Mathematical
Surveys
and
Monographs

Volume 214

Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations

Jared Speck

American Mathematical Society

https://doi.org/10.1090//surv/214

Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations

Mathematical
Surveys
and
Monographs

Volume 214

Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations

Jared Speck

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 35L67; Secondary 35L05, 35L10, 35L72, 35Q31, 35L15.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-214

Library of Congress Cataloging-in-Publication Data

Names: Speck, Jared, 1980–
Title: Shock formation in small-data solutions to 3D quasilinear wave equations / Jared Speck.
Description: Providence, Rhode Island : American Mathematical Society, [2016] | Series: Mathematical surveys and monographs ; volume 214 | Includes bibliographical references and index.
Identifiers: LCCN 2016022109 | ISBN 9781470428570 (alk. paper)
Subjects: LCSH: Wave equation–Numerical solutions. | Shock waves–Mathematics. | Differential equations, Nonlinear–Numerical solutions. | Quasilinearization. | AMS: Partial differential equations – Hyperbolic equations and systems – Shocks and singularities. msc | Partial differential equations – Hyperbolic equations and systems – Wave equation. msc | Partial differential equations – Hyperbolic equations and systems – Second-order hyperbolic equations. msc | Partial differential equations – Hyperbolic equations and systems – Quasilinear second-order hyperbolic equations. msc | Partial differential equations – Equations of mathematical physics and other areas of application – Euler equations. msc | Partial differential equations – Hyperbolic equations and systems – Initial value problems for second-order hyperbolic equations. msc
Classification: LCC QC174.26.W28 S72 2016 | DDC 515/.3535—dc23
LC record available at https://lccn.loc.gov/2016022109

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 21 20 19 18 17 16

To Genevieve, for sleeping (just) long enough
to allow me to complete this project,
and to Teddy, the comeback kid with a heart of steel.

Contents

Preface

A central issue surrounding the study of quasilinear hyperbolic PDEs is that classical solutions, generated by smooth initial conditions, can develop singularities in finite time. In principle, many different kind of singularities are possible. The subject of this monograph is perhaps the most well-known type: shocks. Our work here primarily concerns scalar quasilinear wave equations, the main reasons being **1)** they arise in many important mathematical, physical, and geometric contexts; **2)** the last few decades have led to the development of advanced machinery tailored to such equations; and **3)** their characteristic hypersurfaces[1] are relatively simple[2] and can be analyzed using tools and concepts from the well-developed theory of Lorentzian geometry.

Before the groundbreaking work of S. Alinhac [**2**, **4**–**6**], very little was known about shock formation except in problems that are effectively one (spatial) dimensional. One-dimensional shock formation results are of course classical. We describe some of the most important such results in Chapter 1. As we describe in detail in Chapter 2, Alinhac proved finite-time shock formation for a class of wave equations in two and three spatial dimensions. Roughly, his proof applied to wave equations of the form[3]

$$(h^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$$

whenever the nonlinear terms *fail* to satisfy the null condition, which was first formulated by S. Klainerman [**45**] in three spatial dimensions and later in two spatial dimensions by Alinhac [**1**, **2**]. In the above wave equation, $h = h(\partial\Phi)$ is a Lorentzian metric such that $h(\partial\Phi = 0) = m$, where m is the standard Minkowski metric on $\mathbb{R}^{1+n}$, $n \in \{2, 3\}$. Alinhac's main results concern solutions generated by small initial data that belong to a suitable Sobolev space and that verify a nondegeneracy condition. The foundation of his proof was a new system of *geometric coordinates* tied to an *eikonal function* u, which by definition is a solution to the *eikonal equation*, that is, the hyperbolic PDE

$$(h^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha u\partial_\beta u = 0,$$

supplemented by appropriate initial conditions. The level sets of u are true characteristics (as opposed to approximate ones) corresponding to the nonlinear wave equation.

[1] In the study of wave equations, characteristic hypersurfaces are often referred to as "null hypersurfaces" in view of their connection to the Lorentzian notion of a null vectorfield. More generally, they are often referred to as simply "the characteristics."

[2] Roughly, in our study of wave equations, the characteristics are a family of "true curved cones" corresponding to the dynamic Lorentzian metric of the wave equation.

[3] The equation is to be interpreted as an equation given relative to standard rectangular coordinates, which we describe at the beginning of Chapter 2. Moreover, here and throughout, we use Einstein's summation convention.

Eikonal functions are perhaps best known for the central role they played in Christodoulou-Klainerman's celebrated proof [**19**] of the stability of Minkowski spacetime.[4] That work was the first instance in which eikonal functions were used to prove a global nonlinear result for a hyperbolic PDE. In the study of shock formation, the behavior of u, its properties, and their connection to the behavior of the solution variable[5] lie at the heart of the analysis. This was the case in Alinhac's work and in Christodoulou's work (described two paragraphs below), and it remains true in the present monograph as well. As we will see, in the problem of shock formation, the eikonal function plays an even more important role than it does in the proof of stability of Minkowski spacetime; there is an alternate proof [**59**] of the stability of Minkowski spacetime, due to Lindblad-Rodnianski, that relies on an approximate eikonal function corresponding to the Minkowski metric rather than a true eikonal function. This alternate approach leads to remarkable simplifications in the analysis because the characteristics associated to the Minkowski metric are much simpler.[6] In contrast, in the problem of shock formation, the formation of the shock and the corresponding blow-up of the solution are *exactly tied to the blow-up of the first rectangular coordinate partial derivatives of a true eikonal function.* Thus, there is little hope of finding an alternate proof that avoids the use of a true eikonal function (and, as we will see, the weighty baggage that accompanies it).

