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Homotopy of Operads and Grothendieck–Teichmüller Groups

Part 1: The Algebraic Theory and its Topological Background

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Un souffle ouvre des brèches opéradiques dans les cloisons, – brouille le pivotement des toits rongés, – disperse les limites des foyers, – éclipse les croisées.

Arthur Rimbaud
Nocturne vulgaire. Les Illuminations (1875)

Contents

This work is divided in two volumes, referred to as "Part 1: The Algebraic Theory and its Topological Background" and "Part 2: The Applications of (Rational) Homotopy Theory Methods", as indicated on the cover of the volumes. But our narrative is actually organized into three parts, numbered I-II-III, and three appendices, numbered A-B-C, which define the internal divisions of this book. This first volume comprises Part I, "From Operads to Grothendieck–Teichmüller Groups"; Appendix A, "Trees and the Construction of Free Operads"; and Appendix B, "The Cotriple Resolution of Operads". The second volume comprises Part II, "Homotopy Theory and its Applications to Operads"; Part III, "The Computation of Homotopy Automorphism Spaces of Operads"; and Appendix C, "Cofree Cooperads and the Bar Duality of Operads". Each volume includes a preface, a notation glossary, a bibliography, and a subject index.

References to chapters, sections, paragraphs, and statements of the book are given by $\S x.y.z$ when these cross references are done within a part (I, II, and III) and by $\S P.x.y.z$ where $P = \mathrm{I}, \mathrm{II}, \mathrm{III}$ otherwise. The cross references to the sections, paragraphs, and statements of the appendices are given by $\S P.x.y$ throughout the book, where $\S P = \S\mathrm{A}, \S\mathrm{B}, \S\mathrm{C}$. The preliminary part of the first volume of this book also includes a Foundations and Conventions section, whose paragraphs, numbered §§0.1-0.16, give a summary of the main conventions used in this work.

Preliminaries

Preface

The first purpose of this work is to give an overall reference, starting from scratch, on applications of methods of algebraic topology to the study of operads in topological spaces. Most definitions, notably fundamental concepts of the theory of operads and of homotopy theory, are reviewed in this book in order to make our account accessible to graduate students and to researchers coming from the various fields of mathematics related to our subject. Then our ultimate objective is to give a homotopical interpretation of a deep relationship between operads and Grothendieck–Teichmüller groups. This connection, which has emerged from research on the deformation quantization process in mathematical physics, gives a new approach to understanding internal symmetries of structures that occur in various constructions of algebra and topology.

We review the definition of an operad at the beginning of this monograph. Simply recall for the moment that an operad is a structure, formed by collections of abstract operations, which is used to define a category of algebras. In our study, we mainly consider the example of E_n-operads, $n = 1, 2, \ldots, \infty$, which are used to model a hierarchy of homotopy commutative structures, from fully homotopy associative but not commutative ($n = 1$), up to fully homotopy associative and commutative ($n = \infty$). Let us mention that the notion of an E_1-operad is synonymous to that of an A_∞-operad, which is used in the literature when one deals only with purely homotopy associative structures.

The notion of an E_n-operad formally refers to a class of operads rather than to a singled-out object. This class consists, in the initial definition, of topological operads which are homotopically equivalent to a reference model, the Boardman–Vogt operad of little n-discs D_n. The operad of little n-cubes, which is a simple variant of the little n-discs operad, is also used in the literature to provide an equivalent definition of the class of E_n-operads. We provide a detailed account of the definition of these notions in this book. Nevertheless, as we soon explain, our ultimate purpose is not to study E_n-operads themselves, but homotopy automorphism groups attached to these structures.

Before explaining this goal, we survey some motivating applications of E_n-operads which are not our main subject matter (we only give short introductions to these topics) but illustrate our approach of the subject.

The operads of little n-discs D_n were initially introduced to collect operations acting on iterated loop spaces. The first main application of these operads, which has motivated their definition, is the Boardman–Vogt and May recognition theorems of iterated loop spaces: any space Y equipped with an action of the operad D_n has the homotopy type of an n-fold loop space $\Omega^n X$ up to group-completion (see [27, 28] and [140]). Recall that the set of connected components of an n-fold loop space $\Omega^n X$ is identified with the nth homotopy group $\pi_n(X)$ of the space X. (Recall also

that this group is abelian as soon as $n > 1$.) The action of D_n on $\Omega^n X$ includes a product operation $\mu : \Omega^n X \times \Omega^n X \to \Omega^n X$ which, at the level of connected components, gives the composition operation of the group $\pi_n(X)$ for any $n > 0$. The operad D_n carries the homotopies that make this product associative (and commutative for $n > 1$) and includes further operations, representing homotopy constraints, which we need to form a faithful picture of the structure of the n-fold loop space $\Omega^n X$.

This outline gives the initial topological interpretation of E_n-operads. But this topological picture has also served as a guiding idea for a study of E_n-operads in other domains. Indeed, new applications of E_n-operads, which have initiated a complete renewal of the subject, have been discovered in the fields of algebra and mathematical physics, mostly after the proof of the Deligne conjecture asserting that the Hochschild cochain complex $C^*(A, A)$ of an associative algebra A inherits an action of an E_2-operad. In this context, we deal with a chain version of the previously considered topological little 2-discs operad D_2.

The cohomology of the Hochschild cochain complex $C^*(A, A)$ is identified in degree 0 with the center $Z(A)$ of the associative algebra A. In a sense, the Hochschild cochain complex represents a derived version of this ordinary center $Z(A)$. From this point of view, the construction of an E_2-structure on $C^*(A, A)$ determines, as in the study of iterated loop spaces, the level of homotopical commutativity of the derived center which lies beyond the apparent commutativity of the ordinary center. The first proofs of the Deligne conjecture have been given by Kontsevich-Soibelman [109] and McClure-Smith [142]. The interpretation in terms of derived centers has been emphasized by Kontsevich [108] in order to formulate a natural extension of the conjecture for algebras over E_n-operads, where we now consider any $n \geq 1$ (we also refer to John Francis's work [64] for a solution of this problem in the framework of ∞-category theory).

The verification of the Deligne conjecture has yielded a second generation of proofs, promoted by Tamarkin [173] and Kontsevich [108], of the Kontsevich formality theorem on Hochschild cochains. Recall that this result implies the existence of deformation quantizations of arbitrary Poisson manifolds (we also refer to [38, 148] for higher dimensional generalizations of the deformation quantization problem involving the categories of algebras associated to E_n-operads for all $n \geq 1$). The new approaches of the Kontsevich formality theorem rely on the application of Drinfeld's associators to transport the E_2-structure yielded by the Deligne conjecture on the Hochschild cochain complex to the cohomology. In the final outcome, one obtains that each associator gives rise to a deformation quantization functor. This result has hinted the existence of a deep connection between the deformation quantization problem and the program, initiated in Grothendieck's famous "esquisse" [83], which aims to understand Galois groups through geometric actions on curves. The Grothendieck–Teichmüller groups are devices, introduced in this program, which encode the information that can be captured through the actions considered by Grothendieck. The correspondence between associators and deformation quantizations imply that a rational pro-unipotent version of the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ acts on the moduli space of deformation quantizations. The initial motivation of our work was the desire to understand this connection from a homotopical viewpoint, in terms of homotopical structures associated to E_2-operads. The homotopy automorphisms of operads come into play at this point.

Recently, it has also been discovered that mapping spaces of E_n-operads can be used to compute the homotopy of the spaces of compactly supported embeddings of Euclidean spaces modulo immersions $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ (see notably the works of Dev Sinha [162], Lambrechts-Turchin-Volić [112], Arone-Turchin [9], Dwyer-Hess [59], and Boavida-Weiss [29] for various forms of this statement). We give more details on these developments in the concluding chapter of this book. In a related field, it has been observed that algebras over E_n-operads can be used to define multiplicative analogues of the classical singular homology theory for manifolds (different but equivalent constructions of such multiplicative homology theories are the "topological chiral homology", studied by Jacob Lurie in [127, 128], and the "factorization homology", studied by John Francis in [64] and by Costello-Gwilliam in [50]). These new developments give further motivations for the study of mapping spaces and homotopy automorphism spaces of E_n-operads.

Recall again that an operad is a structure which governs a category of algebras. The homotopy automorphisms of an operad P are transformations, defined at the operad level, which encode natural homotopy equivalences on the category of algebras associated to P. In this interpretation, the group of homotopy automorphism classes of an E_2-operad, which we actually aim to determine, represents the internal symmetries of the first level of homotopy commutative structures which E_2-operads serve to encode. To obtain our result, we mainly work in the setting of rational homotopy theory and we consider a rational version of the notion of an E_2-operad in topological spaces. We precisely establish that the group of rational homotopy automorphism classes of E_2-operads is isomorphic to the pro-unipotent Grothendieck–Teichmüller group $GT(\mathbb{Q})$. This result is new and represents the main outcome of our work.

In the conclusion of this monograph, we will give an overview of a sequel of this research [67], where the author, Victor Turchin, and Thomas Willwacher tackle the computation of the homotopy of the spaces of rational homotopy automorphisms of E_n-operads in dimension $n \geq 2$ (see § III.6). The main outcome of this study is that these spaces of rational homotopy automorphisms of general E_n-operads can be described in terms of the homology of a complex of graphs which, according to an earlier work of Willwacher [184], reduces to the Grothendieck–Teichmüller group in the case $n = 2$. (We have a similar description of the mapping spaces of E_n-operads which occur in the study of embedding spaces.)

To reach all of these results, we have to set up a new rational homotopy theory for topological operads beforehand and to give a sense to the rationalization of operads in topological spaces. We actually define an analogue of the Sullivan model of the rational homotopy of spaces for operads. We then consider cooperads, the structures which are dual to operads in the categorical sense. We precisely show that the rational homotopy of an operad in topological spaces is determined by an associated cooperad in commutative dg-algebras (a Hopf dg-cooperad). We have a small model of the operad of little 2-discs which is given by the Chevalley–Eilenberg cochain complex of the Drinfeld–Kohno Lie algebras (the Lie algebras of infinitesimal braids). We use this model in our proof that the group of rational homotopy automorphism classes of E_2-operads reduces to the pro-unipotent Grothendieck–Teichmüller group. In the course of our study, we also define a cosimplicial analogue of the Sullivan model of operads. This cosimplicial model remains well defined in

the positive characteristic setting and gives, in this context, a model for the homotopy of the completion of topological operads at a prime.

The other main topics considered in our study include the application of homotopy spectral sequences and of Koszul duality techniques for the computation of mapping spaces attached to operads. We aim to give a detailed and comprehensive introduction to the applications of these methods for the study of operads from the point of view of homotopy theory. Besides, we thoroughly review the applications of Hopf algebras to the Malcev completion (the rationalization) of general groups. For the applications to operads, we actually consider an extension of the classical Malcev completion of groups to groupoids. Indeed, we will explain that the pro-unipotent Grothendieck–Teichmüller group can be defined as the group of automorphisms of the Malcev completion of a certain operad in groupoids which governs operations acting on braided monoidal categories. We use this observation and classical constructions of homotopy theory to define our correspondence between the Grothendieck–Teichmüller group and the space of homotopy automorphisms of E_2-operads. The previously mentioned homotopy spectral sequence techniques are used to check that this correspondence induces a bijection in homotopy. This first volume of this monograph is mainly devoted to the fundamental and algebraic aspects of our subject, from the definition of the notion of an operad to the definition of the Grothendieck–Teichmüller group. The applications of homotopy theory to operads and the proof of our isomorphism statement between the Grothendieck–Teichmüller group and the group of homotopy automorphisms classes of E_2-operads are explained in the second volume.

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Mathematical Objectives

The ultimate goal of this work, as we explain in the general introduction, is to prove that the Grothendieck–Teichmüller group represents, in the rational setting at least, the group of homotopy automorphism classes of E_2-operads. This objective can be taken as a motivation to read this book or as a guiding example of an application of our methods.

The definition of an operad is recalled with full details in the first part of this volume. In this introductory section, we only aim to give an idea of our main results. Let us simply recall that an operad P basically consists of a collection $\mathsf{P}(r)$, $r \in \mathbb{N}$, where each object $\mathsf{P}(r)$ parameterizes operations with r inputs $p = p(x_1, \ldots, x_r)$, together with a multiplicative structure which models the composition of such operations. We can define operads in any category equipped with a symmetric monoidal structure $\mathcal{M}$. We then assume $\mathsf{P}(r) \in \mathcal{M}$, and we use the tensor product operation, given with this category $\mathcal{M}$, to define the composition structure attached to our operad. The operads in a base symmetric monoidal category form a category, which we denote by $\mathcal{M}\,\mathcal{O}p$, or more simply, by $\mathcal{O}p = \mathcal{M}\,\mathcal{O}p$, when this ambient category $\mathcal{M}$ is fixed by the context. An operad morphism $f : \mathsf{P} \to \mathsf{Q}$ naturally consists of a collection of morphisms in the base category $f : \mathsf{P}(r) \to \mathsf{Q}(r)$, $r \in \mathbb{N}$, which preserve the composition structure of our operads.

For technical reasons, we have to consider operads P_+ equipped with a distinguished element $* \in \mathsf{P}_+(0)$ (whenever the notion of an element makes sense), which represents an operation with zero input (a unitary operation in our terminology). In the context of sets, we moreover assume that $\mathsf{P}_+(0)$ is a one-point set reduced to this element. In the module context, we assume that $\mathsf{P}_+(0)$ is a one dimensional module over the ground ring. In a general setting, we assume that $\mathsf{P}_+(0)$ is the unit object given with the symmetric monoidal structure of our base category. We then say that P_+ forms a unitary operad. We use the notation $\mathcal{O}p_*$ to refer to the category of unitary operads. The subscript $*$ indicates the fixed arity zero component which we assign to the objects of this category of operads. We usually consider together both a non-unitary operad P, which has no term in arity 0, and an associated unitary operad P_+, where the arity zero term, spanned by the distinguished operation $* \in \mathsf{P}_+(0)$, is added. We therefore follow the convention to use a subscript $+$, marking the addition of this term, for the notation of the unitary operad P_+. We also say that the operad P_+ arises as a *unitary extension* of the non-unitary operad P. We often perform constructions on the non-unitary operad P first, and on the unitary operad P_+ afterwards, by assuming that the additional distinguished element (or unit term) of P_+ is preserved by the operations involved in our construction.

