow[r, "g_1"] & B^2 \arrow[r, "g_2"] & B^3 \arrow[r, "g_3"] & B^4 \arrow[r, "g_4"] & B^5 \end{tikzcd}$$

is commutative, the two rows are exact and $\alpha_1, \alpha_2, \alpha_4$ and α_5 are isomorphisms. Then, the homomorphism α_3 is an isomorphism. ☐

Proof. A standard diagram chase shows that (a) if α_2 and α_4 are surjective and α_5 is injective then α_3 is surjective, and (b) if α_2 and α_4 are injective and α_1 is surjective then α_3 is injective. The lemma then easily follows. For more details, see [**83**, page 129]. □

A.3. Mapping cones

We recall now the *mapping cone* construction from homological algebra. For more on this construction, see [**226**, Chapter 1.5].

Definition A.3.1. Let (C, ∂) and (C', ∂') be two chain complexes over $\mathcal{R}$ and let $f\colon C \to C'$ be a chain map. The ***mapping cone*** $\mathrm{Cone}(f\colon C \to C') = \mathrm{Cone}(f)$ is the chain complex whose underlying module is $C \oplus C'$, and whose differential is

$$D(c, c') = (-\partial(c), \partial(c') + f(c)). \tag{A.5}$$

There are variants of this construction in the presence of gradings. For instance, if the complexes C and C' are graded, so that $f\colon C \to C'$ preserves gradings, then $\mathrm{Cone}(f)$ is graded, by

$$\mathrm{Cone}(f)_d = C_{d-1} \oplus C'_d.$$

With this convention, the differential

$$D_d(c, c') = (-\partial_{d-1} c, \partial'_d(c') + f_{d-1}(c))$$

drops grading by one. More generally, if (C, ∂) and (C', ∂') are two bigraded complexes over $\mathcal{R}$, and $f\colon C \to C'$ is a chain map that is homogeneous of bidegree (m, t), then $\mathrm{Cone}(f)$ is a bigraded chain complex, with the differential of Equation (A.5) and bigrading given by

$$\mathrm{Cone}(f)_{d,s} = C_{d-m-1,s-t} \oplus C'_{d,s}.$$

Lemma A.3.2. *Let C and C' be bigraded chain complexes, and $f\colon C \to C'$ be a bigraded chain map. Then there is a long exact sequence*

$$\dots \longrightarrow H_{d,s}(C) \xrightarrow{H(f)} H_{d,s}(C') \longrightarrow H_{d,s}(\mathrm{Cone}(f)) \longrightarrow H_{d-1,s}(C) \longrightarrow \dots$$

If f is a homogeneous map of bidegree (m, t), then we have the long exact sequence with the following degree shifts

$$\dots \to H_{d,s}(C) \xrightarrow{H(f)} H_{d+m,s+t}(C') \longrightarrow H_{d+m,s+t}(\mathrm{Cone}(f)) \longrightarrow H_{d-1,s}(C) \to \dots$$

Proof. For any chain map $f\colon C \to C'$ there is a short exact sequence of chain complexes

$$0 \longrightarrow C' \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} C \longrightarrow 0, \tag{A.6}$$

where the maps are defined by $i(c') = (0, c')$ and $p(c, c') = (-1)^d c$, when $c \in C_{d,s}$. It follows immediately from Equation (A.5) (and the choice of the sign in the definition of p) that i and p are chain maps, and it is easy to verify that they fit into the above short exact sequence. The grading shifts on the mapping cone are set up so that i preserves bigrading and p shifts it by $(-1, 0)$.

The long exact sequence appearing in the statement of the lemma follows from the long exact sequence associated to the above short exact sequence, once we verify that the connecting homomorphism in this associated long exact sequence is the map induced by f on homology, times $(-1)^d$.

To this end, recall the definition of the connecting homomorphism δ from Lemma A.2.1: given a cycle $c_1 \in C_{d,s}$, find $c_2 \in \mathrm{Cone}_{d+1,s}(f)$ and $c_3 \in C'_{d,s}$ with:

$$q(c_2) = c_1 \qquad \text{and} \qquad i(c_3) = \partial c_2.$$

It follows that c_3 is a cycle; and in fact $\delta([c_1])$ is represented by this cycle. By choosing $c_2 = ((-1)^d c_1, 0)$ and $c_3 = (-1)^d f(c_1)$, the connecting homomorphism is immediately seen to be equal to the map $(-1)^d H(f)$ induced by f on homology. We stated the exact sequence in the lemma without the factor of $(-1)^d$. This is justified, since the kernel (and the image) of $(-1)^d H(f)$ coincides with the kernel (and the image) of $H(f)$, hence the exactness in the lemma follows.

If f is of bidegree (m, t), the same argument provides the long exact sequence with the indicated degree shifts. □

This lemma has the following immediate corollary:

Corollary A.3.3. *A map f is a quasi-isomorphism if and only if* Cone(f) *has trivial homology.* □

As an application, we have the following result, which we state after a definition.

Definition A.3.4. A bigraded chain complex C of $\mathcal{R}$-modules is ***bounded above*** if for all sufficiently large d, $C_{d,s} = 0$.

For example, if the bigraded chain complex C is finitely generated as an $\mathcal{R}$-module, then C is bounded above. If C is a bigraded chain complex that is bounded above, then $H(C)$ is bounded above, as well.

If C is a bigraded chain complex over $\mathcal{R} = \mathbb{K}[V_1, \dots, V_n]$ with $n > 0$, we can construct the quotient complex $\frac{C}{V_1 \cdot C}$, a bigraded chain complex over $\mathcal{R}$, which we abbreviate $\frac{C}{V_1}$.

Proposition A.3.5. *Let C and C' be two bigraded chain complexes over $\mathcal{R} = \mathbb{K}[V_1, \dots, V_n]$ with $n \geq 1$, and suppose that C and C' are both free modules that are bounded above. A bigraded chain map $f\colon C \to C'$ is a quasi-isomorphism if and only if it induces a quasi-isomorphism $\overline{f}\colon \frac{C}{V_1} \to \frac{C'}{V_1}$ over $\mathcal{R}$.*

Before proving this result, we establish the following:

Lemma A.3.6. *Let C be a free, bigraded chain complex over $\mathcal{R} = \mathbb{K}[V_1, \dots, V_n]$ with $n \geq 1$; and suppose that C is bounded above. Then $H(C) \neq 0$ iff $H(\frac{C}{V_1}) \neq 0$.*

Proof. Since C is free, there is a short exact sequence

$$0 \longrightarrow C \xrightarrow{V_1} C \longrightarrow \frac{C}{V_1} \longrightarrow 0.$$

From the associated long exact sequence, it follows that $H(C) = 0$ implies that $H(\frac{C}{V_1}) = 0$. In the other direction, we use the fact that $H(C)$ is bounded above; so if $H(C) \neq 0$, there must be some homogeneous, non-zero element ξ with maximal Maslov grading. This element cannot be in the image of V_1, and hence it must inject into $H(\frac{C}{V_1})$. □

Proof of Proposition A.3.5. Observe that

$$\frac{\mathrm{Cone}(f\colon C \to C')}{V_1} \cong \mathrm{Cone}(\overline{f}\colon \frac{C}{V_1} \to \frac{C'}{V_1}).$$

The map f is a quasi-isomorphism $\iff$ $H(\mathrm{Cone}(f)) = 0$ (by Corollary A.3.3) $\iff$ $H(\frac{\mathrm{Cone}(f)}{V_1}) = 0$ (by Lemma A.3.6) $\iff$ $H(\mathrm{Cone}\overline{f}\colon \frac{C}{V_1} \to \frac{C'}{V_1}) = 0$ (by the above isomorphism) $\iff$ $\overline{f}$ is a quasi-isomorphism (by Corollary A.3.3). $\square$

The long exact sequence of a mapping cone is natural in the following sense:

Lemma A.3.7. *If two bigraded chain maps $f, g\colon C \to C'$ between bigraded complexes of $\mathcal{R}$-modules are chain homotopic, then their mapping cones are isomorphic.*

Proof. If $\partial' \circ h + h \circ \partial = f - g$, define a map $\Phi_h\colon \mathrm{Cone}(f) \to \mathrm{Cone}(g)$ by $\Phi_h(x, x') = (x, h(x) + x')$, and define $\Phi_{-h}\colon \mathrm{Cone}(g) \to \mathrm{Cone}(f)$ analogously, by $\Phi_{-h}(y, y') = (y, -h(y) + y')$. It is straightforward to check that Φ_h and Φ_{-h} are chain maps, with $\Phi_{-h} \circ \Phi_h = \mathrm{Id}_{\mathrm{Cone}(f)}$ and $\Phi_h \circ \Phi_{-h} = \mathrm{Id}_{\mathrm{Cone}(g)}$. $\square$

More generally, maps between mapping cones can be induced as follows:

Lemma A.3.8. *Let C, C', E, E' be four bigraded chain complexes, and suppose that there are chain maps fitting into the square*

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ E & \xrightarrow{g} & E', \end{array}$$

that commutes up to homotopy; i.e. the map $\phi' \circ f$ is chain homotopic to $g \circ \phi$. Suppose moreover that ϕ and ϕ' are bigraded maps, f and g are homogeneous of bidegree (m, t), and the homotopies are compatible with these gradings. Then, there is an induced bigraded chain map $\Phi\colon \mathrm{Cone}(f) \to \mathrm{Cone}(g)$ that fits into the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C' & \xrightarrow{i} & \mathrm{Cone}(f) & \xrightarrow{q} & C & \longrightarrow & 0 \\ & & {\scriptstyle\phi'}\downarrow & & {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\phi} & & \\ 0 & \longrightarrow & E' & \xrightarrow{j} & \mathrm{Cone}(g) & \xrightarrow{p} & E & \longrightarrow & 0. \end{array} \tag{A.7}$$

If ϕ and ϕ' are quasi-isomorphisms, then so is Φ.

Proof. By hypothesis, there is a map $h\colon C \to E'$ with

$$\partial_{E'} \circ h + h \circ \partial_C = \phi' \circ f - g \circ \phi. \tag{A.8}$$

We can now define a bigraded map $\Phi(c, c') = (\phi(c), h(c) + \psi'(c'))$. The verification that this is a chain map easily follows from Equation (A.8). Commutativity of Equation (A.7) is straightforward.

Suppose that ϕ and ϕ' are quasi-isomorphisms and that $(m,t) = (0,0)$. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{d,s}(C) \arrow[r, "\delta"] \arrow[d, "H(\phi)"'] & H_{d,s}(C') \arrow[r] \arrow[d, "H(\phi')"'] & H_{d,s}(\mathrm{Cone}(f)) \arrow[r] \arrow[d, "H(\Phi)"'] & H_{d-1,s}(C) \arrow[r] \arrow[d, "H(\phi)"'] & \cdots \\
\cdots \arrow[r] & H_{d,s}(E) \arrow[r, "\delta'"] & H_{d,s}(E') \arrow[r] & H_{d,s}(\mathrm{Cone}(g)) \arrow[r] & H_{d-1,s}(E) \arrow[r] & \cdots
\end{tikzcd}$$

whose rows are the long exact sequences from Lemma A.2.1. The squares involving δ commute by Lemma A.2.2; the other squares obviously commute; so $H(\Phi)$ is an isomorphism by the five lemma (Lemma A.2.3). The case of general (m,t) follows with minor notational changes. $\square$

The mapping cone of $f\colon C \to C'$ can be thought of as a type of quotient of C' by C, according to the following:

LEMMA A.3.9. *If $f\colon C \to C'$ is an injective chain map, then there is a quasi-isomorphism $\phi\colon \mathrm{Cone}(f) \to \frac{C'}{f(C)}$. This is compatible with gradings: when C and C' are bigraded chain complexes over $\mathcal{R}$ and f is a bigraded chain map, then ϕ is a quasi-isomorphism of bigraded chain complexes over $\mathcal{R}$.*

Proof. For notational convenience, we write out the case where C and C' are $\mathbb{Z}$-graded, and f is a $\mathbb{Z}$-graded chain map. Consider $C'' = \frac{C'}{f(C)}$ and form the short exact sequence

$$0 \longrightarrow C \xrightarrow{f} C' \xrightarrow{q} C'' \longrightarrow 0, \tag{A.9}$$

with $q\colon C' \to C'' = \frac{C'}{f(C)}$ being the projection. Define a chain map $\phi\colon \mathrm{Cone}(f) \to C''$ by the formula $\phi(c,c') = q(c')$. To verify that it is a quasi-isomorphism, we fit together the long exact sequences associated to the short exact sequences in Equations (A.6) and in (A.9), as in the following diagram:
(A.10)

$$\begin{tikzcd}
\arrow[r] & H_{d+1}(C') \arrow[r, "H(i)"] \arrow[d, "\mathrm{Id}"'] & H_{d+1}(\mathrm{Cone}(f)) \arrow[r, "H(p)"] \arrow[d, "H(\phi)"'] & H_d(C) \arrow[r, "(-1)^d\delta"] \arrow[d, "\mathrm{Id}"'] & H_d(C') \arrow[r] \arrow[d, "\mathrm{Id}"'] & {} \\
\arrow[r] & H_{d+1}(C') \arrow[r, "H(q)"] & H_{d+1}(C'') \arrow[r, "(-1)^{d+1}\delta'"] & H_d(C) \arrow[r, "H(f)"] & H_d(C') \arrow[r] & {}
\end{tikzcd}$$

(Note that we multiplied the coboundary map with a $(-1)^d$ to make the rightmost square commute; this does not affect exactness.) Here, $\delta'\colon H_{d+1}(C'') \to H_d(C)$ is the connecting homomorphism for the short exact sequence from Equation (A.9). Once we verify that the squares appearing above commute, the verification that ϕ induces an isomorphism in homology follows from the five lemma (Lemma A.2.3). Commutativity of the left-most square is straightforward: indeed, even on the chain level, it is true that $\phi \circ i = p$. Commutativity of the rightmost square was verified in the proofs of Lemma A.3.2.

Commutativity of the middle square follows once again from a careful look at the definition of the connecting homomorphism. A cycle in $\mathrm{Cone}_{d+1}(f)$ is a pair (c, c') with $\partial c = 0$ and $\partial c' = -f(c)$. Now $\phi(c, c') = q(c')$ and $\delta'[q(c')]$ is defined by finding some cycle c_1 representing $[q(c')]$, and next finding $c_2 \in C_{d+1}$ and $c_3 \in C_d$ so that

$$q(c_2) = c_1 \qquad \text{and } f(c_3) = \partial c_2.$$

Then, c_3 represents $[\delta' q(c')]$. Let $c_1 = q(c')$, $c_2 = c'$, and $c_3 = -c$. With these choices it immediately follows that $(-1)^{d+1}\delta'[q(c')] = (-1)^d c = p(c, c')$, as needed. The bigraded case follows similarly. ☐

The next lemma establishes an exact triangle that contains the homologies of the mapping cones of two maps and their composite. (We will use this result only in the graded setting, so we formulate it in this generality.)

Lemma A.3.10. *Suppose that C, C', and C'' are three $\mathbb{Z}$-graded chain complexes, and $f\colon C' \to C''$ and $g\colon C \to C'$ are chain maps that are homogeneous of degrees a and b respectively. Then, there is a chain map $\Phi\colon \mathrm{Cone}(f) \to \mathrm{Cone}(g)$ which is homogeneous of degree $-a-1$ and whose induced map on homology fits into an exact triangle*

$$\begin{tikzcd}
H(\mathrm{Cone}(f)) \arrow[rr, "-a-1"] & & H(\mathrm{Cone}(g)) \arrow[dl, "a"] \\
& H(\mathrm{Cone}(f \circ g)) \arrow[ul] &
\end{tikzcd} \tag{A.11}$$

where the integers indicate shifts on degree.

Proof. Define $\Phi\colon \mathrm{Cone}(f)_d \to \mathrm{Cone}(g)_{d-a-1}$ by $\Phi(c', c'') = (-1)^d(0, c') \in \mathrm{Cone}(g)_{d-a-1}$.

Obviously, Φ is a chain map which is homogeneous of degree $-a-1$. According to Lemma A.3.2, we have now an exact triangle:

$$\begin{tikzcd}
H(\mathrm{Cone}(f)) \arrow[rr, "-a-1"] & & H(\mathrm{Cone}(g)) \arrow[dl] \\
& H(\mathrm{Cone}(\Phi)) \arrow[ul, "a"] &
\end{tikzcd} \tag{A.12}$$

We will denote elements of $\mathrm{Cone}(\Phi)$ by quadruples $((c_1', c''), (c, c_2'))$, where $(c_1', c'') \in \mathrm{Cone}(f)$ and $(c, c_2') \in \mathrm{Cone}(g)$. Thus, the differential is given by

$$D((c_1', c''), (c, c_2')) = ((\partial' c_1', -f(c_1') - \partial''(c'')), (-\partial c, (-1)^d c_1' + g(c) + \partial' c_2')).$$

Consider the map $\alpha\colon \mathrm{Cone}(f \circ g) \to \mathrm{Cone}(\Phi)$ defined by the formula

$$\alpha(c, c'') = (((-1)^{d-a+1} g(c), (-1)^{d-a} c''), (c, 0)),$$

when c'' is homogeneous of degree d; and the map $\beta\colon \mathrm{Cone}(\Phi) \to \mathrm{Cone}(f \circ g)$ defined by

$$\beta((c_1', c''), (c, c_2')) = (c, (-1)^d c'' - f(c_2'))$$

when c_2' is homogeneous of degree d. Define $h\colon \mathrm{Cone}(\Phi) \to \mathrm{Cone}(\Phi)$ by

$$h((c_1', c''), (c, c_2')) = (((-1)^{d+1} c_2', 0), (0, 0)),$$

when c_2' is homogeneous of degree d. The map α is homogeneous of degree $-a$, β is homogeneous of degree a, and h is homogeneous of degree $a+1$. It is straightforward to verify the identities:

$$0 = \mathrm{Id}_{\mathrm{Cone}(f\circ g)} - \beta \circ \alpha,$$
$$D \circ h + h \circ D = \mathrm{Id}_{\mathrm{Cone}(\Phi)} - \alpha \circ \beta.$$

Thus, we have a homotopy equivalence

$$\mathrm{Cone}(\Phi)[\![a]\!] \simeq \mathrm{Cone}(f \circ g).$$

The exact triangle of Diagram (A.11) is obtained from Diagram (A.12) by applying the above substitution on homology and the appropriate grading shifts. $\square$

We can use mapping cones to show that the notion of "quasi-isomorphic chain complexes" (in the sense of Definition A.1.5) is an equivalence relation.

