Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations

Jared Speck
Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations
Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations

Jared Speck
To Genevieve, for sleeping (just) long enough to allow me to complete this project, and to Teddy, the comeback kid with a heart of steel.
## Contents

Preface xv

Acknowledgments xxiii

Chapter 1. Introduction

1.1. Shock formation in one spatial dimension 1

1.2. New aspects in more than one spatial dimension 11

Chapter 2. Overview of the Two Main Theorems

2.1. First description of the two theorems 33

2.2. The basic structure of the equations 35

2.3. The structure of the equation relative to rectangular coordinates 38

2.4. The (classic) null condition 38

2.5. Basic geometric constructions 41

2.6. The rescaled frame and dispersive sup-norm estimates 44

2.7. The basic structure of the coupled system and sup-norm estimates for the eikonal function quantities 46

2.8. Lower bounds for the rescaled radial derivative of the solution in the case of shock formation 47

2.9. The main ideas behind the vanishing of the inverse foliation density 48

2.10. The role of Theorem 22.1 in justifying the heuristics 49

2.11. Comparison with related work 61

2.12. Outline of the monograph 76

2.13. Suggestions on how to read the monograph 78

Chapter 3. Initial Data, Basic Geometric Constructions, and the Future Null Condition Failure Factor

3.1. Initial data 81

3.2. The eikonal function and the geometric radial variable 82

3.3. First fundamental forms and Levi-Civita connections 83

3.4. Frame vectorfields and the inverse foliation density 84

3.5. Geometric coordinates 87

3.6. Frames 89

3.7. The future null condition failure factor 90

3.8. Contraction and component notation 90

3.9. Projection operators and tensors along submanifolds 91

3.10. Expressions for the metrics and volume form factors 93

3.11. The trace and trace-free parts of tensors 95

3.12. Angular differential 96

3.13. Musical notation 97
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.14</td>
<td>Pointwise norms</td>
<td>97</td>
</tr>
<tr>
<td>3.15</td>
<td>Lie derivatives and projected Lie derivatives</td>
<td>97</td>
</tr>
<tr>
<td>3.16</td>
<td>Second fundamental forms</td>
<td>99</td>
</tr>
<tr>
<td>3.17</td>
<td>Frame components, relative to the nonrescaled frame, of the derivatives of the metric with respect to the solution</td>
<td>100</td>
</tr>
<tr>
<td>3.18</td>
<td>The change of variables map</td>
<td>101</td>
</tr>
<tr>
<td>3.19</td>
<td>Area forms, volume forms, and norms</td>
<td>102</td>
</tr>
<tr>
<td>3.20</td>
<td>Schematic notation</td>
<td>104</td>
</tr>
<tr>
<td>4.1</td>
<td>Re-centered variables and the eikonal function quantities</td>
<td>107</td>
</tr>
<tr>
<td>4.2</td>
<td>Covariant derivatives and Christoffel symbols relative to the rectangular coordinates</td>
<td>108</td>
</tr>
<tr>
<td>4.3</td>
<td>Transport equation for the inverse foliation density</td>
<td>108</td>
</tr>
<tr>
<td>4.4</td>
<td>Transport equations for the rectangular components of the frame vectorfields</td>
<td>109</td>
</tr>
<tr>
<td>4.5</td>
<td>An expression for the re-centered null second fundamental form in terms of other quantities</td>
<td>111</td>
</tr>
<tr>
<td>4.6</td>
<td>Identities involving deformation tensors and Lie derivatives</td>
<td>112</td>
</tr>
<tr>
<td>5.1</td>
<td>Connection coefficients of the rescaled frame</td>
<td>117</td>
</tr>
<tr>
<td>5.2</td>
<td>Connection coefficients of the rescaled null frame</td>
<td>119</td>
</tr>
<tr>
<td>5.3</td>
<td>Frame decomposition of the inverse-foliation-density-weighted wave operator</td>
<td>119</td>
</tr>
<tr>
<td>6.1</td>
<td>Construction of the rotation vectorfields</td>
<td>123</td>
</tr>
<tr>
<td>6.2</td>
<td>Basic properties of the rotation vectorfields</td>
<td>123</td>
</tr>
<tr>
<td>7.1</td>
<td>The commutation vectorfields</td>
<td>127</td>
</tr>
<tr>
<td>7.2</td>
<td>Deformation tensor calculations</td>
<td>128</td>
</tr>
<tr>
<td>8.1</td>
<td>Definitions of various differential operators</td>
<td>135</td>
</tr>
<tr>
<td>8.2</td>
<td>Operator commutator identities</td>
<td>136</td>
</tr>
<tr>
<td>8.3</td>
<td>Notation for repeated differentiation</td>
<td>140</td>
</tr>
<tr>
<td>9.1</td>
<td>Preliminary calculations</td>
<td>143</td>
</tr>
<tr>
<td>9.2</td>
<td>Frame decomposition of the commutation current</td>
<td>146</td>
</tr>
<tr>
<td>10.1</td>
<td>Preliminary calculations</td>
<td>151</td>
</tr>
</tbody>
</table>
10.2. The energy-cone flux integral identities 156
10.3. Integration by parts identities for the top-order square integral estimates 160
10.4. Error integrands arising from the deformation tensors of the multiplier vectorfields 164

Chapter 11. Avoiding Derivative Loss and Other Difficulties via Modified Quantities 167
11.1. Preliminary structural identities 168
11.2. Full modification of the trace of the re-centered null second fundamental form 173
11.3. Partial modification of the trace of the re-centered null second fundamental form 177
11.4. Partial modification of the angular gradient of the inverse foliation density 178

Chapter 12. Small Data, Sup-Norm Bootstrap Assumptions, and First Pointwise Estimates 181
12.1. Restricting the analysis to solutions of the evolution equations 181
12.2. Small data 182
12.3. Fundamental positivity bootstrap assumption for the inverse foliation density 182
12.4. Sup-norm bootstrap assumptions 182
12.5. Basic estimates for the geometric radial variable 184
12.6. Basic estimates for the rectangular spatial coordinate functions 184
12.7. Estimates for the rectangular components of the metrics and the spherical projection tensorfield 185
12.8. The behavior of quantities along the initial data hypersurface 187
12.9. Estimates for the derivatives of rectangular components of various vectorfields and the radial component of the Euclidean rotations 190
12.10. Estimates for the rectangular components of the metric dual of the unit-length radial vectorfield 192
12.11. Precise pointwise estimates for the rotation vectorfields 193
12.12. Precise pointwise differential operator comparison estimates 197
12.13. Useful estimates for avoiding detailed commutators 199
12.14. Estimates for the derivatives of the angular differential of the rectangular spatial coordinate functions 200
12.15. Pointwise estimates for the Lie derivatives of the frame components of the derivative of the rectangular components of the metric with respect to the solution 200
12.16. Crude pointwise estimates for the Lie derivatives of the angular components of the deformation tensors 201
12.17. Two additional crude differential operator comparison estimates 203
12.18. Pointwise estimates for the derivatives of the re-centered null second fundamental form in terms of other quantities 204
12.19. Pointwise estimates for the Lie derivatives of the rotation vectorfields 206
12.20. Pointwise estimates for the angular one-forms and vectorfields corresponding to the commutation vectorfield deformation tensors 207
12.21. Preliminary Lie derivative commutator estimates 209
12.22. Commutator estimates for vectorfields acting on functions and spherical covariant tensorfields 210
12.23. Commutator estimates for vectorfields acting on the covariant angular derivative of a spherical tensorfield 215
12.24. Commutator estimates for vectorfields acting on the angular Hessian of a function 215
12.25. Commutator estimates involving the trace and trace-free parts 218
12.26. Pointwise estimates, in terms of other quantities, for the Lie derivatives of the re-centered null second fundamental form involving an outgoing null differentiation 223
12.27. Improvement of the auxiliary bootstrap assumptions 225
12.28. Sharp pointwise estimates for a frame component of the derivative of the metric with respect to the solution 230
12.29. Pointwise estimates for the angular Laplacian of the derivatives of the rectangular components of the re-centered version of the outgoing null vectorfield 231
12.30. Estimates related to integrals over the spheres 233
12.31. Faster than expected decay for certain wave-variable-related quantities 236
12.32. Pointwise estimates for the vectorfield $X_i$ 239
12.33. Estimates for the components of the commutation vectorfields relative to the geometric coordinates 241
12.34. Estimates for the rectangular spatial derivatives of the eikonal function 243