It is well-known that energy estimates are an unavoidable aspect of the study of quasilinear wave equations in more than one spatial dimension. To close the energy estimates in his proof of shock formation, Alinhac relied on a Nash-Moser iteration scheme featuring a free boundary. The presence of the free boundary is connected to the blow-up time of the iterates, which can vary, albeit slightly. A fundamental aspect of his proof, which is also present in Christodoulou's work and the present monograph, is that *the solution remains regular*[7] *relative to the geometric coordinates* mentioned two paragraphs above. In the equations studied by Alinhac, the singularity occurs in the second rectangular coordinate partial derivatives of Φ and is tied to the degeneration of the geometric coordinate system relative to the rectangular one. At the same time, the degeneracy is tied to the intersection of the characteristics. In discussing Alinhac's results and related ones, we often refer to the intersection of the characteristics as "the formation of a shock." Our intention in using this terminology is to highlight the following fundamentally important aspect of Alinhac's work: he proved that Φ and its first rectangular coordinate partial derivatives $\partial_\alpha \Phi$ remain bounded all the way up to the singularity. In particular, the singularity occurs at the level of the first rectangular coordinate partial derivatives of the Lorentzian metric components $h_{\alpha\beta}(\partial\Phi)$ and not in the $h_{\alpha\beta}(\partial\Phi)$ themselves; this feature is of fundamental importance for closing the proof.

Although Alinhac's approach is compellingly short, it has some limitations, which we describe in detail in Sect. 2.11.1. In particular, his framework allows one

[4]Roughly, [**19**] is a small-data global existence result for the Einstein-vacuum equations.

[5]In the present work, the solution variable is the solution to the wave equation.

[6]In this context, the characteristics associated to the Minkowski metric are the usual flat Minkowski light cones.

[7]Actually, the solution does not necessarily remain regular at the high derivative levels: a fundamental aspect of the proof is that the high-order energies are allowed to blow-up as the shock forms; see three paragraphs below.

to follow the solution only to the first singularity[8] *and not further*. In his 2007 monograph [**17**], D. Christodoulou proved, for a sub-class of Alinhac's wave equations,[9] a breakthrough result that significantly sharpened Alinhac's results and eliminated the drawbacks of his approach. Specifically, Christodoulou's work applies to the wave equations that arise in irrotational relativistic[10] (compressible) fluid mechanics in three spatial dimensions. In this context, the equations are known as the irrotational relativistic Euler equations. Christodoulou assumed that the data have small H^N norm, where N is a sufficiently large (nonexplicit) integer. To deduce the shock formation, he also assumed that the data verify a signed integral inequality, distinct from the nondegeneracy condition of Alinhac mentioned above. Christodoulou's framework allows one to do much more than follow the solution to the first singularity: it provides a complete picture of a portion of the maximal development[11] of the data including a description of the behavior of the solution along the boundary;[12] this is the main advantage of Christodoulou's framework. One of the key reasons that Christodoulou was able to sharpen Alinhac's results is that he was able to close the energy estimates without invoking a Nash-Moser iteration scheme. Moreover, his estimates do not involve a free boundary. Instead, Christodoulou developed a forwards approach relative to a set of geometric coordinates that, like Alinhac's, are tied to a true eikonal function. By forwards approach, we mean that Christodoulou derives traditional global-existence-type estimates for a Cauchy problem in the geometric coordinates, relative to which the solution remains rather smooth (see, however, footnote 7). As in Alinhac's results and those of the present monograph, the blow-up occurs in certain rectangular coordinate partial derivatives of the solution and is tied to the degeneracy of the change of variables map from geometric to rectangular coordinates.

In Christodoulou's framework, the degeneracy mentioned at the end of the previous paragraph and the corresponding blow-up of the solution are mediated by the vanishing of a quantity known as the *inverse foliation density*, which we denote by μ. Roughly, μ is the reciprocal of a derivative of the eikonal function u. Geometrically, $1/\mu$ is a measure of the density of the level sets of u. As we will see starting in Chapter 2, the vanishing of μ is equivalent to the intersection of the characteristics and the blow-up of the eikonal function's first rectangular coordinate partial derivatives, as we mentioned above. Moreover, *these degeneracies are exactly tied to the formation of a singularity in the solution to the wave equation*, much like in the classic example of Burgers' equation (see Sect. 1.1.2). The study of μ and the prospect of its vanishing are the main themes of [**17**] and the present work. For reasons that we describe two paragraphs below, the most compelling advantage of Christodoulou's framework is that it allows one to construct the portion of the set

[8]Roughly speaking, Alinhac's proof works when there is such a unique first singularity; his nondegeneracy conditions on the data ensure that this is the case.