Proposition A.3.11. *If A and B are quasi-isomorphic chain complexes and B and C are quasi-isomorphic chain complexes, then A and C are quasi-isomorphic.*

Proof. By hypothesis, there are chain complexes A' and B', and quasi-isomorphisms $f\colon A' \to A$, $f'\colon A' \to B$, $g\colon B' \to B$, and $g'\colon B' \to C$. Consider the map $F\colon A' \oplus B' \to B$ defined by $F(a', b') = f'(a') - g(b')$. It remains to check that the map $h\colon \mathrm{Cone}(F) \to A[\![-1]\!]$ defined by $h((a', b'), b) = (-1)^d \cdot f(a')$, where d denotes the degree of a' and b', is a quasi-isomorphism; as is the map $h'\colon \mathrm{Cone}(F) \to C[\![-1]\!]$ defined by $h((a', b'), b) = (-1)^d \cdot g'(b')$. To see this, consider the mapping cone of h. There is a natural short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{Cone}(g) \to \mathrm{Cone}(h) \to \mathrm{Cone}(f) \longrightarrow 0,$$

where, by Corollary A.3.3 we have $H(\mathrm{Cone}(g)) = H(\mathrm{Cone}(f)) = 0$. By Lemma A.3.2 this implies $H(\mathrm{Cone}(h)) = 0$ so, again by Corollary A.3.3, h is a quasi-isomorphism. The other map h' is a quasi-isomorphism by a similar argument. $\square$

A.4. On the structure of homology

Proposition A.4.1. *If C is a finitely generated chain complex over the ring $\mathcal{R} = \mathbb{K}[V_1, \dots, V_n]$, then its homology $H(C)$ is also a finitely generated $\mathcal{R}$-module.*

Proof. It is a basic result in commutative algebra (Hilbert's Basis theorem; see for example [**3**, Theorem 7.5]) that $\mathcal{R} = \mathbb{K}[V_1, \dots, V_n]$ is a *Noetherian ring*. This means that every ideal is finitely generated. It follows that every submodule of a finitely generated module is also finitely generated (see for instance [**3**, Theorem 6.4]). In particular, the submodule of cycles in C is finitely generated. Since the quotient of a finitely generated module is finitely generated, $H(C)$ is finitely generated, too. $\square$

Proposition A.4.1 applies to show that grid homology with coefficients in $\mathcal{R} = \mathbb{K}[V_1, \dots, V_n]$ is finitely generated. For the rest of the present section it is crucial to work with the special case where $\mathbb{K}$ is a field (either $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$), and where $\mathcal{R} = \mathbb{K}[U]$ is a polynomial algebra over the field $\mathbb{K}$ in a single variable.

We now state a structure theorem for bigraded modules which is relevant to the structure of grid homology. This result is a variant of the classification theorem for modules over a principal ideal domain, paying special attention to gradings. The statement will use the following notational shorthand. Let $\mathbb{K}[U]/U^n_{(d,s)}$ denote the bigraded cyclic $\mathbb{K}[U]$-module whose generator g has bigrading (d,s). (Note that the non-zero, homogeneous elements of this module have bigradings $(d-2i, s-i)$ for $i = 0, \dots, n-1$.) Furthermore, let $\mathbb{K}[U]_{(d,s)}$ denote the bigraded free module of rank one, whose generator has Maslov grading d and Alexander grading s. The proof will use the following notion:

Definition A.4.2. Let X be a bigraded $\mathbb{K}[U]$-module. An element $\xi \in X$ is called ***U-torsion***, or simply ***torsion***, if for all sufficiently large integers n, $U^n \cdot \xi = 0$. Let $\mathrm{Tors}(X) \subset X$ denote the subset of torsion elements.

Proposition A.4.3. *Suppose that $\mathbb{K}$ is a field, and X is a finitely generated, bigraded $\mathbb{K}[U]$-module in the sense of Definition* A.1.3. *Then, there are collections of triples of integers $\{(d_i, s_i, n_i)\}_{i=1}^k$ and pairs of integers $\{(\delta_j, \sigma_j)\}_{j=1}^N$ so that*

$$X \cong \left(\bigoplus_{i=1}^{k} \mathbb{K}[U]/U^{n_i}_{(d_i,s_i)} \right) \oplus \left(\bigoplus_{j=1}^{N} \mathbb{K}[U]_{(\delta_j,\sigma_j)} \right). \tag{A.13}$$

Proof. Observe first that X/UX is a finite dimensional vector space over $\mathbb{K}$, since X is finitely generated. We prove Equation (A.13) by induction on the dimension of the vector space X/UX.

If the module X is non-zero, then any homogeneous element $x \in X$ with maximal Alexander grading induces a non-zero element of X/UX. It follows that if $X/UX = 0$, then $X = 0$.

If X is non-zero, choose some homogeneous element x in X with maximal Alexander grading a. Consider the cyclic module $\mathbb{K}[U] \cdot x$ generated by x. This submodule is either isomorphic to $\mathbb{K}[U]$, or it is identified with $\mathbb{K}[U]/p(U)$, where $p(U)$ is the polynomial of minimal degree for which $p(U) \cdot x = 0$. In the latter case, we claim that $p(U) = U^i$ for some i: for otherwise we could write $p(U) = U^i + U^{i+1} q(U)$ with $q \neq 0$. Since $U^i x$ has Alexander grading $a-i$ while $U^{i+1}q(U)$ is a sum of terms of Alexander grading $< a-i$, it follows that $U^i \cdot x = 0$, contradicting minimality of the degree of $p(U)$.

Next, let $X' = X/(\mathbb{K}[U] \cdot x)$. The natural projection from X/U to X'/UX' is surjective, and its kernel is generated by x; so $\dim(X'/UX') = \dim(X/U) - 1$, and hence by induction Equation (A.13) applies to X'. The generators of the cyclic summands in X' can be lifted to homogeneous elements $\{y_i\}_{i=1}^m$ of X. Depending on whether or not a generator in X' is torsion, for the lifts y_i we have either $U^{n_i} y_i \in \mathbb{K}[U] \cdot x$ (for some positive integer n_i) or $(\mathbb{K}[U] \cdot y_i) \cap (\mathbb{K}[U] \cdot x) = \{0\}$. In the first case, either $U^{n_i} y_i = 0$ or $U^{n_i} y_i = U^{k_i} x$. Let I be the set of those indices $i \in \{1, \dots, m\}$ for which $U^{n_i} y_i = U^{k_i} x \neq 0$. Observe that for all $i \in I$, $k_i \geq n_i$, following from the maximality of the Alexander grading of x. Next define elements $\{x_i\}_{i=1}^{m+1}$ in X by:

$$x_i = \begin{cases} y_i & \text{if } i \leq m \text{ and } i \notin I \\ y_i - U^{k_i - n_i} x & \text{if } i \in I \\ x & \text{if } i = m+1. \end{cases}$$

It is straightforward to verify that these elements generate X, and each generates a distinct cyclic summand of the stated form. $\square$

The isomorphism from Equation (A.13) is not canonical. A more canonical formulation can be given, in terms of the torsion modules from Definition A.4.2, as follows. Note that the set $\mathrm{Tors}(X) \subset X$ is a $\mathbb{K}[U]$-submodule, and forms the first direct summand in Equation (A.13).

COROLLARY A.4.4. *Let X be a finitely generated bigraded module over $\mathbb{K}[U]$, and $\mathrm{Tors}(X) \subset X$ its torsion submodule. Then, the quotient $X/\mathrm{Tors}(X)$ is isomorphic to $\mathbb{K}[U]^N$ for some N.* □

DEFINITION A.4.5. The number of free summands in a finitely generated module X (i.e. the quantity N appearing in Equation (A.13) and in Corollary A.4.4), is called the ***rank*** of the module.

A.5. Dual complexes

Let X be a module over $\mathcal{R} = \mathbb{K}[V_1, \dots V_n]$. We can consider the *dual module* $\mathrm{Hom}_{\mathcal{R}}(X, \mathcal{R})$ of $\mathcal{R}$-module morphisms. This notion can be extended to chain complexes as follows. Let (C, ∂) be a chain complex over the ring $\mathcal{R}$, then its dual module $\mathrm{Hom}_{\mathcal{R}}(C, \mathcal{R})$ inherits a boundary map $\check{\partial}\colon \mathrm{Hom}_{\mathcal{R}}(C, \mathcal{R}) \to \mathrm{Hom}_{\mathcal{R}}(C, \mathcal{R})$, defined as follows. Given $\phi\colon C \to \mathcal{R}$, the homomorphism $\check{\partial}(\phi)$ is the homomorphism whose value on $c \in C$ is the value of ϕ on ∂c; i.e. $\check{\partial}(\phi)(c) = \phi(\partial c)$. The relation $\partial^2 = 0$ dualizes readily to give $\check{\partial}^2 = 0$. The chain complex $(\mathrm{Hom}_{\mathcal{R}}(C, \mathcal{R}), \check{\partial})$ is called the *dual complex* of (C, ∂).

EXAMPLE A.5.1. If (C, ∂) is a finitely generated, free $\mathcal{R}$-module, given with a basis $\{\mathbf{x}_i\}_{i=1}^N$, then the differential ∂ is specified by the matrix $A = (a_{i,j})$ so that

$$\partial \mathbf{x}_i = \sum_{i,j=1}^{N} a_{i,j} \mathbf{x}_j.$$

In this case, the dual complex is generated by $\{\mathbf{x}_i^*\}_{i=1}^N$, with $x_i^*(x_j) = \delta_{ij}$; and the differential $\check{\partial}$ is specified by the transpose of A:

$$\check{\partial} \mathbf{x}_i^* = \sum_{i,j=1}^{N} a_{j,i} \mathbf{x}_j^*.$$

Suppose now that (C, ∂) is a bigraded chain complex. The dual complex $\mathrm{Hom}_{\mathcal{R}}(C, \mathcal{R})$ inherits a bigrading from the complex C. We explain this in the case where $\mathcal{R} = \mathbb{K}[U]$, which we think of now as a bigraded $\mathcal{R}$-module, where the element U^m has bigrading $(-2m, -m)$. This induces a bigrading on the dual module: an element $\phi \in \mathrm{Hom}_{\mathcal{R}}(C, \mathcal{R})$ has bigrading (m, t) if ϕ maps $C_{d,s}$ to $\mathcal{R}_{d+m,s+t}$. With this convention, the dual of a bigraded chain complex C over $\mathbb{K}[U]$ is also a bigraded chain complex over $\mathbb{K}[U]$, and the induced differential on $\mathrm{Hom}_{\mathcal{R}}(C, \mathcal{R})$ drops grading by one, just as the differential for C did.

Note that the dual complex is equivalent to the usual notion of cohomology, with the understanding that the grading conventions differ by an overall multiplication by -1. We choose our grading conventions exactly so that the dual of a bigraded complex is also bigraded in the same sense.

The *Universal Coefficient Theorem* (for cohomology) provides the link between the homology of a chain complex and the homology of its dual. To state it, consider the pairing $\mathrm{Hom}_{\mathcal{R}}(C,\mathcal{R}) \otimes_{\mathcal{R}} C \to \mathcal{R}$ given by the formula $(f,c) \mapsto f(c)$. This descends to homology, giving the *Kronecker pairing* $H(\mathrm{Hom}_{\mathcal{R}}(C,\mathcal{R}),\check{\partial}) \otimes_{\mathcal{R}} H(C,\partial) \to \mathcal{R}$; and hence a duality map

$$H(\mathrm{Hom}_{\mathcal{R}}(C,\mathcal{R}),\check{\partial}) \to \mathrm{Hom}_{\mathcal{R}}(H(C,\partial),\mathcal{R}). \tag{A.14}$$

The Universal Coefficient Theorem can be conveniently formulated when $\mathcal{R}$ is either a field of a principal ideal domain. When $\mathcal{R}$ is a field (i.e, $n=0$ and $\mathbb{K}$ is $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$), the duality map is an isomorphism, according to the following.

THEOREM A.5.2. *If $\mathcal{R}$ is a field, then for a chain complex (C,∂),*

$$H(\mathrm{Hom}_{\mathcal{R}}(C,\partial),\check{\partial}) \cong \mathrm{Hom}_{\mathcal{R}}(H(C,\partial),\mathcal{R}). \qquad \square$$

If $\mathcal{R}$ is a PID, any module X over $\mathcal{R}$ can be fit into a short exact sequence (called a *free resolution*)

$$0 \longrightarrow F^2 \xrightarrow{\ r\ } F^1 \xrightarrow{\ p\ } X \longrightarrow 0, \tag{A.15}$$

where F^1 and F^2 are free $\mathcal{R}$-modules. Dualizing r gives a map $r^*\colon \mathrm{Hom}_{\mathcal{R}}(F^1,\mathcal{R}) \to \mathrm{Hom}_{\mathcal{R}}(F^2,\mathcal{R})$, whose kernel is identified with $\mathrm{Hom}_{\mathcal{R}}(X,\mathcal{R})$, and whose cokernel, denoted $\mathrm{Ext}(X,\mathcal{R})$, turns out to be independent of the choice of the resolution. (This module is also denoted $\mathrm{Ext}^1(X;\mathcal{R})$ in some texts.)

EXAMPLE A.5.3. The following are the only Ext groups which will be of relevance in this text:

(1) If X is a free module, then we can take $F^1 = X$ and $F^2 = 0$. Thus, $\mathrm{Ext}(X,\mathcal{R}) = 0$.
(2) If $\mathcal{R}$ is a field, then any $\mathcal{R}$-module is free, so $\mathrm{Ext}(X,\mathcal{R}) = 0$.
(3) $\mathrm{Ext}(X \oplus Y,\mathcal{R}) = \mathrm{Ext}(X,\mathcal{R}) \oplus \mathrm{Ext}(Y,\mathcal{R})$.
(4) Let $\mathcal{R} = \mathbb{K}[U]$ and $X = \mathbb{K}[U]/U^n$ for some n. Then, we can take $F^1 = F^2 = \mathbb{K}[U]$, and r to be the map which is multiplication by U^n. In this case, $\mathrm{Ext}(\mathbb{K}[U]/U^n,\mathbb{K}[U]) \cong \mathbb{K}[U]/U^n$.

In fact, we will use a refinement of the Universal Coefficient Theorem, in the case where C is a bigraded chain complex over $\mathbb{K}[U]$, in the sense of Definition A.1.3.

As a first step, observe that if X is a bigraded module over $\mathbb{K}[U]$, then the Ext modules also inherit the structure of a bigraded module over $\mathbb{K}[U]$. To define the bigrading, consider first a free resolution (Equation (A.15)) so that F^1 ane F^2 are bigraded, and the maps p and r preserve bigradings. Such a resolution can be formed by the following tautological construction: F^1 is the free module over $\mathbb{F}[U]$ whose generating set is the set of homogeneous, non-zero elements of X. This comes equipped with a canonical bigraded map $p\colon F^1 \to X$. Its kernel is a free module since $\mathbb{K}[U]$ is a principal ideal domain (hence any submodule of a free module is free), and it is bigraded, since p is bigraded.

Armed with these bigradings, the cokernel of

$$r^*\colon \mathrm{Hom}_{\mathcal{R}}(F^1,\mathcal{R}) \to \mathrm{Hom}_{\mathcal{R}}(F^2,\mathcal{R}),$$

which is $\mathrm{Ext}(X,\mathcal{R})$, naturally inherits a bigrading.

EXAMPLE A.5.4. Consider the bigraded $\mathbb{K}[U]$-module $X = \mathbb{K}[U]/U^n_{(d,s)}$. Then, there is an isomorphism of bigraded modules over $\mathbb{K}[U]$

$$\mathrm{Ext}(X,\mathbb{K}[U]) \cong \mathbb{K}[U]/U^n_{(2n-d,n-s)}.$$

EXAMPLE A.5.5. Consider the bigraded $\mathbb{K}[U]$-module $X = \mathbb{K}[U]_{(d,s)}$, whose generator has bigrading (d,s). Then, there is an isomorphism of bigraded $\mathbb{K}[U]$-modules

$$\mathrm{Hom}_{\mathcal{R}}(X, \mathbb{K}[U]) = \mathrm{Hom}(X, \mathbb{K}[U]) \cong \mathbb{K}[U]_{(-d,-s)}.$$

THEOREM A.5.6. *If C is a bigraded complex of free modules over $\mathbb{K}[U]$, then there is an isomorphism of bigraded modules*

$$H(\mathrm{Hom}(C, \mathbb{K}[U])) \cong \mathrm{Hom}(H(C), \mathbb{K}[U]) \oplus \mathrm{Ext}(H(C), \mathbb{K}[U])[\![1,0]\!]. \tag{A.16}$$

More explicitly, writing

$$H(C) \cong \left(\bigoplus_{i=1}^{k} \mathbb{K}[U]/U^{n_i}_{(d_i,s_i)} \right) \oplus \left(\bigoplus_{j=1}^{N} \mathbb{K}[U]_{(\delta_j,\sigma_j)} \right),$$

we have that

$$H(\mathrm{Hom}(C, \mathbb{K}[U])) \cong \left(\bigoplus_{i=1}^{k} \mathbb{K}[U]/U^{n_i}_{(2n_i-d_i-1,n_i-s_i)} \right) \oplus \left(\bigoplus_{j=1}^{N} \mathbb{K}[U]_{(-\delta_j,-\sigma_j)} \right). \tag{A.17}$$

Proof. The usual $\mathbb{Z}$-graded Universal Coefficient Theorem (see for example [**83**, Chapter 3.1]) adapts to prove Equation (A.16), as follows. Consider the subcomplex $Z \subset C$ of cycles and the subcomplex $B \subset C$ of boundaries. Since C is free, so are B and Z; so the short exact sequence

$$0 \longrightarrow B \xrightarrow{\ i\ } Z \longrightarrow H(C) \longrightarrow 0 \tag{A.18}$$

is a free resolution of $H(C)$. There is a short exact sequence of bigraded chain complexes

$$0 \longrightarrow Z \longrightarrow C \longrightarrow B[\![-1,0]\!] \longrightarrow 0. \tag{A.19}$$

Since B is free, the above short exact sequence dualizes to a short exact sequence of chain complexes

$$0 \longleftarrow \mathrm{Hom}(Z, \mathbb{K}[U]) \longleftarrow \mathrm{Hom}(C, \mathbb{K}[U]) \longleftarrow \mathrm{Hom}(B[\![-1,0]\!], \mathbb{K}[U]) \longleftarrow 0.$$

Here the boundary maps on $\mathrm{Hom}(Z, \mathbb{K}[U])$ and $\mathrm{Hom}(B[\![1,0]\!], \mathbb{K}[U])$ are identically zero. The long exact sequence in homology gives the short exact sequence

$$0 \leftarrow \mathrm{Hom}(H(C), \mathbb{K}[U]) \leftarrow H(\mathrm{Hom}(C, \mathbb{K}[U])) \leftarrow \mathrm{Ext}(H(C), \mathbb{K}[U][\![1,0]\!] \longleftarrow 0, \tag{A.20}$$

since dualizing Equation (A.18) we get that $\mathrm{Hom}(H(C), \mathbb{K}[U]) \cong \mathrm{Ker}(i^*)$, and

$$\frac{\mathrm{Hom}(B[\![-1,0]\!], \mathbb{K}[U])}{\mathrm{Hom}(Z[\![-1,0]\!], \mathbb{K}[U])} \cong \frac{\mathrm{Hom}(B, \mathbb{K}[U])}{\mathrm{Hom}(Z, \mathbb{K}[U])}[\![1,0]\!] \cong \mathrm{Ext}(H(C), \mathbb{K}[U])[\![1,0]\!],$$

in view of our grading conventions on the dual complex (which are opposite to the usual one on cohomology). As usual, a splitting of Equation (A.19) provides a chain map $C \to Z$ which, when composed with the quotient map to $H(C)$, dualizes to a map $\mathrm{Hom}(H(C), \mathbb{K}[U]) \to H(\mathrm{Hom}(C), \mathbb{K}[U])$ that provides a splitting of Equation (A.20).

The graded isomorphism of Equation (A.17) comes from our computation of the bigraded Ext for the relevant modules (see Examples A.5.3, A.5.4 and A.5.5). □

REMARK A.5.7. Note that if the ground ring is not a PID, much of the homological algebra discussed above is slightly more complicated: finitely generated modules might not decompose as direct sums of cyclic modules; and the universal coefficient theorem is much more involved [**226**]. This is already relevant for rings appearing in grid homology, such as $\mathbb{Z}[U]$ (as in Chapter 15) or $\mathbb{K}[U_1, \dots, U_n]$, $n \geq 2$ (as in Chapter 11).