Chapter 13. Sharp Estimates for the Inverse Foliation Density 245
13.1. Basic ingredients in the analysis 245
13.2. Sharp pointwise estimates for the inverse foliation density 249
13.3. Fundamental estimates for time integrals involving the foliation density 266

Chapter 14. Square Integral Coerciveness and the Fundamental Square-Integral-Controlling Quantities 277
14.1. Coerciveness of the energies and cone fluxes 277
14.2. Definitions of the fundamental square-integral-controlling quantities 279
14.3. Coerciveness of the fundamental square-integral-controlling quantities 280

Chapter 15. Top-Order Pointwise Commutator Estimates Involving the Eikonal Function 283
15.1. Top-order pointwise commutator estimates connecting the angular Hessian of the inverse foliation density to the radial Lie derivative of the re-centered null second fundamental form 283
15.2. Top-order pointwise commutator estimates corresponding to the spherical Codazzi equations 290
Chapter 16. Pointwise Estimates for the Easy Error Integrands and Identification of the Difficult Error Integrands Corresponding to the Commuted Wave Equation

16.1. Preliminary analysis and the definition of harmless terms
16.2. The important terms in the top-order derivatives of the deformation tensors of the commutation vectorfields
16.3. Crude pointwise estimates for the below-top-order derivatives of the deformation tensors of the commutation vectorfields
16.4. Pointwise estimates for the top-order derivatives of the outgoing null derivative of the commutation vectorfield deformation tensors
16.5. Proof of Proposition 16.3
16.6. Proof of Corollary 16.5
16.7. Pointwise estimates for the error integrands involving the deformation tensors of the multiplier vectorfields
16.8. Pointwise estimates needed to close the elliptic estimates

Chapter 17. Pointwise Estimates for the Difficult Error Integrands Corresponding to the Commuted Wave Equation

17.1. Preliminary pointwise estimates for the derivatives of the inhomogeneous terms in the transport equations for the fully modified quantities
17.2. Preliminary pointwise estimates for the derivatives of the inhomogeneous terms in the transport equations for the partially modified quantities
17.3. Solving the transport equation satisfied by the fully modified version of the spatial derivatives of the trace of the re-centered null second fundamental form
17.4. Pointwise estimates for the difficult error integrands requiring full modification
17.5. Pointwise estimates for the difficult error integrands requiring partial modification

Chapter 18. Elliptic Estimates and Sobolev Embedding on the Spheres

18.1. Elliptic estimates
18.2. Sobolev embedding

Chapter 19. Square Integral Estimates for the Eikonal Function Quantities that Do Not Rely on Modified Quantities

19.1. Square integral estimates for the eikonal function quantities that do not rely on modified quantities

Chapter 20. A Priori Estimates for the Fundamental Square-Integral-Controlling Quantities

20.1. Bootstrap assumptions for the fundamental square-integral-controlling quantities
20.2. Statement of the two main propositions and the fundamental Gronwall lemma
20.3. Estimates for all but the most difficult error integrals
20.4. Difficult top-order error integral estimates
CONTENTS

B.20. Modified quantities
B.21. Curvature tensors
B.22. Omission of the independent variables in some expressions
B.23. Data and functions relevant for the proof of shock formation

Bibliography

Index
Preface

A central issue surrounding the study of quasilinear hyperbolic PDEs is that classical solutions, generated by smooth initial conditions, can develop singularities in finite time. In principle, many different kind of singularities are possible. The subject of this monograph is perhaps the most well-known type: shocks. Our work here primarily concerns scalar quasilinear wave equations, the main reasons being 1) they arise in many important mathematical, physical, and geometric contexts; 2) the last few decades have led to the development of advanced machinery tailored to such equations; and 3) their characteristic hypersurfaces are relatively simple and can be analyzed using tools and concepts from the well-developed theory of Lorentzian geometry.

Before the groundbreaking work of S. Alinhac, very little was known about shock formation except in problems that are effectively one (spatial) dimensional. One-dimensional shock formation results are of course classical. We describe some of the most important such results in Chapter 1. As we describe in detail in Chapter 2, Alinhac proved finite-time shock formation for a class of wave equations in two and three spatial dimensions. Roughly, his proof applied to wave equations of the form whenever the nonlinear terms fail to satisfy the null condition, which was first formulated by S. Klainerman in three spatial dimensions and later in two spatial dimensions by Alinhac. Alinhac’s main results concern solutions generated by small initial data that belong to a suitable Sobolev space and that verify a non-degeneracy condition. The foundation of his proof was a new system of geometric coordinates tied to an eikonal function, which by definition is a solution to the eikonal equation, that is, the hyperbolic PDE supplemented by appropriate initial conditions. The level sets of $u$ are true characteristics (as opposed to approximate ones) corresponding to the nonlinear wave equation.

1In the study of wave equations, characteristic hypersurfaces are often referred to as “null hypersurfaces” in view of their connection to the Lorentzian notion of a null vectorfield. More generally, they are often referred to as simply “the characteristics.”

2Roughly, in our study of wave equations, the characteristics are a family of “true curved cones” corresponding to the dynamic Lorentzian metric of the wave equation.

3The equation is to be interpreted as an equation given relative to standard rectangular coordinates, which we describe at the beginning of Chapter 2. Moreover, here and throughout, we use Einstein’s summation convention.
Eikonal functions are perhaps best known for the central role they played in Christodoulou-Klainerman’s celebrated proof \cite{19} of the stability of Minkowski spacetime. That work was the first instance in which eikonal functions were used to prove a global nonlinear result for a hyperbolic PDE. In the study of shock formation, the behavior of \( u \), its properties, and their connection to the behavior of the solution variable \( u \) lie at the heart of the analysis. This was the case in Alinhac’s work and in Christodoulou’s work (described two paragraphs below), and it remains true in the present monograph as well. As we will see, in the problem of shock formation, the eikonal function plays an even more important role than it does in the proof of stability of Minkowski spacetime; there is an alternate proof \cite{59} of the stability of Minkowski spacetime, due to Lindblad-Rodnianski, that relies on an approximate eikonal function corresponding to the Minkowski metric rather than a true eikonal function. This alternate approach leads to remarkable simplifications in the analysis because the characteristics associated to the Minkowski metric are much simpler. In contrast, in the problem of shock formation, the formation of the shock and the corresponding blow-up of the solution are exactly tied to the blow-up of the first rectangular coordinate partial derivatives of a true eikonal function. Thus, there is little hope of finding an alternate proof that avoids the use of a true eikonal function (and, as we will see, the weighty baggage that accompanies it).