[9]One has to take into account some simple differences in normalization, described in Sect. 2.11.2, in order to see that Christodoulou's equations fall under the scope of Alinhac's work.

[10]In [**21**], Christodoulou-Miao extended the result to the nonrelativistic case.

[11]Roughly speaking, the maximal development is the largest classical solution that is uniquely determined by the data.

[12]The boundary can be very complicated and, in particular, it is not contained in a constant-time hypersurface.

$\{\mu = 0\}$ that corresponds to the part of the boundary[13] along which the solution blows up.[14] Generally, the set $\{\mu = 0\}$ "evolves" into a spacetime region lying to the future of the constant-time hypersurface of first blow-up and thus it lies to the future of the region that Alinhac was able to probe.

It turns out that Christodoulou's sharp description is accompanied by severe technical difficulties: the high-order energies are allowed to blow-up as $\mu \to 0$. This difficulty is fundamentally tied to the regularity theory of the eikonal function. As was first shown in [**19**], to control the top derivatives of u, one must invoke a nontrivial procedure based on elliptic estimates and modified quantities, the latter being special combinations of terms that satisfy evolution equations with a good structure. In the problem of shock formation, this procedure introduces, at the top order, a difficult factor of $1/\mu$ into the energy identities; *this factor is the reason that the high-order energies are allowed to blow up.* A related but distinct difficulty is that *one needs to show that the low-order energies remain bounded all the way up to* $\{\mu = 0\}$. This latter step is essential for establishing, via Sobolev embedding, the basic uniform L^∞ estimates that allow one to control error terms and to treat the problem as a traditional one in which one derives global-existence-type estimates (relative to the geometric coordinates). These are the main technical difficulties that one encounters in the problem of shock formation à la Christodoulou and they are a primary reason that the work is technical and lengthy. In Chapter 2, we provide an extended overview of these issues, especially in view of the fact that they do not arise in any other context in the study of nonlinear wave equations.

Christodoulou's framework is compelling for the geometric insight it provides into the formation of shocks and for the sharp description that it yields. In addition, his approach is fundamentally important for a related problem: it turns out that his sharp description of the solution near the boundary of the maximal development is an essential ingredient in setting up the *shock development problem*, which is the problem of **1)** continuing the solution to Euler's equations (in a weak sense, subject to appropriate jump and entropy conditions) beyond the first singularity and **2)** at the same time, constructing the shock hypersurface[15] across which the solution is discontinuous.[16] The shock development problem in relativistic fluid mechanics was recently solved in spherical symmetry [**20**]. Away from symmetry, the problem remains open and is expected to be exceptionally difficult.

We mention here an important problem related to the shock development problem, which A. Majda solved [**62**–**64**] in the early 1980s. Specifically, for hyperbolic systems of conservation laws with suitable structure in more than one spatial dimension, Majda solved the *shock front problem.* That is, in a suitable Sobolev

[13] Roughly, a subset of $\{\mu = 0\}$ corresponds to the singular portion of the maximal development of the data.

[14] The boundary of the maximal development also contains another portion, along which the solution does not blow up; see Sect. 2.11.2 and Theorem 2.19 in particular.

[15] Part of the set $\{\mu = 0\}$ turns out to be a hypersurface portion along which, in the classical formulation of the irrotational Euler equations, certain solution derivatives blow up. However, $\{\mu = 0\}$ does not correspond to the physically correct hypersurface of discontinuity. The physically correct hypersurface of discontinuity propagates at supersonic speed starting from the first spacetime point where μ vanishes and develops "before" the set $\{\mu = 0\}$ has a chance to form. The physically correct hypersurface can be derived only by imposing the weak formulation of the full compressible Euler equations (without the assumption of irrotationality) starting from the time of first blow-up and by assuming suitable jump and entropy conditions.

[16] One expects the solution to be smooth on either side of the shock hypersurface.

framework, he proved a local existence result starting from an initial discontinuity given across a smooth hypersurface[17] subset of the Cauchy hypersurface. We stress that the initial hypersurface of discontinuity is prescribed. In contrast, in the shock development problem mentioned in the previous paragraph, the spacetime hypersurface of discontinuity is fully dynamic, emerging from singularity-free initial data. In the shock front problem, the data must verify suitable jump conditions, entropy conditions, and higher-order compatibility conditions. As in the shock development problem, the shock front problem features a free boundary: the shock hypersurface,[18] which is one of the unknowns. Majda's work also required an additional assumption[19] on the data that seems to be necessary for the stability of the corresponding linearized problem.

