The Dynamical Mordell–Lang Conjecture

Jason P. Bell
Dragos Ghioca
Thomas J. Tucker
The Dynamical Mordell–Lang Conjecture
The Dynamical Mordell–Lang Conjecture

Jason P. Bell
Dragos Ghioca
Thomas J. Tucker

American Mathematical Society
Providence, Rhode Island
To Jason’s wife, Jessica, and kids, Chris and Caitlin

To Dragos’ mother, Lidia

To Tom’s wife, Amanda
5.7. The case of Theorem 5.2.0.1 when the polynomials have different degrees

5.8. An alternative proof for the function field case

5.9. Possible extensions

5.10. The case of plane curves

5.11. A Dynamical Mordell-Lang type question for polarizable endomorphisms

Chapter 6. Parametrization of orbits

6.1. Rational maps

6.2. Analytic uniformization

6.3. Higher dimensional parametrizations

Chapter 7. The split case in the Dynamical Mordell-Lang Conjecture

7.1. The case of rational maps without periodic critical points

7.2. Extension to polynomials with complex coefficients

7.3. The case of “almost” post-critically finite rational maps

Chapter 8. Heuristics for avoiding ramification

8.1. A random model heuristic

8.2. Random models and cycle lengths

8.3. Random models and avoiding ramification

8.4. The case of split maps

Chapter 9. Higher dimensional results

9.1. The Herman-Yoccoz method for periodic attracting points

9.2. The Herman-Yoccoz method for periodic indifferent points

9.3. The case of semiabelian varieties

9.4. Preliminaries from linear algebra

9.5. Proofs for Theorems 9.2.0.1 and 9.3.0.1

Chapter 10. Additional results towards the Dynamical Mordell-Lang Conjecture

10.1. A \( v \)-adic analytic instance of the Dynamical Mordell-Lang Conjecture

10.2. A real analytic instance of the Dynamical Mordell-Lang Conjecture

10.3. Birational polynomial self-maps on the affine plane

Chapter 11. Sparse sets in the Dynamical Mordell-Lang Conjecture

11.1. Overview of the results presented in this chapter

11.2. Sets of positive Banach density

11.3. General quantitative results

11.4. The Dynamical Mordell-Lang problem for Noetherian spaces

11.5. Very sparse sets in the Dynamical Mordell-Lang problem for endomorphisms of \( (\mathbb{P}^1)^N \)

11.6. Reductions in the proof of Theorem 11.5.0.2

11.7. Construction of a suitable \( p \)-adic analytic function

11.8. Conclusion of the proof of Theorem 11.5.0.2

11.9. Curves

11.10. An analytic counterexample to a \( p \)-adic formulation of the Dynamical Mordell-Lang Conjecture
Preface

This book originated from the authors’ desire to give an explanation of several recent applications of $p$-adic analysis to number theory and especially to arithmetic geometry. Central to this end has been the work done by several people (including the authors) to prove the Dynamical Mordell-Lang conjecture, which gives predictions about how the orbits of points in a variety under self-maps should intersect subvarieties. As the name suggests, this can be interpreted as a dynamical analogue of the classical Mordell-Lang Conjecture (proved by Faltings and Vojta) concerning intersections between finitely generated subgroups and subvarieties in a semiabelian variety.

Many results working towards this conjecture have used $p$-adic analysis, and we describe all known (to us) partial results up to this point in time—both those using $p$-adic analysis and those using alternative approaches—towards the Dynamical Mordell-Lang Conjecture. In some cases, we present entire proofs of results, while in other cases only a sketch is given, and in certain cases only a brief overview of the idea of the proof is provided. Our choice should not be interpreted as our opinion about the relative importance of the included results, but is instead an editorial choice regarding which material we thought best fits the overarching theme of this book.

We also give other applications of $p$-adic analysis to number theory and arithmetic geometry. In these cases, our list of applications is not meant to be exhaustive, but rather our goal is to show the wide reach of applications and potential applications of $p$-adic analysis to arithmetic geometry. While the uses of $p$-adic analytic methods we give do not always explicitly relate to the Dynamical Mordell-Lang Conjecture, we have generally favored applications of $p$-adic analysis to problems with some relation to the Dynamical Mordell-Lang Conjecture.