It is well-known that energy estimates are an unavoidable aspect of the study of quasilinear wave equations in more than one spatial dimension. To close the energy estimates in his proof of shock formation, Alinhac relied on a Nash-Moser iteration scheme featuring a free boundary. The presence of the free boundary is connected to the blow-up time of the iterates, which can vary, albeit slightly. A fundamental aspect of his proof, which is also present in Christodoulou’s work and the present monograph, is that the solution remains regular relative to the geometric coordinates mentioned two paragraphs above. In the equations studied by Alinhac, the singularity occurs in the second rectangular coordinate partial derivatives of \( \Phi \) and is tied to the degeneration of the geometric coordinate system relative to the rectangular one. At the same time, the degeneracy is tied to the intersection of the characteristics. In discussing Alinhac’s results and related ones, we often refer to the intersection of the characteristics as “the formation of a shock.” Our intention in using this terminology is to highlight the following fundamentally important aspect of Alinhac’s work: he proved that \( \Phi \) and its first rectangular coordinate partial derivatives \( \partial_\alpha \Phi \) remain bounded all the way up to the singularity. In particular, the singularity occurs at the level of the first rectangular coordinate partial derivatives of the Lorentzian metric components \( h_{\alpha\beta}(\partial \Phi) \) and not in the \( h_{\alpha\beta}(\partial \Phi) \) themselves; this feature is of fundamental importance for closing the proof.

Although Alinhac’s approach is compellingly short, it has some limitations, which we describe in detail in Sect. 2.11.1. In particular, his framework allows one

\footnote{Roughly, \cite{19} is a small-data global existence result for the Einstein-vacuum equations.}
\footnote{In the present work, the solution variable is the solution to the wave equation.}
\footnote{In this context, the characteristics associated to the Minkowski metric are the usual flat Minkowski light cones.}
\footnote{Actually, the solution does not necessarily remain regular at the high derivative levels: a fundamental aspect of the proof is that the high-order energies are allowed to blow-up as the shock forms; see three paragraphs below.}
to follow the solution only to the first singularity, and not further. In his 2007 monograph \[17\], D. Christodoulou proved, for a subclass of Alinhac’s wave equations\[9\], a breakthrough result that significantly sharpened Alinhac’s results and eliminated the drawbacks of his approach. Specifically, Christodoulou’s work applies to the wave equations that arise in irrotational relativistic\[10\] (compressible) fluid mechanics in three spatial dimensions. In this context, the equations are known as the irrotational relativistic Euler equations. Christodoulou assumed that the data have small $H^N$ norm, where $N$ is a sufficiently large (nonexplicit) integer. To deduce the shock formation, he also assumed that the data verify a signed integral inequality, distinct from the nondegeneracy condition of Alinhac mentioned above. Christodoulou’s framework allows one to do much more than follow the solution to the first singularity: it provides a complete picture of a portion of the maximal development\[11\] of the data including a description of the behavior of the solution along the boundary\[12\]; this is the main advantage of Christodoulou’s framework. One of the key reasons that Christodoulou was able to sharpen Alinhac’s results is that he was able to close the energy estimates without invoking a Nash-Moser iteration scheme. Moreover, his estimates do not involve a free boundary. Instead, Christodoulou developed a forwards approach relative to a set of geometric coordinates that, like Alinhac’s, are tied to a true eikonal function. By forwards approach, we mean that Christodoulou derives traditional global-existence-type estimates for a Cauchy problem in the geometric coordinates, relative to which the solution remains rather smooth (see, however, footnote\[7\]). As in Alinhac’s results and those of the present monograph, the blow-up occurs in certain rectangular coordinate partial derivatives of the solution and is tied to the degeneracy of the change of variables map from geometric to rectangular coordinates.

In Christodoulou’s framework, the degeneracy mentioned at the end of the previous paragraph and the corresponding blow-up of the solution are mediated by the vanishing of a quantity known as the inverse foliation density, which we denote by $\mu$. Roughly, $\mu$ is the reciprocal of a derivative of the eikonal function $u$. Geometrically, $1/\mu$ is a measure of the density of the level sets of $u$. As we will see starting in Chapter\[2\], the vanishing of $\mu$ is equivalent to the intersection of the characteristics and the blow-up of the eikonal function’s first rectangular coordinate partial derivatives, as we mentioned above. Moreover, these degeneracies are exactly tied to the formation of a singularity in the solution to the wave equation, much like in the classic example of Burgers’ equation (see Sect.\[11]\). The study of $\mu$ and the prospect of its vanishing are the main themes of \[17\] and the present work. For reasons that we describe two paragraphs below, the most compelling advantage of Christodoulou’s framework is that it allows one to construct the portion of the set

---

8 Roughly speaking, Alinhac’s proof works when there is such a unique first singularity; his nondegeneracy conditions on the data ensure that this is the case.

9 One has to take into account some simple differences in normalization, described in Sect.\[4.1.2\], in order to see that Christodoulou’s equations fall under the scope of Alinhac’s work.

10 In \[21\], Christodoulou-Miao extended the result to the nonrelativistic case.

11 Roughly speaking, the maximal development is the largest classical solution that is uniquely determined by the data.

12 The boundary can be very complicated and, in particular, it is not contained in a constant-time hypersurface.
\{ \mu = 0 \} \) that corresponds to the part of the boundary \( 13 \) along which the solution blows up \( 14 \). Generally, the set \( \{ \mu = 0 \} \) “evolves” into a spacetime region lying to the future of the constant-time hypersurface of first blow-up and thus it lies to the future of the region that Alinhac was able to probe.

It turns out that Christodoulou’s sharp description is accompanied by severe technical difficulties: the high-order energies are allowed to blow-up as \( \mu \to 0 \). This difficulty is fundamentally tied to the regularity theory of the eikonal function. As was first shown in \( 19 \), to control the top derivatives of \( u \), one must invoke a nontrivial procedure based on elliptic estimates and modified quantities, the latter being special combinations of terms that satisfy evolution equations with a good structure. In the problem of shock formation, this procedure introduces, at the top order, a difficult factor of \( 1/\mu \) into the energy identities; this factor is the reason that the high-order energies are allowed to blow up. A related but distinct difficulty is that one needs to show that the low-order energies remain bounded all the way up to \( \{ \mu = 0 \} \). This latter step is essential for establishing, via Sobolev embedding, the basic uniform \( L^\infty \) estimates that allow one to control error terms and to treat the problem as a traditional one in which one derives global-existence-type estimates (relative to the geometric coordinates). These are the main technical difficulties that one encounters in the problem of shock formation à la Christodoulou and they are a primary reason that the work is technical and lengthy. In Chapter 2, we provide an extended overview of these issues, especially in view of the fact that they do not arise in any other context in the study of nonlinear wave equations.

Christodoulou’s framework is compelling for the geometric insight it provides into the formation of shocks and for the sharp description that it yields. In addition, his approach is fundamentally important for a related problem: it turns out that his sharp description of the solution near the boundary of the maximal development is an essential ingredient in setting up the shock development problem, which is the problem of 1) continuing the solution to Euler’s equations (in a weak sense, subject to appropriate jump and entropy conditions) beyond the first singularity and 2) at the same time, constructing the shock hypersurface \( 15 \) across which the solution is discontinuous \( 16 \). The shock development problem in relativistic fluid mechanics was recently solved in spherical symmetry \( 20 \). Away from symmetry, the problem remains open and is expected to be exceptionally difficult.

We mention here an important problem related to the shock development problem, which A. Majda solved \( 62 \) \( 64 \) in the early 1980s. Specifically, for hyperbolic systems of conservation laws with suitable structure in more than one spatial dimension, Majda solved the shock front problem. That is, in a suitable Sobolev

---

\( 13 \)Roughly, a subset of \( \{ \mu = 0 \} \) corresponds to the singular portion of the maximal development of the data.