We now describe the origin and motivation behind the present monograph. In his work [**17**], Christodoulou exploited various special structures enjoyed by the wave equations of irrotational relativistic fluid mechanics, structures which Alinhac did not use in his proof of shock formation. In particular, the wave equations in [**17**] derive from a Lagrangian[20] and are invariant under the Poincaré group; just below equation (2.11), we describe some ways in which Christodoulou used these structures in his proof. In studying Christodoulou's work [**17**], the author discovered that it is possible to use his framework to close the proof of shock formation for a larger class of equations and without relying on these special structures. In particular, his framework can be extended to treat all of the wave equations studied by Alinhac. A somewhat surprising fact, which plays a fundamental role in our analysis, is that they all have a special null structure, *even though they fail to satisfy Klainerman's null condition*. This special structure is not visible relative to the standard formulation of the wave equation, *but becomes visible upon reformulating it as a system of geometric wave equations*; see Lemmas A.9 and A.16 for the main results in this direction. It is this realization that led to the present work. Our work here also generalizes and unifies earlier work on singularity formation initiated by F. John in the 1970s and continued by L. Hörmander and many others.

More precisely, in the present monograph, we extend Christodoulou's framework and use it to prove that shock singularities often develop in initially small, regular solutions to two important classes[21] of quasilinear wave equations in three spatial dimensions. Specifically, we study **i)** covariant scalar wave equations of the form $\Box_{g(\Psi)}\Psi = 0$ and **ii)** Alinhac's noncovariant scalar wave equations, that is, wave equations of the form $(h^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$. Our main result shows that whenever the nonlinear terms fail Klainerman's (classic) null condition,[22] shocks develop in solutions arising from an open set of small data. Hence, within the classes **i)** and **ii)**, our work can be viewed as a sharp converse to a fundamental result, due separately to Christodoulou [**15**] and Klainerman [**47**], which showed that

[17] This hypersurface is co-dimension two when viewed as a subset of spacetime.

[18] The shock hypersurface is a co-dimension one subset of spacetime.

[19] The assumption is automatically verified for the nonrelativistic Euler equations under the adiabatic equations of state $p = A\rho^\gamma$, where $A > 0$ and $\gamma > 1$ are constants.

[20] That is, the equations in [**17**] are Euler-Lagrange equations.

[21] It turns out that the two classes of equations are more closely related than one might expect; see the discussion below equation (2.9) and in Appendix A.

[22] Readers should take care not to confuse Klainerman's null condition with the future strong null condition and past strong null condition introduced in Appendix A and mentioned in Remark 2.3.

when the null condition is verified in three spatial dimensions, small-data global existence holds. Roughly, we give the same sharp description of the solution that Christodoulou gave in [**17**]. However, to avoid lengthening the monograph, we did not give a full description of the boundary of the maximal development nor the behavior of the solution along it. For readers interested in those details, we remark that the estimates proved in our main Theorem 22.1 are sufficient for invoking the arguments of [**17**, Chapter 15] in which Christodoulou reveals properties of the maximal development. That is, with modest additional effort, our results could be extended to give the same sharp description of the maximal development that Christodoulou gave in [**17**, Chapter 15].

In proving our main results, we have taken substantial steps that go beyond replicating the proofs given by Christodoulou in [**17**]. This is partly out of necessity, as the general class of equations that we treat leads to new kinds of error terms that are not present in [**17**]. However, we have also developed alternate strategies that greatly simplify certain aspects of the proof. One big simplification is that we are more selective in our use of geometry. That is, we use sharp, fully geometric decompositions only for treating the most delicate terms. Another simplification is that our bootstrap argument is very straightforward in view of the fact that we have organized the monograph in a linear fashion (see two paragraphs below). We have also developed alternate approaches to deriving some of the difficult top-order estimates by reducing them to other top-order estimates. This spares one a great deal of effort; see, for example, the discussion at the beginning of Sect. 15.1 in which we describe a simplified approach for obtaining estimates for the top-order derivatives of μ.

We now give an overview of the content and organization of the monograph. Chapter 1 sets the stage for the rest of monograph but is independent of the remaining chapters. It contains historical background, a discussion of shock formation in solutions to Burgers' equation, a discussion of singularity formation in 2×2 strictly hyperbolic genuinely nonlinear systems, an overview of wave dispersion in higher dimensions and its connection to global and almost global existence results, an overview of the vectorfield method (including the multiplier and commutator methods) for deriving generalized energy estimates, and a discussion of the null condition. In Chapter 2, we describe the main results of the monograph, place them in context, and provide an extended overview of the most important aspects of the proofs. In the remaining chapters, we develop the machinery and estimates needed to prove the two theorems of the monograph, which are located in Chapters 22 and 23. Roughly, in the first theorem (the main one, which is difficult to prove), we show that the solution must persist unless μ vanishes, and we derive sharp a priori estimates that hold as long as μ remains positive. In the second theorem, which is a relatively easy consequence of the first one, we exhibit an open set of data such that μ does in fact vanish in finite time, thus yielding a shock singularity. Our analysis in Chapters 3–23 applies to covariant wave equations of the form $\Box_{g(\Psi)}\Psi = 0$, while in Appendix A, we outline how to extend the results to the class of noncovariant wave equations studied by Alinhac. In Appendix B, we summarize the notation and conventions used in Chapters 2–23.

Chapters 3–23 are interdependent and are designed to be read consecutively. That is, this part of the monograph constitutes one long bootstrap-type proof, presented in chronological order. In Sect. 2.13, we give an overview of the contents

of each chapter and provide suggestions on how to read the monograph, both for expert and novice readers.