We thank all our colleagues with whom we wrote many of the papers whose results are detailed in this book; obviously, without the joint efforts we put towards solving the Dynamical Mordell-Lang Conjecture we would not have had a topic for this book. So, we thank Rob Benedetto, Ben Hutz, Par Kurlberg, Jeff Lagarias, Tom Scanlon, Yu Yasufuku, Umberto Zannier, and Mike Zieve. We are also grateful to the referees for their careful reading of a previous version of this book, and for suggesting many improvements for our work. Last, but definitely not least, we thank our families for their love and support while writing this book.
Notation

We let \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \text{ and } \mathbb{C} \) be the sets of integer, rational, real, respectively, complex numbers. \( \mathbb{N}_0 \) is the set of all nonnegative integers, while \( \mathbb{N} \) is the set of all positive integers.

An arithmetic progression is a set of the form \( \{a + rn\}_{n \in \mathbb{N}_0} \), where the common difference \( r \) may be equal to 0 (in which case the set consists of a single element). If the common difference \( r \) is nonzero, then the arithmetic progression is infinite. Note that in the literature, sometimes one calls such a sequence a one-sided arithmetic progression in order to distinguish it from a two-sided arithmetic progression, which is a set of the form \( \{a + rn\}_{n \in \mathbb{Z}} \). However, since in this book we mainly encounter one-sided arithmetic progressions and only occasionally encounter two-sided arithmetic progressions, our convention is to call arithmetic progression a sequence \( \{a + rn\}_{n \in \mathbb{N}_0} \), while a sequence \( \{a + rn\}_{n \in \mathbb{Z}} \) is called a two-sided arithmetic progression.

For a matrix \( A \), we denote by \( A^t \) its transpose. For a set \( U \), we denote by \( id_U \) the identity function on \( U \).

For any field \( K \), we denote by \( \text{char}(K) \) its characteristic. By \( \overline{K} \) we denote a fixed algebraic closure of \( K \).

For any subfield \( K \subseteq \mathbb{Q} \), we denote by \( o_K \) the ring of algebraic integers contained in \( K \). If \( K \) is a number field, and \( p \) is a prime ideal of \( K \), then \( k_p \) is the residue field corresponding to \( p \), i.e., \( k_p \cong o_K/p \).

The usual affine space of dimension \( m \) is denoted by \( \mathbb{A}^m \); for any field \( K \), we have that \( \mathbb{A}^m(K) \) consists of all \( m \)-tuples of points with coordinates in \( K \). Similarly, we denote by \( \mathbb{P}^m \) the projective space of dimension \( m \); for any field \( K \), we have that \( \mathbb{P}^m(K) \) consists of all equivalence classes of \((m+1)\)-tuples of points with coordinates in \( K \) not all equal to 0, under the equivalence relation

\[
[x_0 : x_1 : \cdots : x_m] \sim [y_0 : y_1 : \cdots : y_m]
\]

if and only if there exists a nonzero scalar \( c \in K \) such that

\[
y_i = cx_i \quad \text{for all } i = 1, \ldots, m.
\]

By affine variety we mean a subset of an affine space defined by a set of algebraic equations. Note that we do not ask a priori the variety be irreducible. Similarly, by projective variety we mean a subset of a projective space defined by a set of algebraic equations. We endow both the affine space and the projective space with the Zariski topology where the closed sets are precisely the (affine, respectively projective) varieties. We say that \( X \) is a quasiprojective variety if it is the open subset of a projective subvariety of some projective space. We say that a variety \( X \) is defined over a field \( K \) if it may be defined by a set of equations with coefficients in \( K \). For a variety \( X \) defined over a field \( K \), we denote by \( X(K) \) the set of \( K \)-rational points of \( X \).

We denote by \( \mathbb{G}_a \) the affine line \( \mathbb{A}^1 \) endowed with the additive group law; we extend this law coordinatewise to \( \mathbb{G}_a^n \). We denote by \( \mathbb{G}_m \) the (Zariski open subset of the affine line) \( \mathbb{A}^1 \setminus \{0\} \), i.e., the affine line without the origin, endowed with the multiplicative group law. Similarly to \( \mathbb{G}_a^n \), we extend the multiplicative group law to \( \mathbb{G}_m^n \).

An abelian variety is an irreducible projective variety which has the structure of an algebraic group.
For a set $X$, a map $\Phi : X \to X$ is called a self-map. In general, for a self-map $\Phi : X \to X$ and for any integer $n \geq 0$, we denote by $\Phi^n$ the $n$-th compositional iterate of $\Phi$, i.e. $\Phi^n = \Phi \circ \cdots \circ \Phi$ ($n$ times), with the convention that $\Phi^0$ is the identity map. The orbit of a point $x \in X$ is denoted as $O_\Phi(x)$ and it is the set of all $\Phi^n(x)$ for $n \in \mathbb{N}_0$.

A dynamical system consists of a topological space $X$ endowed with a continuous self-map $\Phi$.