\( 14 \)The boundary of the maximal development also contains another portion, along which the solution does not blow up; see Sect. 2112 and Theorem 249 in particular.

\( 15 \)Part of the set \( \{ \mu = 0 \} \) turns out to be a hypersurface portion along which, in the classical formulation of the irrotational Euler equations, certain solution derivatives blow up. However, \( \{ \mu = 0 \} \) does not correspond to the physically correct hypersurface of discontinuity. The physically correct hypersurface of discontinuity propagates at supersonic speed starting from the first spacetime point where \( \mu \) vanishes and develops “before” the set \( \{ \mu = 0 \} \) has a chance to form. The physically correct hypersurface can be derived only by imposing the weak formulation of the full compressible Euler equations (without the assumption of irrotationality) starting from the time of first blow-up and by assuming suitable jump and entropy conditions.

\( 16 \)One expects the solution to be smooth on either side of the shock hypersurface.
framework, he proved a local existence result starting from an initial discontinuity given across a smooth hypersurface subset of the Cauchy hypersurface. We stress that the initial hypersurface of discontinuity is prescribed. In contrast, in the shock development problem mentioned in the previous paragraph, the spacetime hypersurface of discontinuity is fully dynamic, emerging from singularity-free initial data. In the shock front problem, the data must verify suitable jump conditions, entropy conditions, and higher-order compatibility conditions. As in the shock development problem, the shock front problem features a free boundary: the shock hypersurface which is one of the unknowns. Majda’s work also required an additional assumption on the data that seems to be necessary for the stability of the corresponding linearized problem.

We now describe the origin and motivation behind the present monograph. In his work [17], Christodoulou exploited various special structures enjoyed by the wave equations of irrotational relativistic fluid mechanics, structures which Alinhac did not use in his proof of shock formation. In particular, the wave equations in [17] derive from a Lagrangian and are invariant under the Poincaré group; just below equation (2.11), we describe some ways in which Christodoulou used these structures in his proof. In studying Christodoulou’s work [17], the author discovered that it is possible to use his framework to close the proof of shock formation for a larger class of equations and without relying on these special structures. In particular, his framework can be extended to treat all of the wave equations studied by Alinhac. A somewhat surprising fact, which plays a fundamental role in our analysis, is that they all have a special null structure, even though they fail to satisfy Klainerman’s null condition. This special structure is not visible relative to the standard formulation of the wave equation, but becomes visible upon reformulating it as a system of geometric wave equations; see Lemmas A.9 and A.16 for the main results in this direction. It is this realization that led to the present work. Our work here also generalizes and unifies earlier work on singularity formation initiated by F. John in the 1970s and continued by L. Hörmander and many others.

More precisely, in the present monograph, we extend Christodoulou’s framework and use it to prove that shock singularities often develop in initially small, regular solutions to two important classes of quasilinear wave equations in three spatial dimensions. Specifically, we study i) covariant scalar wave equations of the form $\square_g(\Psi) = 0$ and ii) Alinhac’s noncovariant scalar wave equations, that is, wave equations of the form $(h^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$. Our main result shows that whenever the nonlinear terms fail Klainerman’s (classic) null condition shocks develop in solutions arising from an open set of small data. Hence, within the classes i) and ii), our work can be viewed as a sharp converse to a fundamental result, due separately to Christodoulou [15] and Klainerman [47], which showed that

---

17 This hypersurface is co-dimension two when viewed as a subset of spacetime.
18 The shock hypersurface is a co-dimension one subset of spacetime.
19 The assumption is automatically verified for the nonrelativistic Euler equations under the adiabatic equations of state $p = A\rho^\gamma$, where $A > 0$ and $\gamma > 1$ are constants.
20 That is, the equations in [17] are Euler-Lagrange equations.
21 It turns out that the two classes of equations are more closely related than one might expect; see the discussion below equation (2.9) and in Appendix A.
22 Readers should take care not to confuse Klainerman’s null condition with the future strong null condition and past strong null condition introduced in Appendix A and mentioned in Remark 2.3.
when the null condition is verified in three spatial dimensions, small-data global existence holds. Roughly, we give the same sharp description of the solution that Christodoulou gave in [17]. However, to avoid lengthening the monograph, we did not give a full description of the boundary of the maximal development nor the behavior of the solution along it. For readers interested in those details, we remark that the estimates proved in our main Theorem 22.1 are sufficient for invoking the arguments of [17, Chapter 15] in which Christodoulou reveals properties of the maximal development. That is, with modest additional effort, our results could be extended to give the same sharp description of the maximal development that Christodoulou gave in [17, Chapter 15].

In proving our main results, we have taken substantial steps that go beyond replicating the proofs given by Christodoulou in [17]. This is partly out of necessity, as the general class of equations that we treat leads to new kinds of error terms that are not present in [17]. However, we have also developed alternate strategies that greatly simplify certain aspects of the proof. One big simplification is that we are more selective in our use of geometry. That is, we use sharp, fully geometric decompositions only for treating the most delicate terms. Another simplification is that our bootstrap argument is very straightforward in view of the fact that we have organized the monograph in a linear fashion (see two paragraphs below). We have also developed alternate approaches to deriving some of the difficult top-order estimates by reducing them to other top-order estimates. This spares one a great deal of effort; see, for example, the discussion at the beginning of Sect. 15.1 in which we describe a simplified approach for obtaining estimates for the top-order derivatives of $\mu$.

We now give an overview of the content and organization of the monograph. Chapter 1 sets the stage for the rest of the monograph but is independent of the remaining chapters. It contains historical background, a discussion of shock formation in solutions to Burgers’ equation, a discussion of singularity formation in $2 \times 2$ strictly hyperbolic genuinely nonlinear systems, an overview of wave dispersion in higher dimensions and its connection to global and almost global existence results, an overview of the vectorfield method (including the multiplier and commutator methods) for deriving generalized energy estimates, and a discussion of the null condition. In Chapter 2, we describe the main results of the monograph, place them in context, and provide an extended overview of the most important aspects of the proofs. In the remaining chapters, we develop the machinery and estimates needed to prove the two theorems of the monograph, which are located in Chapters 22 and 23. Roughly, in the first theorem (the main one, which is difficult to prove), we show that the solution must persist unless $\mu$ vanishes, and we derive sharp a priori estimates that hold as long as $\mu$ remains positive. In the second theorem, which is a relatively easy consequence of the first one, we exhibit an open set of data such that $\mu$ does in fact vanish in finite time, thus yielding a shock singularity. Our analysis in Chapters 3–23 applies to covariant wave equations of the form $\Box_g(\Psi)\Psi = 0$, while in Appendix A, we outline how to extend the results to the class of noncovariant wave equations studied by Alinhac. In Appendix B, we summarize the notation and conventions used in Chapters 2–23.

Chapters 3–23 are interdependent and are designed to be read consecutively. That is, this part of the monograph constitutes one long bootstrap-type proof, presented in chronological order. In Sect. 24, we give an overview of the contents
of each chapter and provide suggestions on how to read the monograph, both for expert and novice readers.

This monograph is mostly self-contained but relies extensively on basic concepts from Lorentzian and Riemannian geometry such as Levi-Civita connections, fundamental forms, curvature, pullbacks, etc. We anticipate that there are readers with knowledge in fluid mechanics, conservation laws, and/or PDEs but who are unfamiliar with those geometric concepts. Such readers can find introductory geometric material, suitable for reading almost all\textsuperscript{23} of the present monograph, in select portions of the books \cite{66,67,78}. Novice geometers should bear in mind that at the end of the day, one aims to derive estimates, and that the geometry merely provides a framework for organizing calculations and revealing analytic structural features that would otherwise be difficult to detect.