A.6. On filtered complexes

In Chapter 13, grid diagrams are used to go beyond a bigraded homology group; they are used to define a quasi-isomorphism class of $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complexes, in the sense of Definition 13.1.1. We refer the reader back to Section 13.1 for the necessary definitions and algebraic constructions. The filtered complexes we consider in this section will be defined over $\mathbb{K}[V_1, \dots, V_n]$; $\mathbb{K}$ can be any base ring.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complexes over $\mathbb{K}[V_1, \dots, V_n]$, and let $\phi\colon \mathcal{C} \to \mathcal{C}'$ be a filtered chain map, in the sense of Definition 13.1.4. Such a map induces a bigraded chain map $\mathrm{gr}(\phi)$ on the associated graded chain complex; and ϕ is called a filtered quasi-isomorphism when $\mathrm{gr}(\phi)$ is a quasi-isomorphism.

PROPOSITION A.6.1. *Let $\phi\colon \mathcal{C} \to \mathcal{C}'$ be a filtered quasi-isomorphism. Then, ϕ induces isomorphisms $H(\mathcal{F}_i\mathcal{C}) \cong H(\mathcal{F}_i\mathcal{C}')$ and $H(\mathcal{C}) \cong H(\mathcal{C}')$.*

The proof will follow from the following special case:

LEMMA A.6.2. *If $\mathcal{C}$ is a filtered complex with $H(\mathrm{gr}(\mathcal{C})) = 0$, then $H(\mathcal{F}_i\mathcal{C}) = 0$ and $H(\mathcal{C}) = 0$.*

Proof. Fix an integer d. We show that $H_d(\mathcal{F}_i\mathcal{C}) = 0$ by increasing induction on i. The case where i is small is always true, because our filtered complexes are assumed to be bounded below; i.e. for sufficiently small i, $\mathcal{F}_i\mathcal{C}_d = 0$. For the inductive step, assume that $H_d(\mathcal{F}_i\mathcal{C}) = 0$, and note that $H_d(\frac{\mathcal{F}_{i+1}\mathcal{C}}{\mathcal{F}_i\mathcal{C}}) = 0$, by the hypothesis that $H(\mathrm{gr}(\mathcal{C})) = 0$. Thus, the vanishing of $H_d(\mathcal{F}_{i+1}\mathcal{C})$ follows from the long exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{F}_i\mathcal{C} \longrightarrow \mathcal{F}_{i+1}\mathcal{C} \longrightarrow \frac{\mathcal{F}_{i+1}\mathcal{C}}{\mathcal{F}_i\mathcal{C}} \longrightarrow 0. \qquad \square$$

Proof of Proposition A.6.1. The *filtered mapping cone* of ϕ, $\mathcal{C}one(\phi)$, is the mapping cone of ϕ endowed with the filtration where

$$\mathcal{F}_i\mathcal{C}one(\phi) \cong \mathrm{Cone}(\phi|_{\mathcal{F}_i\mathcal{C}}). \tag{A.21}$$

Observe that there is an isomorphism of chain complexes $\mathrm{gr}(\mathcal{C}one(\phi)) \cong \mathrm{Cone}(\mathrm{gr}(\phi))$. By Corollary A.3.3, $H(\mathrm{Cone}(\mathrm{gr}(\phi))) = 0$, and hence by Lemma A.6.2,

$$H(\mathcal{F}_i\mathcal{C}one(\phi)) = 0 = H(\mathcal{C}one(\phi)).$$

In view of Equation (A.21), Corollary A.3.3 ensures that ϕ induces isomorphisms $H(\mathcal{F}_i\mathcal{C}) \cong H(\mathcal{F}_i\mathcal{C}')$. Similarly, $H(\mathcal{C}one(\phi)) = 0$ implies that $H(\mathcal{C}) \cong H(\mathcal{C}')$. □

Remark A.6.3. Recall that our filtered complexes are required to be bounded below, by definition. Without this hypothesis, the conclusion of Proposition A.6.1 does not hold. For example, consider the filtered complex $\mathcal{C}$ generated over $\mathbb{K}$ by two sequences of elements $\{x_i, y_i\}_{i\leq 0}$, with $x_i \in \mathcal{F}_i\mathcal{C}_1 \setminus \mathcal{F}_{i-1}\mathcal{C}_1$, $y_i \in \mathcal{F}_i\mathcal{C}_0 \setminus \mathcal{F}_{i-1}\mathcal{C}_{-1}$, and with a differential $\partial x_i = y_i + y_{i-1}$. Clearly, $H(\mathrm{gr}(\mathcal{C})) = 0$, but $H(\mathcal{C}) \cong \mathbb{K}$.

As a corollary to the above discussion, there are numerical invariants associated to the filtered quasi-isomorphism types of filtered complexes. Suppose that $\mathcal{C}$ is a $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{F}$, and suppose that $H(\mathcal{C})$ is finite dimensional. Then, we claim that for sufficiently small i, the image of $H(\mathcal{C}_i)$ in $H(\mathcal{C})$ is trivial: $H(\mathcal{C})$ is finite dimensional, so since the filtration is bounded below, we can choose i small enough that for each d for which $H_d(\mathcal{C}) \neq 0$, $\mathcal{F}_i\mathcal{C}_d = 0$. Similarly, for i sufficiently large, $H(\mathcal{C}_i) \to H(\mathcal{C})$ is non-trivial; if $z \in \mathcal{C}$ is a non-trivial cycle representing some homology class, then since the filtration exhausts $\mathcal{C}$, we know that $z \in \mathcal{F}_i\mathcal{C}$ for sufficiently large i; i.e. $[z] \in H(\mathcal{C})$ is the image of a homology class in $H(\mathcal{F}_i\mathcal{C})$. Thus, we can make the following;

Definition A.6.4. Let $t(\mathcal{C})$ be the minimal i so that $H(\mathcal{F}_i\mathcal{C}) \to H(\mathcal{C})$ is non-zero.

Corollary A.6.5. *Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are quasi-isomorphic complexes, and that $H(\mathcal{C}) \cong H(\mathcal{C}') \neq 0$. Then, $t(\mathcal{C}) = t(\mathcal{C}')$.*

Proof. It suffices to show that if $\phi\colon \mathcal{C} \to \mathcal{C}'$ is a filtered quasi-isomorphism, then $t(\mathcal{C}) = t(\mathcal{C}')$. To this end, consider the commutative square

$$\begin{tikzcd} H(\mathcal{F}_s\mathcal{C}) \arrow[r, "H(i_s)"] \arrow[d, "H(\phi|_{\mathcal{F}_s\mathcal{C}})"'] & H(\mathcal{C}) \arrow[d, "H(\phi)"] \\ H(\mathcal{F}_s\mathcal{C}') \arrow[r, "H(i'_s)"] & H(\mathcal{C}'), \end{tikzcd}$$

where the horizontal maps are induced by inclusions. Both vertical maps are isomorphisms by Proposition A.6.1, so $H(i_s)$ is non-trivial exactly when $H(i'_s)$ is. $\square$

Exercise A.6.6. Let $\mathcal{C}$ be a $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{F}[U]$, and suppose that $H(\mathcal{C}) \cong \mathbb{F}[U]$. For any integer $i \geq 0$, let

$$\tau_i(\mathcal{C}) = \min\{s \big| \text{ the image of } H(\mathcal{F}_s\mathcal{GC}^-) \text{ in } H(\mathcal{GC}^-) \cong \mathbb{F}[U] \text{ contains } U^i\}.$$

Show that if $\mathcal{C}'$ is a $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{F}[U]$ that is filtered quasi-isomorphic to $\mathcal{C}$, then for all i, $\tau_i(\mathcal{C}) = \tau_i(\mathcal{C}')$.

A.7. Small models for filtered grid complexes

There are infinitely many different chain complexes in a fixed quasi-isomorphism class of a $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex; and the chain complexes coming from the filtered grid complex tend to have many generators. Thus, it is useful to get an economical complex representing a fixed filtered quasi-isomorphism class. Such model complexes were used in Section 14.2 to describe examples.

In Section 14.2 we represented $\mathcal{GC}^-(\vec{K})$ (the filtered quasi-isomorphism class of the filtered chain complexes $\mathcal{GC}^-(\mathbb{G})$ for grid diagrams representing $\vec{K}$) by a free chain complex over $\mathbb{F}[U]$ whose rank coincides with the dimension of the $\mathbb{F}$-vector space $\widehat{GH}(\vec{K})$. The existence of such a representative was stated in Lemma 14.2.1. Our aim here is to prove that lemma, after restating the result in more abstract terms. For this purpose, throughout this section we fix a field $\mathbb{K}$. We start with a special case.

LEMMA A.7.1. *Fix a field $\mathbb{K}$, and let $\mathcal{C}$ be a $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{K}[U]$ that is a finitely generated, free module over $\mathbb{K}[U]$. Then, $\mathcal{C}$ is filtered homotopy equivalent to a free, finitely generated chain complex $\mathcal{C}'$ over $\mathbb{K}[U]$ whose induced differential on $\mathrm{gr}(\frac{\mathcal{C}'}{U})$ vanishes; i.e. $\dim_{\mathbb{K}} H(\mathrm{gr}(\frac{\mathcal{C}'}{U})) = \dim_{\mathbb{K}} \mathrm{gr}(\frac{\mathcal{C}'}{U})$.*

Proof. Suppose that the differential on $\mathrm{gr}(\frac{\mathcal{C}}{U})$ does not vanish. This means that there is a basis $\{[x_i]_{i=1}^m\}$ for $\mathrm{gr}(\frac{\mathcal{C}}{U})$ so that $\partial[x_1] = [x_2]$ with $[x_2] \neq 0$. Find lifts x_i of the $[x_i]$; i.e. if $[x_i] \in \mathcal{F}_{a_i}/\mathcal{F}_{a_i-1}$, choose $x_i \in \mathcal{F}_{a_i}$ whose projection is $[x_i]$. Clearly, $\{x_i\}_{i=1}^m$ generate $\mathcal{C}$ as a $\mathbb{K}[U]$-module. Thus, there is a $\mathbb{K}[U]$-module map $T\colon \mathcal{C} \to \mathcal{C}$ defined by

$$T(x_i) = \begin{cases} x_1 & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The map T is a filtered map; and so $\phi = \mathrm{Id} - \partial T - T\partial$ is a filtered chain map. Clearly $T^2 = 0$, and so it follows that $\phi \circ T = T \circ \phi$; hence the subcomplex $\phi(\mathcal{C})$ of $\mathcal{C}$ is chain homotopy equivalent to $\mathcal{C}$. Note that $\phi(x_1) = 0$, and so the dimension of $\mathrm{gr}(\phi(\frac{\mathcal{C}}{U}))$ is smaller than the the dimension of $\mathrm{gr}(\frac{\mathcal{C}}{U})$. This procedure must terminate at some point, giving a chain homotopy equivalent subcomplex $\mathcal{C}'$ whose rank agrees with $H(\mathrm{gr}(\frac{\mathcal{C}'}{U}))$. □

PROPOSITION A.7.2. *Fix a field $\mathbb{K}$ and let $\mathcal{C}$ be a $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex over $\mathbb{K}[U]$, that is a free module over $\mathbb{K}[U]$, and suppose that $H(\mathrm{gr}(\frac{\mathcal{C}}{U}))$ is a finite dimensional $\mathbb{K}$-vector space. Then, $\mathcal{C}$ is filtered homotopy equivalent to a chain complex $\mathcal{C}'$ that is a finitely generated, free module over $\mathbb{K}[U]$ whose rank coincides with $\dim_{\mathbb{K}} H(\mathrm{gr}(\frac{\mathcal{C}}{U}))$.*

Proof. If $\mathcal{C}$ is infinitely generated, choose a generating set $\{[x_i]_{i=1}^{\infty}\}$ of $\mathrm{gr}(\frac{\mathcal{C}}{U})$, and fix some integer $d \geq 0$. By boundedness, there are only finitely many x_i with grading $\geq d$. Using the method from Lemma A.7.1, we can find a sequence of filtered subcomplexes $\mathcal{C}^{(i)}$ of $\mathcal{C}$, with the following properties:

- $\mathcal{C}^{(i)} \subseteq \mathcal{C}^{(i+1)}$
- $\mathcal{C}^{(i)} = \mathcal{C}$ for sufficiently large i,
- $\mathrm{gr}(\frac{\mathcal{C}^{(i)}}{U})$ has vanishing differentials in all gradings $\geq i$.
- There are maps $T^{(i)}\colon \mathcal{C}^{(i)} \to \mathcal{C}^{(i)}$, with $T^{(i)} \circ \phi^{(i)} = \phi^{(i)} \circ T^{(i)}$, so that $\mathcal{C}^{(i-1)}$ is the image of $\phi^{(i)}\colon \mathcal{C}^{(i)} \to \mathcal{C}^{(i)}$ by the map $\phi^{(i)} = \mathrm{Id} - \partial \circ T^{(i)} - T^{(i)} \circ \partial$.
- For any fixed d, $i \leq d-1$, $T^{(i)}$ vanishes on $\mathcal{C}^{(i)}_d$.

It follows from the above properties that the inclusion of $\mathcal{C}^{(i-1)}$ in $\mathcal{C}^{(i)}$ is a filtered chain homotopy equivalence. For each d, $T^{(i)}$ vanishes on $\mathcal{C}^{(i)}_d$ for all but finitely many i; and so, $\phi^{(i)}|_{\mathcal{C}^{(i)}}$ is the identity map for all but finitely many i. Thus, for a fixed d, there is an i_d so that $\mathcal{C}^{(i_d)}_d = \mathcal{C}^{(i_d-1)}_d = \dots$, so can form the infinite composite $\phi = \dots \circ \phi^{(i)} \circ \phi^{(i+1)} \circ \dots$. The image of ϕ is our desired subcomplex. □

A.8. Filtered quasi-isomorphism versus filtered homotopy type

In Section 13.1, we discussed two equivalence relations on filtered chain complexes: filtered quasi-isomorphism, and filtered chain homotopy equivalence. Two filtered homotopy equivalent complexes are necessarily filtered quasi-isomorphic (Exercise 13.1.10), but filtered quasi-isomorphic complexes need not be filtered chain homotopy equivalent (Example 13.1.8). Since our grid complexes are free modules over $\mathbb{K}[V_1, \dots, V_n]$, this distinction disappears, according to the present results.

Throughout this section, we fix once again $\mathbb{K}$ to be a field. We show the following:

Proposition A.8.1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complexes over $\mathbb{K}[U]$. Then, these complexes are filtered chain homotopy equivalent over $\mathbb{K}[U]$ if and only if they are filtered quasi-isomorphic over $\mathbb{K}[U]$.*

We return to the proof after reducing to a special case (where $\mathcal{C}' = 0$).

Definition A.8.2. A $\mathbb{Z}$-graded, $\mathbb{Z}$-filtered chain complex $\mathcal{C}$ over a ring $\mathcal{R}$ is ***chain contractible*** if there is an $\mathcal{R}$-module homomorphism $H\colon \mathcal{C} \to \mathcal{C}$, the ***contraction map***, that raises grading by 1 and satisfies the equation $\partial \circ H + h \circ \partial = \mathrm{Id}_{\mathcal{C}}$, so that $\mathcal{C}$ is filtered chain homotopy equivalent to the trivial chain complex 0.

Lemma A.8.3. *Let $f\colon \mathcal{C} \to \mathcal{C}'$ be a $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain map between two $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complexes over $\mathbb{K}[V_1, \dots, V_n]$. Then, $\mathcal{C}one(f)$ is contractible if and only if f is a filtered chain homotopy equivalence.*

Proof. Recall that $\mathcal{C}one(f)$ is the direct sum $\mathcal{C} \oplus \mathcal{C}'$, equipped with the differential

$$D = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix}.$$

A null-homotopy $H\colon \mathcal{C}one(f) \to \mathcal{C}one(f)$ of the identity map satisfies the equation

$$D \circ H + H \circ D - \mathrm{Id}_{\mathcal{C} \oplus \mathcal{C}'} = 0. \tag{A.22}$$

Writing H in its components, we get $H = \begin{pmatrix} h & g \\ i & h' \end{pmatrix}$, where $g\colon \mathcal{C}' \to \mathcal{C}$, $h\colon \mathcal{C} \to \mathcal{C}$, $h'\colon \mathcal{C}' \to \mathcal{C}'$. Equation (A.22) can be interpreted as the vanishing of a 2×2 matrix; i.e. it gives four relations; and three of these are:

$$\partial \circ g - g \circ \partial' = 0, \qquad -\partial \circ h - h \circ \partial = \mathrm{Id}_{\mathcal{C}} - g \circ f, \qquad \partial' \circ h' + h' \circ \partial' = \mathrm{Id}_{\mathcal{C}'} - f \circ g.$$

In words, g is a chain map, and f and g are homotopy inverses to one another. In particular, f is a homotopy equivalence.

Conversely, given f, g, h, and h' satisfying the above, it follows that if we let

$$H = \begin{pmatrix} h & g \\ 0 & h' \end{pmatrix},$$

then $\Phi = D \circ H + H \circ D$ is a lower triangular chain map, and hence invertible, and so $\Phi^{-1} \circ H$ is the desired contraction. □

Lemma A.8.4. *Let $\mathcal{C}$ be a $\mathbb{Z}$-filtered, $\mathbb{Z}$-graded chain complex that is free over $\mathbb{K}[U]$, where $\mathbb{K}$ is a field. If $H(\mathrm{gr}(\mathcal{C})) = 0$, then $\mathcal{C}$ is filtered homotopy equivalent over $\mathbb{K}[U]$ to the 0 complex.*

Proof. This is an immediate consequence of Proposition A.7.2: under our present hypotheses, the subcomplex is the 0 complex. □

Proof of Proposition A.8.1 One direction is easy: a filtered homotopy equivalence f induces a homotopy equivalence on the associated graded level, which is therefore a quasi-isomorphism, and so f is a filtered quasi-isomorphism.

For the converse direction, recall that for any filtered chain map

$$H(\mathrm{gr}(\mathcal{C}one(f))) \cong H(\mathrm{Cone}(\mathrm{gr}(f))),$$

so if f a quasi-isomorphism, then $H(\mathrm{Cone}(\mathrm{gr}(f)) = 0$ by the long exact sequence for the mapping cone of $\mathrm{gr}(f)$. Thus, by Lemma A.8.4, we conclude that $\mathcal{C}one(f)$ is chain contractible over $\mathbb{K}[V_i]$. The result now follows from Lemma A.8.3. □

APPENDIX B

Basic theorems in knot theory

This appendix covers some of the foundational material on knot theory used in this book. We start, in Section B.1, with a classical theorem of Reidemeister that characterizes, via local moves, those link diagrams that represent isotopic links. In Section B.2, we discuss the analogue of this theorem in the contact context, for transverse and Legendrian knots and links. In Section B.3 we prove the theorem of Reidemeister-Singer that relates different Seifert surfaces for the same link. Next, we prove Cromwell's theorem in Section B.4 that characterizes, via local moves, those grid diagrams that represent isotopic links. The Legendrian version of this theorem is also proved in this section. Finally, in Section B.5 we define the normal form of a cobordism between knots, and show that any cobordism can be isotoped to such a form.