For two real-valued functions $f$ and $g$, we write $f(x) = o(g(x))$ if $\lim_{x \to \infty} f(x)/g(x) = 0$. Similarly, we write $f(x) = O(g(x))$ if the function $x \mapsto f(x)/g(x)$ is bounded as $x \to \infty$.

In a metric space $(X, d(\cdot, \cdot))$, for $x \in X$ and $r \in \mathbb{R}_{>0}$ we denote by $D(x, r)$ the open disk

$$D(x, r) = \{ y \in X : d(x, y) < r \}.$$  

We denote by $\overline{D}(x, r)$ the closure of $D(x, r)$. 
Bibliography


BIBLIOGRAPHY


Abelian variety, \( \text{xii, 5, 17} \)
endomorphism, \( \text{5, 17} \)
heights, \( \text{15} \)
is abelian, \( \text{17} \)
translation by a point, \( \text{5, 181} \)

Absolute value, \( \text{31} \)
archimedean, \( \text{31} \)
equivalence, \( \text{31} \)
extension, \( \text{31} \)
non-archimedean, \( \text{31} \)
norm, \( \text{31} \)
p-adic, \( \text{24} \)
place, see also Place
triangle inequality, \( \text{31} \)
ultrametric inequality, \( \text{31} \)
valuation, \( \text{31} \)

Affine space, \( \text{511, 11} \)
Algebraic group, \( \text{17} \)
Algebraic power series, \( \text{238} \)
Algebraically stable, \( \text{176} \)
Analytic variety, \( \text{169} \)
Analytic Zariski dense, \( \text{169} \)
André-Oort Conjecture, \( \text{59} \)
Arithmetic progression, \( \text{511} \)
common difference, \( \text{511} \)
infinite, \( \text{511} \)
one-sided, \( \text{511} \)
two-sided, \( \text{511} \)

Automata theory, \( \text{231, 233} \)
Automatic sequence, \( \text{232, 238, 243} \)
Automatic set, \( \text{238} \)
Autormorphism, \( \text{58, 74} \)

Bad reduction, see also Good reduction
Banach density, \( \text{170} \)
Banach density, \( \text{52, 170, 183, 185} \)
is subadditive, \( \text{184} \)
Berkovich space, \( \text{190} \)
Birational equivalence, \( \text{13} \)
Birthday paradox, \( \text{143} \)
Bogomolov Conjecture, \( \text{59} \)

Canonical height, see also Height, canonical polarizable endomorphism, see also Height, polarizable endomorphism
Chebotarev Density Theorem, \( \text{32, 144} \)
Chebyshev polynomial, \( \text{102, 133, 255} \)
homogeneous, \( \text{93} \)
Codimension, \( \text{15} \)
Cohen Structure Theorem, \( \text{76} \)
Coherent backward orbit, \( \text{251} \)
Compactification, \( \text{176} \)
Coordinate ring, \( \text{14} \)
Critical point, see also Ramification point, \( \text{127, 206} \)

Definable sets, \( \text{172} \)
Deformed torus, \( \text{170} \)
Degree
rational map, \( \text{14} \)
Denis-Manin-Mumford conjecture, \( \text{219} \)
Denis-Mordell-Lang Conjecture, \( \text{61} \)
Difference operator, \( \text{29} \)
Differential, \( \text{16} \)
1-form, \( \text{16} \)
Differential equations, \( \text{52} \)
Dimension, \( \text{16} \)
Divisor, \( \text{17} \)
Cartier, \( \text{18} \)
effective, \( \text{14} \)
linear equivalence, \( \text{18} \)
principal, \( \text{18} \)
support of, \( \text{17} \)
Weil, \( \text{18} \)
Drinfeld module, \( \text{61, 217, 218} \)
algebraic submodule, \( \text{220} \)
exponential, \( \text{223} \)
generic characteristic, \( \text{218} \)
logarithm, \( \text{223} \)
special characteristic, \( \text{218, 219} \)
 submodule, \( \text{218} \)
torsion point, \( \text{218} \)
torsion submodule, \( \text{218} \)
Dynamical degree, see also Degree, dynamical
Dynamical Mordell-Lang Conjecture, \( \text{2, 15} \)