We close by highlighting the following wide-open question:

In more than one spatial dimension, to what extent can the results of this monograph be generalized to quasilinear systems featuring multiple speeds of propagation, such as the equations of elasticity, the equations of magnetohydrodynamics, the equations of crystal optics, the Euler–Einstein equations of cosmology, or even coupled systems of wave equations featuring two or more metrics with strictly separated\textsuperscript{24} speeds of propagation?

Jared Speck

---

\textsuperscript{23}Our short proof of geometric Sobolev embedding, presented in Chapter \textsuperscript{18}, relies on a handful of more advanced results from geometry.

\textsuperscript{24}This roughly corresponds to the presence of two or more distinct families of characteristic hypersurfaces.
Acknowledgments

I am grateful for the support offered by NSF grant #DMS-1162211 and by a Solomon Buchsbaum grant administered by the Massachusetts Institute of Technology. I would like to thank the American Institute of Mathematics for funding three SQuaREs workshops on the formation of shocks, which greatly aided the development of many of the ideas in this monograph. I would like to thank Gustav Holzegel, Sergiu Klainerman, Jonathan Luk, Willie Wong, and Shiwu Yang for participating in the workshops and for their helpful contributions. I offer special thanks to Willie Wong for creating Figure 4. I am grateful for the support of my PhD advisors, Michael Kiessling and Shadi Tahvildar-Zadeh, who encouraged me to read Christodoulou’s monograph on shock formation. I would also like to thank Hans Lindblad for sharing his insight on Alinhac’s work.
Bibliography


Index

2 × 2 strictly hyperbolic genuinely nonlinear system, 6
$C^0$ norm
definition, 103
$C^k(\Omega)$
definition, 103
$Harmless^{\leq N}$ terms
definition, 296
main features, 296
$L^2$ estimates
hierarchy for $\Psi$, 456
hierarchy for the eikonal function quantities, 456
$L^2$-controlling quantities
definition, 279
initial smallness, 282
quantification of coerciveness, 280
$L^p$ norm
geometric version, 103
over subsets, 103
$S_{t,u}$ projection
basic properties, 113
definition, 91
frame covariant derivatives of, 131
occasional redundancy, 113
$S_{t,u}$ tensor, 92
$S_{t,u}$-projected Lie derivative
connection to $\pi$, 128
definition, 98
$\Sigma_t^U_0$ projection
definition, 91
$\Sigma_t^U_0$ tensor, 92
$\Sigma_t^U_0$-projected Lie derivative, 98
a posteriori estimate, 253, 254, 255
a priori energy estimates
for symmetric hyperbolic systems, 15
in the shock formation problem, provided by the fundamental Gronwall lemma, 274
abuse of notation regarding the symbol $\vartheta$, 88
algebraic topology, 450, 462
Alinhac, S.
shock formation results, 32, 61
almost global existence, 28
angular differential
definition, 96
meaning of the rectangular component, 96
of $L^2$, 109
of $L^2_{\text{Small}}$, 109
of $R^t_{\text{Small}}$, 109
of the rectangular spatial coordinates, 96
angular divergence
definition, 132
area form on $S_{t,u}$
definition, 102
size, 151
arrays of frame components
notation for derivatives, 104
pointwise norm, 100
atlas
standard atlas on $S^2$, 88
auxiliary bootstrap assumptions
derivation of improvements, 290
statement of, 153
blow-up
caused by the vanishing of the inverse foliation density, 16
first known example of shock type, 2
for 2 × 2 strictly hyperbolic genuinely nonlinear systems, 10
for solutions to Burgers’ equation, 3
for the compressible Euler equations, 164
bootstrap assumptions
auxiliary sup-norm type, 183
fundamental sup-norm assumptions for $\Psi$, 182
overview, 15
positivity of the inverse foliation density, 182
bounded $L^2$ curvature conjecture, 19
boxed constants, 366, 485
Burgers’ equation, 3
causal vector, 20