This monograph is mostly self-contained but relies extensively on basic concepts from Lorentzian and Riemannian geometry such as Levi-Civita connections, fundamental forms, curvature, pullbacks, etc. We anticipate that there are readers with knowledge in fluid mechanics, conservation laws, and/or PDEs but who are unfamiliar with those geometric concepts. Such readers can find introductory geometric material, suitable for reading almost all[23] of the present monograph, in select portions of the books [**66**, **67**, **78**]. Novice geometers should bear in mind that at the end of the day, one aims to derive estimates, and that the geometry merely provides a framework for organizing calculations and revealing analytic structural features that would otherwise be difficult to detect.

We close by highlighting the following wide-open question:

> In more than one spatial dimension, to what extent can the results of this monograph be generalized to quasilinear systems featuring multiple speeds of propagation, such as the equations of elasticity, the equations of magnetohydrodynamics, the equations of crystal optics, the Euler–Einstein equations of cosmology, or even coupled systems of wave equations featuring two or more metrics with strictly separated[24] speeds of propagation?

Jared Speck

[23] Our short proof of geometric Sobolev embedding, presented in Chapter 18, relies on a handful of more advanced results from geometry.

[24] This roughly corresponds to the presence of two or more distinct families of characteristic hypersurfaces.

Acknowledgments

I am grateful for the support offered by NSF grant #DMS-1162211 and by a Solomon Buchsbaum grant administered by the Massachusetts Institute of Technology. I would like to thank the American Institute of Mathematics for funding three SQuaREs workshops on the formation of shocks, which greatly aided the development of many of the ideas in this monograph. I would like to thank Gustav Holzegel, Sergiu Klainerman, Jonathan Luk, Willie Wong, and Shiwu Yang for participating in the workshops and for their helpful contributions. I offer special thanks to Willie Wong for creating Figure 4. I am grateful for the support of my PhD advisors, Michael Kiessling and Shadi Tahvildar-Zadeh, who encouraged me to read Christodoulou's monograph on shock formation. I would also like to thank Hans Lindblad for sharing his insight on Alinhac's work.

Bibliography

[1] S. Alinhac, *The null condition for quasilinear wave equations in two space dimensions I*, Invent. Math. **145** (2001), no. 3, 597–618. MR1856402 (2002i:35127)

[2] ———, *The null condition for quasilinear wave equations in two space dimensions. II*, Amer. J. Math. **123** (2001), no. 6, 1071–1101. MR1867312 (2003e:35193)

[3] Serge Alinhac, *Blowup for nonlinear hyperbolic equations*, Progress in Nonlinear Differential Equations and their Applications, 17, Birkhäuser Boston, Inc., Boston, MA, 1995. MR1339762 (96h:35109)

[4] ———, *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II*, Acta Math. **182** (1999), no. 1, 1–23. MR1687180 (2000d:35148)

[5] ———, *Blowup of small data solutions for a quasilinear wave equation in two space dimensions*, Ann. of Math. (2) **149** (1999), no. 1, 97–127. MR1680539 (2000d:35147)

[6] ———, *A minicourse on global existence and blowup of classical solutions to multidimensional quasilinear wave equations*, Journées "Équations aux Dérivées Partielles" (Forges-les-Eaux, 2002), 2002, pp. Exp. No. I, 33. MR1968197 (2004b:35227)

[7] ———, *An example of blowup at infinity for a quasilinear wave equation*, Astérisque **284** (2003), 1–91. Autour de l'analyse microlocale. MR2003417 (2005a:35197)

[8] Vladimir I. Arnold, *Ordinary differential equations*, Universitext, Springer-Verlag, Berlin, 2006. Translated from the Russian by Roger Cooke, Second printing of the 1992 edition. MR2242407

[9] Thierry Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 252, Springer-Verlag, New York, 1982. MR681859 (85j:58002)

[10] Frederick Bloom, *Mathematical problems of classical nonlinear electromagnetic theory*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 63, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993. MR1369088 (98c:78001)

[11] James Challis, *On the velocity of sound*, Philos. Magazine **32** (1848), 494–499.

[12] Jeff Cheeger and David G. Ebin, *Comparison theorems in Riemannian geometry*, AMS Chelsea Publishing, Providence, RI, 2008. Revised reprint of the 1975 original. MR2394158 (2009c:53043)

[13] Geng Chen, Robin Young, and Qingtian Zhang, *Shock formation in the compressible Euler equations and related systems*, J. Hyperbolic Differ. Equ. **10** (2013), no. 1, 149–172. MR3043493

[14] Bennett Chow, Peng Lu, and Lei Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, vol. 77, American Mathematical Society, Providence, RI, 2006. MR2274812 (2008a:53068)

[15] Demetrios Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), no. 2, 267–282. MR820070 (87c:35111)

[16] ———, *The action principle and partial differential equations*, Annals of Mathematics Studies, vol. 146, Princeton University Press, Princeton, NJ, 2000. MR1739321 (2003a:58001)

[17] ———, *The formation of shocks in 3-dimensional fluids*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2007. MR2284927 (2008e:76104)

[18] Demetrios Christodoulou and Sergiu Klainerman, *Asymptotic properties of linear field equations in Minkowski space*, Comm. Pure Appl. Math. **43** (1990), no. 2, 137–199. MR1038141 (91a:58202)

[19] ———, *The global nonlinear stability of the Minkowski space*, Princeton Mathematical Series, vol. 41, Princeton University Press, Princeton, NJ, 1993. MR1316662 (95k:83006)

[20] Demetrios Christodoulou and André Lisibach, *Shock development in spherical symmetry*, Annals of PDE **2** (2016), no. 1, 1–246.