277
INDEX

p-adic analytic version, 209
positive characteristic, 231
rational maps, 53, 131
split case, 55, 57, 85, 110, 112, 113, 127
Dynamical Mordell-Lang principle, 171
Dynamical system, xiii
Eigenvalue, 124, 150, 158
Endomorphism
automorphism, see also Automorphism
conjugated, 49
étale, 55, 113
generic, 53, 251
polarizable, 24, 44, 113, 261
split, 56
Exceptional point, 120, 127
Fiber, 21
Finite-state automata, 238, 243, 244
First-order language, 172
Flat
map, 18
module, 18
Formal diffeomorphism, 124
Forward difference operator, 29
F-set, 233
Function field, 13
Gaussian distribution, 145
Generic fiber, 21
Genus, 16
Global field, 40
Global sections, 22
Good reduction, 100, 118, 119, 223
Grand orbit, 251
Group scheme
additive, 131, 137
multiplicative, 131, 137
Height, 40, 41
abelian variety, 45
canonical, 42
local canonical, 42
polarizable endomorphism, 45
projective space, 44
Weil, 41, 125, 196
Hensel’s Lemma, 25, 26
Hilbert’s Irreducibility Theorem, 33
Homothety, 154
Hypersurface, 11
Independent events, 149
Indeterminacy locus, 13, 176
Iterational special variety, 169
Iterational variety, 169
Jacobian, 15
Jordan block, 160
Jordan matrix, 160
Laurent’s theorem, 104
Line bundle, 20
ample, 22
very ample, 22, 44
Linear recurrence sequence, 1, 2, 33, 36
characteristic equation, 36
characteristic roots, 36
extceptional zeros, 39
Fibonacci, 2
length, 35
non-degenerate, 35
recurrence relation, 1
simple, 35
Local parameters
at a smooth point, 155
Local ring, 13
Locally free sheaf, see also Sheaf
rank of, 20
Locally ringed space, 20
Mahler series, 25
analytic, 20
Manin-Mumford Conjecture, 55
Monomializable map, 176
Mordell-Lang Conjecture, 57, 60
Morphism, 13
automorphism, 68
birational, 13
closed, 16
dominant, 14
endomorphism, see also Endomorphism, 13
étale, 8, 13, 40, 52
finite, 14
flat, 18
immersion, 24
isomorphism, 13
of presheaves, 19
of ringed spaces, 21
of schemes, 21
of S-schemes, 21
unramified, 15
Néron-Severi group, 175
Natural density, 179, 183
Noetherian space, 181, 189
Nonsingular point, see also Smooth point
Normal form, 138
Normalization, 16
Northcott’s Theorem, 11
o-Minimal, 172
o-minimal structure, 171, 172, 245
Orbit
of a point, xiii, 22
of a subvariety, 24
p-adic
  analytic curve, 209
  analytic function, 28, 120, 156, 160, 199
  analytic manifold, 156
  integer, 23
  logarithm, 28
  unit, 25
  p-adic arc lemma, 8, 55, 57, 60, 62, 68, 73, 129, 145, 150, 153, 156, 158, 167, 182, 205, 209, 221, 258
  approximation, 209
p-adic lemma, 143
Perfect field, 230
Periodic
cycle, see also Periodic point, cycle point, see also Periodic point
subvariety, 22, 23, 50, 111, 179, 258
Periodic point, 119
  attracting, 117, 119, 120, 126, 153, 169
  cycle, 205
  indifferent, 117, 125, 158, 205
  minimal period, 22
  period, 22
  super-attracting, 117, 119, 128, 169, 206
Picard group, 18
Pink-Zilber Conjectures, 55
Place, 31
  finite, 223
  infinite, 223
  lying above, 31
  lying below, 31
Polynomial
  additive, see also Drinfeld module
  Chebyshev, see also Chebyshev
    polynomial
    isotrivial, 43
    normal form, 125, 133
Polynomial-exponential equation, 3, 237
Preperiodic
  point, 22, 43, 49
  subvariety, 22
Product formula, 40
  field, 40
Projective space, 11
  Projective variety, 11
    is complete, 16
Projectively linearizable map, 173
Push-forward, 20
Quasiperiodic
domain, 122, 127, 129, 131, 137, 207
  map, 120
Quasiperiodic disk, see also Quasiperiodic domain
Quasiperiodic map, 207
Quasiprojective variety, see also Variety
Ramification, 13, 80
  locus, 143, 144
  point, 120
Random map behavior, see also Random model
Random model, 145, 146, 148, 150
Rational map, 13
  post-critically finite, 128, 156, 257
Regular function, see also Regular map
  at a point, 12
Residue field, 31
Rigid analytic space, 197
Rigid analytic topology, see also Rigid analytic space
Ringed space, 20
Ritt’s theorem, 90, 91, 102
Scheme, 20, 21
  S-scheme, 21
  affine, 21
  morphism, 21
Self-map, 313
Semiabelian variety, 17, 50
  characteristic p, 232
  endomorphism, 17
  translation by a point, 231
Sheaf, 18
  invertible, 20
  locally free, 20
  of regular functions, 19
  presheaf, 18
  restricted, 20
  stalk, 19
  tensor power, 20
  tensor product, 20
Siegel curve, see also Siegel factor, 93, 103
  Siegel factor, 32, 103, 104, 113
  Siegel’s Theorem, 32, 98
  S-integer, 32
  S-integral, 258
Skolem’s method, 36, 58
  see also p-adic arc lemma
Skoelen-Mahler-Lech Theorem, 243, 248, 250
Smooth point, 115, 130
  set of all smooth points, 116
Smooth variety, 15
  is normal, 16
  Special fiber, 175
  Specialization, 95
  Spectral radius, 175
  Spectrum of a ring, 20
  Szeméredi’s theorem, 182
Tangent space, 115, 154
Tate algebra, 189, 210
T-module, 210
Torus
  algebraic, 170, 171
deformed, 170