B.1. The Reidemeister Theorem

The Reidemeister Theorem 2.1.4 allows us to study knots and links in $\mathbb{R}^3$ in terms of their diagrams. We restate this theorem as follows:

THEOREM B.1.1 (Reidemeister, [**196**]). *Two link diagrams represent equivalent links if and only if these diagrams can be transformed into each other by a finite sequence of Reidemeister moves (shown on Figure* 2.2*) and planar isotopies.*

One direction of the equivalence in the theorem is straightforward: Reidemeister moves and planar isotopies clearly preserve the isotopy type of the link.

There are a number of proofs of this fundamental result; below we will describe a proof using singularity theory from [**199**]. (For a proof of the piecewise linear version of the theorem, see [**18**].) Throughout the discussion, we will assume familiarity with standard transversality results in differential topology, as presented, for example in [**87, 151**]. We will concentrate on the case of knots; the general case of links can be proved along the same lines.

Suppose that two knots K_0 and K_1 are smoothly isotopic. An isotopy gives rise to a one-parameter family of knot diagrams. Keeping track of the isotopy parameter, this gives a map from the two-manifold $[0,1]\times S^1$ to the three-manifold $[0,1]\times\mathbb{R}^2$. A key local result we will use is the following theorem of Whitney:

THEOREM B.1.2 (Whitney, [**228**]). *Let W be a smooth two-manifold and Y a smooth three-manifold. Any smooth map $g_0\colon W\to Y$ can be approximated arbitrarily closely (in the C^2 topology) by a smooth map $g\colon W\to Y$ with the following property. Around each point $p\in W$, there are local coordinates (x,y) so that p corresponds to $(0,0)$, and there are local coordinates (u,v,z) around $g(p)\in Y$, so that* $(0,0,0)$

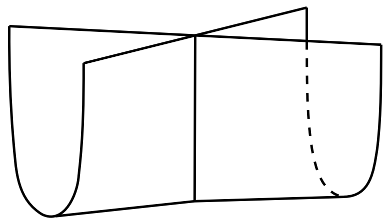

FIGURE B.1. **The Whitney umbrella.** It is the image of the singularity $(x, y) \mapsto (x^2, xy, y)$.

corresponds to $g(p)$, with respect to which the function g has the form

- $(x, y) \mapsto (x, y, 0)$ *or*
- $(x, y) \mapsto (x^2, xy, y)$.

More intrinsically, consider the Jacobian $J_g(p)\colon T_pW \to T_{g(p)}Y$ of g at $p \in W$. Points of the first kind are those for which the Jacobian is injective; at those points g is an *immersion*. Points of the second kind are the *singular points* of g, where the Jacobian has 1-dimensional kernel. Note that for a generic choice of g (as above), the rank of $J_g(p)$ is non-zero for all $p \in W$. Thus, Whitney's theorem gives a canonical form for the neighborhood of the singular points of g; such a singularity is called a *Whitney umbrella*. Its image is the locus of points $(u, v, z) \in \mathbb{R}^3$ with $v^2 - uz^2 = 0$ (and $u \geq 0$); see Figure B.1. A proof of Theorem B.1.2 is given in Subsection B.1.2.

Suppose that $\mathcal{D}_0$ and $\mathcal{D}_1$ are diagrams of the smoothly embedded knots K_0 and K_1, and assume that the two knots are isotopic. Regard the isotopy as a smoothly embedded surface-with-boundary $f\colon [0, 1] \times S^1 \to [0, 1] \times \mathbb{R}^3$ with the following properties:

- $\mathrm{Im}(f) \cap (\{i\} \times \mathbb{R}^3) = \{i\} \times K_i$ for $i = 0, 1$, and
- the intersection $f([0, 1] \times S^1) \cap (\{t\} \times \mathbb{R}^3)$ is transverse for all $t \in [0, 1]$.

Let $\mathrm{pr}_1\colon [0, 1] \times \mathbb{R}^3 \to [0, 1]$ denote the projection onto the first factor. The second condition above is equivalent to requiring that $\mathrm{pr}_1 \circ f\colon [0, 1] \times S^1 \to [0, 1]$ has no critical points; it is clearly an open condition.

Extend the projection map $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ defining the knot diagram to the map $P = \mathrm{Id} \times \pi\colon [0, 1] \times \mathbb{R}^3 \to [0, 1] \times \mathbb{R}^2$. Compose the embedding f of the annulus $[0, 1] \times S^1$ with P to get a map $\phi = P \circ f\colon [0, 1] \times S^1 \to [0, 1] \times \mathbb{R}^2$. Theorem B.1.1 is proved by applying Theorem B.1.2 to this map ϕ, as follows.

Proof of Theorem B.1.1. Consider two isotopic knots K_0, K_1 with the isotopy $f\colon [0, 1] \times S^1 \to [0, 1] \times \mathbb{R}^3$ between them, and with knot diagrams $\mathcal{D}_0$ and $\mathcal{D}_1$, and let $\phi\colon [0, 1] \times S^1 \to [0, 1] \times \mathbb{R}^2$ be the map defined as above. Since P is a submersion, ϕ can be put into general position by slightly perturbing the map f, so that it remains an isotopy between K_0 and K_1.

Theorem B.1.2 shows that there are finitely many points $B \subset [0, 1] \times S^1$ with a Whitney umbrella singularity, away from which the map is an immersion. By a further general position argument, we can assume that every point in $[0, 1] \times \mathbb{R}^2$ has at most three preimages. In fact, we can arrange that ϕ has only finitely many triple points T (i.e. points in $[0, 1] \times \mathbb{R}^2$ with three preimages), and a union of one-dimensional submanifolds $D \subset [0, 1] \times \mathbb{R}^2$ of double points. The closure of the set

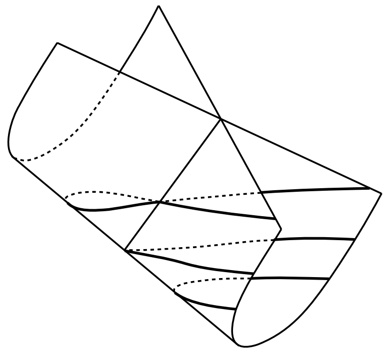

FIGURE B.2. **The Whitney umbrella and the first Reidemeister move.** We can assume that the singularity is transverse to the projection.

of double points includes the set of triple points and the set of Whitney umbrella singularities; its boundary also includes the double points $D \cap (\{i\} \times \mathbb{R}^2)$ $(i = 0, 1)$ of the two original knot diagrams $\mathcal{D}_0$ and $\mathcal{D}_1$.

Regard the intersections $\phi([0,1] \times S^1) \cap (\{t\} \times \mathbb{R}^2)$ as a one-parameter family of knots diagrams. Again, by general position, there are finitely many special $t \in [0, 1]$ where these diagrams are not generic, and where exactly one of the following occurs:

- $\phi^{-1}(\{t\} \times \mathbb{R}^2)$ contains a Whitney umbrella singularity;
- $\phi^{-1}(\{t\} \times \mathbb{R}^2)$ contains a triple point;
- $\{t\} \times \mathbb{R}^2$ is tangent to D.

Consider $0 \leq t_1 < t_2 \leq 1$, and suppose that there are no special values of $t \in [t_1, t_2]$. In this case, the knot projections at t_1 and t_2 are planar isotopic.

Assume next that the interval $[t_1, t_2]$ contains a single special value, which corresponds to a Whitney umbrella singularity. Furthermore, assume that at that value t the slice $\{t\} \times \mathbb{R}^2$ is transverse to the one-dimensional image of the Jacobian. Then the diagrams $\{t_1\} \times \mathbb{R}^2$ and $\{t_2\} \times \mathbb{R}^2$ differ by a single Reidemeister 1 move; see Figure B.2. For example, intersecting the surface $v^2 - uz^2 = 0$ with the one-parameter family of planes $\{u = z + a\}$ with $|a| < \epsilon < 1$, gives the one-parameter family of plane curves given by the parametric equation $v^2 - z^2(z + a) = 0$ with $z + a \geq 0$, and these plane curves at $a = -\epsilon$ and $a = \epsilon$ differ by a local Reidemeister 1 move.

As the $\{t\} \times \mathbb{R}^2$ slice passes through a point of tangency with D, the knot projection undergoes a Reidemeister 2 move; see Figure B.3.

A triple point is locally modelled on three intersecting planes; intersecting the surface with transverse level sets immediately before and immediately after the triple point gives two diagrams that differ by a Reidemeister 3 move. This completes the proof of Reidemeister's theorem. □

B.1.1. Morse functions. The above proof involves a careful analysis of the possible local forms of generic functions, as described in Theorem B.1.2. A similar analysis appears in *Morse theory*, where one considers real-valued functions on n-dimensional manifolds. Although we do not use Morse theory for the proof of Whitney's Theorem B.1.2, since we will use facts on Morse functions later, we

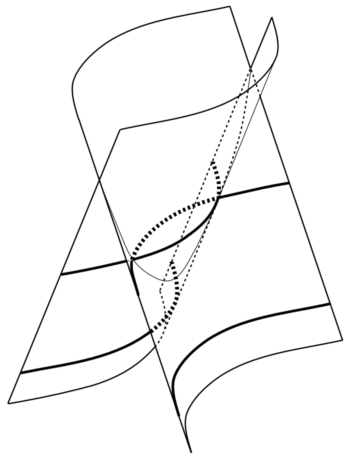

FIGURE B.3. **The double point set of the projection.** When crossing the double point set with a plane, we get crossings in the knot projection. When the plane $\{t\} \times \mathbb{R}^2$ is tangent to the double point set, we get a Reidemeister 2 move.

recall here some of the basic notions of this theory; for a thorough treatment of this beautiful subject, see [**142, 143**], cf. also [**77**].

Let $f\colon M \to \mathbb{R}$ be a smooth function on a compact, n-dimensional manifold. A *critical point* of f is a point $p \in M$ where the Jacobi matrix, written in a local coordinate system $(x_1, \dots, x_n)$ as $J_f(p) = (\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p))$, is zero. A critical point p corresponding to $(0, \dots, 0)$ in the coordinate system $(x_1, \dots, x_n)$ is called *non-degenerate* if the *Hessian matrix*

$$H(f)(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$$

has non-zero determinant. (This property is independent of the choice of local coordinate system around p.) A function is called a *Morse function* if every critical point is non-degenerate. Morse functions form an open and dense set among smooth functions [**143**, Theorem 2.7].

The *Morse Lemma* [**142**, Lemma 2.2] states that around each non-degenerate critical point p, there are local coordinates $(x_1, \dots, x_n)$ around p, so that $(0, \dots, 0)$ corresponds to p and

$$f(x_1, \dots, x_n) = f(p) - \sum_{i=1}^{\lambda_p} x_i^2 + \sum_{i=\lambda_p+1}^{n} x_i^2,$$

for some integer λ_p. The integer λ_p is the *index* of f at a critical point p, and it is independent of the chosen local coordinate system.

B.1.2. The proof of Theorem B.1.2. In our presentation we will follow Whitney's original argument from [**228**]; for a more modern treatment see [**75**, Theorem VI.4.6].

Since the theorem concerns the local behaviour of a generic map between a two- and a three-dimensional manifold, we will concentrate on maps $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ with

$g(0,0) = (0,0,0)$, which we abbreviate $g\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. For a generic choice of g the Jacobian is not equal to the zero matrix (which is of codimension six among all 2×3 matrices) in any point p, hence rk $J_g(p)$ is at least 1 for all $p \in \mathbb{R}^2$.

The Jacobian $J_g(0)$ of g at the origin is a 2×3 matrix, and if its rank is 2, then the implicit function theorem gives coordinate systems where $g(x,y) = (x,y,0)$. In the following we will examine the case when rk $J_g(0) = 1$.

Choose a coordinate system (x,y) so that $\frac{\partial g}{\partial x}(0) = (0,0,0)$. Consider the 3×3 matrix

$$Q_g(0) = \begin{pmatrix} \frac{\partial g_1}{\partial y}(0) & \frac{\partial^2 g_1}{\partial x^2}(0) & \frac{\partial^2 g_1}{\partial x \partial y}(0) \\ \frac{\partial g_2}{\partial y}(0) & \frac{\partial^2 g_2}{\partial x^2}(0) & \frac{\partial^2 g_2}{\partial x \partial y}(0) \\ \frac{\partial g_3}{\partial y}(0) & \frac{\partial^2 g_3}{\partial x^2}(0) & \frac{\partial^2 g_3}{\partial x \partial y}(0) \end{pmatrix}. \tag{B.1}$$

Lemma B.1.3. *For $g\colon (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ with* rk $J_g(0) = 1$ *there are functions h arbitrarily close to g (in the C^2 topology) with $J_g(0) = J_h(0)$ and* $\det Q_h(0) \neq 0$. *If* $\det Q_g(0) \neq 0$ *then there is an open neighbourhood U of* $0 \in \mathbb{R}^2$ *with the property that* rk $J_g(p) = 2$ *for $p \in U \setminus \{0\}$.*

Proof. Let g_i $(i = 1,2,3)$ denote the coordinate functions of g. Choose a coordinate system (x,y) around 0 so that $\frac{\partial}{\partial x}$ spans the kernel of $J_g(0)$; i.e. for $i = 1, \ldots, 3$,

$$\frac{\partial g_i}{\partial x}(0,0) = 0. \tag{B.2}$$

Since rk $J_g(0) = 1$, there is a linear transformation of the target so that

$$\left(\frac{\partial g_1}{\partial y}(0,0), \frac{\partial g_2}{\partial y}(0,0), \frac{\partial g_3}{\partial y}(0,0)\right) = (0,0,1). \tag{B.3}$$

Now either adding $\epsilon \cdot x^2$ to g_1 or $\delta \cdot xy$ to g_2 (or both) for sufficiently small ϵ and δ, we get a function h close to g with $J_h(0) = J_g(0)$ and $\det Q_h(0) \neq 0$.

For the second statement, let V be a neighborhood of the origin so that for $p \in V$, $\det Q_g(p) \neq 0$. Let L_p denote the line spanned by the first column of $Q_g(p)$ and let T_p be the plane spanned by the second and third columns, all viewed as vectors in $\mathbb{R}^3$. The condition on $Q_g(p)$ ensures that for $p \in V$, the line L_p and the plane T_p span $\mathbb{R}^3$; in fact, after passing to a smaller neighborhood $V' \subset V$ of 0 if needed, we can assume that for $p, p' \in V'$ the line L_p and the plane $T_{p'}$ intersect transversely. For such a sufficiently small neighborhood (by the mean value theorem) the vector $(\frac{\partial g_i}{\partial x}(p') - \frac{\partial g_i}{\partial x}(0))_{i=1}^3$ is in a plane close to $T_{p'}$, and hence is not in $L_{p'}$. Since $\frac{\partial g_i}{\partial x}(0) = 0$ for $i = 1,2,3$, this shows that $(\frac{\partial g_i}{\partial x}(p'))_{i=1}^3$ and $(\frac{\partial g_i}{\partial y}(p'))_{i=1}^3$ are linearly independent, hence rk $J_g(p') = 2$, concluding the proof. $\square$

Proof of Theorem B.1.2. Suppose that $g\colon W^2 \to Y^3$ is a generic map, hence by our discussion above we can assume that rk $J_g(w) > 0$ for every point w of W. If this rank is equal to 2, then the implicit function theorem concludes the proof and shows that locally g is of the form $(x,y) \mapsto (x,y,0)$. If the rank is equal to 1 at a point $w \in W$, then again by genericity (and by Lemma B.1.3) we can assume that the determinant of Equation (B.1) is non-zero and w has a neighbourhood U with rk $J_g(p) = 2$ for all $p \in U \setminus \{w\}$ (that is, w is isolated).

Since we are working on coordinate charts, from now on we will work with g as a function $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. As in the proof of Lemma B.1.3, we can choose our local coordinates (x, y) on $\mathbb{R}^2$ so that Equation (B.2) and (B.3) holds. By the implicit function theorem, we can choose our coordinate system on $\mathbb{R}^3$ so that $g_3(x, y) = y$. The non-degeneracy of $Q_g(0)$ then implies that one of $\frac{\partial^2 g_i}{\partial x^2}(0)$ for $i = 1, 2$ is non-zero. Changing coordinates on $\mathbb{R}^3$ if necessary, we can arrange that

$$\frac{\partial^2 g_1}{\partial x^2}(0, 0) \neq 0. \tag{B.4}$$

In the next step we apply a coordinate transformation so that $g_1(x, y) = x^2$. To achieve this, let $F(x, y) = \frac{\partial g_1}{\partial x}(x, y)$, and consider the function $x = \phi(y)$ solving

$$F(\phi(y), y) = 0 \tag{B.5}$$

in a neighborhood of the origin. This solution can be found using the implicit function theorem, thanks to Equation (B.4). Consider the coordinate change on $\mathbb{R}^2$ specified by

$$x' = x - \phi(y), \quad y' = y.$$

In this new coordinate system, by the chain rule, Equation (B.3), and Equation (B.5),

$$\frac{\partial g_1}{\partial x'}(0, y') = \frac{\partial g_1}{\partial x}(\phi(y'), y') = 0.$$

Thus, when expanding g_1 in x' we get $g_1(x', y') = \psi_0(y') + (x')^2\psi_2(x', y')$. By Equation (B.4), it follows that $\psi_2(0, 0) \neq 0$. With the coordinate change

$$x'' = |\psi_2(x', y)|^{\frac{1}{2}} \cdot x', \quad y'' = y'$$

on $\mathbb{R}^2$, and the coordinate change

$$g_1' = g_1 - \psi_0(g_3), \qquad g_2' = g_2 \qquad g_3' = g_3,$$

on $\mathbb{R}^3$, after dropping primes, we get new coordinate systems where

$$g_1(x, y) = x^2, \qquad g_3(x, y) = y.$$

It remains to bring g_2 to the required form.

Since $\det Q_g(0) \neq 0$, the third column of the matrix from Equation (B.1) must be non-zero; but the top and the bottom entry clearly vanish, so it follows that $\frac{\partial^2 g_2}{\partial x \partial y}(0) \neq 0$; while $\frac{\partial g_2}{\partial x}(0, 0)$ and $\frac{\partial g_2}{\partial y}(0, 0) = 0$ by Equations (B.2) and (B.3). It follows that $g_2(x, y)$ can be written as

$$g_2(x, y) = xy + R(x, y),$$

where $\frac{\partial R}{\partial x}(0, 0) = \frac{\partial R}{\partial y}(0, 0) = \frac{\partial^2 R}{\partial x \partial y}(0, 0) = 0$. Let $g_2' = g_2 - R(0, g_3)$. Observe that $R(x, y) - R(0, y)$ is divisible by x, so dropping primes, we get $g_2(x, y) = x(y + R_1(x, y))$. Let

$$R_2(x, y) = \frac{R_1(x, y) + R_1(-x, y)}{2} \qquad \text{and} \qquad R_3(x, y) = \frac{R_1(x, y) - R_1(-x, y)}{2},$$

so

$$R_1(x, y) = R_2(x, y) + R_3(x, y),$$

with $R_2(-x, y) = R_2(x, y)$ and $R_3(x, y) = -R_3(-x, y)$.