In topology, an E_2-operad is usually defined as an operad in the category of spaces which is equivalent to Boardman–Vogt's operad of little 2-discs D_2 in the

homotopy category of operads. The spaces $\mathsf{D}_2(r)$ underlying this operad have a trivial homotopy in dimension $* \neq 1$, and for $* = 1$, we have $\pi_1 \mathsf{D}_2(r) = P_r$, where P_r denotes the pure braid group on r strands. Thus, the space $\mathsf{D}_2(r)$ is an Eilenberg-MacLane space $K(P_r, 1)$ associated to the pure braid group P_r. For our purpose, we consider a rationalization of the little 2-discs operad $\widehat{\mathsf{D}_2}$, for which we have $\pi_1 \widehat{\mathsf{D}_2}(r) = \hat{P}_r$, where $\hat{P}_r$ denotes the Malcev completion of the group P_r. We give a general definition of the rationalization for operads in topological spaces in the second volume of this book. In the case of the little 2-discs operad D_2, we have a simple model of the rationalization $\widehat{\mathsf{D}_2}$ which relies on the Eilenberg-MacLane space interpretation of the little 2-discs spaces. We give a brief outline of this approach soon.

Homotopy automorphisms can be defined in the general setting of model categories which provides a suitable axiomatic framework for the application of homotopy theory concepts to operads. In order to introduce our subject, we first explain a basic interpretation of the general definition of a homotopy automorphism in the context of topological operads.

We have a natural notion of homotopy $\sim$ for morphisms of operads in topological spaces. To a topological operad Q, we associate the collection of path spaces $\mathsf{Q}^{\Delta^1}(r) = \mathtt{Map}_{\mathcal{T}op}([0,1], \mathsf{Q}(r))$, which inherits an operad structure from Q and defines a path-object associated to Q in the category of topological operads. We explicitly define a homotopy between operad morphisms $f, g : \mathsf{P} \to \mathsf{Q}$ as an operad morphism $h : \mathsf{P} \to \mathsf{Q}^{\Delta^1}$ which satisfies $d_0 h = f$, $d_1 h = g$, where $d_0, d_1 : \mathsf{Q}^{\Delta^1} \to \mathsf{Q}$ are the natural structure morphisms (evaluation at the origin and at the end point) associated with our path-object Q^{Δ^1}. This homotopy h is intuitively equivalent to a continuous family of operad morphisms $h_t : \mathsf{P} \to \mathsf{Q}$ going from $h_0 = f$ to $h_1 = g$.

In a first approximation, we take the sets of homotopy classes of operad morphisms as the morphism sets of the homotopy category $\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)$ which we associate to the category of topological operads $\mathcal{T}op\,\mathcal{O}p$. In principle, we have to deal with a suitable notion of cofibrant object in the category of operads and to replace any operad by a cofibrant resolution in order to use this definition of morphism set. But we will explain this issue later on. We focus on the basic definition of the morphism sets of the homotopy category for the moment.

The groups of homotopy automorphism classes, which we aim to determine, are the groups of automorphisms of the homotopy category $\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)$. The automorphism group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(\mathsf{P})$ associated to a given operad $\mathsf{P} \in \mathcal{T}op\,\mathcal{O}p$ accordingly consists of homotopy classes of morphisms $f : \mathsf{P} \to \mathsf{P}$, which have a homotopy inverse $g : \mathsf{P} \to \mathsf{P}$ such that $fg \sim id$ and $gf \sim id$. We consider the operadic homotopy relation at each step of this definition.

Note that a topological operad P gives rise to an operad object in the homotopy category of topological spaces $\mathtt{Ho}(\mathcal{T}op)$, and we could also study the automorphism group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op)\,\mathcal{O}p}(\mathsf{P})$ formed in this naive category of homotopy operads $\mathtt{Ho}(\mathcal{T}op)\,\mathcal{O}p$. But these automorphism groups differ from our groups of homotopy automorphisms and do not give the appropriate structure for the homotopy version of usual constructions of group theory (like homotopy fixed points). Indeed, an automorphism of the operad P in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$ is just a collection of homotopy classes of maps $f \in [\mathsf{P}(r), \mathsf{P}(r)]$, invertible in the homotopy category of spaces, and which preserve the operadic

structures up to homotopy, unlike our homotopy automorphisms that preserve operadic structures strictly. Moreover, actual operad morphisms $f, g : P \to Q$ define the same morphism of operads in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$ as soon as we have a homotopy between the individual maps $f : P(r) \to P(r)$ and $g : P(r) \to P(r)$, for all $r \in \mathbb{N}$ (regardless of operad structures). Thus, operad morphisms which are homotopic in the strong operadic sense determine the same morphism of operads in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$, but the converse implication does not hold. By associating the collection of homotopy classes of maps $f : P(r) \to P(r)$ to a homotopy automorphism $f \in \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(P)$, we obtain a mapping $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(P) \to \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op)\,\mathcal{O}p}(P)$, from the group of homotopy classes of homotopy automorphisms towards the group of automorphisms of the operad in the homotopy category of spaces, but this mapping is neither an injection nor a surjection in general.

To apply methods of algebraic topology, we associate to any operad P a whole simplicial set of homotopy automorphisms $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(P)$ rather than a single group of homotopy automorphism classes. To be more precise, this group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(P)$, which we primarily aim to determine, is identified with the set of connected components of our homotopy automorphism space $\pi_0(\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(P))$. In the second volume of this work, we explain the definition of these homotopy automorphism spaces in the general context of simplicial model categories. For the moment, we simply give a short outline of the definition in the context of topological operads.

First, we extend the definition of our path object and we consider, for each $n \in \mathbb{N}$, an operad P^{Δ^n} which is defined by the collection of function spaces $P^{\Delta^n}(r) = \mathtt{Map}_{\mathcal{T}op}(\Delta^n, P(r))$ on the n-simplex Δ^n. This operad sequence P^{Δ^n} inherits a simplicial structure from the topological simplices Δ^n. In particular, since we obviously have $P = P^{\Delta^0}$, we have a morphism $v^* : P^{\Delta^n} \to P$ associated to each vertex v of the n-simplex Δ^n. The simplicial set $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(P)$ precisely consists, in dimension $n \in \mathbb{N}$, of the morphisms of topological operads $f : P \to P^{\Delta^n}$ such that the composite $v^* f$ defines a homotopy equivalence of the operad $P = P^{\Delta^0}$, for each vertex $v \in \Delta^n$. From this definition, we immediately see that the 0-simplices of the simplicial set $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(P)$ are the homotopy equivalences of the operad P, the 1-simplices are the operadic homotopies $h : P \to P^{\Delta^1}$ between homotopy equivalences, and therefore, we have a formal identity $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(P) = \pi_0\,\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(P)$ between our group of homotopy automorphism classes $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(P)$ and the set of connected components of $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(P)$.

In what follows, we often consider simplicial sets as combinatorial models of topological spaces. In this situation, we adopt a common usage of homotopy theory to use the name 'space' for a simplicial set. Therefore we refer to the simplicial set $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(P)$ as the homotopy automorphism space associated to the operad P.

Besides homotopy equivalences, we consider a class of morphisms, called weak-equivalences, which are included in the definition of a model structure on the category of operads. We adopt the standard notation of the theory of model categories $\xrightarrow{\sim}$ to refer to this class of distinguished morphisms. The notion of a model category also includes the definition of a class of cofibrant objects, generalizing the cell complexes of topology, and which are well suited for the homotopy constructions we aim to address.

To be more specific, recall that a map of topological spaces $f : X \to Y$ is a weak-equivalence when this map induces a bijection on connected components $f_* : \pi_0(X) \xrightarrow{\simeq} \pi_0(Y)$ and an isomorphism on homotopy groups $f_* : \pi_*(X, x) \xrightarrow{\simeq} \pi_*(Y, f(x))$, for all $* > 0$ and for any choice of base point $x \in X$. We define a weak-equivalence of operads as an operad morphism $f : P \to Q$ of which underlying maps $f : P(r) \to Q(r)$ are weak-equivalences of topological spaces. In the context of topological spaces, a classical result asserts that any weak-equivalence between cell complexes is homotopically invertible as a map of topological spaces. In the context of topological operads, we similarly obtain that any weak-equivalence between cofibrant operads $f : P \xrightarrow{\sim} Q$ is homotopically invertible as an operad morphism: we have an operad morphism $g : Q \to P$ in the converse direction as our weak-equivalence $f : P \xrightarrow{\sim} Q$ such that $fg \sim id$ and $gf \sim id$, where we now consider the operadic homotopy relation (as in the definition of a homotopy automorphism for operads).

The proof of the model category axioms for operads includes the construction of a cofibrant resolution functor, which assigns a cofibrant operad R equipped with a weak-equivalence $R \xrightarrow{\sim} P$ to any given operad P. The definition of the homotopy category of topological operads in terms of homotopy class sets of morphisms is actually the right one when we replace each operad P by such a cofibrant model $R \xrightarrow{\sim} P$. In particular, when we form the group of homotopy automorphism classes of an operad $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(P)$, we have to assume that P is cofibrant as an operad, otherwise we tacitely assume that we apply our construction to a cofibrant resolution of P. The general theory of model categories ensures that the obtained group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\,\mathcal{O}p)}(P)$ does not depend, up to isomorphism, on the choice of this cofibrant resolution. We have similar results and we apply similar conventions for the homotopy automorphism spaces $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(P)$. To be precise, in the general context of the theory of model categories, we have a notion of fibrant object, which is dual to the notion of a cofibrant object, and we actually have to consider objects which are both cofibrant and fibrant when we use the above definition of the group of homotopy automorphism classes of an object. We have a similar observation for the definition of homotopy automorphism spaces. But we can neglect this issue for the moment, because all objects of our model category of topological operads are fibrant.

We go back to the case of the little 2-discs operad. We aim to determine the homotopy groups of the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(D^{\widehat{}}_{2+})$ associated to the rationalization of D_2 and in the unitary operad context, which we mark by the addition of the subscript $+$ in our notation. Recall that the connected components of this space $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(D^{\widehat{}}_{2+})$ correspond to homotopy classes of operad homotopy equivalences $f : \hat{R}_{2+} \xrightarrow{\sim} \hat{R}_{2+}$, where $\hat{R}_2$ denotes a cofibrant model of the rationalized little 2-discs operad $D^{\widehat{}}_2$. Our result reads:

THEOREM A. *The homotopy automorphism space of the rationalization of the little 2-discs operad $D^{\widehat{}}_{2+}$ satisfies*

$$\pi_*\,\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(D^{\widehat{}}_{2+}) = \begin{cases} GT(\mathbb{Q}), & \textit{for } * = 0, \\ \mathbb{Q}, & \textit{for } * = 1, \\ 0, & \textit{otherwise}, \end{cases}$$

where $GT(\mathbb{Q})$ denotes the rational pro-unipotent version of the Grothendieck–Teichmüller group, such as defined by V. Drinfeld in [57].

The identity established in this theorem is a new result. The ultimate goal of this work precisely consists in proving this statement. In fact, we will more precisely prove that the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\widehat{D_{2+}})$ is weakly-equivalent (as a monoid) to a semi-direct product $GT(\mathbb{Q}) \ltimes SO(2)\widehat{\ }$, where we regard the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ as a discrete group in topological spaces and $SO(2)\widehat{\ }$denotes a rationalization of the space of rotations of the plane $SO(2)$ which naturally acts on the little 2-discs spaces by operad automorphisms. We have $\pi_1\, SO(2) = \mathbb{Z} \Rightarrow \pi_1\, SO(2)\widehat{\ } = \mathbb{Q}$ and the module $\mathbb{Q}$ in the above theorem comes from this factor $SO(2)\widehat{\ }$ of our semi-direct product decomposition of the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\widehat{D_{2+}})$.

We mentioned, at the beginning of this survey, that the operad of little 2-discs D_2 consists of Eilenberg-MacLane spaces $K(P_r, 1)$, where P_r denotes the pure braid group on r strands, and the associated rationalized operad $\widehat{D_2}$ consists of Eilenberg-MacLane spaces $K(\hat{P}_r, 1)$, where we now consider the Malcev completion of the group P_r. We have standard models of the Eilenberg-MacLane spaces $K(P_r, 1)$ which are given by the classifying spaces of the groups P_r. But these spaces do not form an operad. Nevertheless, we can adapt this classifying space approach to give a simple model of E_2-operad. Instead of the pure braid group P_r, we consider the classifying space of a groupoid of parenthesized braids $PaB(r)$. The morphisms of this groupoid are braids on r strands indexed by elements of the set $\{1, \ldots, r\}$. The parenthesization refers to an extra structure, added to the contact points of the braids, which define the object sets of our groupoid. Unlike the pure braid groups P_r, the collection of groupoids $PaB(r)$ forms an operad in the category of groupoids, and the associated collection of classifying spaces $\mathtt{B}(PaB)(r) = \mathtt{B}(PaB(r))$ forms an operad in topological spaces. We check, by relying on an argument of Zbigniew Fiedorowicz, that this operad $\mathtt{B}(PaB)$ is weakly-equivalent to the operad of little 2-discs D_2 and, hence, forms an E_2-operad.

We can perform the Malcev completion of the groupoids $PaB(r)$ underlying the parenthesized braid operad PaB. We get an operad in groupoids $PaB\widehat{\ }$. We will see that the collection of classifying spaces $\mathtt{B}(PaB\widehat{\ })(r) = \mathtt{B}(PaB(r)\widehat{\ })$ associated to this operad $PaB\widehat{\ }$forms an operad in topological spaces which defines a model for the rationalization of the operad of little 2-discs $\widehat{D_2}$.