[21] Demetrios Christodoulou and Shuang Miao, *Compressible flow and Euler's equations*, Surveys of Modern Mathematics, vol. 9, International Press, Somerville, MA; Higher Education Press, Beijing, 2014. MR3288725

[22] Constantine M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Third, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 325, Springer-Verlag, Berlin, 2010. MR2574377

[23] K. O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. **7** (1954), 345–392. MR0062932 (16,44c)

[24] Lars Gårding, *Solution directe du problème de Cauchy pour les équations hyperboliques*, La théorie des équations aux dérivées partielles. Nancy, 9-15 avril 1956, 1956, pp. 71–90. MR0116142 (22 #6937)

[25] ———, *Cauchy's problem for hyperbolic equations*, Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18-23 août 1957, 1958, pp. 104–109. MR0116143 (22 #6938)

[26] Yan Guo and A. Shadi Tahvildar-Zadeh, *Formation of singularities in relativistic fluid dynamics and in spherically symmetric plasma dynamics*, Nonlinear partial differential equations (Evanston, IL, 1998), 1999, pp. 151–161. MR1724661

[27] Sigurdur Helgason, *The Radon transform*, Second, Progress in Mathematics, vol. 5, Birkhäuser Boston, Inc., Boston, MA, 1999. MR1723736

[28] Gustav Holzegel, Sergiu Klainerman, Jared Speck, and Willie Wai-Yeung Wong, *Small-data shock formation in solutions to 3d quasilinear wave equations: An overview*, Journal of Hyperbolic Differential Equations **13** (2016), no. 01, 1–105, available at `http://www.worldscientific.com/doi/pdf/10.1142/S0219891616500016`.

[29] Lars Hörmander, *The lifespan of classical solutions of nonlinear hyperbolic equations*, Pseudodifferential operators (Oberwolfach, 1986), 1987, pp. 214–280. MR897781 (88j:35024)

[30] ———, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 26, Springer-Verlag, Berlin, 1997. MR1466700 (98e:35103)

[31] A. Jeffrey, *The formation of magnetoacoustic shocks*, J. Math. Anal. Appl. **11** (1965), 139–150. MR0197036 (33 #5220)

[32] A. Jeffrey and M. Teymur, *Formation of shock waves in hyperelastic solids*, Acta Mech. **20** (1974), 133–149. MR0356643 (50 #9113)

[33] Alan Jeffrey and Viktor P. Korobeinikov, *Formation and decay of electromagnetic shock waves*, Zeitschrift für angewandte Mathematik und Physik ZAMP **20** (1969), no. 4, 440–447. MR0197036 (33 #5220)

[34] F. John and S. Klainerman, *Almost global existence to nonlinear wave equations in three space dimensions*, Comm. Pure Appl. Math. **37** (1984), no. 4, 443–455. MR745325 (85k:35147)

[35] Fritz John, *Formation of singularities in one-dimensional nonlinear wave propagation*, Comm. Pure Appl. Math. **27** (1974), 377–405. MR0369934 (51 #6163)

[36] ———, *Blow-up for quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math. **34** (1981), no. 1, 29–51. MR600571 (83d:35096)

[37] ———, *Formation of singularities in elastic waves*, Trends and applications of pure mathematics to mechanics (Palaiseau, 1983), 1984, pp. 194–210. MR755727 (85h:73018)

[38] ———, *Blow-up of radial solutions of $u_{tt} = c^2(u_t)\Delta u$ in three space dimensions*, Mat. Apl. Comput. **4** (1985), no. 1, 3–18. MR808321 (87c:35114)

[39] ———, *Existence for large times of strict solutions of nonlinear wave equations in three space dimensions for small initial data*, Comm. Pure Appl. Math. **40** (1987), no. 1, 79–109. MR865358 (87m:35128)

[40] ———, *Solutions of quasilinear wave equations with small initial data. The third phase*, Nonlinear hyperbolic problems (Bordeaux, 1988), 1989, pp. 155–184. MR1033282 (91e:35132)

[41] ———, *Nonlinear wave equations, formation of singularities*, University Lecture Series, vol. 2, American Mathematical Society, Providence, RI, 1990. Seventh Annual Pitcher Lectures delivered at Lehigh University, Bethlehem, Pennsylvania, April 1989. MR1066694 (91g:35001)

[42] Jürgen Jost, *Riemannian geometry and geometric analysis*, Fifth, Universitext, Springer-Verlag, Berlin, 2008. MR2431897 (2009g:53036)