Solve the implicit equation

$$y + R_2(x, y) = 0$$

for $y = y(x)$. Since $\frac{\partial g_2}{\partial y}(0,0) = 0$, it follows that the above implicit equation has a solution. By the symmetry $x \mapsto -x$ and local uniqueness, this solution can be written as $y = \eta(x^2)$, for a smooth function η. Define

$$\zeta(x^2) = xR_3(x, \eta(x^2)),$$

and consider the new coordinates

$$x' = x, \quad y' = y - \eta(x^2), \quad g_1' = g_1, \quad g_2' = g_2 - \zeta(g_1), \quad g_3' = g_3 - \eta(g_1).$$

Once again, this is a valid coordinate transformation, which keeps the shape of g_1 and g_3, and rewrites g_2 (again, after dropping primes) as $g_2(x,y) = x(y + R_4(x,y))$ with $R_4(x,0) = 0$, hence

$$g_2(x,y) = xy(1 + R_5(x,y)).$$

Decomposing $R_5(x,y) = P(x^2,y) + xQ(x^2,y)$ into its even and odd parts in x (as we did with R_1), we find

$$g_2(x,y) = xy(1 + P(x^2,y)) + x^2yQ(x^2,y). \tag{B.6}$$

The coordinate transformation $g_2 = g_2 - g_1g_3Q(g_1,g_3)$ can be used to eliminate second term in Equation (B.6), hence we are left with $g_2(x,y) = xy(1 + P(x^2,y))$.

Let $G(x,y) = \frac{P(x,y)}{1+P(x,y)}$ and apply the following coordinate transformation:

$$g_1' = g_1, \quad g_2' = g_2 - G(g_1,g_3)g_2, \quad g_3' = g_3.$$

Since $G(0,0) = 0$, this last transformation still gives coordinates on $\mathbb{R}^3$, and now

$$g_2'(x,y) = g_2(x,y) \cdot (1 - G(x^2,y)) = g_2(x,y) \cdot \frac{1}{1 + P(x^2,y)} = xy,$$

hence (after dropping primes) we get the desired from of $g = (g_1, g_2, g_3)$ around the origin. □

B.2. Reidemeister moves in contact knot theory

Recall that Legendrian and transverse knots can also be dealt with via their (front) projections, and there is a similar set of Reidemeister moves for these diagrams. In this section we adapt the argument from the smooth case to prove these analogues of the Reidemeister theorem in the contact case.

B.2.1. Transverse Reidemeister moves. Since the transverse condition is open, we can assume that the transverse isotopy between two transverse knots has only generic singularities. This leads quickly to the verification of the *transverse Reidemeister theorem*:

Proof of Theorem 12.5.4. Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two transversely isotopic transverse knots, with front diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$. Let $f\colon [0,1] \times S^1 \to [0,1] \times \mathbb{R}^3$ be the surface given by the transverse isotopy. Since transversality is an open condition, if we take a small perturbation of f, we can assume that the composition of f with the projection map P from Section B.1 is a generic map $\phi = P \circ f\colon [0,1] \times S^1 \to [0,1] \times \mathbb{R}^2$.

Recall that the self-linking number of a transverse knot is equal to the writhe of its front projection, so Reidemeister 1 moves (which change the writhe) cannot occur in the diagram-change along a transverse isotopy. This shows that the projection of a transverse isotopy cannot have Whitney umbrella singularities, and so for transverse knots we need only two types of Reidemeister moves: the adaptations of the second and of the third Reidemeister moves of Figure 2.2, as given in Figure 12.14. (Configurations from Figure 12.13 are disallowed by the transverse condition.) □

While transverse isotopies do not contain Reidemeister 1 moves, the converse is not true: an isotopy between two transverse knots giving a map without any Whitney umbrella singularity is not necessarily a transverse isotopy. In fact, using results from [**219**], two transverse representatives of the same knot type with the same self-linking number always can be connected by a smooth isotopy whose projection has no Whitney umbrella singularities, so their diagrams can be connected by a sequence of Reidemeister 2 and 3 moves and planar isotopies. As the transversely non-simple knot types demonstrate, such an isotopy may not be a transverse isotopy, and so the planar isotopies or the Reidemeister moves involve disallowed configurations (shown by Figure 12.13).

B.2.2. Legendrian Reidemeister moves. Since the Legendrian condition is not open, the appropriate adaptation of Reidemeister's theorem requires additional care. Our treatment here follows the discussion of Światkowski [**214**]. We start by restating the theorem:

THEOREM B.2.1 (Światkowski, [**214**]). *Two front projections correspond to Legendrian isotopic Legendrian links if and only if the projections can be connected by Legendrian planar isotopies and by Legendrian Reidemeister moves (shown in Figure* 12.2*).*

This subsection is devoted to a proof of this result; although the theorem holds for links, for simplicity in the proof we will deal with the case of knots only.

First we make more precise what we mean by generic Legendrian knot.

DEFINITION B.2.2. A Legendrian knot $\mathcal{K}$ is ***front generic*** if its front projection (projection to the (x, z) plane) has the following properties:

- (fg-1) The only singularities of the projection are cusps, which can be given as $t \mapsto (t^2, t^3)$ in appropriate local coordinates;
- (fg-2) the vertices of the cusps are distinct and are not on any other strand of the projection;
- (fg-3) the strands of the projection meet transversely, without triple intersections.

As in Section 12.1, we denote the coordinate functions of a parametrization of the given Legendrian knot $\vec{\mathcal{K}}$ by $x(t), y(t), z(t)$. The Legendrian condition can be expressed as

$$z'(t) = y(t) \cdot x'(t). \tag{B.7}$$

LEMMA B.2.3. *Suppose that $\vec{\mathcal{K}}$ is an oriented Legendrian knot. The front projection of $\vec{\mathcal{K}}$ has singularities only as described in Property* (fg-1) *if and only if $x(t)$ is a Morse function.*

Proof. Notice first that the projection has a singularity at t if $x'(t) = 0$, since by Equation (B.7), this implies $z'(t) = 0$. If all singularities have the local form described in Property (fg-1), then $x''(t) \neq 0$ at the critical points of x, and hence $x(t)$ is obviously Morse.

Conversely, suppose that $x(t)$ is Morse, so at a critical point in local coordinates it is equal to t^2. (In these coordinates, the critical point corresponds to $t = 0$.) From Equation (B.7) we have

$$z''(t) = y'(t) \cdot x'(t) + y(t) \cdot x''(t),$$

which at the singular point $t = 0$ (that is, at a critical point of $x(t)$) gives $z''(0) = y(0) \cdot x''(0)$. If we choose the line parallel to the x axis, passing through the singularity as the first coordinate axis and the line with slope $y(0) = \frac{z''(0)}{x''(0)}$ (also passing through the singularity) as the second coordinate axis around the singularity, we get a local coordinate system in the (x, z) plane such that the projection is given by the map $t \mapsto (t^2, f(t))$, with $f(0) = f'(0) = f''(0) = 0$. Since $x'(0) = z'(0) = 0$ implies $y'(0) \neq 0$ (since $\vec{\mathcal{K}}$ is given by a smooth embedding), we have that $f'''(t) \neq 0$, so $f(t)$ can be written as $t^3(1 + h(t))$ with $h(0) = 0$. Decompose $h(t)$ into even and odd parts, as in the proof of Theorem B.1.2 so that $h(t) = g_1(t^2) + tg_2(t^2)$. Then, $f(t) = t^3(1 + g_1(t^2)) + t^4 g_2(t^2)$. The second term is a function of the first coordinate, hence can be easily eliminated, leaving us with $f(t) = t^3(1 + g_1(t^2))$. With the function $F(t) = \frac{g_1(t)}{1+g_1(t)}$ the new local coordinate

$$f_{new}(t) = f(t) - F(t) \cdot f(t)$$

shows that the singularity can be written in the form $t \mapsto (t^2, t^3)$. □

Front generic Legendrian knots form an open and dense set among Legendrian knots. In verifying this statement, we will use another projection of a Legendrian knot (usually called the *Legendrian projection*): the projection to the (x, y)-plane. Since all contact planes embed under the differential of this projection, it can be shown that (unlike the front projection, containing cusps) the Legendrian projection of a smooth Legendrian knot is a smooth immersion. Since the form $dz - ydx$ vanishes on the tangents of a Legendrian knot, the third coordinate function $z(t)$ can be recovered from the Legendrian projection by the formula

$$z(t) = z_0 + \int_0^t y(\theta) x'(\theta) d\theta.$$

Such a lift is not uniquely determined by the Legendrian projection (since it depends on the choice of z_0); also, not every smooth, immersed closed curve in the (x, y) plane gives rise to a Legendrian knot. Indeed, by choosing z_0 over a point (x_0, y_0) of the immersed closed curve, and parametrizing the diagram in the (x, y)-plane by $[0, 2\pi]$ (with the understanding that 0 and 2π both map to (x_0, y_0)), the value $z(2\pi)$ is not necessarily equal to z_0. If the integral from 0 to 2π vanishes, the diagram lifts to a closed, immersed curve, which furthermore is embedded if the two z-coordinates over any double points are different; see also [**45**]. This construction will be used in the proof of the next lemma, which justifies Definition B.2.2.

LEMMA B.2.4. ([**214**, Theorem A]) *The set of front generic Legendrian knots is open and dense in the set of all Legendrian knots.*

Proof. First we would like to arrange that the front projection has only cusp singularties. Let $\lambda\colon S^1 \to \mathbb{R}$ be a Morse function close to $x(t)$. Using the curve $C =$

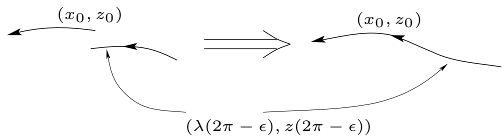

FIGURE B.4. **Closing up the front projection after replacing $x(t)$ by a nearby Morse function $\lambda(t)$.**

$(\lambda(t), y(t))$ in the (x, y)-plane as a Legendrian projection, we recover a Legendrian arc, with z-coordinate function (for $t \in [0, 2\pi]$) given by

$$z(t) = z_0 + \int_0^t y(\theta)\lambda'(\theta)d\theta. \tag{B.8}$$

Suppose that the function λ is sufficiently close to $x(t)$ and choose the point (x_0, y_0, z_0) on the Legendrian knot so that at this point the t-derivative of the z-coordinate is non-zero. Then the point (x_0, y_0, z_0) maps to a smooth point in the front projection, and we can close up the front diagram with a local modification near this image, as shown in Figure B.4. In this procedure we take the curve given by the integral for the values $t \in [0, 2\pi - \epsilon]$ and close up the resulting front projection. This then defines a Legendrian knot which is close to the given one and satisfies Property (fg-1) of Definition B.2.2.

The resulting front diagram now has the required cusp singularities. The smooth segments can be easily isotoped in the (x, z) plane using standard transversality arguments for maps of intervals to the plane.

Since Morse functions form an open and dense subset of all C^∞ functions, and the further properties (not having triple intersections and cusps on other branches) are also open and dense, the claims of the statement follow. □

Notice that the above procedure provides a Legendrian isotopy from any Legendrian knot to one which is front generic: just connect the function $x(t)$ to the chosen Morse function $\lambda(t)$ with a family of functions $\lambda^s(t)$ (with $x(t) = \lambda^0(t)$ and $\lambda(t) = \lambda^1(t)$) such that $\lambda^s(t)$ is Morse for $s > 0$, and do the final deformation (of closing the Legendrian arcs) parametrically.

In the next step we clarify what we mean by a generic isotopy in the Legendrian context. Suppose that H is a Legendrian isotopy between two Legendrian knots. The first step in achieving a generic front was to modify the coordinate function $x(t)$ to be generic (simply meaning Morse). For an isotopy, the similar step requires the genericity of a map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ derived from the first coordinate function of the isotopy, cf. Definition B.2.7. The following classification result of Whitney (similar in spirit to Theorem B.1.2) will be of crucial importance.

THEOREM B.2.5 (Whitney, [**229**]). *Let W and X be two smooth two-dimensional manifolds. A map $g_0\colon W \to X$ can be approximated arbitrarily close (in the C^2 topology) with a smooth map $g\colon W \to X$ with the following property: around each point $p \in W$ there are local coordinates (x, y) (so that p corresponds to $(0, 0)$) and local coordinates (u, v) around $g(p)$ in such a way that g has the form*

- $(x, y) \mapsto (x, y)$ *so g is a local diffeomorphism near p or*
- $(x, y) \mapsto (x^2, y)$ *so p is a fold point of g or*
- $(x, y) \mapsto (xy + x^3, y)$ *so p is a simple cusp.*

□

Remark B.2.6. The proof of the above theorem can be given by adapting the methods of the proof of Theorem B.1.2. Indeed, for a generic map $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ we can assume that the rank of the Jacobian J_g is non-zero at every point. At points when $\operatorname{rk} J_g = 2$, we have a local diffeomorphism, providing the first case in Theorem B.2.5. If $\operatorname{rk} J_g(p) = 1$ at a point p (say, $p = 0$), we can choose coordinates (x, y) so that x spans the kernel of $J_g(0)$, and the coordinate functions can be chosen so that $g_2(x, y) = y$. Assuming $\frac{\partial^2 g_1}{\partial x^2}(0) \neq 0$, the argument from the proof of Theorem B.1.2 normalizing the first coordinate function there (see text around Equation (B.5)) shows that, after appropriate coordinate changes, we get $(x, y) \mapsto (x^2, y)$, the second option of the theorem; see [**229**, Section 15]. In the final case, when $\frac{\partial^2 g_1}{\partial x^2}(p) = 0$, by genericity we can assume that $\frac{\partial^3 g_1}{\partial x^3}(p) \neq 0$, and a slightly longer argument (again, in the spirit of the proof of Theorem B.1.2, detailed in [**229**, Section 16]) concludes the proof of Theorem B.2.5. (For a more modern treatment of this result see [**75**, Chapter VI, Section 2].)

Definition B.2.7. Suppose that $H = H(s, t)$ is a Legendrian isotopy between the two Legendrian knots $\vec{\mathcal{K}}_1$ and $\vec{\mathcal{K}}_2$ and write $H = (H_x, H_y, H_z)$, where the components denote the coordinate functions (now from $S^1 \times [0, 1]$ to $\mathbb{R}$). The isotopy H is a ***front isotopy*** if it satisfies the following conditions:

- The singularities of the map $F\colon S^1 \times [0, 1] \to \mathbb{R}^2$ given by $F(s, t) = (H_x(s, t), t)$ are folds and simple cusps,
- the double point set D of the composition $\phi = P \circ H$ of H with $P = \mathrm{Id} \times \pi$ (where $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the projection to the (x, z) plane) is a one-dimensional submanifold and the restriction of the projection pr_1 to $[0, 1]$ is a Morse function,
- the triple points of the map ϕ are isolated, and
- the values of the projection pr_1 at the cusp singularities, at the critical points of pr_1 on D and on the triple points are all distinct.

Proposition B.2.8. ([**214**, Lemma 3.3]) *If $\vec{\mathcal{K}}_1$ and $\vec{\mathcal{K}}_2$ are Legendrian isotopic Legendrian knots with generic fronts, then there is a front isotopy between $\vec{\mathcal{K}}_1$ and $\vec{\mathcal{K}}_2$.*

Proof. The proof is similar in spirit to the proof of Lemma B.2.4. Let H be a Legendrian isotopy between $\vec{\mathcal{K}}_1$ and $\vec{\mathcal{K}}_2$ and suppose that $H = (H_x, H_y, H_z)$ are the coordinate functions of the isotopy. Consider the function $F(s, t) = (H_x(s, t), t)$, mapping from $S^1 \times [0, 1]$ to $\mathbb{R}^2$. Consider a generic perturbation F_1 of this map, which (by an appropriate coordinate transformation) can be assumed to be of the form $F_1(s, t) = (G_x(s, t), t)$.

The pair (G_x, H_y) now provides a family of closed curves (parametrized by $t \in [0, 2\pi]$) on the (x, y) plane, hence we can use the integral formula of Equation (B.8) to lift this planar isotopy to an isotopy of knots in the three-space. The same problem as in the proof of Lemma B.2.4 arises: the resulting Legendrian curves might not close up. By performing the same closing operation (possibly in different places), we get an isotopy which now satisfies the first constraint given in Definition B.2.7. The usual genericity arguments conclude the proof. □

We can now prove the Legendrian Reidemeister theorem.

Proof of Theorem 12.1.7. As always, one direction of the theorem is easy: if two front projections differ by Legendrian Reidemeister moves and Legendrian planar isotopies, then the corresponding two knots are Legendrian isotopic.

Suppose now that two Legendrian knots are isotopic via a Legendrian isotopy H. By Lemma B.2.4 we can assume that the knots are front generic, and by Proposition B.2.8 we can assume that the isotopy is a front isotopy.

Suppose that $(s_0, t_0) \in S^1 \times [0,1]$ is a point where the function $F(s,t) = (H_x(s,t), t)$ has a simple cusp singularity (here $s \in S^1$ and $t \in [0,1]$). By Theorem B.2.5 the local model of the map F at such a point is given by $(st + s^3, t)$. Since $\frac{\partial H_y(s,t)}{\partial s}$ is non-zero at such a cusp point (as the Legendrian projection to the (x,y) plane is still an immersion), we get that the ratio $\frac{\partial H_z(s,t)}{\partial s} / \frac{\partial H_x(s,t)}{\partial s}$ is monotone there, providing a local model which shows that near the point (s_0, t_0) the diagrams given by the slices $\{t\} \times \mathbb{R}^2$ undergo a Legendrian Reidemeister 1 move. Indeed, the slice with $\{t\} \times \mathbb{R}^2$ gives the x-coordinate function $s \mapsto st + s^3$, which has no critical point for $t > 0$ and two critical points (giving rise to two cusps in the front projection) for $t < 0$.

As in the smooth case, we need to examine the points where $\{t\} \times \mathbb{R}^2$ is tangent to the double point set D. It is easy to see that in these points only a smooth point and a cusp can meet: by our genericity assumption two cusps cannot project to the same point, and two smooth branches of the projection cannot be tangent to each other, since then the Legendrian knot in the isotopy has a self-intersection. This means that the tangencies of the double point set give Legendrian Reidemeister 2 moves, as shown by Figure 12.2.

We can handle the case when the plane $\{t\} \times \mathbb{R}^2$ crosses a triple point exactly as in the smooth case, providing the Legendrian Reidemeister 3 moves. □

B.2.3. Approximations. In Section 12.5.2 we described how to approximate Legendrian knots by transverse knots, and conversely, transverse knots by Legendrian ones. We constructed the transverse push-off of a Legendrian knot, giving a transverse knot unique up to transverse isotopy associated to the Legendrian knot. (See Proposition 12.5.5.) A concrete description of this operation in terms of front projections is illustrated in Figure 12.15. Similarly, we explained how to construct a Legendrian approximation of a given transverse knot. The construction was defined in terms of front projections, using an algorithm which transformed a front projection of the transverse knot to the front projection of a Legendrian knot. This algorithm constructs a diagram that is defined uniquely up to negative stabilization.

According to Theorem 12.5.9, these constructions are inverses to one another, giving a one-to-one correspondence between transverse isotopy classes of transverse knots and equivalence classes of Legendrian isotopy classes of Legendrian knots modulo negative stabilization. Our present goal is to prove Theorem 12.5.9. The arguments below are based on [**43**, Section 2]. (Note that our convention differs from the one of [**43**]: we work with the contact structure given by the one-form $\alpha = dz - ydx$, while in the reference the contact structure is given by $ydx - dz = -\alpha$, giving different orientations in certain statements.)