507
ceiling function, 210
Challis, J., 1
change of variables map
behavior up to the shock, 155
definition, 101
Jacobian determinant, 101
sufficient conditions for it to be a global diffeomorphism, 161
characteristic curve, 10
characteristic hypersurface, xv, 41
characteristic vectorfields, 6
classical lifespan, 453
co-area formula, 14
Codazzi equations
identities connected to, 291
commutation current, 144
frame decomposition of its divergence, 136
commutation vectorfield
commutation identity with the covariant wave operator, 143
definition, 51, 127
Euclidean rotation subset, 128
expression for its deformation tensor, 129
for the linear wave equation, 25
geometric rotation subset, 127
good properties of, 144
spatial subset, 127
commutator
$S_{t,u}$ tangent nature of $[L, \tilde{R}]$, 58
expression for $[L, \tilde{R}]$, 179
expression for $[O_{\gamma \text{m}}, O_{\text{in}}]$, 124
of two operators, 126, 136
properties of vectorfield commutators, 142
commutator method, 19, 29, 28, 51
commutator property
of commutator vectorfields with the covariant wave operator, 144
of the elements of $L_{\text{Flat}}$ with the Minkowskian wave operator, 23
of vectorfields with the Minkowskian wave operator, 26
comparison estimate
$\nabla$ in terms of $\zeta$, 197
$\nabla^2$ in terms of $\zeta^2$, 201
$\zeta$ in terms of $\zeta^2$, 209
compatible current
correction current, 155
definition, 155
expression for its divergence, 155
for symmetric hyperbolic systems, 110
for wave equations, 20
cone flux
coerciveness, 50, 277
corresponding to the Morawetz multiplier, 59
corresponding to the timelike multiplier, 59
definition, 156
conformal Killing field
doing the Minkowski metric, 22, 25
connection coefficients
decomposition into singular and regular pieces, 117
of the rescaled frame, 117
of the rescaled null frame, 119
connections, 83
constant time slices, 83
constants
dependence of constants on $U_0$, 392
importance of the boxed constants, 366
sharp constants, 280
structural constant, 57
the boxed constants are perhaps affected by non-optimal definitions, 485
continuation criteria
for quasilinear wave equations, 1
for symmetric hyperbolic systems, 15
relevant for the main results of the monograph, 447
contraction notation
abbreviations, 91
covariant derivative
expressed relative to rectangular coordinates, 108
covariant Laplacian, 54
covariant wave operator, 15, 83
commutation identity with commutation vectorfields, 143
expression relative to the rescaled frame, 143
frame decomposition of, 119
covering map, 462
curvature
Gaussian curvature of $\hat{g}$, 353
Ricci curvature of $\hat{g}$, 353
Ricci curvature of $g$, 144
Riemann curvature of $\hat{g}$, 160
Riemann curvature of $g$, 169
scalar curvature of $\hat{g}$, 353
decay faster than expected, 183
rates of decay, 183
definition of a commutation vectorfield expressions for the frame components, 129
timelike multiplier, 147, 236
tensorfields, 302
of a deformation tensor
definition, 20, 112
important terms in \((\hat{R})\pi\), 299
important terms in \((\hat{\rhoL})\pi\), 307
important terms in \((O_1)\pi\), 302
of a commutation vectorfield expressions for the frame components, 129
importance of \((Z)\pi\), 147
important structural features, 128
role in generating inhomogeneous terms in the wave equation, 127
special structure, 148
the null components of the deformation tensors of the multiplier vectorfields, 148
derivative loss avoiding it leads to degenerate energy estimates, 40
how to avoid it, 54
naive estimates lead to it, 40, 53
permissible below top order, 60
descent scheme for the below-top-order energy estimates, 59
differential structures, 89
L^2-type, 16, 22
decay rates in the shock formation problem, 183
pointwise type for the linear wave equation, 24
divergence angular divergence, 135
of a vectorfield, expressed in terms of rescaled frame derivatives, 148
divergence theorem for symmetric hyperbolic systems, 15
in the shock formation problem, 115
with important cancellations, 150
dominant energy condition, 20
Duhamel’s principle, 474
eikonal equation, xv
definition, 41
eikonal function, xv
appearance of error terms depending on its top-order derivatives, 148
definition, 41, 82
identities involving rectangular spatial derivatives, xv
in the context of Burgers’ equation, 4
necessity of in the proof of shock formation, 12
role in the proof of shock formation, xv
role in the proof of the stability of Minkowski spacetime, xv
ways in which it has been used in nonlinear problems, 12
eikonal function quantities, 17, 107
L^2 estimates not relying on the modified quantities, 361
elliptic estimates for solutions to Poisson’s equation, 366
for symmetric trace-free type \((\mu)_{\tau u}\) tensorfields, 356
role in controlling \(\mu \nabla \chi\), 55
terms that need to be treated with them, 149
energy coerciveness, 50, 277
coresponding to the timelike multiplier, 50
definition, 150
for symmetric hyperbolic systems
definition, 13
energy estimates bootstrap assumptions, 365
descent scheme for the below-top-order energy estimates, 59
high-order degeneracy, 54
main difficulties in obtaining a priori estimates, 44
nondegeneracy at the low orders, 50
overview of the hierarchy, 55
overview of the top-order estimates, 50
proof of the main below-top-order energy-cone flux integral inequalities, 129
proof of the main top-order energy-cone flux integral inequalities, 126
quantities that bound difficult top-order integrals generated by \(\tilde{K}\), 369
quantities that bound difficult top-order integrals generated by \(T\), 367
quantities that bound easy integrals generated by \(\tilde{K}\), 369
quantities that bound easy integrals generated by \(T\), 368
sharp L^2 estimates for \(\tilde{L}\) applied to the partially modified version of \(\tilde{\mu} - N-1\mu\), 401
sharp L^2 estimates for \(\tilde{L}\) applied to the partially modified version of \(\tilde{
\rho}_{N-1}\chi\) \(\text{Small}\), 402
sharp L^2 estimates for the partially modified version of \(\tilde{\mu} - N-1\mu\), 395
sharp L^2 estimates for the partially modified version of \(\tilde{
\rho}_{N-1}\chi\) \(\text{Small}\), 400
statement of the main estimates for the difficult top-order error integrals corresponding to \(\tilde{K}\), 430
statement of the main estimates for the
difficult top-order error integrals
corresponding to $T$, 388
the main below-top-order energy-cone
flux integral inequalities, 371
the main top-order energy-cone flux
integral inequalities, 370
energy identity
 corresponding to the timelike multiplier, 50
for symmetric hyperbolic systems, 153
for the linear wave equation, 22
in the shock formation problem, 157
with important cancellations, 158
of Morawetz type for the linear wave
equation, 22
energy method, 11
classes of equations to which it has been
applied, 12
energy-momentum tensorfield
definition, 20, 152
expression for its divergence, 20, 152
expressions for its null components, 153
Euclidean rotation, 25
 definition, 123
insufficiency in the shock formation
problem, 51
Euler-Lagrange equation, 2
irrotational fluid mechanics, xix
the irrotational compressible Euler
equations, 17
first fundamental form
definition, 83
its volume form factor relative to
global existence
in higher dimensions, 28
in three spatial dimensions under the
null condition, 28
geometric coordinates
definition, 88
overview of their construction, 41
solution remains regular relative to them, 88
geometric radial variable
definition, 82
silent use of a basic estimate, 184
global existence
in three spatial dimensions under the
null condition, 28
Gronwall lemma
proof of the fundamental Gronwall
lemma, 431
statement of the fundamental Gronwall
lemma, 371
Huygens’ principle, 255
immersion, 462
inherent $S_{t,u}$ tensors vs. $S_{t,u}$ tensors
embedded in spacetime, 92
inhomogeneous terms in the commuted
wave equation
structure of inhomogeneous terms after
one commutation, 144
inhomogeneous terms in the commuted
wave equation
basic structure, 294
error terms that never appear, 54
explanation of the various types, 148
identification of the difficult factors, 158
reduction of the analysis, 207
initial data
Alinhac’s criteria for shock formation, 65
assumptions on the support, 31
Christodoulou’s criteria for shock formation, 71
conditions that lead to shock formation, 72
definition, 81
definition of their size, 182
estimates for the $\mathcal{Z}$-derivatives of $\Psi$ along $\Sigma^1_0$, 189
estimates for the eikonal function quantities along $\Sigma^1_0$, 188
estimates of $\Psi$ along $\Sigma^1_0$, 187
initial smallness of the $L^2$-controlling quantities, 282
nearly spherically symmetric assumption, 48
shrinking the amplitude to deduce shock formation, 73
smallness assumption, 182
treating a larger set of data, 35
injectivity radius, 461
integration by parts
identity on $S_{t,u}$, 161
identity used for top-order estimates, 160, 161, 163
inverse foliation density, xvii
a posteriori estimates, 245
absence of its reciprocal in the Codazzi-type identities, 291
as a weight in the energies and cone fluxes, 52
auxiliary quantities used to analyze it, 245
connection between $\nabla^2\mu$ and $\mathcal{L}_{R\chi}^{(Small)}$, 283
connection to the Jacobian determinant of the change of variables map, 43
definition, 43, 84
fundamental estimates for time integrals involving its reciprocal, 260
gaining powers by integrating in time, 60
geometric interpretation, 33
heuristic model of its behavior, 55
in the context of Burgers’ equation, 5
its role in time-integral estimates, 245
overview of why it can vanish in finite time, 49, 467
point of no return smallness, 53
regions of distinct behavior, 249
relationship to the blow-up of various quantities, 84
role as a weight for the covariant wave operator, 51, 153
role in the monograph, 49
sharp pointwise estimates, 249
smaller-than-expected acceleration, 58
smallness implies quantified negativity of its outgoing null derivative, 53
the transport equation that it satisfies, 108
the transport equation that it satisfies in the case of non-covariant wave equations, 185
vanishing is precisely tied to singularity formation, 43
what happens when it vanishes, 455
inverse function theorem, 161
Jacobi identity, 98
John’s conjecture, 62, 63, 66, 72
proof sketch of the conjecture, 73
John’s criterion for shock formation, 73
John, F.
nonconstructive proof of blow-up, 52
Killing field, 133
of the Minkowski metric, 25
Klainerman-Sobolev inequality, 26
Lagrangian coordinates, 4
Lagrangians
exceptional Lagrangian, 70
in irrotational relativistic fluid mechanics, 68
Lebesgue norm
Euclidean version, 15
geometric version, 103
over subsets, 103
Leibniz rule
Jacobi identity, 98
Levi-Civita connections, 83
Lie bracket, 97
Lie derivative
$S_{t,u}$-projected Lie derivatives
basic commutation formula, 113
definition, 98, 113
$\Sigma^1_0$-projected Lie derivatives
definition, 98
alternate expression, 98
coordinate invariant property, 98
definition, 97
definition of trace-free $S_{t,u}$-projected Lie derivatives, 135
Leibniz rule, 98
pullback formulation, 98
lifespan lower bound of John and Hörmander, 98
linear wave equation, 21
linear wave operator, 33
local well-posedness
for quasilinear wave equations, 18
for symmetric hyperbolic system, 15
relevant for the main results of the monograph, 447
long-time existence
localized very-long-time existence, 72, 79
Lorentz boost, 25
Lorentzian metric, xv, 18