The Grothendieck–Teichmüller group $GT(\mathbb{Q})$ can actually be identified with a group of automorphisms associated to the (unitary) operad in groupoids $PaB\widehat{\ }_+$ and an automorphism of topological operads $\mathtt{B}(\phi) : \mathtt{B}(PaB\widehat{\ }_+) \xrightarrow{\sim} \mathtt{B}(PaB\widehat{\ }_+)$ can be associated to any element of this group $\phi \in GT(\mathbb{Q})$ by functoriality of the classifying space construction. We can lift any such automorphism to any chosen cofibrant model $R\widehat{\ }_{2+}$ of the rationalization of the operad of little 2-discs $\widehat{D_{2+}}$. We accordingly associate a well-defined rational homotopy automorphism of the E_2-operad $R\widehat{\ }_{2+}$ to our element of the Grothendieck–Teichmüller group $\phi \in GT(\mathbb{Q})$. Our main theorem asserts that, when we work in the rational setting, this construction gives exactly all homotopy automorphism classes of E_2-operads.

We consider a pro-unipotent version of the Grothendieck–Teichmüller group in our theorem. We mostly study this group for the applications in the rational homotopy theory of operads, but we also have a pro-finite version of the

Grothendieck–Teichmüller group which is better suited for the purposes of the original Grothendieck's program in Galois theory. In fact, a pro-finite analogue of our result, which relates this pro-finite Grothendieck–Teichmüller group to a pro-finite completion of E_2-operads, has been obtained by Horel [90] during the writing of this monograph. We will give more explanations on this statement and on other generalizations of the result of Theorem A in the concluding chapter of the second volume of this monograph (§ III.6).

The (rational) homology of the little 2-discs spaces $\mathtt{H}_*(D_2(r)) = \mathtt{H}_*(D_2(r), \mathbb{Q})$, $r \in \mathbb{N}$, forms an operad in graded modules $\mathtt{H}_*(D_2)$. We will see that this homology operad $\mathtt{H}_*(D_2)$ is identified with an operad, defined in terms of generating operations and relations, and which we can associate to the category of Gerstenhaber algebras (a graded version of the notion of a Poisson algebra). We therefore use the notation Gerst_2 for this operad such that $\mathit{Gerst}_2 = \mathtt{H}_*(D_2)$. In what follows, we consider the (rational) cohomology of the little 2-discs spaces $\mathtt{H}^*(D_2(r)) = \mathtt{H}^*(D_2(r), \mathbb{Q})$ rather than the homology $\mathtt{H}_*(D_2(r))$. We use that the cohomology of a space inherits a unitary commutative algebra structure and that the collection of the cohomology algebras $\mathtt{H}^*(D_2(r))$ associated to the little 2-discs spaces $D_2(r)$ forms a cooperad in the category of unitary commutative algebras in graded modules, where the name cooperad obviously refers to the structure dual to an operad in the categorical sense (we go back to this concept later on). We also use the phrase '(graded) Hopf cooperad' to refer to this particular case of the structure defined by a cooperad in the category of unitary commutative algebras (in graded modules).

For our purpose, we actually need a counterpart, in the category of graded Hopf cooperads, of the category of unitary operads. We have to adapt the definition of a cooperad in this case, because the consideration of an arity zero term in the context of a cooperad creates convergence difficulties in the definition of cofree objects. We work out this problem by integrating this part of the composition structure of our cooperads into a diagram structure. We will be more precise later on. For the moment, we just use the notation $gr\,\mathcal{H}opf\,\mathcal{O}p^c_*$, with our distinguishing $*$ mark, for this category of (graded) Hopf cooperads which we associate to unitary operads. We will adopt another notation as soon as we will be able to make the definition of this category more precise.

We have a general approach to compute the homotopy of mapping spaces in the category of operads $\mathtt{Map}_{\mathcal{T}op\,\mathcal{O}p}(P, Q)$. In short, the idea is to determine the homotopy of mapping spaces on free simplicial resolutions of our objects P, Q in the category of operads in topological spaces $\mathcal{T}op\,\mathcal{O}p$. We then get a spectral sequence whose second page reduces to the cohomology of a deformation complex associated to the cohomology cooperads $\mathtt{H}^*(P), \mathtt{H}^*(Q) \in gr\,\mathcal{H}opf\,\mathcal{O}p^c_*$. The definition of this deformation complex involves both the commutative algebra structure and the cooperad structure of these graded Hopf cooperads $\mathtt{H}^*(P), \mathtt{H}^*(Q) \in gr\,\mathcal{H}opf\,\mathcal{O}p^c_*$. To be explicit, at the deformation complex level, we deal with a free resolution of the Hopf cooperad $\mathtt{H}^*(Q)$ in the commutative algebra direction and with a cofree resolution of the object $\mathtt{H}^*(P)$ in the cooperad direction. Then our deformation complex precisely consists of modules of biderivations associated to these resolutions in the category of graded Hopf cooperads.

In fact, we only use this general approach in a follow-up (see [68]) as we can use a simplification of the free commutative algebra resolution when we establish our result about the homotopy automorphism space of the operad of little 2-discs.

To be explicit, instead of a resolution in the category of commutative algebras, we use a counterpart, for operads, of the classical Postnikov decomposition of spaces. To ease the definition of this Postnikov decomposition for the rationalization of the operad of little 2-discs $\widehat{D_2}$, we actually consider a classifying space $\mathtt{B}(\widehat{CD})$ on an operad of chord diagrams $\widehat{CD}$ which is equivalent to the (Malcev completion of the) operad of parenthesized braids $\widehat{PaB}$.

To perform our computation, we moreover decompose the general homotopy spectral sequences of operadic mapping spaces in two intermediate spectral sequences, where we deal with the resolution in the operad direction in a first step and with the obstruction problems associated to the Postnikov decomposition of our target object in the second step. We will see that these spectral sequences vanish in degree $* > 1$, reduce to the module of rank one $\mathbb{Q}$ in degree $* = 1$, and reduce to a graded Lie algebra $\mathfrak{grt}$ associated to the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ in degree $* = 0$. We use this correspondence to check that all classes of degree $* = 0$ in the second page of our spectral sequence are hit by homotopy automorphisms of our operad which come from the Grothendieck–Teichmüller group. We conclude from this result that our mapping from the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ to the space of homotopy automorphisms of the operad $\widehat{D_{2+}}$ induces a bijection in homotopy.

We tackle this verification in the third part of this work, in the second volume of this monograph. We review Drinfeld's definition of the pro-unipotent Grothendieck–Teichmüller group first. We explain that the pro-unipotent Grothendieck–Teichmüller group can be defined as a group of automorphisms associated to the Malcev completion of the parenthesized braid operad $\widehat{PaB}$. We then develop a new rational homotopy theory of operads before tackling the computation of the homotopy of the homotopy automorphism space of rational E_2-operads.

For this purpose, we notably define an analogue of the Sullivan model for the rational homotopy of operads in topological spaces. Briefly recall that the classical Sullivan model of a simplicial set X is defined by a commutative cochain differential graded algebra $\Omega^*(X)$ (a cochain commutative dg-algebra for short), which consists of piecewise linear differential forms on X. We consider cooperads in commutative cochain dg-algebras to define the Sullivan model of operads in topological spaces. We use the notation $dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*$ for this category of cooperads and, for short, we also call 'Hopf cochain dg-cooperads' the objects of this category. We also adopt the notation $dg^*\,\mathcal{C}om_+$ for the category of unitary commutative cochain dg-algebras in what follows. We already mentioned that a cooperad is a structure dual to an operad in the categorical sense. Briefly say for the moment that a cooperad C in a category $\mathcal{M}$ essentially consists of a collection of objects $C(r) \in \mathcal{M}$ together with a comultiplicative structure of a form opposite to the composition operations of an operad. We simply take $\mathcal{M} = dg^*\,\mathcal{C}om_+$ in this general definition when we form our category of Hopf cochain dg-cooperads. Recall that we temporarily use the subscript mark $*$ in our notation of the category of Hopf cochain dg-cooperads $dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*$ in order to indicate that we actually consider a counterpart, in the cooperad context, of our category of unitary operads.

The Sullivan dg-algebra $\Omega^*(X)$ does not preserve operad structures, but we will explain in the second volume of this monograph that we can define an operadic enhancement of the Sullivan functor in order to assign a well-defined Hopf cochain dg-cooperad $\Omega^*_\sharp(P)$ to any operad in simplicial sets P. We will prove that the

commutative cochain dg-algebras $\Omega^*_\sharp(P)(r)$ which define the components of this Hopf cochain dg-cooperad $\Omega^*_\sharp(P)$ are weakly-equivalent (quasi-isomorphic) to the Sullivan dg-algebras $\Omega^*(P(r))$ associated to the spaces $P(r)$ which underlie our operad P. We use this observation to check that this Hopf dg-cooperad $\Omega^*_\sharp(P)$ determines the operad in simplicial sets P up to rational equivalence.

We will also see that our functor $\Omega^*_\sharp : P \to \Omega^*(P)$, from the category of operads in simplicial sets to the category of Hopf cochain dg-cooperads, admits a left adjoint $\mathtt{G}_\bullet(-) : K \to \mathtt{G}_\bullet(K)$, which assigns an operad in simplicial sets $\mathtt{G}_\bullet(K)$ to any Hopf cochain dg-cooperad K. We consider the image of the Hopf cochain dg-cooperad $K = \Omega^*_\sharp(P)$ associated to an operad in simplicial sets P under a left derived functor of this left adjoint $\mathtt{L}\,\mathtt{G}_\bullet : K \to \mathtt{L}\,\mathtt{G}_\bullet(K)$. We will prove that this operad in simplicial sets $P\widehat{\ }= \mathtt{L}\,\mathtt{G}(\Omega^*_\sharp(P))$ forms, under mild finiteness assumptions, a suitable model for the rationalization of the operad P in the sense that the components of this operad $P\widehat{\ }(r)$ are equivalent, in the homotopy category of spaces, to the Sullivan rationalization $X\widehat{\ }= P(r)\widehat{\ }$of the simplicial sets $X = P(r)$.

The Sullivan dg-algebra functor (and our operadic enhancement of this functor similarly) is defined on the category of simplicial sets $s\mathcal{S}et$. But we can use the classical singular complex functor $\mathtt{Sing}_\bullet(-)$, from topological spaces to simplicial sets, and the geometric realization functor which goes the other way round, in order to prolong our constructions on operads in simplicial sets to operads in topological spaces.

The category $dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*$ inherits a model structure, like the category of topological operads, and we can therefore apply the general theory of model categories to associate homotopy automorphism groups $\mathtt{Aut}_{\mathtt{Ho}(dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*)}(A)$, as well as homotopy automorphisms spaces $\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(A)$, to any object of the category of Hopf cochain dg-cooperads $A \in dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*$. In the case of topological operads, we already mentioned that homotopy automorphisms spaces are well defined for cofibrant objects only. In the case of Hopf cochain dg-cooperads, we have to perform both cofibrant and fibrant resolutions before applying the homotopy automorphism construction.

The results obtained in our study of the rational homotopy of operads imply that the group of homotopy automorphisms attached to the model $\Omega^*_\sharp(P)$ of an operad in spaces P is isomorphic to the group of homotopy automorphisms attached to the rational completion of this operad $P\widehat{\ }$. Theorem A is therefore equivalent to the following statement:

THEOREM B. *Let E_2 be a (cofibrant) model of E_2-operad in the category of topological space. Let $K_2 = \Omega^*_\sharp(E_2)$ be the Hopf cochain dg-cooperad associated to this operad E_2. The homotopy automorphism space of this object in the category of Hopf cochain dg-cooperads $\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(K_2)$ has trivial homotopy groups*

$$\pi_*(\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(K_2)) = 0$$

in dimension $* > 1$, the $\mathbb{Q}$-module of rank one as homotopy group

$$\pi_1(\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(K_2)) = \mathbb{Q}$$

in dimension $* = 1$, and we have an isomorphism of groups

$$GT(\mathbb{Q})^{op} \xrightarrow{\simeq} \pi_0(\mathtt{Aut}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_*}(K_2))$$

in dimension $* = 0$.

The assertions of this theorem have been foreseen by M. Kontsevich in [108]. First results in the direction of Theorem B also occur in articles of D. Tamarkin [174] and T. Willwacher [184]. But these authors deal with operads within the category of differential graded modules, after forgetting about commutative algebra structures, and their results actually give a stable version (in the sense of homotopy theory) of our statements. The definition of a setting, where we can combine a model for operadic structures and a commutative algebra model for the topology underlying our objects, is a new contribution of this monograph. The proof of Theorem B in this context is also a new outcome of our work, like the result of Theorem A.

Recall that E_2-operads only give the second layer of a full sequence of homotopy structures, ranging from E_1, fully homotopy associative but non-commutative, up to E_∞, fully homotopy associative and commutative. In a follow-up [67], the author, Victor Turchin and Thomas Willwacher give a computation, in terms of graph complexes, of the homotopy of the homotopy automorphism spaces of E_n-operads for $2 < n < \infty$. We give an overview of these results in the concluding chapter of the second volume of this monograph. Let us mention that the group of homotopy automorphism classes of E_1-operads can easily be determined, but the result is trivial in this case. The group of homotopy automorphisms of an E_∞-operad is trivial too.

The proof of our result requires the elaboration of new theories, like the definition of a model for the rational homotopy of topological operads, and the first aim of this monograph is to work out such general problems. The purpose of this book is also to give a comprehensive introduction to our subject, heading to our main theorems as straight as possible and with minimal background, for mathematicians coming from other domains and for graduate students.