PROPOSITION B.2.9. *Suppose that the two transverse knots $\mathcal{T}_1$ and $\mathcal{T}_2$ are transverse isotopic. Then their Legendrian approximations become isotopic after suitable negative stabilizations.*

Proof. Consider front projections $\mathcal{D}(\mathcal{T}_1)$ and $\mathcal{D}(\mathcal{T}_2)$ of $\mathcal{T}_1$ and $\mathcal{T}_2$. The algorithm described in Section 12.5.2 provides diagrams $\mathcal{D}(\vec{\mathcal{K}}_1)$ and $\mathcal{D}(\vec{\mathcal{K}}_2)$ of the Legendrian approximations, both determined up to negative stabilizations and Legendrian planar isotopy.

Since the two transverse knots are transverse isotopic, by the transverse Reidemeister theorem there is a sequence of transverse Reidemeister moves and transverse planar isotopies (so avoiding the disallowed configurations of Figure 12.13) transforming one diagram into the other. We will show that the Legendrian approximations of the diagrams are Legendrian isotopic, possibly after negative stabilizations.

A transverse planar isotopy either directly translates to a Legendrian planar isotopy, or it introduces a vertical tangency (pointing necessarily up), or it contains an isolated moment where a non-vertical segment crosses a vertical one, see Figure B.5. The figure also shows how these moves translate to Legendrian isotopies and negative stabilizations. In the first move (pictured on the left part of Figure B.5) there is one further case to analyze (when the strand points up and to the left), while in the second move (shown on the right part of Figure B.5) there are eight possibilities (depending on the orientation of the horizontal segment, the nature of the crossing, and direction of the vertical segment; i.e. whether the extremal point is on the left or the right). In the figure we only show one case, the remaining ones can be handled by similar means. The diagrams show how the Legendrian approximations of the results of the transverse planar isotopies can be realized by Legendrian Reidemeister moves and negative stabilizations.

In a similar manner, a transverse Reidemeister 2 move translates to a Legendrian Reidemeister 2 move — after possibly applying negative stabilizations. There are eight cases here, depending on the orientations of the strands and on the choice of the over-passing strand, but two of them contains a (transversally) disallowed

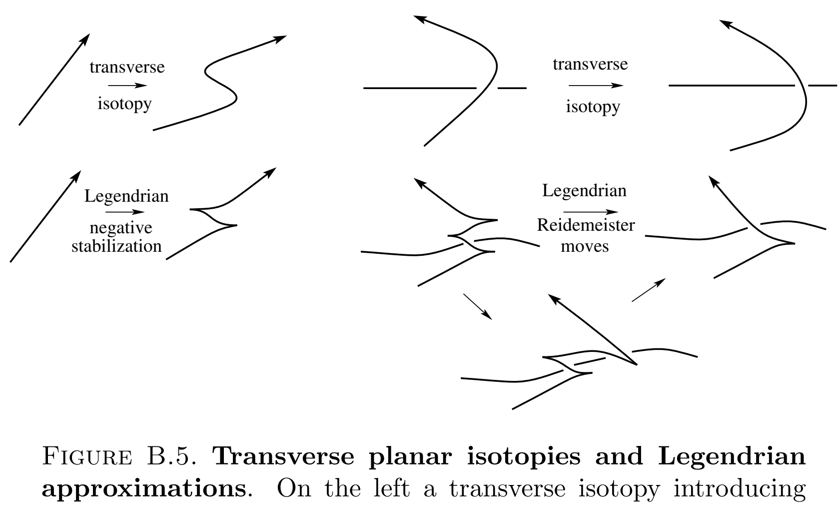

FIGURE B.5. **Transverse planar isotopies and Legendrian approximations**. On the left a transverse isotopy introducing (upward pointing) vertical tangencies is translated into negative stabilizations of the Legendrian approximation. On the right (one case of) an exchange of a vertical tangency and an intersection point is translated to the Legendrian approximation. The seven further cases are handled similarly.

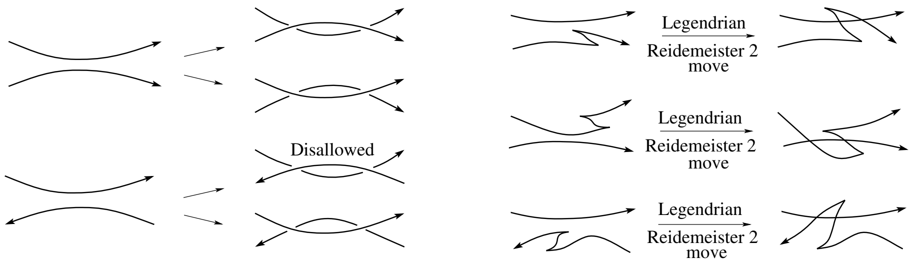

FIGURE B.6. **Reidemeister 2 moves and Legendrian approximations.** In the left we show four cases of the eight possible transverse Reidemeister 2 moves (one of which is disallowed), and on the right their Legendrian approximations, which require the application of a negative stabilization and a Legendrian Reidemeister 2 move.

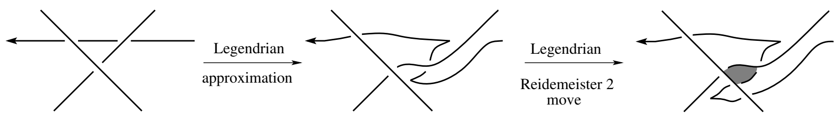

FIGURE B.7. **The effect of a transverse Reidemeister 3 move on the Legendrian approximation in case $p = 1$.** The left diagram shows the transverse projection, the middle one depicts the front projection of the Legendrian approximation and (after a Legendrian Reidemeister 2 move) we get a new diagram in which a Legendrian Reidemeister 3 move can be performed (indicated by the gray triangle); just as in the $p = 0$ case.

portion. Figure B.6 verifies the statement for four cases; simple modifications provide the result for the remaining configurations.

In a transverse Reidemeister 3 move we have three crossings; let p denote the number of crossings that are disallowed in a Legendrian front projection (so in the Legendrian approximation, a negative stabilization is needed; see Figure 12.16). We will group the various configurations according to the value of p.

If $p = 0$ then the transverse Reidemeister 3 move translates directly into a Legendrian Reidemeister 3 move on the Legendrian approximation.

When $p = 1$, then when translating the transverse diagram to a Legendrian diagram, we need to apply the modification of Figure 12.16 at one of the crossings. This case can be reduced to the $p = 0$ case by a Legendrian Reidemeister 2 move. This is illustrated in Figure B.7 in one case; the further $p = 1$ possibilities can be handled similarly.

When $p = 2$, let s denote the strand passing through the two crossings disallowed in the Legendrian front diagram. In turning the transverse diagram into the Legendrian front, we need to stabilize twice. Depending on the local configuration there are two cases. When s points from left to right, we stabilize it twice, while if it points from right to left, we stabilize on the other two strands. In both cases a Legendrian Reidemeister 2 move reduces the problem to the $p = 0$ case, concluding the argument. For illustration, see Figure B.8.

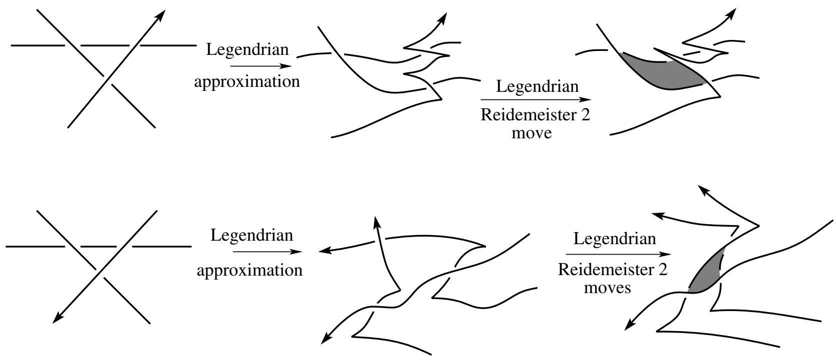

FIGURE B.8. **The effect of a transverse Reidemeister 3 move on the Legendrian approximation in case $p = 2$.** The two cases (depending on the orientation of the strand s containing the two crossings) are given in the two rows.

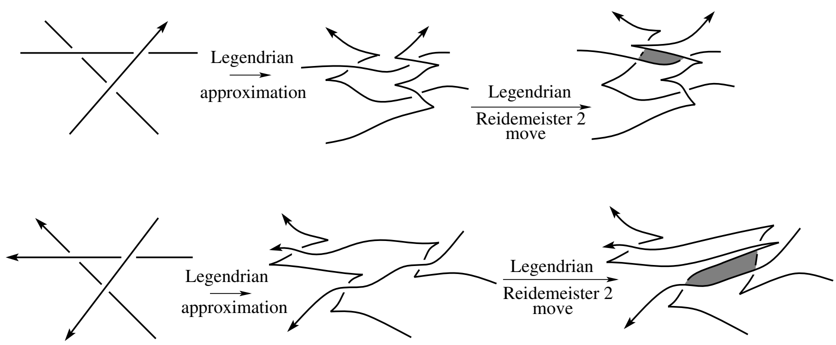

FIGURE B.9. **The effect of a transverse Reidemeister 3 move on the Legendrian approximation in case $p = 3$.** We choose a strand on which two modifications of Figure 12.16 can be applied, and then apply appropriate Legendrian Reidemeister moves to reduce the problem to the $p = 0$ case.

Finally we consider the case when $p = 3$. It is easy to see that we can always choose the three negative stabilizations in such a way that one of the strands is stabilized twice. Once again, a Legendrian Reidemeister 2 move reduces to the $p = 0$ case; see Figure B.9. □

Proof of Theorem 12.5.9. Consider the front diagram $\mathcal{D}(\mathcal{T})$ of the transverse knot $\mathcal{T}$. Approximate it by a Legendrian diagram (as shown in Figure 12.16) and then consider the transverse push-off (as shown in Figure 12.15). In this way get a transverse front diagram transverse planar isotopic to $\mathcal{D}(\mathcal{T})$, verifying the existence statement of the theorem.

One direction of the equivalence is simple. Indeed, if two Legendrian knot diagrams differ by a single negative stabilization, then their transverse push-offs are transverse isotopic, since in the diagram the smoothing of the new cusps can be straightened by a transverse planar isotopy. Furthermore, if two Legendrian knots

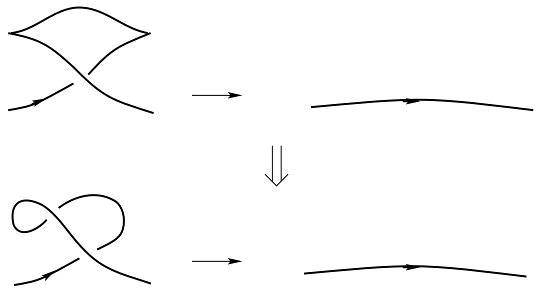

FIGURE B.10. **The effect of a Legendrian Reidemeister 1 move on the transverse push-off.** The Legendrian move gives rise to a transverse Reidemeister 2 move. Similar diagrams show the change for the other orientation of the strand.

are Legendrian isotopic, then their transverse push-offs are transverse isotopic. Indeed, a Legendrian planar isotopy translates to a transverse planar isotopy, the Legendrian Reidemeister 2 and 3 moves immediately translate to transverse Reidemeister moves while the transformation of a Legendrian Reidemeister 1 move requires a transverse Reidemeister 2 moves, as shown in Figure B.10. (An alternative argument was given in the proof of Proposition 12.5.5.)

The proof of the converse direction is the content of Proposition B.2.9. □

B.3. The Reidemeister-Singer Theorem

In Chapter 2 we met three invariants of knots and links derived using an auxiliary choice of a Seifert surface for K: the Alexander polynomial, the signature and the determinant. The independence of these quantities from the choice of the Seifert surface was established using the *Reidemeister-Singer Theorem*, which relates various Seifert surfaces of a given link. This section is devoted to the proof of this theorem; our approach here follows [**9**].

Fix an oriented link $\vec{L}$. From a Seifert surface Σ of $\vec{L}$ further Seifert surfaces can be obtained by stabilizing Σ (see Figure 2.12): fix an arc in $\mathbb{R}^3$ connecting two points in Σ, approaching it from the same side, called a *stabilizing arc*, and attach a handle to Σ supported in a neighborhood of this path, to obtain a stabilization Σ' of Σ. Also, we say that Σ is a *destabilization* of Σ'. With this language in place, we can state the Reidemeister-Singer Theorem:

THEOREM B.3.1 (Reidemeister-Singer, [**195, 212**]). *Any two Seifert surfaces of a given link in S^3 become ambient isotopic after an appropriate sequence of stabilizations and destabilizations.*

We describe first an important ingredient in this proof, *Seifert's algorithm* for constructing a Seifert surface associated to a link diagram for an oriented link $\vec{L}$. This algorithm proceeds as follows. Form the oriented resolution of each crossing in the diagram. The resulting configuration on the plane will consist of a collection of disjoint oriented circles, called *Seifert circles* (or Seifert circuits). Regard the plane as the subset $\{z = 0\} \subset \mathbb{R}^3$, and lift each disk bounded by a Seifert circle to a parallel copy of the disk in $\mathbb{R}^3$, lifting those disks contained inside $k \geq 0$ nested

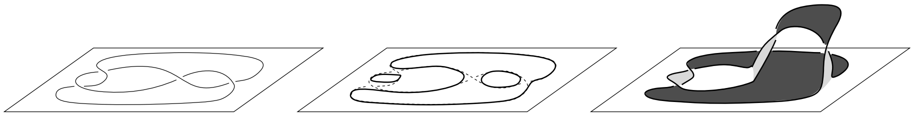

FIGURE B.11. **Seifert's algorithm.** The knot diagram on the left is resolved to obtain the collection of Seifert circles in the middle (where the crossing points are still indicated by dotted lines), which are turned into a Seifert surface as illustrated on the right.

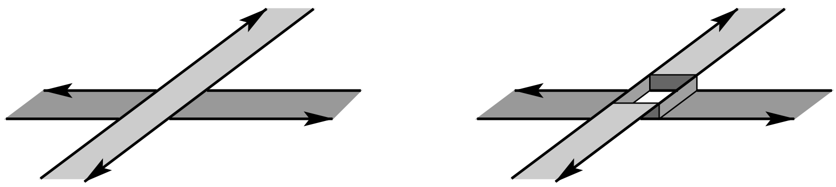

FIGURE B.12. **Crossings of bands in the projection.** By attaching an appropriate tube, locally we can turn the surface to an algorithmic surface.

Seifert circles into the affine plane $\{z = k\}$. Restore the crossings by attaching half-twisted bands that connect various translated disks, as specified by the crossings in the given diagram. This procedure gives a possibly disconnected, oriented surface with the given link as oriented boundary. We call such a surface an *algorithmic surface*. See Figure B.11 for an example. It is straightforward to see that this surface can be stabilized to get a Seifert surface.

LEMMA B.3.2. *Any Seifert surface of a given oriented link $\vec{L}$ can be isotoped and stabilized until it is an algorithmic surface.*

Proof. Consider a Seifert surface Σ for the link $\vec{L}$. Since a Seifert surface is connected, Σ can be realized as a single disk D, and a union of bands attached along the boundary of D. We can isotop Σ so that the disk projects injectively to the plane, and we can think of the bands as thin neighborhoods of their core curves, which core curves immerse into the plane minus the image of D. (Compare the proof of Proposition 3.4.11.) By general position, the core curves intersect each other in double points. By shrinking the bands and then twisting them if necessary, we can assume that projections of the bands meet each other in squares that immerse into the plane with opposite orientation, as shown in Figure B.12.

Stabilize Σ' at each crossing between the bands, as shown on the right diagram of Figure B.12. The resulting surface Σ' is the algorithmic surface Σ_{alg} associated to the projection of $\partial\Sigma'$. □

A link diagram naturally gives rise to another surface in $\mathbb{R}^3$ whose boundary is $\vec{L}$, the *black surface* (Definition 2.7.1), obtained by gluing the black domains in the chessboard coloring together along half-twisted bands to restore the crossings. In Section 2.7, a diagram is called *special* if F_b is a Seifert surface for $\vec{L}$. The above proof gives the following result of independent interest, which was stated earlier as Lemma 2.7.6:

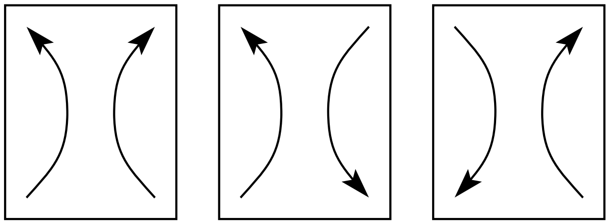

FIGURE B.13. **Three different orientation types for Reidemeister 2 moves.** Perform a Reidemeister 2 move locally in the square. The middle and the one on the right are "oppositely oriented" as in Lemma B.3.5.

PROPOSITION B.3.3. *Any link $\vec{L}$ admits a special diagram.*

Proof. For the projection $\mathcal{D}$ constructed in the proof of Lemma B.3.2, $\Sigma_{alg}(\mathcal{D})$ is F_b. □

Returning to the proof of the Reidemeister-Singer theorem, it remains to check that the algorithmic surfaces change by isotopies, stabilizations, and destabilizations under the various Reidemeister moves. We start with some special cases.

LEMMA B.3.4. *If $\mathcal{D}$ and $\mathcal{D}'$ are two knot projections that differ by a Reidemeister 1 move, then their algorithmic surfaces $\Sigma_{alg}(\mathcal{D})$ and $\Sigma_{alg}(\mathcal{D}')$ are isotopic.*

Proof. Suppose that $\mathcal{D}'$ has one more crossing than $\mathcal{D}$. Observe that $\Sigma_{alg}(\mathcal{D}')$ is obtained from $\Sigma_{alg}(\mathcal{D})$ by adding one more disk, connected to $\Sigma_{alg}(\mathcal{D})$ by a half twisted band. Untwist the band to get the isotopy to $\Sigma_{alg}(\mathcal{D})$. □

LEMMA B.3.5. *Suppose that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a single Reidemeister 2 move supported over a disk U, so that the two strands crossing each other (twice) are oriented oppositely. (See Figure B.13.) Then their respective algorithmic surfaces $\Sigma_{alg}(\mathcal{D})$ and $\Sigma_{alg}(\mathcal{D}')$ are either isotopic, or they become isotopic after a stabilization.*

Proof. Label the diagrams so that $\mathcal{D}'$ has two more crossings than $\mathcal{D}$. Consider a disk U in the link diagram that meets the two strands in $\mathcal{D}$, oriented oppositely. There are four combinatorially distinct cases, according to how the two arcs close up in the Seifert circles: (1) the two arcs are part of the same Seifert circle that bounds a disk D meeting U in one component, (2) the two arcs are part of the same Seifert circle that bounds a disk D meeting U in two components, (3) the two arcs are part of two distinct Seifert circles that are not nested, and finally (4) the two arcs are part of two distinct Seifert circles that are nested. See Figure B.14.

In case (1), $\Sigma_{alg}(\mathcal{D})$ and $\Sigma_{alg}(\mathcal{D}')$ are isotopic, via an isotopy that introduces two canceling twists in the band corresponding to D, as illustrated in the top row of Figure B.15. In cases (2)-(4), $\Sigma_{alg}(\mathcal{D})$ can be stabilized to obtain a surface which is isotopic to $\Sigma_{alg}(\mathcal{D}')$. The local pictures for cases (2) and (3) look the same, as in the second row of Figure B.15; the local picture for case (4) is shown in the third row of Figure B.15. □

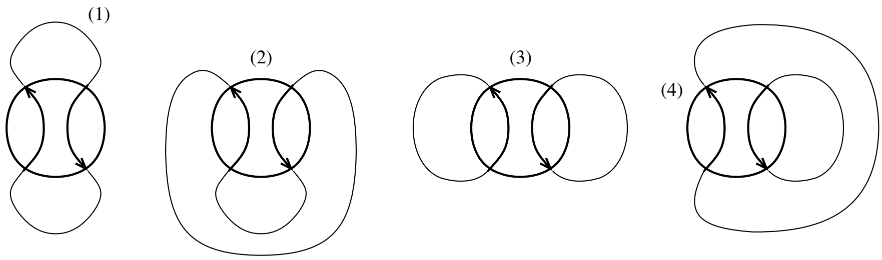

FIGURE B.14. **Cases of Reidemeister 2 moves with oppositely oriented strands.** We separate how the two local arcs complete to Seifert circles, as shown.