Valuation, 24

\( p \)-adic, 24

Vanishing ideal, 11

Variety, 12

abelian, see also Abelian variety

affine, xii see also Affine variety

as a scheme, 24

complete, 16

defined over a field, xii

degree of, 15

dimension of, 15

geometrically irreducible, 12

irreducible, xii 12

model of, 24

normal, 16

projective, xii see also Projective variety

quasiprojective, xii 12

rational points of, xii

semiabelian, see also Semiabelian variety

smooth locus, see also Smooth point, 154

subvariety, 12

Very dense set, 68 180 183

Very sparse set, 179 181 195 197 207 209

Weil bounds, 144 148

Zariski closure, 12

Zariski topology, 11
Selected Published Titles in This Series

209 Steve Y. Oudot, Persistence Theory: From Quiver Representations to Data Analysis, 2015
208 Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó, Grid Homology for Knots and Links, 2015
207 Vladimir I. Bogachev, Nicolai V. Krylov, Michael Röckner, and Stanislav V. Shaposhnikov, Fokker–Planck–Kolmogorov Equations, 2015
205 Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik, Tensor Categories, 2015
204 Victor M. Buchstaber and Taras E. Panov, Toric Topology, 2015
203 Donald Yau and Mark W. Johnson, A Foundation for PROPs, Algebras, and Modules, 2015
201 Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill, Editors, Topological Modular Forms, 2014
200 Nikolaï Nadirashvili, Vladimir Tkachev, and Serge Vlăduț, Nonlinear Elliptic Equations and Nonassociative Algebras, 2014
198 Jörn Jahnel, Brauer Groups, Tamagawa Measures, and Rational Points on Algebraic Varieties, 2014
197 Richard Evan Schwartz, The Octagonal PETs, 2014
196 Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou, Geometry of Isotropic Convex Bodies, 2014
195 Ching-Li Chai, Brian Conrad, and Frans Oort, Complex Multiplication and Lifting Problems, 2014
194 Samuel Herrmann, Peter Imkeller, Ilya Pavlyukevich, and Dierk Peithmann, Stochastic Resonance, 2014
193 Robert Rumely, Capacity Theory with Local Rationality, 2013
192 Messoud Efendiev, Attractors for Degenerate Parabolic Type Equations, 2013
191 Grégoire Berhuy and Frédérique Oggier, An Introduction to Central Simple Algebras and Their Applications to Wireless Communication, 2013
190 Aleksandr Pukhlikov, Birationally Rigid Varieties, 2013
189 Alberto Elduque and Mikhail Kochetov, Gradings on Simple Lie Algebras, 2013
188 David Lannes, The Water Waves Problem, 2013
186 Gregory Berkolaiko and Peter Kuchment, Introduction to Quantum Graphs, 2013
185 Patrick Iglesias-Zemmour, Diffeology, 2013
184 Frederick W. Gehring and Kari Hag, The Ubiquitous Quasidisk, 2012

For a complete list of titles in this series, visit the AMS Bookstore at www.ams.org/bookstore/survseries/.
The Dynamical Mordell-Lang Conjecture is an analogue of the classical Mordell-Lang conjecture in the context of arithmetic dynamics. It predicts the behavior of the orbit of a point $x$ under the action of an endomorphism $f$ of a quasiprojective complex variety $X$. More precisely, it claims that for any point $x$ in $X$ and any subvariety $V$ of $X$, the set of indices $n$ such that the $n$-th iterate of $x$ under $f$ lies in $V$ is a finite union of arithmetic progressions. In this book the authors present all known results about the Dynamical Mordell-Lang Conjecture, focusing mainly on a $p$-adic approach which provides a parametrization of the orbit of a point under an endomorphism of a variety.