We heavily use the formalism of Quillen's model categories [152] which we apply to operads in order to form our model for the rational homotopy of topological operads. We rely on the modern reference books by Hirschhorn [89] and by Hovey [91] for the subject of Quillen's model categories. We also refer to the book [61] for a comprehensive introduction to the rational homotopy theory and to Bousfield-Gugenheim's memoir [36] for an interpretation of the Sullivan model in the formalism of model categories. We also refer to Sullivan's article [170] for the applications of rational homotopy theory to the study of homotopy automorphisms of spaces. We review these subjects thoroughly before tackling our own constructions.

We first explain the connections between little 2-discs operads and braided structures, as well as the definition of the Grothendieck–Teichmüller group in terms of automorphisms of operads in groupoids. We give a comprehensive account of these topics in the first volume of this monograph, after an introduction to the general theory of operads. We notably give an operadic formulation of the classical coherence theorems of monoidal categories, of braided monoidal categories, and of symmetric monoidal categories. We will explain that the previously considered operad of parenthesized braids PaB, which we define by using the fundamental groupoid of the little 2-discs operad, actually governs the structure of a braided monoidal category.

We also review the applications of the theory of Hopf algebras to the Malcev completion of groups with the aim of extending this completion process to

groupoids and to operads. We focus on the study of a pro-unipotent Grothendieck–Teichmüller group in this work and we actually rely on the operadic Malcev completion construction in our definition of the Grothendieck–Teichmüller group. In passing, we will explain an operadic interpretation of the notion of a Drinfeld associator which was used by Tamarkin in order to prove the rational formality of E_2-operads.

We address the definition of our model for the rational homotopy of operads in the second volume of this monograph, after a survey of the general theory of model categories and of the rational homotopy theory of spaces. We mentioned in the introduction of this work that the Chevalley–Eilenberg cochain complex of the Drinfeld–Kohno Lie algebras (the Lie algebras of infinitesimal braids) can be used to define a small model of a rational E_2-operad. We also explain this construction in the second volume of this monograph. We review the already alluded proof of the rational formality of E_2-operads in the second volume too. We actually explain the definition of small models of E_n-operads for all $n \geq 2$ by using a graded version of the Drinfeld–Kohno Lie algebras. We tackle the computation of the homotopy automorphism space of rational E_2-operads afterwards, in the concluding part of the second volume.

Foundations and Conventions

The reader is assumed to be familiar with the language of category theory and to have basic knowledge about fundamental concepts like adjoint and representable functors, colimits and limits, categorical duality, which we freely use throughout this work. The reader is also assumed to be aware on the applications of colimits and limits in basic examples of categories (including sets, topological spaces, and modules). Nonetheless, we will review some specialized topics, like reflexive coequalizers and filtered colimits, which are considered in applications of category theory to operads.

We use single script letters (like $\mathcal{C}$, $\mathcal{M}$, ...) as general notation for abstract categories. We use script expressions (like $\mathcal{M}od$, $\mathcal{A}s$, $\mathcal{O}p$, ...) for particular instances of categories which we consider in this work (like modules, associative algebras, operads, ...). We are going to explain that the formal definition of many algebraic structures remains the same in any instance of base category $\mathcal{M}$ and essentially depends on a *symmetric monoidal structure* given with this category $\mathcal{M}$. We usually assume that the category $\mathcal{M}$, to which we assign the role of a base category, is equipped with enriched hom-bifunctors $\mathtt{Hom}_{\mathcal{M}}(-,-)$. We give a more detailed reminder on this notion in §§0.12-0.13.

In practice, we take our base category $\mathcal{M}$ among the category of sets $\mathcal{S}et$, the category of simplicial sets $s\mathcal{S}et$, the category of topological spaces $\mathcal{T}op$, a category of $\Bbbk$-modules $\mathcal{M}od$ (where $\Bbbk$ refers to a fixed ground ring), or among a variant of these categories. To be precise, besides plain $\Bbbk$-modules, we have to consider categories formed by differential graded modules $dg\,\mathcal{M}od$ (we usually speak about *dg-modules* for short), graded modules $gr\,\mathcal{M}od$, simplicial modules $s\,\mathcal{M}od$, and cosimplicial modules $c\,\mathcal{M}od$. The first purpose of this preliminary chapter is to quickly recall the definition of these categories (at least, in order to fix our conventions). By the way, we also recall the definition of the category of simplicial sets $s\mathcal{S}et$, which we use along with the familiar category of topological spaces $\mathcal{T}op$.

To complete our account, we recall the general definition of a symmetric monoidal category and we explain some general constructions which we associate to this notion. But we postpone our reminder on the definition of the monoidal structure of the category of dg-modules, simplicial modules and cosimplicial modules until the moment where we tackle the applications of these base categories.

In the module context, we assume that a ground ring $\Bbbk$ is given and is fixed once and for all. In certain constructions, we have to assume that this ground ring $\Bbbk$ is a field of characteristic 0.

0.1. *Graded and differential graded modules.* The category of *differential graded modules* $dg\,\mathcal{M}od$ (*dg-modules* for short) consists of $\Bbbk$-modules equipped with a decomposition $C = \bigoplus_{n\in\mathbb{Z}} C_n$, which ranges over $\mathbb{Z}$, and with a morphism $\delta : C \to C$ (the *differential*) such that we have $\delta^2 = 0$ and $\delta(C_n) \subset C_{n-1}$, for all $n \in \mathbb{Z}$. We

obviously define a morphism of dg-modules as a morphism of $\Bbbk$-modules $f : C \to D$ which intertwines differentials and which satisfies the relation $f(C_n) \subset D_n$, for all $n \in \mathbb{Z}$.

In textbooks of homological algebra (like [181]), authors mostly deal with an equivalent notion of *chain complex*, of which components are split off into a sequence of $\Bbbk$-modules C_n linked by the differentials $\delta : C_n \to C_{n-1}$ rather than being put together in a single object. The idea of a dg-module (used for instance in [129]) is more natural for our purpose and is also more widely used in homotopy theory. In what follows, we rather reserve the phrase 'chain complex' for certain specific constructions of dg-modules.

The category of *graded modules* $gr\,\mathcal{M}od$ consists of $\Bbbk$-modules equipped with a decomposition $C = \bigoplus_{n\in\mathbb{Z}} C_n$, which ranges over $\mathbb{Z}$, but where we have no differential. We obviously define a morphism of graded modules as a morphism of $\Bbbk$-modules $f : C \to D$ which satisfies the relation $f(C_n) \subset D_n$, for all $n \in \mathbb{Z}$.

We have an obvious functor $(-)_\flat : dg\,\mathcal{M}od \to gr\,\mathcal{M}od$ defined by retaining the underlying graded structure of a dg-module and by forgetting about the differential. We notably consider the underlying graded module of dg-modules, which this forgetful process formalizes, when we define the notion of a quasi-free object (in the category of commutative algebras, in the category of operads, ...). The other way round, we can embed the category of graded modules $gr\,\mathcal{M}od$ into the category of dg-modules $dg\,\mathcal{M}od$ by viewing a graded module as a dg-module equipped with a trivial differential $\delta = 0$. We use this identification at various places.

Recall that the homology of a dg-module C is defined by the quotient $\Bbbk$-module $\mathtt{H}_*(C) = \ker\delta/\operatorname{im}\delta$ which inherits a natural grading from C. The homology defines a functor $\mathtt{H}_*(-) : dg\,\mathcal{M}od \to gr\,\mathcal{M}od$. In most references of homological algebra, authors use the phrase '*quasi-isomorphism*' for the class of morphisms of dg-modules which induce an isomorphism in homology. In what follows, we rather use the name '*weak-equivalence*' which we borrow from the general formalism of model categories (see §II.1 for this notion).

We generally use the mark $\xrightarrow{\sim}$ to refer to the class of weak-equivalences in a model category (see §II.1) and we naturally use the same notation in the dg-module context. We mostly use the notions introduced in this paragraph in the second part of this book and we review these definitions with full details in §II.5.

0.2. *Degrees and signs of dg-algebra.* The component C_n of a dg-module (respectively, graded module) C defines the homogeneous component of degree n of C. To specify the degree of a homogeneous element $x \in C_n$, we use the expression $\deg(x) = n$. We adopt the standard convention of dg-algebra to associate a sign $(-1)^{\deg(x)\deg(y)}$ to each transposition of homogeneous elements (x, y). We do not specify such a sign in general and we simply use the notation $\pm$ to refer to it. We will see that the introduction of these signs is forced by the definition of the symmetry isomorphism of the tensor product of dg-modules (see §II.5.2).

We usually consider lower graded dg-modules, but we also have a standard notion of dg-module equipped with a decomposition in upper graded components $C = \bigoplus_{n\in\mathbb{Z}} C^n$ such that the differential satisfies $\delta(C^n) \subset C^{n+1}$. Certain constructions (like the duality of $\Bbbk$-modules and the conormalized complex of cosimplicial spaces) naturally produce upper graded dg-modules. In what follows, we apply the relation $C_{-n} = C^n$ to identify an upper graded with a lower graded dg-module. We also review these concepts in §II.5.

0.3. *Simplicial objects and cosimplicial objects in a category.* The *simplicial category* Δ, which models the structure of simplicial and cosimplicial objects in a category, is defined by the collection of finite ordinals $\underline{n} = \{0 < \cdots < n\}$ as objects together with the non-decreasing maps between finite ordinals $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$ as morphisms. We define a *simplicial object* X in a category $\mathcal{C}$ as a contravariant functor $X : \Delta^{op} \to \mathcal{C}$ which assigns an object $X_n \in \mathcal{C}$ to each $n \in \mathbb{N}$ and a morphism $u^* : X_n \to X_m$ to each non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$. We similarly define a *cosimplicial object* in $\mathcal{C}$ as a covariant functor $X : \Delta \to \mathcal{C}$ which assigns an object $X^n \in \mathcal{C}$ to each $n \in \mathbb{N}$ and a morphism $u_* : X^m \to X^n$ to each non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$. Naturally, we define a morphism of simplicial objects $f : X \to Y$ (and a morphism of cosimplicial object similarly) as a sequence of morphisms $f : X_n \to Y_n$ in the ambient category $\mathcal{C}$ which intertwine the action of the simplicial operators u^* on our objects X and Y.

We use the notation $s\,\mathcal{C}$ for the category of simplicial objects in a given ambient category $\mathcal{C}$ and the notation $c\,\mathcal{C}$ for the category of cosimplicial objects. The category of *simplicial sets* $s\mathcal{S}et$, for instance, formally consists of the simplicial objects in the category of sets $X : \Delta^{op} \to \mathcal{S}et$.

The *simplices* Δ^n, $n \in \mathbb{N}$, are the fundamental examples of simplicial sets which are given by the representable functors $\mathtt{Mor}_\Delta(-, \underline{n}) : \Delta^{op} \to \mathcal{S}et$, where we use the notation $\mathtt{Mor}_\Delta(\underline{m}, \underline{n})$ to refer to the morphism sets of the simplicial category Δ. The collection of the n-simplices Δ^n, $n \in \mathbb{N}$, forms a cosimplicial object in the category of simplicial sets itself, with the covariant action of non-decreasing maps $u_* : \Delta^m \to \Delta^n$ defined by the composition on the target in these morphism sets $\Delta^n = \mathtt{Mor}_\Delta(-, \underline{n})$.

In the case of a simplicial set X, an element $\sigma \in X_n$ is called an *n-dimensional simplex* (or more simply an *n-simplex*) in X. The definition of the n-simplex Δ^n as a representable functor $\Delta^n = \mathtt{Mor}_\Delta(-, \underline{n})$ implies that we have the relation $\mathtt{Mor}_{s\mathcal{S}et}(\Delta^n, X) = X_n$, for any simplicial set $X \in s\mathcal{S}et$, where we use the notation $\mathtt{Mor}_{s\mathcal{S}et}(-,-)$ for the morphism set of the category $s\mathcal{S}et$. To make this correspondence explicit, we consider the n-simplex, denoted by $\iota_n \in (\Delta^n)_n$, which corresponds to the identity of the object $\underline{n}$ in the simplicial category Δ. The morphism $\sigma_* : \Delta^n \to X$, associated to any n-simplex $\sigma \in X_n$, is characterized by the relation $\sigma_*(\iota_n) = \sigma$.

The topological simplices $\mathbb{\Delta}^n = \{(t_0, \ldots, t_n) | 0 \leq t_i \leq 1,\, t_0 + \cdots + t_n = 1\}$ form another fundamental instance of a cosimplicial object which is defined in the category of topological spaces. The cosimplicial structure map $u_* : \mathbb{\Delta}^m \to \mathbb{\Delta}^n$ associated to a morphism of the simplicial category $u \in \mathtt{Mor}_\Delta(\underline{m}, \underline{n})$ sends any element $(s_0, \ldots, s_m) \in \mathbb{\Delta}^m$ to the point $(t_0, \ldots, t_n) \in \mathbb{\Delta}^n$ such that $t_i = \sum_{u(k)=i} s_k$.

We mainly use simplicial objects and cosimplicial objects in the second volume of this book and we go back to the definitions of this paragraph in §§II.1-3. We also study simplicial and cosimplicial modules (simplicial and cosimplicial objects in module categories) in depth in §II.5.