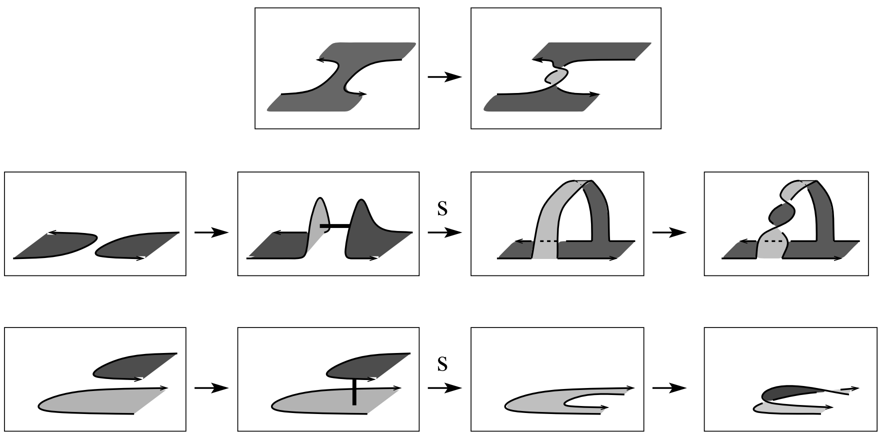

FIGURE B.15. **Algorithmic surfaces and Reidemeister** 2 **moves.** We have indicated sequences of isotopies and stabilizations that connect algorithmic surfaces under oppositely oriented Reidemeister 2 moves. Unmarked arrows between pictures indicate isotopies, and those labelled with "S" indicate stabilizations. Stabilizing arcs are indicated before each stabilization.

Suppose that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a Reidemeister 3 move supported over a disk U containing the three strands with three crossings. The oriented resolution of the three crossings give three disjoint strands in $\mathcal{D}$ (and possibly a further circle). The strands are said to be *compatibly oriented* if the three incoming ends of the strands in U (and hence also the three out-going ends) are consecutively ordered in the cyclic ordering of ∂U. Clearly, the three strands for $\mathcal{D}$ in U are oriented compatibly if an only if the corresponding three strands for $\mathcal{D}'$ in U are oriented compatibly. In this case, we call the Reidemeister move *compatibly oriented*; otherwise, we say it is *incompatibly oriented*. See Figure B.16.

Locally, inside U, if we consider an unoriented Reidemeister 3 move, there are 8 different orientations we could introduce, corresponding to the orientations of the 3 strands. Of these 8 orientations, 6 are compatibly oriented, and 2 are not.

LEMMA B.3.6. *If $\mathcal{D}$ and $\mathcal{D}'$ differ from each other by a compatibly oriented Reidemeister 3 move, then the surfaces $\Sigma_{alg}(\mathcal{D})$ and $\Sigma_{alg}(\mathcal{D}')$ are isotopic.*

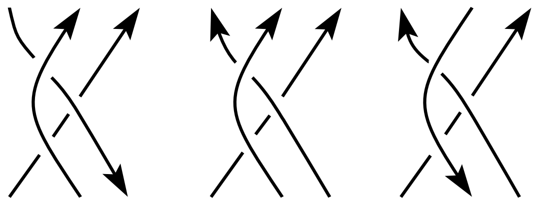

FIGURE B.16. **Compatibly oriented Reidemeister 3 moves.** The first two are compatibly oriented; the last one is not.

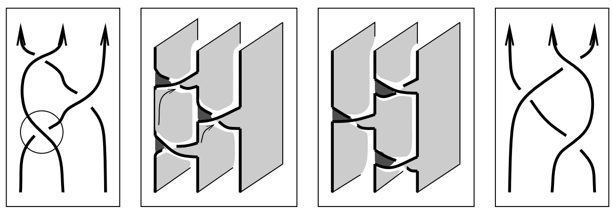

FIGURE B.17. **Sliding bands to effect Reidemeister 3 moves.** In the picture we perform a coherently oriented Reidemeister 3 move on the diagram (in the first picture). We have circled the crossing whose associated band, in the algorithmic surface (second picture) is to be slid over the other two bands (as indicated by the two arrows of the second picture), to give the algorithmic surface (third picture) for the diagram after the Reidemeister move (fourth picture).

Proof. Seifert's algorithm starts from a collection of disks that bound the oriented resolution $\mathcal{D}_0$ of $\mathcal{D}$. $\mathcal{D}_0$ meets U in three parallel, compatibly oriented strands. First add bands for the crossings outside U, and then attach three half-twisted bands to reintroduce the three crossings in U. Sliding one of these three bands across the other two gives the isotopy between $\Sigma_{alg}(\mathcal{D})$ and $\Sigma_{alg}(\mathcal{D}')$. We can identify the moving band as follows. Thinking of the strands as pointing upwards, there is a middle strand and two strands that cut across it, that we call *cross strands*. We claim that at least one of the two cross strands crosses the other two strands with the same sign. (If this were not the case, the diagram in $\mathcal{D} \cap U$ would be alternating, and hence the Reidemeister 3 move would not be possible.) Slide the band for the remaining crossing over the two bands attached to this cross strand. See Figure B.17. □

Proof of Theorem B.3.1. Lemma B.3.2 reduces the problem to show that any two algorithmic surfaces become isotopic after stabilizations. Theorem B.1.1 further reduces to the verification that algorithmic surfaces remain isotopic, up to stabilizations, under the three Reidemeister moves.

Isotopy invariance of the algorithmic surface up to stabilizations, for most cases of the Redemeister moves, was verified in Lemmas B.3.5, B.3.5 and B.3.6. The remaining two cases, i.e. Reidemeister 2 moves where the strands are oriented in the same direction and incompatibly oriented Reidmeister 3 moves, can be reduced to the earlier cases, as illustrated in Figure B.18. □

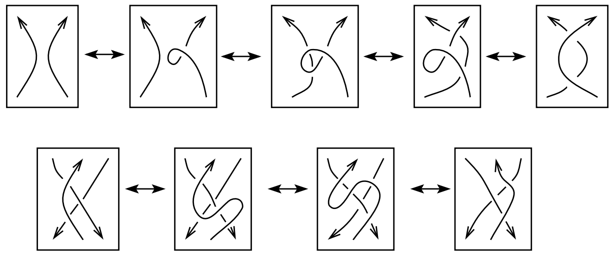

FIGURE B.18. **Reducing moves.** Expressing the remaining Reidemeister moves in terms of those studied previously.

B.4. Cromwell's Theorem

In this section we give a proof of Cromwell's Theorem 3.1.9 that identifies the local moves needed to connect any two grid diagrams representing isotopic links. Our proof is similar in spirit to Dynnikov's proof [**37**].

Recall that the *grid moves* on a planar grid diagram refer to the commutation moves (Definition 3.1.6) and the stabilizations and destabilizations (Definition 3.1.7).

THEOREM B.4.1 (Cromwell, [**27**]). *Two planar grid diagrams represent equivalent links if and only if there is a finite sequence of grid moves that transform one into the other.*

We will prove Theorem B.4.1 by approximating knot projections by grid diagrams. More formally:

DEFINITION B.4.2. Let $\mathcal{D}$ be a diagram for an oriented link. A ***grid approximation to*** $\mathcal{D}$ is a planar grid diagram whose associated oriented link diagram is planar isotopic to $\mathcal{D}$.

A key step in the proof is the following proposition, which we prove after some preparatory lemmas and definitions.

PROPOSITION B.4.3. *Any diagram for an oriented link has a grid approximation. Furthermore, this grid approximation is unique up to grid moves.*

We study knot diagrams via their projection to the y-axis $\mathbb{R} \subset \mathbb{R}^2$.

DEFINITION B.4.4. The projection of a diagram $\mathcal{D}$ to the y axis has two kinds of special points: critical points and crossings between two arcs. A diagram $\mathcal{D}$ is called a ***bridge diagram*** if the following conditions are met:

- crossings are not critical points of the projection;
- the critical points of the projection are isolated minima and maxima;
- no two special points project to the same value.

A diagram $\mathcal{D}$ in general position is a bridge diagram; in particular, any diagram can be approximated by an arbitrarily small (C^2) perturbation by bridge diagrams. The special points in a bridge diagram inherit an ordering induced by their y values.

DEFINITION B.4.5. A planar grid diagram G is called ***bridge-like*** if the following conditions are satisfied:

- Each horizontal segment contains at most one crossing.

- A horizontal segment is a local maximum or a local minimum if and only if it contains no crossings.

Tilting the horizontal segments without introducing new local minima or maxima and then smoothing out the corners transforms a bridge-like grid diagram into a bridge diagram. The resulting bridge diagram is called the *smoothing* of the given bridge-like grid diagram.

Conversely, given a bridge diagram, we will construct a bridge-like grid diagram in Lemma B.4.10. This construction will involve an intermediate object.

DEFINITION B.4.6. A ***pre-grid diagram*** is a planar diagram with the following properties:

- all the arcs are composed of vertical or horizontal segments, and
- all the vertical segments cross over the horizontal ones.

It is called ***generic*** if all the horizontal segments project to different y values, and all the vertical segments project to different x values. Two generic pre-grid diagrams $\mathcal{D}_0$ and $\mathcal{D}_1$ are said to be ***combinatorially equivalent*** if they can be connected by a one-parameter family of generic pre-grid diagrams.

Obviously, a grid diagram gives a generic pre-grid diagram in the above sense.

LEMMA B.4.7. *For each generic pre-grid diagram $\mathcal{E}$, there is a unique grid diagram whose associated projection is combinatorially equivalent to $\mathcal{D}$. Moreover, if two generic pre-grid diagrams can be connected by a one-parameter family of (not necessarily generic) pre-grid diagrams with the same number of segments, then their associated grid diagrams differ by commutation moves.*

Proof. If $\mathcal{E}$ is a generic pre-grid diagram, it has n horizontal segments, whose y-coordinates are $y_1 < \cdots < y_n$ and n vertical segments, whose x-coordinates are $x_1 < \cdots < x_n$. Move the horizontal segment whose y-coordinate is y_i to one whose y-coordinate is $i - \frac{1}{2}$. Move the vertical ones to half-integral coordinates similarly. The result is a grid diagram.

In a generic one-parameter family of pre-grid diagrams, there are finitely many values where two horizontal arcs project to the same y-coordinate, or two vertical ones project to the same x-coordinate. As we pass through each of these values, the associated grid diagrams undergo a single commutation move. □

LEMMA B.4.8. *Given any planar grid diagram G, there is a bridge-like grid diagram G' that can be obtained from G by a finite sequence of grid moves.*

Proof. The diagram G is not bridge-like if (1) there are horizontal segments that are local maxima or local minima that contain crossings; or (2) there are horizontal segments that contain no crossings that are not local maxima or minima; or (3) there are horizontal segments that contain more than one crossing.

Segments of the first kind can be stabilized so that they become two segments, where one is a local maximum or local minimum and the other contains the crossings. Segments of the second kind can be stabilized so that they become two segments, one of which is a local maximum and the other which is a local minimum. Segments of the third kind can be eliminated as follows. Suppose a horizontal segment contains k crossings. Stabilize $k - 1$ times at one of the two corners of this

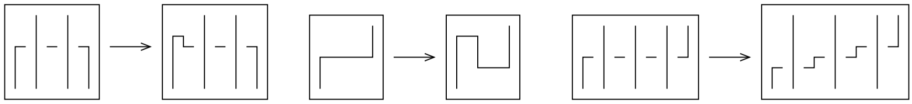

FIGURE B.19. **Turning a grid diagram into a bridge-like grid diagram.**

segment, and then commute the newly-created short vertical segments so that they separate the crossings. These three steps are illustrated in Figure B.19. ☐

We are now ready to associate a bridge-like grid diagram to a bridge diagram. To express the uniqueness of this construction, it is useful to have the following:

DEFINITION B.4.9. Two bridge diagrams $\mathcal{D}_0$ and $\mathcal{D}_1$ are ***bridge isotopic*** if they can be connected by a smoothly varying one-parameter family $\{\mathcal{D}_s\}_{s\in[0,1]}$ of bridge diagrams.

LEMMA B.4.10. *Given a bridge diagram $\mathcal{D}$, there is a bridge-like grid diagram whose smoothing is bridge isotopic to $\mathcal{D}$; and any two such bridge-like grid diagrams can be connected by a sequence of commutation moves.*

Proof. Associate first to any bridge diagram $\mathcal{D}$ a pre-grid diagram $\mathcal{E}$, as follows. Decompose $\mathcal{D}$ into vertical slices $y_0 < y_1 < \ldots < y_n$ so that $\mathcal{D}$ projects to $[y_0, y_n]$ and each interval $[y_i, y_{i+1}]$ contains the y coordinate of at most one special point. The diagram $\mathcal{E}$ is built out of pieces constructed from the slices $\mathbb{R} \times [y_i, y_{i+1}]$, as follows. If there are no special points in $\mathbb{R} \times [y_i, y_{i+1}]$, replace the segments in $\mathcal{D}$ by vertical segments; otherwise replace it with a planar isotopic picture containing exactly one horizontal segment and all other vertical segments, with the constraint that vertical segments cross over horizontal ones. See Figure B.20. Fit the pieces together, starting from the bottom piece, and successively attaching higher pieces, stretched horizontally as needed so that the vertical strands going off the bottom of the new piece match with the vertical strands going off the top of the previous one. The resulting pre-grid diagram $\mathcal{E}$ might not be generic (different horizontal

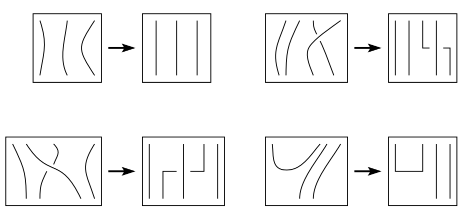

FIGURE B.20. **Approximating a bridge diagram by a grid diagram.**

arcs have distinct y values by construction; but different vertical arcs need not have distinct x values); but it has a small perturbation $\mathcal{E}'$ that is. Let G be a grid diagram whose associated projection is combinatorially equivalent to the generic pre-grid diagram $\mathcal{E}'$; this exists by Lemma B.4.7. If $\mathcal{E}'$ is sufficiently close to $\mathcal{E}$, the smoothing of G is bridge isotopic to $\mathcal{D}$, verifying the existence statement in our lemma.

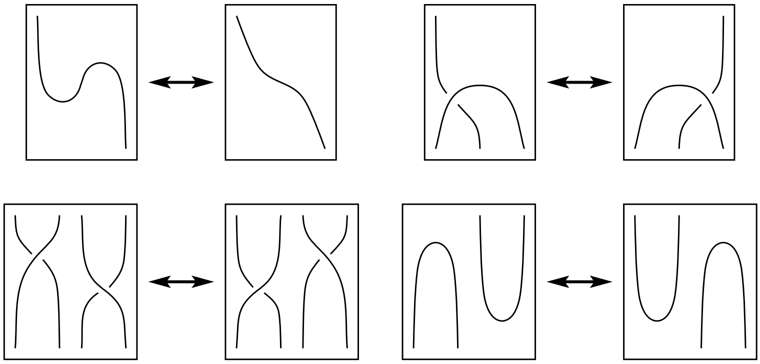

FIGURE B.21. **The three types of bridge moves needed to connect bridge diagrams.** Top left is a birth/death move; top right is a crossing slide; the bottom row shows two of the 16 possible special point commutations (between crossings and critical points).

The grid diagram G constructed above is uniquely defined up to commutation moves (independent of the choice of $\mathcal{E}'$) according to Lemma B.4.7. In fact, if G is any bridge-like grid diagram whose smoothing is bridge isotopic to $\mathcal{D}$, the diagram G itself can be obtained from the above procedure. The claimed uniqueness statement (up to commutations) follows. $\square$

Lemma B.4.10 associates to a bridge isotopy class of bridge diagram a grid diagram, determined uniquely up to commutation moves. We will now examine how the bridge isotopy class changes under a generic planar isotopy. Note that under a planar isotopy the number of crossings does not change, but the maxima and minima can interact with each other, or with the crossings, and the ordering on the special points can change. We formalize these changes as follows:

DEFINITION B.4.11. Two bridge diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ are said to be related by a ***bridge move*** if $\mathcal{D}_2$ is obtained from $\mathcal{D}_1$ by one of the three possible moves:

- creation of a pair of a local maximum and a local minimum, or the cancellation of such a pair; either is called a ***birth/death move***,
- sliding a crossing through a minimim or maximum, called a ***crossing slide***,
- commuting a pair of special points (each of which can be a maximum, minimum, or a crossing), called a ***special point commutation***.

See Figure B.21 for illustrations of these bridge moves.

LEMMA B.4.12. *Any two planar isotopic knot diagrams can be transformed into each other by a finite sequence of bridge moves.*

Proof. Fix two bridge diagrams $\mathcal{D}_0$ and $\mathcal{D}_1$. By hypothesis, these two bridge diagrams can be connected by a planar isotopy $\{\mathcal{D}_s\}_{s\in[0,1]}$. It follows from a general position argument that if the planar isotopy $\{\mathcal{D}_s\}_{s\in[0,1]}$ is chosen generically, then there are finitely many values of $s \in (0,1)$ for which $\mathcal{D}_s$ is not a bridge diagram; at each of these special values, for sufficiently small ϵ, the bridge diagrams $\mathcal{D}_{s-\epsilon}$ and $\mathcal{D}_{s+\epsilon}$ are related by one of the bridge moves enumerated in Definition B.4.11. $\square$

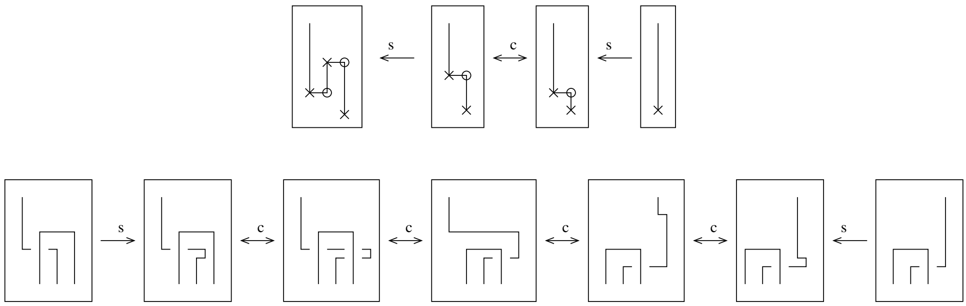

FIGURE B.22. **Realizing bridge moves as grid moves.** In the top row, we realize a birth/death move as grid moves; in the bottom, we realize a cross slide. In the top row, we mark the break points by X- and O-markings arbitrarily; in the bottom row, these markings are dropped. Arrows marked with c indicate commutation moves, and those with s indicate stabilizations. Note that the fourth step in the second row is a repeated application of commutation moves.