Main theorem
its role in the proof of shock formation, 49
overview, 33
statement and proof of the sharp
classical lifetime theorem, 453
maximal development, xvii, 68, 71
Christodoulou’s sharp description, 71
maximal future development, 33
modified quantity, 54
$L^2$ estimates that do not rely on the
modified quantities, 361
definition of the higher-order fully
modified versions of $tr\chi^{(Small)}$, 175
definition of the higher-order partially
modified versions of $\phi_u$, 179
definition of the higher-order partially
modified versions of $tr\chi^{(Small)}$, 176
definition of the lowest-order fully
modified version of $tr\chi^{(Small)}$, 173
definition of the lowest-order partially
modified version of $\phi_u$, 178
definition of the lowest-order partially
modified version of $tr\chi^{(Small)}$, 177
difficult pointwise estimates involving the
fully modified quantities, 340
difficult pointwise estimates involving the
partially modified quantities, 351
role of the fully modified quantities in
avoiding derivative loss, 167
role of the partially modified quantities in
avoiding unfavorable error integrals, 168
solving the transport equation satisfied
by the fully modified version of
$tr\chi^{(Small)}$, 337
the transport equation satisfied by the
higher-order fully modified versions of $tr\chi^{(Small)}$, 175
the transport equation satisfied by the
higher-order partially modified versions of $\phi_u$, 179
the transport equation satisfied by the
higher-order partially modified versions of $tr\chi^{(Small)}$, 176
the transport equation satisfied by the
lowest-order partially modified version of $\phi_u$, 178
the transport equation satisfied by the
lowest-order partially modified version of $tr\chi^{(Small)}$, 177
the transport equation satisfied by the
once-differentiated fully modified
version of $tr\chi^{(Small)}$, 178
Morawetz energy and cone flux
coerciveness, 52
Morawetz identity
for the linear wave equation
coerciveness, 23
Morawetz integral
bounds for error integrals that rely on it,
376
definition, 92, 280
origin, 280
quantified coerciveness, 52, 280
Morawetz multiplier
connection to good $t$-weights, 22
definition, 153
error integrands associated to, 164
for the linear wave equation, 22
simplified version for the linear wave
equation, 23
Morawetz term
in the energy-cone flux identities, 165
isolating it, 387
multiplier method, 119, 28
multiplier vectorfields, 49
definition, 153
error integrands associated to them, 164
expressions for the null components of
their deformation tensors, 153
null decomposition of the corresponding
error integrands, 162
musical notation, 97

Nash-Moser iteration
not part of Christodoulou’s framework, xvii
role in Alinhac’s work, xvii
Noether’s theorem, 21
nondegenerate behavior up to the shock
of $\Psi$, 454
of the eikonal function quantities, 455
of the rectangular metric component
functions, 454
norm
$C^0$ norm, 103
Euclidean Lebesgue norm, 15
Euclidean Sobolev norm, 15
geometric $L^p$ norm, 105
pointwise norm of an $S_{\ell,u}$ tensor, 97
pointwise norms vs. the size of
rectangular components, 187
normal
the timelike unit normal to $\Sigma_t$
basic properties, 80
definition, 86
notation
clarification of the schematic notation,
$2^N-1$, 213, 286
component notation, 91
contraction notation, 89, 91
musical notation, 97
INDEX 513

schematic notation, 104
schematic notation for repeated
differentiation, 140
suppression of the independent variables, 181
null condition, 38
connection to global existence, 39, 457
connection to the main results of the
monograph, 32, 36
connection to the structure of quadratic
terms decomposed relative to a
Minkowskian frame, 39
definition of Klainerman’s (classic) null
condition, 39
future strong null condition, 43, 484
Klainerman’s (classic) null condition, 28
past strong null condition, 484
strong null condition, 34
null cones
outgoing null cones, 83
null form
standard null form
definition, 482
important property, 482
null frame
Minkowskian null frame, 25
nonrescaled null frame, 89
rescaled null frame, 89
null generator, 42
null hypersurface, 83
in the context of wave equations, 21
null second fundamental form
alternate expression, 99
another alternate expression, 99
definition, 99
expression for the re-centered null second
fundamental form, 111
overview of, 106
null vectorfield
basic properties, 85, 86
ingoing, 86
ingoing and μ-weighted, 85
Minkowski-null vectorfield, 22
outgoing null geodesic vectorfield, 84
rescaled outgoing null vectorfield, 85
definition, 43
past null condition failure factor, 90
for non-covariant wave equations, 481
past strong null condition, 183
plane symmetry, 1
pointwise norm
definition, 97
shorthand notation, 140
projection
$S_{t,u}$ projection
basic properties, 113
definition, 97
$\Sigma_{t,\nu}^\circ$ projection
definition, 91
of tensors, 92
projected Lie derivative, 98
rectangular components of the $S_{t,u}$
projection tensorfield, 123
pullback, 98
pullback formulation of Lie
differentiation, 98
quasilinear terms
relationship to singularity formation, 40
radial coordinate
Euclidean radial coordinate, 56
radial vectorfield
basic properties, 86
definition, 56
key lower bound verified by the
solution’s rescaled radial derivative, 17
overview, 14
standard Euclidean-type, 22
Radon transform, 92
Raychaudhuri-type equation, 173
re-centered variables
definition, 107
rescaled frame
decomposition of a vectorfield relative to
it, 91
definition, 143
Riccati-type equation, 9
Riccati-type ODE, 3
Ricci curvature
of $g$, 354
of $g$, 144
Riemann curvature
definition, 169
key identity, 171
rectangular components of, 169
symmetry properties, 169
Riemann invariant, 2
rotation vectorfield
$S_{t,u}$ components of, 124
basic properties of, 129
decomposition of, 124
definition of the Euclidean rotations, 123
definition of the geometric rotations, 123
expressions for commutators of rotations,
124
expressions for its $S_{t,u}$ covariant
derivatives, 134
rectangular components of, 124
scalar curvature of $g$, 354
first variation, 354
scaling vectorfield, 25
schematic notation, 104
for pointwise norms, 140
second fundamental form
alternate expression, 99
another alternate expression, 99
decomposition into singular and regular
pieces, 117
definition, 99
semilinear term
eliminating the dangerous one, 46
sharp classical lifespan theorem, 453
sharp notation, 97
shock formation
an open set of small nearly spherically
symmetric data that lead to shock
formation, 174
beyond nearly spherically symmetric
data, 177
in the context of Burgers’ equation, 5
overview of the main shock formation
result, 54
statement and proof of the main shock
formation result, 177
the data-dependent function that drives
it, 469
the key estimate that drives it, 176
simply connected, 162
Sobolev embedding, 58
its role in the proof of shock formation,
183
on the $S_{t,u}$, 358
sup-norm bounds in terms of the
energies, 359
Sobolev regularity
behavior of the solution under additional
regularity assumptions on the data,
157
number of derivatives needed to close the
estimates, 150
number of derivatives needed to treat
noncovariant wave equations, 358
number of derivatives used by
Christodoulou, 36
Sobolev space
Euclidean type vs. geometric type, 15
spacelike hypersurface
for symmetric hyperbolic systems, 13
in the context of wave equations, 21
spacetime metric
components relative to the geometric
coordinates, 18
decomposition into the Minkowski metric
plus a perturbation, 35
expression in terms of the frame
vectorfields, 94
its volume form factor relative to
gauge coordinates, 94
spheres, 83
standard atlas on $S^2$, 88
stationary phase, 25
strictly hyperbolic system, 61
7
sup-norm
convention, 103
definition, 104
suppression of coordinate charts, 88
symmetric hyperbolic system, 12
symmetrized trace-free part of a tensor, 96
time function, 13
timelike multiplier
definition, 153
error integrands associated to, 164
trace
$\hat{g}$-trace of a tensor, 95
$g$-trace of a tensor, 95
trace-free part of a tensor, 96
translation vectorfield, 25
transport equation
in the case of non-covariant wave
equations, 185
for $L^1$, 174
for $L^2$, 199
for $L^1_S$, 109
for $\Sigma$, 199
for the Gaussian curvature of $\hat{g}$, 354
for the higher-order fully modified
versions of $tr_{\hat{g}}X^{(Small)}$, 175
for the higher-order partially modified
versions of $\hat{L}$, 179
for the higher-order partially modified
versions of $tr_{\hat{g}}X^{(Small)}$, 178
for the inverse foliation density, 174
for the lowest-order partially modified
version of $\hat{L}$, 178
for the lowest-order partially modified
version of $tr_{\hat{g}}X^{(Small)}$, 174
for the once-differentiated fully modified
version of $tr_{\hat{g}}X^{(Small)}$, 174
nature of the wave equation in one
spatial dimension, 17
role in the derivation of $C^0$ estimates for
the eikonal function quantities, 88
solving the transport equation satisfied
by the fully modified version of
$tr_{\hat{g}}X^{(Small)}$, 337
vectorfield method, 19
volume form
definition of the (rescaled) forms on $\Sigma^u$, $C^u$, and $M_{t,u}$, 102
expressions for the geometric volume
form factors, 104
wave equation
covariant
approximately reduces to a transport equation, 468
basic example, 17
expressed relative to rectangular coordinates, 38
expressed relative to the rescaled frame, 46
structure of the inhomogeneous terms after one commutation, 144
the type treated in the monograph, 35
non-covariant
  commuted with one rectangular derivative, 151
  contrasting against covariant wave equations, 479
main new estimate needed at the top order, 185, 188
reformulated as a system of covariant wave equations, 152
  the type to which the main results apply, 179
noncovariant
  the type to which the main results apply, 37
the two classes treated in the monograph, 31
## Selected Published Titles in This Series