0.4. *Faces and degeneracies in a simplicial object.* The maps $d^i : \{0 < \cdots < n-1\} \to \{0 < \cdots < n\}$, $i = 0, \ldots, n$, such that

$$d^i(x) = \begin{cases} x, & \text{for } x < i, \\ x+1, & \text{for } x \geq i, \end{cases} \tag{1}$$

and the maps $s^j : \{0 < \cdots < n+1\} \to \{0 < \cdots < n\}$, $j = 0, \ldots, n$, such that

$$s^j(x) = \begin{cases} x, & \text{for } x \leq j, \\ x-1, & \text{for } x > j, \end{cases} \tag{2}$$

generate the simplicial category in the sense that any non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$ can be written as a composite of maps of that form. Moreover, any relation between these generating morphisms can be deduced from the following generating relations:

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, \text{ for } i < j, \\ s^j d^i &= \begin{cases} d^i s^{j-1}, & \text{for } i < j, \\ id, & \text{for } i = j, j+1, \\ d^{i-1} s^j, & \text{for } i > j+1, \end{cases} \\ s^j s^i &= s^i s^{j+1}, \text{ for } i \leq j. \end{aligned} \tag{3}$$

The structure of a cosimplicial object is, as a consequence, fully determined by a sequence of objects $X^n \in \mathcal{C}$ together with morphisms $d^i : X^{n-1} \to X^n$, $i = 0, \ldots, n$, and $s^j : X^{n+1} \to X^n$, $j = 0, \ldots, n$, for which these relations (3) hold. The morphisms $d^i : X^{n-1} \to X^n$, $i = 0, \ldots, n$, which represent the image of the maps d^i under the functor defined by X, are the *coface operators* of the cosimplicial object X (we may also speak about the *cofaces* of X for short). The morphisms $s^j : X^{n+1} \to X^n$, $j = 0, \ldots, n$, which represent the image of the maps s^j are the *codegeneracy operators* of X (or, more simply, the *codegeneracies* of X).

Dually, the structure of a simplicial object is fully determined by a sequence of objects $X_n \in \mathcal{C}$ together with morphisms $d_i : X_n \to X_{n-1}$, $i = 0, \ldots, n$, and $s_j : X_n \to X_{n+1}$, $j = 0, \ldots, n$, for which relations

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, \text{ for } i < j, \\ d_i s_j &= \begin{cases} s_{j-1} d_i, & \text{for } i < j, \\ id, & \text{for } i = j, j+1, \\ s_j d_{i-1}, & \text{for } i > j+1, \end{cases} \\ s_i s_j &= s_{j+1} s_i, \text{ for } i \leq j, \end{aligned} \tag{4}$$

opposite to (3), hold. The morphisms $d_i : X_n \to X_{n-1}$, $i = 0, \ldots, n$, which represent the image of the maps d^i under the contravariant functor defined by X, are the *face operators* of the simplicial object X, and the morphisms $s_j : X_n \to X_{n+1}$, $j = 0, \ldots, n$, which represent the image of the maps s^j, are the *degeneracy operators* of X. We also recall the definition of these operators in the course of our study (in §§II.1-3).

0.5. *Geometric realization of simplicial sets and singular complex of topological spaces.* Recall that a topological space $|K|$, traditionally called the geometric realization of K, is naturally associated to each simplicial set $K \in s\mathcal{S}et$. This space is defined by the coend

$$|K| = \int^{\underline{n} \in \Delta} K_n \times \mathbb{\Delta}^n .$$

where each set K_n is viewed as a discrete space and we consider the topological n-simplices $\mathbb{\Delta}^n$ (of which definition is recalled in §0.3). The coend which we consider in

this construction is equivalent to the quotient object of the coproduct $\coprod_n K_n \times \mathbb{\Delta}^n = \coprod_n \{\coprod_{\sigma \in K_n} \{\sigma\} \times \mathbb{\Delta}^n\}$ under the relations

$$(u^*(\sigma), (t_0, \dots, t_m)) \equiv (\sigma, u_*(t_0, \dots, t_m)),$$

for $u \in \mathtt{Mor}_\Delta(\underline{m}, \underline{n})$, $\sigma \in K_n$, and $(t_0, \dots, t_m) \in \mathbb{\Delta}^m$. The definition of the map $u_* : \mathbb{\Delta}^m \to \mathbb{\Delta}^n$ associated to each $u \in \mathtt{Mor}_\Delta(\underline{m}, \underline{n})$ involves the cosimplicial structure of the topological n-simplices $\mathbb{\Delta}^n$. One easily checks that the realization of the n-simplex $\Delta^n = \mathtt{Mor}_\Delta(-, \underline{n})$ is identified with the topological n-simplex $\mathbb{\Delta}^n$.

In the converse direction, we can use the singular complex construction to associate a simplicial set $\mathtt{Sing}_\bullet(X)$ to any topological space X. This simplicial set $\mathtt{Sing}_\bullet(X)$ consists in dimension n of the set of continuous maps $\sigma : \mathbb{\Delta}^n \to X$ going from the topological n-simplex $\mathbb{\Delta}^n$ to X. The composition of simplices $\sigma : \mathbb{\Delta}^n \to X$ with the cosimplicial operator $u_* : \mathbb{\Delta}^m \to \mathbb{\Delta}^n$ associated to any $u \in \mathtt{Mor}_\Delta(\underline{m}, \underline{n})$ yields a map $u^* : \mathtt{Sing}_n(X) \to \mathtt{Sing}_m(X)$ so that the collection of sets $\mathtt{Sing}_n(X) = \mathtt{Mor}_{\mathcal{T}op}(\mathbb{\Delta}^n, X)$, $n \in \mathbb{N}$, inherits a natural simplicial structure.

The geometric realization obviously gives a functor $| - | : s\mathcal{S}et \to \mathcal{T}op$. The singular complex construction gives a functor in the converse direction $\mathtt{Sing}_\bullet : \mathcal{T}op \to s\mathcal{S}et$, which is actually a right adjoint of the geometric realization functor (see [79, §I.2]). We study generalizations of these constructions in §II.3.

0.6. *Simplicial modules, cosimplicial modules, and homology.* The category of *simplicial modules* $s\,\mathcal{M}od$ is the category of simplicial objects in the category of $\Bbbk$-modules $\mathcal{M}od$. Thus, a simplicial module K can be defined either as a contravariant functor from the simplicial category Δ to the category of $\Bbbk$-modules $\mathcal{M}od$, or, equivalently, as a collection of $\Bbbk$-modules K_n, $n \in \mathbb{N}$ equipped with face operators $d_i : K_n \to K_{n-1}$, $i = 0, \dots, n$, and degeneracy operators $s_j : K_n \to K_{n+1}$, $j = 0, \dots, n$, which satisfy the simplicial relations.

The category of *cosimplicial modules* $c\,\mathcal{M}od$ similarly consists of the cosimplicial objects in the category of $\Bbbk$-modules.

To any simplicial module K, we associate a dg-module $\mathtt{N}_*(K)$, called the *normalized complex* of K, and defined by the quotient $\mathtt{N}_n(K) = K_n / s_0 K_{n-1} + \dots + s_{n-1} K_{n-1}$ in degree n, together with the differential $\delta : \mathtt{N}_n(K) \to \mathtt{N}_{n-1}(K)$ such that $\delta = \sum_{i=0}^n (-1)^i d_i$. This normalized chain complex construction naturally gives a functor $\mathtt{N}_* : s\,\mathcal{M}od \to dg\,\mathcal{M}od$. The homology of a simplicial module K is defined as the homology of the associated normalized complex $\mathtt{N}_*(K)$. For simplicity, we use the same notation for the homology functor on simplicial modules and on dg-modules. Hence, we set $\mathtt{H}_*(K) = \mathtt{H}_*(\mathtt{N}_*(K))$, for any $K \in s\,\mathcal{M}od$. We study simplicial and cosimplicial modules in depth in §II.5 and we recall the definition of the normalized complex construction at this moment.

0.7. *Normalized complex and homology of simplicial sets.* We will consider the functor $\Bbbk[-] : s\mathcal{S}et \to s\,\mathcal{M}od$ which maps a simplicial set X to the simplicial module $\Bbbk[X]$ generated by the set X_n in dimension n, for any $n \in \mathbb{N}$, and which inherits an obvious simplicial structure. We also have a contravariant functor $\mathtt{A} : s\mathcal{S}et^{op} \to c\,\mathcal{M}od$ which maps a simplicial set X to the cosimplicial module $\mathtt{A}(X) = \Bbbk^X$, dual to $\Bbbk[X]$, and defined in dimension n by the $\Bbbk$-module of functions $u : X_n \to \Bbbk$ on the set $X_n \in \mathcal{S}et$.

We use the notation $\mathtt{N}_*(X)$ for the normalized complex of the simplicial $\Bbbk$-module $\Bbbk[X]$ associated to a simplicial set X. We retrieve the classical homology of simplicial sets by considering the homology of these simplicial modules. We also

use the notation $\mathtt{H}_*(-)$ for the homology functor on simplicial sets. We accordingly have the formula $\mathtt{H}_*(X) = \mathtt{H}_*(\mathtt{N}_*(X))$, for any $X \in s\mathcal{S}et$.

The normalized complexes of the simplices Δ^n, $n \in \mathbb{N}$, naturally form a simplicial object in the category of dg-modules $\mathtt{N}_*(\Delta^\bullet)$. For a given simplicial module K, we have a coend formula

$$\mathtt{N}_*(K) = \int^{\underline{n}\in\Delta} K_n \otimes \mathtt{N}_*(\Delta^n),$$

and the normalized complex construction of §0.6 can be regarded as a dg-module version of the geometric realization of simplicial sets (we explain this idea in §5.0.11).

0.8. *Symmetric monoidal categories and the structure of base categories.* In the introduction of this chapter, we briefly mentioned that our base categories, let $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \ldots$, are all instances of *symmetric monoidal categories.*

By definition, a symmetric monoidal category is a category $\mathcal{M}$ equipped with a tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ that satisfies natural unit, associativity and symmetry relations which we express as follows:

(a) We have a unit object $\mathbb{1} \in \mathcal{M}$ such that we have a natural isomorphism $A \otimes \mathbb{1} \simeq A \simeq \mathbb{1} \otimes A$, for $A \in \mathcal{M}$.

(b) We have a natural isomorphism $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$, for every triple of objects $A, B, C \in \mathcal{M}$, which satisfies a pentagonal coherence relation (Mac Lane's pentagon relation) when we put together the associativity isomorphisms associated to a 4-fold tensor product, and additional triangular coherence relations with respect to the unit isomorphism when we assume that one of our objects is the unit object of our category (we refer to [130, §XI.1] for the expression of these constraints).

(c) We have a natural symmetry isomorphism $A \otimes B \simeq B \otimes A$, associated to every pair of objects $A, B \in \mathcal{M}$, which satisfies hexagonal coherence relations (Drinfeld's hexagon relation) when we apply the symmetry isomorphism to a 3-fold tensor product, and additional triangular coherence relations with respect to the unit isomorphism when we assume that one of our objects is the unit object (see again [130, §XI.1] for details).

In the case of $\Bbbk$-modules $\mathcal{M}od$, the monoidal structure is given by the usual tensor product of $\Bbbk$-modules, taken over the ground ring, together with the ground ring itself as unit object. The definition of the tensor product of dg-modules, simplicial modules, cosimplicial modules is reviewed later on, when we tackle applications of these base categories. In the category of sets $\mathcal{S}et$ (respectively, topological spaces $\mathcal{T}op$, simplicial sets $s\mathcal{S}et$), the tensor product is simply given by the cartesian product $\otimes = \times$ together with the one-point set $\mathbb{1} = pt$ as unit object. In what follows, we also use the general notation $*$ for the terminal object of a category, and we may write $pt = *$ when we want to stress that the one point-set actually represents the terminal object of the category of sets (respectively, topological spaces, simplicial sets).

The unit object and the isomorphisms that come with the unit, associativity and commutativity relations of a symmetric monoidal category are part of the structure. Therefore, these morphisms have, in principle, to be given with the definition. But, in our examples, we can assume that the unit and associativity relations are identities, and in general, we just make explicit the definition of the symmetry isomorphism $c = c(A, B) : A \otimes B \xrightarrow{\simeq} B \otimes A$.

We make explicit the coherence constraints for the unit, associativity, and symmetry isomorphisms of symmetric monoidal categories in §I.6 (we use a braided analogue of the structure of a symmetric monoidal category in our definition of the Grothendieck–Teichmüller group). We also review the definition of several notions of structure preserving functors between symmetric monoidal categories in §I.3.

0.9. *Tensor products and colimits.* In many constructions, we consider symmetric monoidal categories $\mathcal{M}$ equipped with colimits and limits and whose the tensor product *distributes over colimits* in the sense that:

(a) The canonical morphism $\operatorname{colim}_{\alpha\in\mathcal{I}}(A_\alpha \otimes B) \to (\operatorname{colim}_{\alpha\in\mathcal{I}} A_\alpha) \otimes B$ associated to a diagram $A_\alpha \in \mathcal{M}$, $\alpha \in \mathcal{I}$, is an isomorphism for all $B \in \mathcal{M}$, and similarly as regards the canonical morphism $\operatorname{colim}_{\beta\in\mathcal{J}}(A \otimes B_\beta) \to A \otimes (\operatorname{colim}_{\beta\in\mathcal{J}} B_\beta)$ associated to a diagram $B_\beta \in \mathcal{M}$, $\beta \in \mathcal{J}$, where we now fix the object $A \in \mathcal{M}$.

This requirement is fulfilled by all categories which we take as base symmetric monoidal categories $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \ldots$ and is required for the application of categorical constructions to operads and to algebras over operads. On the other hand, we will also consider instances of symmetric monoidal categories which do not satisfy this distribution relation. One simple example is given by taking the direct sum $\oplus : \mathcal{M}od \times \mathcal{M}od \to \mathcal{M}od$ (instead of the ordinary tensor product) as the tensor product operation of a symmetric monoidal structure on the category of $\Bbbk$-modules. We use this additive monoidal structure when we study a counterpart of the Postnikov decomposition of spaces in the category of operads.

0.10. *Symmetric groups and tensor permutations.* We use the notation Σ_r for the group of permutations of $\{1, \ldots, r\}$. Depending on the context, we regard a permutation $s \in \Sigma_r$ as a bijection $s : \{1, \ldots, r\} \to \{1, \ldots, r\}$ or as a sequence $s = (s(1), \ldots, s(r))$ equivalent to an ordering of the set $\{1, \ldots, r\}$. In any case, we will use the notation $id = id_r$ for the identity permutation in Σ_r. We omit the subscript r which indicates the cardinal of our permutation when we do not need to specify this information.