[43] Joseph B. Keller and Lu Ting, *Periodic vibrations of systems governed by nonlinear partial differential equations*, Comm. Pure Appl. Math. **19** (1966), 371–420. MR0205520 (34 #5347)

[44] S. Klainerman, *On "almost global" solutions to quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math. **36** (1983), no. 3, 325–344. MR697468 (84e:35102)

[45] Sergiu Klainerman, *Long time behaviour of solutions to nonlinear wave equations*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 1984, pp. 1209–1215. MR804771

[46] ———, *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. Math. **38** (1985), no. 3, 321–332. MR784477 (86i:35091)

[47] ———, *The null condition and global existence to nonlinear wave equations*, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 1986, pp. 293–326. MR837683 (87h:35217)

[48] ———, *A commuting vectorfields approach to Strichartz-type inequalities and applications to quasi-linear wave equations*, Internat. Math. Res. Notices **5** (2001), 221–274. MR1820023 (2001k:35210)

[49] Sergiu Klainerman and Andrew Majda, *Formation of singularities for wave equations including the nonlinear vibrating string*, Comm. Pure Appl. Math. **33** (1980), no. 3, 241–263. MR562736 (81f:35080)

[50] Sergiu Klainerman and Igor Rodnianski, *Improved local well-posedness for quasilinear wave equations in dimension three*, Duke Math. J. **117** (2003), no. 1, 1–124. MR1962783 (2004b:35233)

[51] ———, *Rough solutions of the Einstein-vacuum equations*, Ann. of Math. (2) **161** (2005), no. 3, 1143–1193. MR2180400 (2007d:58051)

[52] Sergiu Klainerman, Igor Rodnianski, and Jeremie Szeftel, *The bounded L^2 curvature conjecture*, Invent. Math. **202** (2015), no. 1, 91–216. MR3402797

[53] Steven G. Krantz and Harold R. Parks, *Geometric integration theory*, Cornerstones, Birkhäuser Boston, Inc., Boston, MA, 2008. MR2427002 (2009m:49075)

[54] Peter D. Lax, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations*, J. Mathematical Phys. **5** (1964), 611–613. MR0165243 (29 #2532)

[55] ———, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. MR0350216 (50 #2709)

[56] Jean Leray, *Hyperbolic differential equations*, The Institute for Advanced Study, Princeton, N. J., 1953. MR0063548 (16,139a)

[57] Hans Lindblad, *Counterexamples to local existence for quasilinear wave equations*, Math. Res. Lett. **5** (1998), no. 5, 605–622. MR1666844 (2000a:35171)

[58] ———, *Global solutions of quasilinear wave equations*, Amer. J. Math. **130** (2008), no. 1, 115–157. MR2382144 (2009b:58062)

[59] Hans Lindblad and Igor Rodnianski, *The global stability of Minkowski space-time in harmonic gauge*, Annals of Mathematics **171** (2010), no. 3, 1401–1477.

[60] Tai Ping Liu, *Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations*, J. Differential Equations **33** (1979), no. 1, 92–111. MR540819 (80g:35075)

[61] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984. MR748308 (85e:35077)

[62] Andrew Majda, *The existence and stability of multidimensional shock fronts*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), no. 3, 342–344. MR609047 (82g:35074)

[63] ———, *The existence of multidimensional shock fronts*, Mem. Amer. Math. Soc. **43** (1983), no. 281, v+93. MR699241 (85f:35139)

[64] ———, *The stability of multidimensional shock fronts*, Mem. Amer. Math. Soc. **41** (1983), no. 275, iv+95. MR683422 (84e:35100)

[65] J.R. Munkres, *Topology*, Second, Prentice-Hall, 2000.

[66] Barrett O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983. With applications to relativity. MR719023 (85f:53002)

[67] Peter Petersen, *Riemannian geometry*, Second, Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR2243772 (2007a:53001)

[68] Ivan Petrovskii, *Über das Cauchysche Problem für Systeme von partiellen Differentialgleichungen*, Rec. Math. [Mat. Sbornik] N.S. **44** (1937), no. 2, 815–870.

[69] Siméon D. Poisson, *Mémoire sur la théorie du son*, J. Ecole Polytechnique **7** (1808), 319–392.

[70] Amalkumar Raychaudhuri, *Relativistic cosmology. i*, Phys. Rev. **98** (1955May), 1123–1126.

[71] Bernhard Riemann, *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Abhandlungen der Kniglichen Gesellschaft der Wissenschaften in Göttingen **8** (1860), 43-66.

[72] Manuel Salas, *The curious events leading to the theory of shock waves*, Shock Waves **16** (July 2007), no. 6, 477–487.

[73] Julius Schauder, *Das Anfangswertproblem einer quasilinearen hyperbolischen Differentialgleichung zweiter Ordnung in beliebiger Anzahl von unabhängigen veränderlichen* (ger), Fundamenta Mathematicae **24** (1935), no. 1, 213–246.