<table>
<thead>
<tr>
<th>Page</th>
<th>Title</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>214</td>
<td>Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations</td>
<td>Jared Speck</td>
</tr>
<tr>
<td>213</td>
<td>Beurling Generalized Numbers</td>
<td>Harold G. Diamond and Wen-Bin Zhang (Cheung Man Ping)</td>
</tr>
<tr>
<td>212</td>
<td>Ramsey Theory for Product Spaces</td>
<td>Pandelis Dodos and Vassilis Kanellopoulos</td>
</tr>
<tr>
<td>211</td>
<td>Galois Theories of Linear Difference Equations: An Introduction</td>
<td>Charlotte Hardouin, Jacques Sauloy, and Michael F. Singer</td>
</tr>
<tr>
<td>210</td>
<td>The Dynamical Mordell–Lang Conjecture</td>
<td>Jason P. Bell, Dragos Ghica, and Thomas J. Tucker</td>
</tr>
<tr>
<td>209</td>
<td>Persistence Theory: From Quiver Representations to Data Analysis</td>
<td>Steve Y. Oudot</td>
</tr>
<tr>
<td>208</td>
<td>Grid Homology for Knots and Links</td>
<td>Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó</td>
</tr>
<tr>
<td>207</td>
<td>Fokker–Planck–Kolmogorov Equations</td>
<td>Vladimir I. Bogachev, Nicolai V. Krylov, Michael Röckner, and Stanislav V. Shaposhnikov</td>
</tr>
<tr>
<td>205</td>
<td>Tensor Categories</td>
<td>Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik</td>
</tr>
<tr>
<td>204</td>
<td>Toric Topology</td>
<td>Victor M. Buchstaber and Taras E. Panov</td>
</tr>
<tr>
<td>203</td>
<td>A Foundation for PROPs, Algebras, and Modules</td>
<td>Donald Yau and Mark W. Johnson</td>
</tr>
<tr>
<td>202</td>
<td>Asymptotic Geometric Analysis, Part I</td>
<td>Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D. Milman</td>
</tr>
<tr>
<td>201</td>
<td>Topological Modular Forms</td>
<td>Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill, Editors</td>
</tr>
<tr>
<td>200</td>
<td>Nonlinear Elliptic Equations and Nonassociative Algebras</td>
<td>Nikolai Nadirashvili, Vladimir Tkachev, and Serge Vlăduţ</td>
</tr>
<tr>
<td>199</td>
<td>Foundations of Free Noncommutative Function Theory</td>
<td>Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov</td>
</tr>
<tr>
<td>198</td>
<td>Brauer Groups, Tamagawa Measures, and Rational Points on Algebraic Varieties</td>
<td>Jörg Jahnel</td>
</tr>
<tr>
<td>197</td>
<td>The Octagonal PETs</td>
<td>Richard Evan Schwartz</td>
</tr>
<tr>
<td>196</td>
<td>Geometry of Isotropic Convex Bodies</td>
<td>Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou</td>
</tr>
<tr>
<td>195</td>
<td>Complex Multiplication and Lifting Problems</td>
<td>Ching-Li Chai, Brian Conrad, and Frans Oort</td>
</tr>
<tr>
<td>194</td>
<td>Stochastic Resonance</td>
<td>Samuel Herrmann, Peter Imkeller, Ilya Pavlyukevich, and Dierk Peithmann</td>
</tr>
<tr>
<td>193</td>
<td>Capacity Theory with Local Rationality</td>
<td>Robert Rumely</td>
</tr>
<tr>
<td>192</td>
<td>Attractors for Degenerate Parabolic Type Equations</td>
<td>Messoud Efendiev</td>
</tr>
<tr>
<td>191</td>
<td>An Introduction to Central Simple Algebras and Their Applications to Wireless Communication</td>
<td>Grégory Berhuy and Frédérique Oggier</td>
</tr>
<tr>
<td>190</td>
<td>Birationally Rigid Varieties</td>
<td>Aleksandr Pukhlikov</td>
</tr>
<tr>
<td>189</td>
<td>Gradings on Simple Lie Algebras</td>
<td>Alberto Elduque and Mikhail Kochetov</td>
</tr>
<tr>
<td>188</td>
<td>The Water Waves Problem</td>
<td>David Lannes</td>
</tr>
</tbody>
</table>

For a complete list of titles in this series, visit the AMS Bookstore at [www.ams.org/bookstore/survseries/](http://www.ams.org/bookstore/survseries/).
In 1848 James Challis showed that smooth solutions to the compressible Euler equations can become multivalued, thus signifying the onset of a shock singularity. Today it is known that, for many hyperbolic systems, such singularities often develop. However, most shock-formation results have been proved only in one spatial dimension. Serge Alinhac’s groundbreaking work on wave equations in the late 1990s was the first to treat more than one spatial dimension. In 2007, for the compressible Euler equations in vorticity-free regions, Demetrios Christodoulou remarkably sharpened Alinhac’s results and gave a complete description of shock formation.

In this monograph, Christodoulou’s framework is extended to two classes of wave equations in three spatial dimensions. It is shown that if the nonlinear terms fail to satisfy the null condition, then for small data, shocks are the only possible singularities that can develop. Moreover, the author exhibits an open set of small data whose solutions form a shock, and he provides a sharp description of the blow-up. These results yield a sharp converse of the fundamental result of Christodoulou and Klainerman, who showed that small-data solutions are global when the null condition is satisfied.

Readers who master the material will have acquired tools on the cutting edge of PDEs, fluid mechanics, hyperbolic conservation laws, wave equations, and geometric analysis.