In a symmetric monoidal category equipped with a strictly associative tensor product, we can form r-fold tensor products $T = X_1 \otimes \cdots \otimes X_r$ without care and omit unnecessary bracketing. Then we also have a natural isomorphism

$$X_1 \otimes \cdots \otimes X_r \xrightarrow{s^*} X_{s(1)} \otimes \cdots \otimes X_{s(r)},$$

associated to each permutation $s \in \Sigma_r$, and such that the standard unit and associativity relations $id^* = id$ and $t^*s^* = (st)^*$ hold. To construct this action, we use the classical presentation of Σ_r with the transpositions $t_i = (i\ i+1)$ as generating elements and the identities

$$t_i^2 = id, \quad \text{for } i = 1, \ldots, r-1, \tag{1}$$

$$t_i t_j = t_j t_i, \quad \text{for } i, j = 1, \ldots, r-1, \text{ with } |i-j| \geq 2, \tag{2}$$

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad \text{for } i = 1, \ldots, r-2, \tag{3}$$

as generating relations. We assign the morphism

$$X_1 \otimes \cdots \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_r \xrightarrow{\simeq} X_1 \otimes \cdots \otimes X_{i+1} \otimes X_i \otimes \cdots \otimes X_r,$$

induced by the symmetry isomorphism $c(X_i, X_{i+1}) : X_i \otimes X_{i+1} \xrightarrow{\simeq} X_{i+1} \otimes X_i$, to the transposition $t_i = (i\ i+1)$. The axioms of symmetric monoidal categories imply that these morphisms satisfy the relations (1-3) attached to the elementary

transpositions in Σ_r. Hence, we can use the presentation of Σ_r to coherently extend the action of the transpositions $t_i \in \Sigma_r$ on tensor powers to the whole symmetric group.

0.11. *Tensor products over arbitrary finite sets.* In our constructions, we often deal with tensor products $\bigotimes_{i\in\underline{r}} X_i$ that range over an arbitrary set $\underline{r} = \{i_1, \dots, i_r\}$ (not necessarily equipped with a canonical ordering). In fact, we effectively realize such a tensor product $\bigotimes_{i\in\underline{r}} X_i$ as an ordered tensor product $X_{u(1)} \otimes \cdots \otimes X_{u(r)}$, which we associate to the choice of a bijection $u : \{1 < \cdots < r\} \xrightarrow{\simeq} \underline{r}$. The tensor products associated to different bijection choices $u, v : \{1 < \cdots < r\} \xrightarrow{\simeq} \underline{r}$ differ by a canonical isomorphism $s^* : X_{u(1)} \otimes \cdots \otimes X_{u(r)} \xrightarrow{\simeq} X_{v(1)} \otimes \cdots \otimes X_{v(r)}$ which we determine from the permutation $s \in \Sigma_r$ such that $v = us$ by using the just defined action of symmetric groups on tensors.

In principle, the tensor product $\bigotimes_{i_k\in\underline{r}} X_{i_k}$ is only defined up to these canonical isomorphisms. However, we can adapt the general Kan extension process to make this construction more rigid. Formally, we define the unordered tensor product as the colimit $\bigotimes_{i_k\in\underline{r}} X_{i_k} = \operatorname{colim}_{u:\{1<\cdots<r\}\xrightarrow{\simeq}\underline{r}} X_{u(1)} \otimes \cdots \otimes X_{u(r)}$ that ranges over the category formed by the bijections $u : \{1 < \cdots < r\} \xrightarrow{\simeq} \underline{r}$ as objects and the permutations $s \in \Sigma_r$ such that $v = us$ as morphisms. The colimit process automatically performs the identification of the tensors associated to different bijection choices.

This construction can be regarded as an instance of a Kan extension process which we will apply to structures, called symmetric sequences, underlying operads (see §I.2.5).

0.12. *Enriched category structure of base categories.* The morphism sets of a category $\mathcal{C}$ will always be denoted by $\mathtt{Mor}_{\mathcal{C}}(A, B)$. But many categories which we consider come equipped with a hom-bifunctor $\mathtt{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$ with values in one of our base symmetric monoidal categories $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \dots$, and which provides $\mathcal{C}$ with an *enriched category structure*.

The structure of an enriched category includes operations that extend the classical composition operations attached to the morphism sets of ordinary categories. In the usual setting, the units of the composition are given by identity morphisms $id_A \in \mathtt{Mor}_{\mathcal{C}}(A, A)$ associated to all objects $A \in \mathcal{C}$. In the case of an enriched category, the units of the composition are morphisms

$$id_A : \mathbb{1} \to \mathtt{Hom}_{\mathcal{C}}(A, A), \tag{1}$$

given for all objects $A \in \mathcal{C}$, and defined on the tensor unit of the base category $\mathbb{1}$. The composition products are morphisms

$$\circ : \mathtt{Hom}_{\mathcal{C}}(B, C) \otimes \mathtt{Hom}_{\mathcal{C}}(A, B) \to \mathtt{Hom}_{\mathcal{C}}(A, C), \tag{2}$$

given for all $A, B, C \in \mathcal{C}$, and where we consider the tensor product of hom-objects in the base category instead of the cartesian product of morphism sets. These composition products are assumed to satisfy obvious analogues, now expressed in terms of commutative diagrams, of the unit and associativity relations of the composition in ordinary categories. Each of our base categories $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \dots$ is enriched over itself. In the case of sets $\mathcal{S}et$, we trivially take $\mathtt{Hom}_{\mathcal{S}et}(-,-) = \mathtt{Mor}_{\mathcal{S}et}(-,-)$. In the case of topological spaces $\mathcal{T}op$, the hom-objects $\mathtt{Hom}_{\mathcal{T}op}(A, B)$ are given by the morphism sets $\mathtt{Mor}_{\mathcal{T}op}(A, B)$ equipped with the usual compact-open topology. In the case of modules $\mathcal{M}od$, the hom-objects $\mathtt{Hom}_{\mathcal{M}od}(A, B)$ are similarly given by the morphism sets of the category $\mathtt{Hom}_{\mathcal{M}od}(A, B) = \mathtt{Mor}_{\mathcal{M}od}(A, B)$, which

come naturally equipped with a module structure (the usual one). In our remaining fundamental examples $\mathcal{M} = s\mathcal{S}et, dg\,\mathcal{M}od, \ldots$, the hom-objects $\mathtt{Hom}_{\mathcal{M}}(A, B)$ consist of maps which are given by an extension of the definition of the morphisms of our category $\mathcal{M}$. (We give the explicit definition of these hom-objects later on, when we begin to use these categories.)

In all these examples, we actually take hom-objects which fit an adjunction relation with respect to the symmetric monoidal structure (authors say that our base categories are instances of *closed monoidal categories*). We review this connection in a next paragraph.

0.13. *The general notion of an enriched category, morphisms and homomorphisms.* In what follows, we actually use enriched categories both as a natural framework to perform constructions on objects and as examples of structured objects. The base categories $\mathcal{M} = \mathcal{S}et, \mathcal{M}od, s\mathcal{S}et, dg\,\mathcal{M}od, \ldots$ correspond to the first usage of enriched category structures, while the Hopf categories, which we consider in our definition of the Malcev completion of groupoids (see §I.9.0), correspond to the second form of applications of enriched categories.

In the first case, an enriched category structure is often given as an extra structure associated with an ordinary category $\mathcal{C}$. Then we deal with both morphism sets $\mathtt{Mor}_{\mathcal{C}}(-,-)$ and with hom-objects $\mathtt{Hom}_{\mathcal{C}}(-,-)$ with values in a given symmetric monoidal category $\mathcal{M}$ (not necessarily a base category). We say that our category $\mathcal{C}$ is *enriched over* $\mathcal{M}$ when we need to specify this category where our hom-objects are defined. We assume that the hom-objects are equipped with unit and composition morphisms §0.12(1-2) formed within our symmetric monoidal category $\mathcal{M}$.

In this context, where enriched categories arise as extra-structures associated with an underlying ordinary category $\mathcal{C}$, we also naturally assume that the hom-objects form a bifunctor $\mathtt{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$ so that we have morphisms

$$(1) \qquad f_* : \mathtt{Hom}_{\mathcal{C}}(-, A) \to \mathtt{Hom}_{\mathcal{C}}(-, B) \quad \text{and} \quad f^* : \mathtt{Hom}_{\mathcal{C}}(B, -) \to \mathtt{Hom}_{\mathcal{C}}(A, -),$$

for every $f \in \mathtt{Mor}_{\mathcal{C}}(A, B)$. The unit morphisms and the composition operations §0.12(1-2) have to be invariant under these actions of morphisms on hom-objects.

In our basic examples, where hom-objects are made from point-sets, we can identify the actual morphisms of the category $f \in \mathtt{Mor}_{\mathcal{C}}(A, B)$ with particular elements of the hom-objects $\mathtt{Hom}_{\mathcal{C}}(A, B)$. The general elements $u \in \mathtt{Hom}_{\mathcal{C}}(A, B)$ are conversely identified with maps $u : A \to B$ which satisfy some mild requirements, and these hom-objects $\mathtt{Hom}_{\mathcal{C}}(A, B)$ are generally given by an extension of the morphism sets of our category $\mathtt{Mor}_{\mathcal{C}}(A, B)$. In this setting, we use the name '*homomorphism*' to refer to the general elements of the hom-objects $\mathtt{Hom}_{\mathcal{C}}(A, B)$ as opposed to the '*morphisms*', which refer to the elements of the morphism sets $\mathtt{Mor}_{\mathcal{C}}(A, B)$. We may however use the arrow notation $u : A \to B$ when we want to regard such a homomorphism $u \in \mathtt{Hom}_{\mathcal{C}}(A, B)$ as a map. In this case, the belonging category of the arrow u is specified by the context. The composition on hom-objects also usually extends the composition on morphisms, and the morphisms (1), which make the hom-objects into a bifunctor, are generally identified with the left (respectively, right) composition with the homomorphism which we associate to any morphism $f \in \mathtt{Mor}_{\mathcal{C}}(A, B)$.

In a general setting, we can define a correspondence between morphisms and homomorphisms by using a natural transformation

$$(2) \qquad \iota_\sharp : \mathbb{1}[\mathtt{Mor}_{\mathcal{C}}(A, B)] \to \mathtt{Hom}_{\mathcal{C}}(A, B),$$

where the expression $\mathbb{1}[\mathtt{Mor}_{\mathcal{C}}(A,B)]$ denotes the coproduct, ranging over the set of morphisms $f \in \mathtt{Mor}_{\mathcal{C}}(A,B)$, of copies of the unit object $\mathbb{1}$.

0.14. *Closed symmetric monoidal categories.* In the case of our base categories $\mathcal{M} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \ldots$, we actually take hom-bifunctors that fit in an adjunction relation $\mathtt{Mor}_{\mathcal{M}}(A \otimes B, C) \simeq \mathtt{Mor}_{\mathcal{M}}(A, \mathtt{Hom}_{\mathcal{M}}(B,C))$ with respect to the symmetric monoidal structure of the category[1]. The bijection which gives this adjunction relation is also assumed to be natural in $A, B, C \in \mathcal{M}$.

We generally say that a symmetric monoidal category $\mathcal{M}$ is *closed* when the tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ has a right adjoint $\mathtt{Hom}_{\mathcal{M}}(-,-) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$ so that we have an isomorphism

$$\mathtt{Mor}_{\mathcal{M}}(A \otimes B, C) \simeq \mathtt{Mor}_{\mathcal{M}}(A, \mathtt{Hom}_{\mathcal{M}}(B,C)), \tag{1}$$

for any $A, B, C \in \mathcal{M}$. Note that the existence of this adjoint implies that our tensor product distributes over colimits as we require in §0.9. The other way round, the hom-object bifunctor $\mathtt{Hom}_{\mathcal{M}}(-,-)$, which we define by an adjunction relation of this form (1), automatically satisfies the same distribution relations with respect to colimits and limits as the general morphism set bifunctor of our base category. Namely, we have the identity $\mathtt{Hom}_{\mathcal{M}}(\operatorname{colim}_\alpha A_\alpha, B) = \lim_\alpha \mathtt{Hom}_{\mathcal{M}}(A_\alpha, B)$, when we take a colimit on the source of our hom-object $A = \operatorname{colim}_\alpha A_\alpha$, and the identity $\mathtt{Hom}_{\mathcal{M}}(A, \lim_\beta B_\beta) = \lim_\beta \mathtt{Hom}_{\mathcal{M}}(A, B_\beta)$, when we consider a limit on the target $B = \lim_\beta B_\beta$.

The hom-objects $\mathtt{Hom}_{\mathcal{M}}(A,B)$ defined by an internal hom-functor naturally inherit an evaluation morphism

$$\epsilon : \mathtt{Hom}_{\mathcal{M}}(A,B) \otimes A \to B \tag{2}$$

which represents the augmentation of the adjunction (1) and which generalizes the usual evaluation of maps in the category of sets. The unit of our adjunction is given by a morphism

$$\iota : A \to \mathtt{Hom}_{\mathcal{M}}(B, A \otimes B) \tag{3}$$

associated to each pair of objects in our category $A, B \in \mathcal{M}$.

The hom-objects of a closed symmetric monoidal category automatically inherit composition units $id_A : \mathbb{1} \to \mathtt{Hom}_{\mathcal{C}}(A,A)$, given by the right adjoint of the unit isomorphisms $\mathbb{1} \otimes A \xrightarrow{\simeq} A$ of the symmetric monoidal structure, as well as composition operations $\circ : \mathtt{Hom}_{\mathcal{C}}(B,C) \otimes \mathtt{Hom}_{\mathcal{C}}(A,B) \to \mathtt{Hom}_{\mathcal{C}}(A,C)$, given by the right adjoint of the composite evaluation morphisms $\mathtt{Hom}_{\mathcal{M}}(B,C) \otimes \mathtt{Hom}_{\mathcal{M}}(A,B) \otimes A \xrightarrow{id \otimes \epsilon} \mathtt{Hom}_{\mathcal{M}}(B,C) \otimes B \xrightarrow{\epsilon} C$. Thus, any closed symmetric monoidal category is automatically enriched in the sense of §0.12.