[74] Thomas C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Comm. Math. Phys. **101** (1985), no. 4, 475–485. MR815196 (87d:35127)

[75] Hart F. Smith and Daniel Tataru, *Sharp local well-posedness results for the nonlinear wave equation*, Ann. of Math. (2) **162** (2005), no. 1, 291–366. MR2178963 (2006k:35193)

[76] Christopher D. Sogge, *Lectures on non-linear wave equations*, Second, International Press, Boston, MA, 2008. MR2455195 (2009i:35213)

[77] Jared Speck, *The non-relativistic limit of the Euler-Nordström system with cosmological constant*, Rev. Math. Phys. **21** (2009), no. 7, 821–876. MR2553428

[78] Robert M. Wald, *General relativity*, University of Chicago Press, Chicago, IL, 1984. MR757180 (86a:83001)

[79] Q. Wang, *A geometric approach for sharp local well-posedness of quasilinear wave equations*, ArXiv e-prints (August 2014), available at `https://arxiv.org/abs/1408.3780`.

Index

Selected Published Titles in This Series

214 **Jared Speck,** Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations, 2016

213 **Harold G. Diamond and Wen-Bin Zhang (Cheung Man Ping),** Beurling Generalized Numbers, 2016

212 **Pandelis Dodos and Vassilis Kanellopoulos,** Ramsey Theory for Product Spaces, 2016

211 **Charlotte Hardouin, Jacques Sauloy, and Michael F. Singer,** Galois Theories of Linear Difference Equations: An Introduction, 2016

210 **Jason P. Bell, Dragos Ghioca, and Thomas J. Tucker,** The Dynamical Mordell–Lang Conjecture, 2016

209 **Steve Y. Oudot,** Persistence Theory: From Quiver Representations to Data Analysis, 2015

208 **Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó,** Grid Homology for Knots and Links, 2015

207 **Vladimir I. Bogachev, Nicolai V. Krylov, Michael Röckner, and Stanislav V. Shaposhnikov,** Fokker–Planck–Kolmogorov Equations, 2015

206 **Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni,** The Ricci Flow: Techniques and Applications: Part IV: Long-Time Solutions and Related Topics, 2015

205 **Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik,** Tensor Categories, 2015

204 **Victor M. Buchstaber and Taras E. Panov,** Toric Topology, 2015

203 **Donald Yau and Mark W. Johnson,** A Foundation for PROPs, Algebras, and Modules, 2015

202 **Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D. Milman,** Asymptotic Geometric Analysis, Part I, 2015

201 **Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill, Editors,** Topological Modular Forms, 2014

200 **Nikolai Nadirashvili, Vladimir Tkachev, and Serge Vlăduţ,** Nonlinear Elliptic Equations and Nonassociative Algebras, 2014

199 **Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov,** Foundations of Free Noncommutative Function Theory, 2014

198 **Jörg Jahnel,** Brauer Groups, Tamagawa Measures, and Rational Points on Algebraic Varieties, 2014

197 **Richard Evan Schwartz,** The Octagonal PETs, 2014

196 **Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou,** Geometry of Isotropic Convex Bodies, 2014

195 **Ching-Li Chai, Brian Conrad, and Frans Oort,** Complex Multiplication and Lifting Problems, 2014

194 **Samuel Herrmann, Peter Imkeller, Ilya Pavlyukevich, and Dierk Peithmann,** Stochastic Resonance, 2014

193 **Robert Rumely,** Capacity Theory with Local Rationality, 2013

192 **Messoud Efendiev,** Attractors for Degenerate Parabolic Type Equations, 2013

191 **Grégory Berhuy and Frédérique Oggier,** An Introduction to Central Simple Algebras and Their Applications to Wireless Communication, 2013

190 **Aleksandr Pukhlikov,** Birationally Rigid Varieties, 2013

189 **Alberto Elduque and Mikhail Kochetov,** Gradings on Simple Lie Algebras, 2013

188 **David Lannes,** The Water Waves Problem, 2013

For a complete list of titles in this series, visit the AMS Bookstore at **www.ams.org/bookstore/survseries/**.

In 1848 James Challis showed that smooth solutions to the compressible Euler equations can become multivalued, thus signifying the onset of a shock singularity. Today it is known that, for many hyperbolic systems, such singularities often develop. However, most shock-formation results have been proved only in one spatial dimension. Serge Alinhac's groundbreaking work on wave equations in the late 1990s was the first to treat more than one spatial dimension. In 2007, for the compressible Euler equations in vorticity-free regions, Demetrios Christodoulou remarkably sharpened Alinhac's results and gave a complete description of shock formation.

In this monograph, Christodoulou's framework is extended to two classes of wave equations in three spatial dimensions. It is shown that if the nonlinear terms fail to satisfy the null condition, then for small data, shocks are the only possible singularities that can develop. Moreover, the author exhibits an open set of small data whose solutions form a shock, and he provides a sharp description of the blow-up. These results yield a sharp converse of the fundamental result of Christodoulou and Klainerman, who showed that small-data solutions are global when the null condition is satisfied.

Readers who master the material will have acquired tools on the cutting edge of PDEs, fluid mechanics, hyperbolic conservation laws, wave equations, and geometric analysis.

ISBN 978-1-4704-2857-0

9 781470 428570

SURV/214