Proof of Proposition B.4.3. First isotop the diagram to a bridge diagram (which is unique up to bridge moves). Lemma B.4.10 associates to each bridge diagram $\mathcal{D}$ a grid diagram G, which is well-defined up to commutation moves. We claim that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are planar isotopic diagrams, then their associated planar grid diagrams can also be connected by a sequence of commutation and stabilization moves. Lemma B.4.12 reduces this to a verification in the case where $\mathcal{D}_1$ and $\mathcal{D}_2$ are related by a birth/death move, a crossing slide, or a special point commutation, where the verification is easy. The first two are illustrated in Figure B.22; special point commutations can be readily realized by commutation moves.

In effect, we have now defined a map F which associates to a link diagram $\mathcal{D}$ modulo planar isotopies a grid diagram $F(\mathcal{D})$, modulo grid moves. To complete the proof, we verify that if G is a grid diagram, and $\mathcal{D}(G)$ is its associated link diagram, then $F(\mathcal{D}(G))$ and G are equivalent under grid moves. In the special case where G is bridge-like, this follows from Lemma B.4.10. The general case can be reduced to this case by Lemma B.4.8. $\square$

Proof of Theorem B.4.1. It is straightforward to see that grid moves represent equivalent links: a commutation give rise to either a planar isotopy or a sequence of Reidemeister 2 and 3 moves; while a stabilization gives either a planar isotopy or a Reidemeister 1 move. Turning to the converse direction, in view of Theorem B.1.1 and Proposition B.4.3, it suffices to show that if $\mathcal{D}_1$ and $\mathcal{D}_2$ differ by any Reidemeister move, we can find grid approximations to $\mathcal{D}_1$ and $\mathcal{D}_2$ that differ by grid moves. This is illustrated in Figure B.23. $\square$

REMARK B.4.13. Cromwell's original proof [**27**] uses the interpretation of grid diagrams as braids. (See Section 12.8.) A theorem of Markov [**11**] (see also [**220**]) gives the basic moves that connect braids that determine isotopic links. Cromwell's proof then follows Markov's moves with grid moves.

B.4.1. Legendrian knots and Cromwell's theorem. The methods described above can be adapted to the Legendrian context. According Theorem 12.1.7,

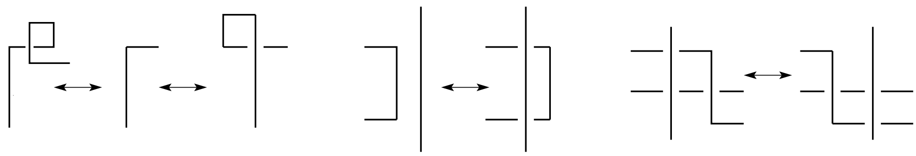

FIGURE B.23. **Reidemeister moves in grid diagrams.** Each move can be realized by (de)stabilizations and commutations.

Legendrian knots and links can be studied via their front projection, using the Legendrian Reidemeister moves (shown in Figure 12.2).

Adapting the proof of Cromwell's theorem in combination with the above result, we can express Legendrian knot theory in S^3 in terms of grid diagrams. (As usual, we will discuss the case of knots; the general case requires only minor modifications.)

It will be useful to have the following terminology:

DEFINITION B.4.14. For a planar grid diagram, we call commutation moves and (de)stabilizations of type *X:NW*, *X:SE*, *O:SE*, and *O:NW* ***Legendrian grid moves***.

As explained in Chapter 12, a planar grid diagram can be used to construct a Legendrian front projection. The adaptation of Theorem B.4.1 to the Legendrian case states:

THEOREM B.4.15. *The Legendrian knots $\vec{\mathcal{K}}_1$ and $\vec{\mathcal{K}}_2$ associated to the planar grid diagrams G_1 and G_2 are Legendrian isotopic if and only if G_1 and G_2 can be connected by a sequence of Legendrian grid moves.*

The proof is given after some preliminaries. We start by showing how to associate a grid diagram to a Legendrian front projection. Recall from Definition B.2.2 that a Legendrian front projection has two kinds of special points: cusps and crossings. A Legendrian knot was called front generic if its singularities are cusps, no cusp is on another branch, and different branches meet transversally, without triple intersections.

Start from a generic front projection, which we denote by $\mathcal{D}_{\vec{\mathcal{K}}}$, and rotate it 90°, so that the left cusps become local minima and the right cusps become local maxima, which we now smooth out. Switch all the crossings (note that a grid diagram G associates a Legendrian knot in the mirror of the knot type represented by G, cf. Definition 12.2.1). The result is a bridge diagram, in the sense of Definition B.4.4. In a typical bridge diagram, there are two types of allowed crossings; but in the bridge diagram arising from a Legendrian front projection, there is only one type of allowed crossing. Lemma B.4.10 in turn associates to the bridge diagram a grid diagram, unique up to commutation moves.

LEMMA B.4.16. *If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two front generic projections of Legendrian knots that differ by a Legendrian planar isotopy, then their associated grid diagrams G_1 and G_2 differ by a sequence of commutations.*

Proof. If two Legendrian front projections are Legendrian planar isotopic, then their associated bridge diagrams differ by a sequence of special point commutations, in the sense of Definition B.4.11: none of the other two types of bridge moves can

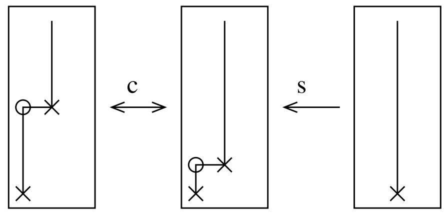

FIGURE B.24. **Eliminating horizontal segments that are not local maxima or minima.** Eliminate crossingless intervals (with northwest and southeast corners) by a sequence of commutation moves and a (Legendrian) destabilization.

occur. These special point commutations can be followed by commutation moves in the associated grid diagrams, as in the proof of Proposition B.4.3. ☐

Observe that the construction of a grid diagram from a Legendrian knot projection produces a bridge-like grid diagram. In fact, it is a special bridge-like grid diagram, in the following sense (compare with Definition B.4.5):

DEFINITION B.4.17. A grid diagram G is called ***Legendrian bridge-like*** if the following conditions are satisfied:

- A horizontal segment with a southwest corner on it is a local minimum, and one with a northeast corner on it is a local maximum.
- Each horizontal segment contains at most one crossing.
- A horizontal segment is a local maximum or a local minimum if and only if it contains no crossings.

We will need the following refinement of Lemma B.4.8:

LEMMA B.4.18. *For any grid diagram G, there is a Legendrian bridge-like diagram G' that can be obtained from G by a finite sequence of Legendrian grid moves.*

Proof. We adapt the proof of Lemma B.4.8. At each southwest corner in G that either contains crossings or is not a local minimum, we can stabilize G in a Legendrian manner, so that the new southwest corner occurs at a local minimum without crossings. Further Legendrian stabilizations can be done to ensure that the northeast corners are at local maxima, without crossings.

Next, we eliminate the horizontal segments that are not local maxima or local minima and that contain no crossings. Note that the corners of these segments are northwest and southeast. In the proof of Lemma B.4.8, we stabilized to eliminate these segments; but such a stabilization is not Legendrian, so we eliminate them now differently. If the horizontal segment is of length one, then we apply row commutations and commute it down until it meets the other marking in the column of its left endpoint. A Legendrian destabilization of *X:SE* or *O:SE* eliminates this horizontal segment; see Figure B.24.

If the segment has length $d > 1$, we can reduce d by commutation moves as follows. Suppose that the right endpoint of the horizontal segment connects to a vertical segment above the horizontal segment, so the left endpoint connects to a vertical segment below the horizontal. Take the leftmost vertical segment above the horizontal segment, if it exists. Apply repeated commutation moves to this segment, moving it to the left, until it is no longer above the distinguished horizontal segment, to reduce to a case where the horizontal segment has length less than d. If there is no leftmost vertical segment above the horizontal segment, take a rightmost vertical segment below the horizontal segment, and commute it to the right to reduce d. Note that if there is no vertical segment above or below the given horizontal one, then $d = 1$.

Eliminating all the horizontal segments with no crossings that are not maxima or minima, we obtain a new grid diagram G', which still may have horizontal segments that have more than one crossing. These segments are eliminated as in the proof of Lemma B.4.8 by stabilizations and commutations. Since the endpoints of these segments are northwest and southeast corners, the stabilizations we use are Legendrian. $\square$

With these preliminaries in place, we turn to the proof of the Legendrian analogue of Cromwell's theorem:

Proof of Theorem B.4.15. Any oriented Legendrian knot $\vec{\mathcal{K}}$ admits a generic front projection; fix one and call it $\mathcal{D}_{\vec{\mathcal{K}}}$. Lemma B.4.16 associates a grid diagram, up to commutation moves, to a Legendrian front projection, up to Legendrian planar isotopies. As the diagram undergoes Legendrian Reidemeister moves, observe that the associated grid diagram undergoes Legendrian grid moves: Reidemeister 2 and 3 moves can be realized by commutation moves, and Legendrian Reidemeister 1 moves are realized by Legendrian stabilizations and destabilizations. (Compare Figure B.23.) Thus, by Theorem 12.1.7, we have defined a map f from Legendrian knots (up to isotopy) to grid diagrams, up to Legendrian grid moves.

In the other direction, Definition 12.2.1 associates a Legendrian knot to a planar grid diagram, and Lemma 12.2.4 shows that this descends to a well-defined map g from grid diagrams up to Legendrian grid moves to Legendrian knots.

We verify that these two maps are inverses to one another. The fact that $\vec{\mathcal{K}}$ is Legendrian isotopic to $g(f(\vec{\mathcal{K}}))$ is straightforward. It remains to see that for a planar grid diagram G, $f(g(G))$ is equivalent to G under Legendrian grid moves. If G is Legendrian bridge-like in the sense of Definition B.4.17, then its associated Legendrian projection has the property that its associated grid diagram (according to Lemma B.4.16) differs from G by commutation moves. Thus, $f(g(G))$ and G are equivalent, as desired. Lemma B.4.18 reduces the general case to this special case, concluding the proof. $\square$

B.5. Normal forms of cobordisms between knots

In the proof of the slice genus bound provided by the knot signature and by the τ-invariant (Theorem 2.6.6 and Corollary 8.1.2 respectively) the normal form theorem (stated as Proposition 2.6.11 in Section 2.6) played a crucial role. In this section we verify Proposition 2.6.11, which we restate below:

PROPOSITION B.5.1. *Suppose that two knots $K_1, K_2 \subset S^3$ can be connected by a genus g oriented cobordism $W \subset [0,1] \times S^3$. Then, there are knots $K_1', K_2' \subset S^3$ and integers b and d with the following properties:*

(1) $\mathcal{U}_b(K_1)$ can be obtained from K_1' by b simultaneous oriented saddle moves.
(2) K_1' and K_2' can be connected by a sequence of $2g$ oriented saddle moves.
(3) $\mathcal{U}_d(K_2)$ can be obtained from K_2' by d simultaneous oriented saddle moves.

Recall that $\mathcal{U}_n(K)$ is the $(n+1)$-component link with one component K and n further unknotted, unlinked components; the definition of oriented saddle moves (and simultaneous oriented saddle moves) was given in Definition 2.6.10.

In the proof of Proposition B.5.1 we will appropriately isotop the surface W in $[0,1]\times S^3$. We will appeal to standard arguments and concepts from Morse theory [**142**]. (See also Section B.1.1.)

Let $f\colon M\to\mathbb{R}$ be a Morse function on a compact n-dimensional manifold M, and suppose that for some value $c\in\mathbb{R}$ the level set $f^{-1}(c)$ contains a unique critical point of index λ. Then for sufficiently small ϵ the sublevel set $f^{-1}((-\infty,c+\epsilon])$ can be constructed by adding an n-dimensional λ-handle to $f^{-1}((-\infty,c-\epsilon])$; see for example [**142**, Theorem 3.1]. For the definition of a λ-handle in general, see [**77**, Definition 4.1.1].

In fact, we will need this result only in dimension $n=2$. In this case, the Morse lemma shows that we can find local coordinates (x_1,x_2) in a local chart $U=(x_1,x_2)$ around a critical point p, corresponding to $x_1=x_2=0$, with respect to which the function takes the form

- $f(x)=c+x_1^2+x_2^2$ and so $c=f(p)$ is a local minimum, or
- $f(x)=c-x_1^2-x_2^2$ and so $f(p)$ is a local maximum, or
- $f(x)=c-x_1^2+x_2^2$. In this case, $f^{-1}((-\infty,c+\epsilon])$ is obtained from $f^{-1}((-\infty,c-\epsilon])$ by attaching a 2-dimensional 1-handle, that is, a band.

A Morse function is called *self-indexing* if its value at any critical point is equal to its index. As [**143**, Theorem 4.8] shows, every smooth manifold admits a self-indexing Morse function.

We will need a variant of the above theory, associated to cobordisms between knots in the product of an interval with S^3. To make notation somewhat simpler, from now on the cobordism between the copies of S^3 containing K_1 and K_2 will be identified with $[-1,3]\times S^3$. The embedding of W into $[-1,3]\times S^3$, followed by the projection onto the $[-1,3]$ factor defines a function, $f_W\colon W\to[-1,3]$.

Definition B.5.2. The cobordism $W\subset[-1,3]\times S^3$ is in ***normal form*** if the function f_W is Morse and it maps all the index-0 critical points on W to 0, the index-1 critical points to 1, and the index-2 critical points to 2; that is, f_W is a self-indexing Morse function on W.

Lemma B.5.3. ([**100**, Theorem 13.1.8], see also [**101**]) *A smooth cobordism W can be isotoped into normal form.*

Proof. We start by dealing with index-0 (and symmetrically index-2) critical points of f_W. We can assume that all critical points map into the interval $[0,2]$. For an index-0 critical point (if it is not already in $f_W^{-1}(0)$) consider an arc in $[0,2]\times S^3$ starting at the critical point and ending in $\{0\}\times S^3$, chosen so that the restriction of the the projection function to the arc has no critical points, and away from its starting point the arc is disjoint from W. (The existence of such an arc follows from a general position argument.) There is a local isotopy of W supported in a neighborhood of the arc that pushes the critical point down into $\{0\}\times S^3$. Index-2 critical points can be handled symmetrically by arcs starting at the critical point and ending in $\{2\}\times S^3$.

Next we deal with the index-1 critical points. Our aim is to show that two index-1 critical points can be pushed into the same level. Suppose that $t_1<t_2$

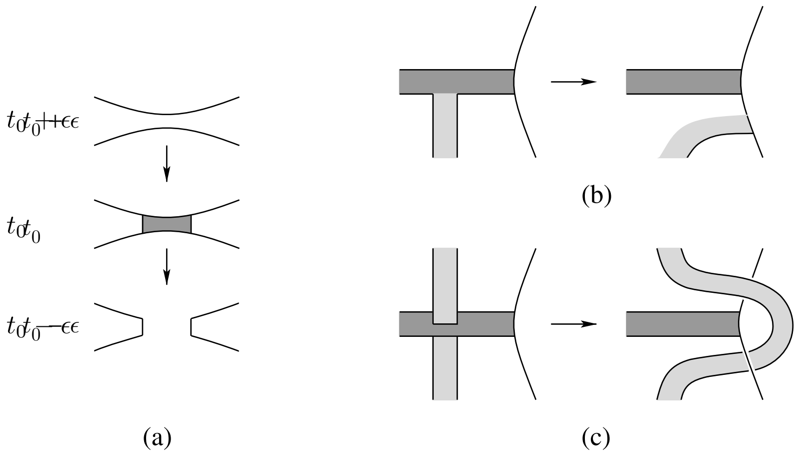

FIGURE B.25. **Turning a cobordism into normal form.** The figure in (a) shows how an index-1 critical point can be converted to a saddle band. Figures in (b) and (c) show how to arrange that the transported bands become disjoint from the bands in the new level. In (b) and (c) the lighter shading denotes the transported band from level t_1.

are two levels containing index-1 critical points with the additional property that $f_W^{-1}((t_1, t_2))$ contains no critical points. By isotoping W we replace the critical points by embedded bands, called *saddle bands*, as shown by Figure B.25(a). Isotop the surface so that the entire band is in a level set. (After this isotopy, the projection is no longer Morse.)

After this modification, the cobordism becomes trivial between the levels t_1 and t_2, since there are no critical values between t_1 and t_2. This means that after an isotopy (keeping $f_W^{-1}((-\infty, t_1])$ fixed) it can be assumed that the cobordism is the product cobordism between these two levels. Along the product cobordism, however, we can transport the saddle bands from level t_1 to t_2. If a transported band (from level t_1) is disjoint from all the bands in level t_2, then it will serve as a saddle band there (and can be converted back to an index-1 critical point, now on level t_2). It can happen, however, that a transported band B intersects some other bands in level t_2. The band we wish to transport can be viewed as a slight two-dimensional thickening of its one-dimensional core arc, so we examine how the core arc of the band can meet the other bands. The intersection is either at an endpoint of the core arc, or at an interior point. In case the intersection is at an endpoint, isotop the transported band slightly away, as illustrated in Figure B.25(b). In case the intersection point is an interior point of the core, we isotop the transported band (and hence the cobordism W) as indicated in Figure B.25(c). After these modifications we have a cobordism isotopic to the original one with the additional property that all index-1 critical points from level t_1 are moved to level t_2. Repeating this procedure for all the finitely many critical points of index 1, we get a surface isotopic to the original cobordism with the desired properties. $\square$

EXERCISE B.5.4. Using the normal form, show that a knot K is slice if and only if there is a ribbon knot R such that $K\#R$ is ribbon. (For the definition of slice and ribbon knots see Section 2.4; cf. the slice-ribbon conjecture from Remark 2.6.3.)

Proposition B.5.1 now follows from a slight modification of the normal form:

Proof of Proposition B.5.1. Suppose that $W \subset [-1,3] \times S^3$ is a cobordism between the knots K_1 and K_2, and assume (by Lemma B.5.3) that it is in normal form. Then the link $f_W^{-1}(0.1)$ can be identified with $\mathcal{U}_b(K_1)$ (for some $b \in \mathbb{N}$) and similarly $f_W^{-1}(1.9)$ can be identified with $\mathcal{U}_d(K_2)$ (for some $d \in \mathbb{N}$), where b and d are given by the number of index-0, resp. index-2 critical points of the Morse function f_W. Index-1 critical points define bands (i.e. 1-handles), and since W is connected, there are b index-1 critical points such that the corresponding handles turn $f_W^{-1}(0.1) = \mathcal{U}_b(K_1)$ into a knot K_1', and there are d further index-1 critical points such that the corresponding handles turn $f_W^{-1}(1.9) = \mathcal{U}_d(K_2)$ into a knot K_2'. Transporting the b index-1 critical points which connect $\mathcal{U}_b(K_1)$ to level 0.2, and the d index-1 critical points making $\mathcal{U}_d(K_2')$ connected to level 1.8, and denoting the resulting Morse function by g_W, the level sets $g_W^{-1}(0.5)$ and $g_W^{-1}(1.5)$ provide the desired knots K_1' and K_2'. (Note that since all the transported index-1 critical points are on levels 0.2 or 1.8, we get simultaneous saddle moves from them.) The remaining index-1 critical points provide a sequence of n oriented saddle bands between K_1' and K_2'. Since the Euler characteristic of F, which we assumed to be $-2g$, is given by $b + d$ minus the total number of saddles, it follows that there are $n = 2g$ saddles in the sequence of 1-handles from K_1' to K_2', completing the proof of the proposition. $\square$

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Index

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