Besides, we have tensor product operations $\mathtt{Hom}_{\mathcal{M}}(A,B) \otimes \mathtt{Hom}_{\mathcal{M}}(C,D) \xrightarrow{\otimes} \mathtt{Hom}_{\mathcal{M}}(A \otimes C, B \otimes D)$, which are given by the right adjoint of the composites $\mathtt{Hom}_{\mathcal{M}}(A,B) \otimes \mathtt{Hom}_{\mathcal{M}}(C,D) \otimes A \otimes C \simeq \mathtt{Hom}_{\mathcal{M}}(A,B) \otimes A \otimes \mathtt{Hom}_{\mathcal{M}}(C,D) \otimes C \xrightarrow{\epsilon \otimes \epsilon} B \otimes D$ where we apply the symmetry operator of $\mathcal{M}$ and we form the tensor product of the evaluation morphisms associated to the hom-objects. This tensor product operation gives an extension, at the level of enriched hom-objects, of the tensor product of morphisms and satisfies the same unit, associativity, and symmetry relations.

[1]In the case $\mathcal{M} = \mathcal{T}op$, we just need to restrict ourselves to a good category of topological spaces, such as the usual category of compactly generated spaces, in order to ensure that such an adjunction relation holds (see for instance [130, §VII.8] for an overview of this subject).

The existence of the hom-object bifunctor $\mathtt{Hom}_{\mathcal{M}}(-,-)$ is notably useful for the study of algebras over operads and we give a short account of these applications in the first chapter of this book §I.1. In this book, we mostly study operad themselves and we mainly deal with the usual internal hom-objects associated to our fundamental examples of base model categories. To be specific, we use simplicial hom-objects, which we deduce from the standard internal hom-objects of the category of simplicial sets, when we study mapping spaces of operads in the second part of this work and we use hom-objects of dg-modules as an auxiliary device to compute the homotopy of these mapping spaces on the category of operads.

0.15. *Concrete (symmetric monoidal) categories.* Recall that a category $\mathcal{C}$ is *concrete* when we have a faithful functor $U : \mathcal{C} \to \mathcal{S}et$ from this category $\mathcal{C}$ to the category of sets $\mathcal{S}et$. Most usual categories, which are defined in terms of point sets equipped with extra structures, are naturally equipped with the structure of a concrete category.

In what follows, we will say that a base symmetric monoidal category $\mathcal{M}$ is *concrete (as a symmetric monoidal category)* when such a faithful functor $U : \mathcal{M} \to \mathcal{S}et$ is given by the representable functor $U = \mathtt{Mor}_{\mathcal{M}}(\mathbb{1},-)$ associated to the unit object of our category $\mathbb{1} \in \mathcal{M}$. The category of sets $\mathcal{M} = \mathcal{S}et$, the category of topological spaces $\mathcal{M} = \mathcal{T}op$, the category of modules $\mathcal{M} = \mathcal{M}od$, ... are examples of concrete symmetric monoidal categories. In this situation, we regard the morphism set $U(X) = \mathtt{Mor}_{\mathcal{M}}(\mathbb{1}, X)$ as a set of points, which we faithfully associate to any object of our category $X \in \mathcal{C}$, and we have a natural pointwise tensor product operation $x \otimes y \in U(X \otimes Y)$, which we define by the obvious composition operation $\mathbb{1} \xrightarrow{\simeq} \mathbb{1} \otimes \mathbb{1} \xrightarrow{x \otimes y} X \otimes Y$, for any $x \in U(X)$ and $y \in U(Y)$, where we consider the unit isomorphism $\mathbb{1} \simeq \mathbb{1} \otimes \mathbb{1}$ of our category $\mathcal{M}$. We just have $x \otimes y = (x, y) \in X \times Y$ in the case $\mathcal{M} = \mathcal{S}et, \mathcal{T}op$, and we retrieve the standard notion of a tensor product of elements in the case $\mathcal{M} = \mathcal{M}od$.

We mainly use the concept of a concrete symmetric monoidal category informally, in order to give a sense to set-theoretic tensor products which we define to illustrate some constructions of the theory. Let us mention that we can still form such set-theoretic tensor products (with some restriction) in the category of dg-modules, in the category of graded modules, in the category of simplicial modules and in the category of cosimplicial modules though these categories do not form concrete symmetric monoidal categories in the sense of our definition.

0.16. *The notation of colimits, limits and universal objects.* We adopt the following conventions for the notation of colimits, limits, and universal objects in categories. We generally use the unbased set notation $\varnothing$ for the initial object of a base category, the notation $\amalg$ for coproducts, and the notation $*$ for the terminal object. In certain situations, we use the empty set notation and we write $A = \varnothing$ to assert that an object A is undefined.

We use additive category notation when we deal with additive structures, or when our base category consists of modules. We then write 0 for the initial object of the category (the zero object). We also use $\oplus$ as a generic notation for the coproduct in the additive case.

When we deal with a category of objects equipped with a multiplicative structure (algebras, operads, ...), we generally adopt the base set notation $\vee$ for the coproduct, but we do not have any general convention for the notation of the initial object in this setting. In fact, we usually keep the notation of a particular object

of the base category which we use to effectively realize the initial object of our categories of structured objects. We can use a similar convention for coproducts when we can deduce the definition of this categorical operation from structure operations of our base category. For instance, we generally use the tensor product notation to refer to a coproduct of unitary commutative algebras, because we will observe in §I.3.0.3 that the coproduct is realized by the tensor product in this case.

Reading Guide and Overview of this Volume

This monograph comprises three main parts, referred to as Part I–III, which form a progression up to our ultimate mathematical goal. Part I, "From Operads to Grothendieck–Teichmüller Groups", is mainly devoted to the algebraic foundations of our subject. In Part II, "Homotopy Theory and its Applications to Operads", we develop our rational homotopy theory of operads after a comprehensive review of the applications of methods of homotopy theory. In Part III, "The Computation of Homotopy Automorphism Spaces of Operads", we work out our problem of giving a homotopy interpretation of the Grothendieck–Teichmüller group.

These parts are widely independent from each others. Each part of this book is also divided into subparts which, by themselves, form self-contained groupings of chapters, devoted to specific topics, and organized according to an internal progression of the level of the chapters each. There is a progression in the level of the parts of the book too, but the chapters are written so that a reader with a minimal background could tackle any of these subparts straight away in order to get a self-contained reference and an overview of the literature on each of the subjects addressed in this monograph.

This volume comprises the first named part of the book, "From Operads to Grothendieck-Teichmüller Groups", and two appendices, "Trees and the Construction of Free Operads" and "The Cotriple Resolution of Operads", where we revisit with full details the definition of operads in terms of composition operations shaped on trees and we explain the applications of trees to the definition of universal objects in the category of operads.

The following overview is not intended for a linear reading but should serve as a guide each time the reader tackles new parts of this volume.

Part I. From Operads to Grothendieck–Teichmüller Groups. The first part of this book includes: an introduction to the fundamental concepts of the theory of operads; a survey on the definition of the little discs operads and of E_n-operads together with a detailed study of the connections between the little 2-disc operad and braided category structures; an introduction to the theory of Hopf algebras together with a study of the applications of Hopf algebras to the Malcev completion of groups, groupoids and operads; and a detailed account of the definition of the Grothendieck-Teichmüller group from the viewpoint of the theory of algebraic operads.

Part I(a). The General Theory of Operads. We give a detailed survey of the general definitions of the theory of operads in this part. The first chapter §1 is introductory and does not contain any original idea. We mainly explain the relationship between operads and algebras. In the second chapter §2, we explain our working definition of the notion of an operad and we give a new approach

to handle unitary operads (the operads equipped with a distinguished arity zero operation which can be used to model categories of algebras with unit). In the third chapter §3, we study the applications of general concepts of the theory of monoidal categories to operads.

Chapter 1. *The Basic Concepts of the Theory of Operads.* In this first chapter, we explain May's definition of the notion of an operad as an object which governs the structure defined by collections of operations $p = p(x_1, \ldots, x_r)$, where $r \in \mathbb{N}$ (see §1.1). We examine the applications of usual categorical constructions to operads and we study the categories of algebras associated to operads afterwards (see §§1.2-1.3). We also recall the definition of particular instances of colimits (filtered colimits and reflexive coequalizers) which we heavily use in the theory of operads in an appendix section of this chapter (§1.4).

Chapter 2. *The Definition of Operadic Composition Structures Revisited.* The definition of an operad depends on composition schemes which we associate to operations $p = p(x_1, \ldots, x_r)$. In May's definition, which we recall in §1, we consider composition products where we can plug operations $q_i = q_i(x_1, \ldots, x_{n_i})$ in all inputs $i = 1, \ldots, r$ of a given operation $p = p(x_1, \ldots, x_r)$. This definition is perfectly suited for an introduction of the subject and for the study of algebras associated to operads. However, to work with operads themselves, we need to revisit the definition of our objects in order to get more insights into the structure of the composition products. We devote this second chapter to this subject. In a first step (§2.1), we check that the composition products of an operad, are, according to an observation of Martin Markl, fully determined by composition products on two factors (these operations are also called the partial composition products in the operad literature). In a second step (§§2.2-2.4), we explain a new approach to handle unitary operads. In short, we will see that the compositions with an extra operation of arity zero of a unitary operad can be encoded in a diagram structure associated to our object. We crucially use this observation in our study of the (rational) homotopy of operads in the second part of this book.

In general, we assume that an operad consists of a sequence of terms $\mathsf{P}(r)$, indexed by non-negative integers $r \in \mathbb{N}$, and whose elements intuitively represent operations with r inputs indexed by the ordinal $\underline{\mathsf{r}} = \{1 < \cdots < r\}$. To complete the account of this chapter, we explain an extension of the definition of an operad where terms $\mathsf{P}(\underline{\mathsf{r}})$ indexed by arbitrary finite sets $\underline{\mathsf{r}} = \{i_1, \ldots, i_r\}$ are considered (§2.5). In general, we can use bijections $\{1 < \cdots < r\} \xrightarrow{\simeq} \{i_1, \ldots, i_r\}$ to make the indexing by an arbitrary finite set $\{i_1, \ldots, i_r\}$ equivalent to an indexing by an ordinal $\{1 < \cdots < r\}$. Nevertheless, certain constructions on operads produce operations with no canonical input numbering and the extension of the input indexing to arbitrary finite sets becomes useful in this case. (The construction of the free operad in §A gives a motivating application of this concept.) These ideas go back to Joyal's theory of species [98] and were first applied to operads in Getzler–Jones's paper [77] and in Ginzburg–Kapranov's paper [78].

Chapter 3. *Symmetric Monoidal Categories and Operads.* The third chapter of this part is devoted to applications of the theory of symmetric monoidal categories to the study of operads. The ideas of this chapter are not original, apart when we tackle the applications of our constructions to the new model of unitary operads which we introduced in the previous chapter.

We devote a preliminary section of the chapter (§3.0) to a survey of the definition of unitary commutative algebras and of counitary cocommutative coalgebras in symmetric monoidal categories. We examine the definition of operads in general symmetric monoidal categories afterwards (§3.1). We notably study the image of operads under functors between symmetric monoidal categories. We also survey the definition of the category of Hopf operads, which we define as the category of operads in the symmetric monoidal category of counitary cocommutative coalgebras (§3.2). The Hopf cooperads, which we consider in the summary of our mathematical objectives, are the dual structures of these Hopf operads.

We have various notions of functors associated to symmetric monoidal categories. We devote an appendix section of this chapter to a survey of this subject (§3.3).

Part I(b). Braids and E_2-operads. The main purpose of this part is to recall the general definition of an E_n-operad (by using the model of the little n-discs operads) and to study the connections between E_n-operads, Artin's braid groups, and braided monoidal categories in the case $n = 2$.

Chapter 4. *The Little Discs Model of E_n-operads.* We recall the definition of the little n-discs operad and we make explicit the definition of an E_n-operad in the first section of this chapter (§4.1). We devote the next section of the chapter (§4.2) to a survey on the computation of the cohomology and of the homology of the little n-discs operad. We then give an overview of several variants of the little discs operads in an outlook section (§4.3). This chapter is mostly a survey of the literature and does not contain any original result.

We will see that the homology of an operad in topological spaces forms an operad in graded modules. In the second part of this book, we will also use that the cohomology of a topological operad with coefficients in a field inherits the dual structure of a cooperad in graded modules (when our objects satisfy mild finiteness assumptions). We also have a natural unitary commutative algebra structure on the cohomology of a space. We actually get that the cohomology of a topological operad with coefficients in a field forms a cooperad in unitary commutative algebras in graded modules (a graded Hopf cooperad) when we put this unitary commutative algebra structure and the cooperad structure together. We just make explicit the definition of the homology operad associated to the operad of little n-discs and the commutative algebra structure of the spaces of little n-discs in the second section of this chapter. (We will see that the homology of the operad of little n-discs is identified with an operad governing graded Poisson algebra structures.)

We use that the category of graded modules inherits a symmetric monoidal structure to give a sense to these notions of commutative algebras and of operads in graded modules. We devote an appendix section of this chapter to an account of our conventions on graded modules and to a survey of the definition of this symmetric monoidal structure on the category of graded modules (§4.4).

Chapter 5. *Braids and the Recognition of E_2-operads.* We tackle the study of the relationship between E_2-operads and braids in this chapter. We recall the definition of the Artin braid groups and some conventions on braids in a preliminary section of the chapter (§5.0).

We then give an account of Fiedorowicz's definition of models of E_2-operads from contractible operads endowed with an action of braid groups (§5.1). We use this approach to prove that the classifying spaces of a certain operad in groupoids,

the colored braid operad, define an E_2-operad (§5.2). This operad of colored braids is closely related to the operad of parenthesized braids which we consider in the summary of our mathematical objectives. For the moment, simply say that the operad of colored braids is formed by groupoids whose morphism sets are identified with cosets of the pure braid groups inside the full Artin braid groups.

We prove in a second part of the chapter that the operad of colored braids is equivalent to the operad in groupoids formed defined by the fundamental groupoids of the underlying spaces of the little 2-discs operad (§5.3). We then use the adjunction between the fundamental groupoid and the classifying space construction in homotopy theory to give a second proof that the classifying spaces of the colored braid operad define an operad which is weakly-equivalent to the operad of little 2-discs, and hence, define a model of E_2-operad.

We can regard Fiedorowicz construction as a recognition criterion for the class of E_2-operads in topological spaces. We give an overview of more general recognition methods, which address the problem of giving an intrinsic definition of E_n-operads for all $n \geq 1$, in the concluding section of this chapter (§5.4).

Chapter 6. *The Magma and Parenthesized Braid Operads.* In the introductory chapter (§1), we recalled a general correspondence between operads and categories of algebras. In the case of an operad in the category of small categories (or groupoids), like the operad of colored braids considered in §5, the algebras are objects of the category of categories, and our operad therefore governs a class of monoidal structures which can be associated to a category. The operad of colored braids of §5 actually encodes the structure of a strict braided monoidal category, where we have a tensor product which is associative in the strict sense. The main purpose of this chapter is to explain this correspondence with full details and to give the definition of a variant of the colored braid operad, the operad of parenthesized braids, which we associate to braided monoidal categories with general associativity isomorphisms.

We first give a definition of an operad governing general monoidal categories (where the tensor product is just associative up to coherently defined isomorphisms) by elaborating on the classical Mac Lane Coherence Theorem of which we give an operadic interpretation (§6.1). We explain the definition of the operad of parenthesized braids afterwards (§6.2). To complete the account of this chapter, we also explain the definition of an operad of parenthesized symmetries which is an analogue for symmetric monoidal categories of the operad of parenthesized braids (§6.3).

Part I(c). Hopf Algebras and the Malcev Completion. In this part, we revisit the fundamental results of the theory of Hopf algebras and we study the applications of Hopf algebras to the definition of a rationalization process, the Malcev completion, which extends the classical rationalization of abelian groups to general (possibly non-abelian) groups. We then check that the Malcev completion process applies to groupoids and to operads in groupoids. We use the Malcev completion of the parenthesized braid operad in our definition of the Grothendieck–Teichmüller group (in the next part).

Chapter 7. *Hopf Algebras.* We review the foundations of the theory of Hopf algebras first and we devote this first chapter of the part to this subject. We explain the definition of a Hopf algebra in the general context of additive symmetric monoidal categories enriched in $\mathbb{Q}$-modules and we check that the main results of the theory remain valid in this framework. In what follows, we mainly apply

our constructions to Hopf algebras in a category of modules over a characteristic zero field and to Hopf algebras in complete filtered modules, but our framework covers other examples of categories where Hopf algebras are usually defined in the literature (for instance, the categories of motives).

We explain the general definition of a Hopf algebra in the first section of the chapter (§7.1). We review the relationship between Hopf algebras and Lie algebras in the second section of the chapter (§7.2). We recall the definition of the enveloping algebra of a Lie algebra in the course of this study and we revisit the proof of the classical structure theorems of the theory of Hopf algebras, namely the Poincaré-Birkhoff-Witt Theorem and the Milnor-Moore Theorem. We devote the third section of chapter (§7.3) to a thorough study of the structure of Hopf algebras in complete filtered modules. We then use the phrase 'complete Hopf algebras' to refer to a subcategory of the category of Hopf algebras in complete filtered modules formed by objects which satisfy a natural connectedness condition. We notably consider this subcategory of complete Hopf algebras when we define our Malcev completion functor on groups.

Chapter 8. *The Malcev Completion for Groups.* We just examine the applications of Hopf algebras to the Malcev completion of groups in this chapter.

We first explain the definition of a general completion process on Hopf algebras. We apply this completion process to group algebras in order to get a completed group algebra functor from the category of groups to the category of complete Hopf algebras. We have a natural (complete) group-like element functor which goes the other way round, from complete Hopf algebras to groups. We precisely define the Malcev completion of a group as the group of complete group-like elements in the completed group algebra of our group. We devote the first section of the chapter to the definition of these functors (§8.1).

We also say that a group is Malcev complete when this group occurs as the image of a complete Hopf algebra under the group-like element functor. We study the category of Malcev complete groups and the properties of the Malcev completion process in the second and third sections of the chapter (§§8.2-8.3). We will notably explain that the elements of a Malcev complete group can be represented as the exponential of elements of a complete Lie algebra associated to our group. We use this correspondence to check that, in a Malcev complete group, we can define power operations g^a with exponents in an arbitrary field of coefficients $a \in \Bbbk$.

We devote the rest of the chapter to the study of the Malcev completion of free groups and of semi-direct products (§§8.4-8.5). In the course of this study, we also recall the definition of a counterpart, for Hopf algebras and for complete Hopf algebras, of the classical semi-direct product of groups.

Chapter 9. *The Malcev Completion for Groupoids and Operads.* The (complete) Hopf algebras of §§7-8 can be identified with group objects in the category of (complete) counitary cocommutative algebras. In this chapter, we introduce a generalization of this notion, which we call (complete) Hopf groupoids (see §§9.0-9.1), in order to extend the Malcev completion process of the previous chapter from groups to groupoids (§9.1). Then we check that this Malcev completion functor on groupoids preserves symmetric monoidal category structures, and as a consequence, gives rise to a Malcev completion functor on the category of operads in groupoids (§9.2). By the way, we explain the definition of an operadic version

of the classical notion of a local coefficient system. We naturally get such structures, which we call 'local coefficient system operads', when we study the tower decomposition of the Malcev completion of operads in groupoids.

We devote an appendix section of the chapter (§8.5) to the study of the existence of group-like elements in the complete counitary cocommutative algebras underlying a complete Hopf groupoid. We use the results of this appendix to formulate connectedness hypothesis which we naturally need for our study of complete Hopf groupoids.

Most of the statements explained in this chapter are new, though the completion of Hopf groupoids was already considered for the definition of motivic fundamental groupoids by Deligne and Deligne–Goncharov (see [52, 54]), and this chapter could also serve as a basic reference for the algebraic background of this subject.

Part I(d). The Operadic Definition of the Grothendieck–Teichmüller Group. The main goal of this part is to explain the definition of the pro-unipotent Grothendieck-Teichmüller group as a group of automorphisms associated to a Malcev completion of the parenthesized braid operad. We devote the first chapter of the part to a preliminary study of this Malcev complete operad of parenthesized braids. By the way, we explain the definition of a related operad, the operad of chord diagrams, and we give an operadic interpretation of the notion of a Drinfeld associator, which we use as equivalences of operads in Malcev complete groupoids between the Malcev completion of the parenthesized braid operad and the chord diagram operad. We also explain the definition of a graded version of the Grothendieck–Teichmüller group as a group of automorphisms associated to a parenthesized version of the chord diagram operad. We tackle the definition of the pro-unipotent Grothendieck-Teichmüller group itself in the second chapter of the part.

The Grothendieck–Teichmüller group has a pro-finite version too, which we do not really use in this work, but which is the version to be considered for the applications to the Grothendieck proposal, where the goal is to study of the absolute Galois group through geometric actions on curves. We just give an overview of this program in the concluding chapter of this part.

Most of the ideas used in this part are known to experts, but only partial references on the operadic interpretation of the pro-unipotent Grothendieck–Teichmüller group and on the Drinfeld associators were available in the literature so far.

Chapter 10. *The Malcev Completion of the Braid Operads and Drinfeld's Associators.* The operad of parenthesized braids is an operad in groupoids whose morphism sets consist (like the morphism sets of the colored braid operad of §5) of cosets of the pure braid groups inside the full Artin braid groups. We therefore study the Malcev completion of the pure braid groups in the first section of this chapter (§10.0) before studying the Malcev completion of the parenthesized braid operad (and of the colored braid operad). We notably recall the definition of Lie algebra counterparts of the braid groups, which we call the Drinfeld–Kohno Lie algebras and which we denote by $\mathfrak{p}(r)$ (some authors call these Lie algebras the 'Lie algebras of infinitesimal braids'). We will see that, by results of Drinfeld and Kohno, the Malcev completion $\hat{P}_r$ of the pure braid group on r-strands P_r is isomorphic to the group of exponential elements which we associate to a completion $\hat{\mathfrak{p}}(r)$ of the Drinfeld–Kohno Lie algebra $\mathfrak{p}(r)$.

We study the Malcev completion of the operad of parenthesized braids (and of the colored braid operad) afterwards (in §10.1). We then explain the definition of the chord diagram operad and we revisit the definition of a Drinfeld associator (§10.2). In short, we will see that the Drinfeld–Kohno Lie algebras form an operad and the chord diagram operad is an operad in Malcev complete groups which we associate to this operad in Lie algebras.

We devote the rest of the chapter to the definition of a graded version of the Grothendieck–Teichmüller group (§10.3) and to the applications of this object for the study of natural tower decompositions of the set of Drinfeld's associators (§10.4). In the course of this study, we explain the definition of the graded Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ which we associate to this graded Grothendieck–Teichmüller group $GRT(\Bbbk)$. We use this analysis of the tower decomposition of the set of Drinfeld's associators when we examine the outcome of the homotopy spectral sequence associated to the space of homotopy automorphisms of E_2-operads in the third part of this book.

Chapter 11. *The Grothendieck–Teichmüller Group.* We explain the operadic definition of the pro-unipotent Grothendieck–Teichmüller group in the first section of this chapter (§11.1). We precisely check that this group $GT(\Bbbk)$, such as defined by Drinfeld, can be identified with a group of operad automorphisms associated to the Malcev completion of the operad of parenthesized braids. We study a natural action of the group $GT(\Bbbk)$ on the set of Drinfeld's associators afterwards (in §11.2). We use this action to check, after Drinfeld, that the pro-unipotent Grothendieck–Teichmüller group $GT(\Bbbk)$ is isomorphic to the graded Grothendieck–Teichmüller group studied in the previous chapter $GRT(\Bbbk)$.

Then we explain the definition of a tower decomposition of the Grothendieck–Teichmüller group $GT(\Bbbk) = \lim_m GT_{\langle m\rangle}(\Bbbk)$ (§11.3). This tower decomposition is associated to a filtration of the group $GT(\Bbbk)$ by normal subgroups $\mathtt{F}_m\, GT(\Bbbk)$. We explicitly have $GT_{\langle m\rangle}(\Bbbk) = GT(\Bbbk)/\mathtt{F}_m\, GT(\Bbbk)$ for all $m \geq 0$. We actually have a pro-unipotent structure on the first layer of this filtration $GT^1(\Bbbk) = \mathtt{F}_1\, GT(\Bbbk)$ (not on the whole group $GT(\Bbbk)$), which we can also identify with the kernel of a natural character map $\lambda : GT(\Bbbk) \to \Bbbk^{\times}$ on the group $GT(\Bbbk)$. To complete the account of this chapter, we check that the subquotients $\mathtt{E}^0_m\, GT(\Bbbk) = \mathtt{F}_m\, GT(\Bbbk)/\mathtt{F}_{m+1}(\Bbbk)$, $m \geq 1$, form a weight graded Lie algebra which is isomorphic to the graded Grothendieck–Teichmüller Lie algebra of the previous chapter $\mathfrak{grt}$. (We address this subject in §11.4.)

Chapter 12. *A Glimpse at the Grothendieck Program.* This chapter serves as a conclusion for this volume. We provide a brief introduction to the Grothendieck program in Galois theory and we give an overview of the literature about the connections between Grothendieck–Teichmüller groups, motivic Galois groups, and multizetas.

Appendix A. Trees and the Construction of Free Operads. In this appendix, we explain the applications of trees to the definition of universal constructions in the category of operads. We make our definition of a tree precise in a preliminary section (§A.1). We explain the definition of general treewise composition operations associated to operads afterwards (in §A.2). We then explain the applications of trees to the definition of free objects in the category of operads (§§A.3-A.4) and to the definition of coproducts with free objects (§A.5).

Most of the ideas which we use in this appendix are not original, as the applications of trees for the study of operads go back to Stasheff's work on the recognition of loop spaces [167] and to Boardman–Vogt's work on homotopy invariant structures [28]. The definition of free operads in terms of trees, in particular, is due to Ginzburg–Kapranov [78]. We just give a new definition of reduced free objects in the context of unitary operads.

Appendix B. The Cotriple Resolution of Operads. In this appendix, we explain the definition of a simplicial resolution functor on the category of operads, the cotriple resolution, which we use in our study of the homotopy of operads in the next parts of this book. We restrict our analysis to the case of connected operads for simplicity. In short, we prove that the cotriple resolution of an operad has an explicit description which we obtain by inserting extra structures, modelled by chains of tree morphisms, in our previous construction of free operads. We explain the definition of our notion of a tree morphism in the first section of this appendix (§B.0) and we tackle the applications to the cotriple construction afterwards (§B.1).

The free operad functor inherits a natural composition operation which makes this object a monad on the category of symmetric sequences (underlying the category of operads). We recall the definition of the concept of a monad in §B.2, and we prove that the structure of an operad can also be defined in terms of an action of this free operad monad on a symmetric sequence. In the language of category theory, this result asserts that the category of operads is monadic.

Most of the results explained in this appendix are known to experts (like the constructions of the previous appendix). We still just give a new definition of a reduced version of the cotriple resolution for unitary operads.

Glossary of Notation

Background

Fundamental objects

$\Bbbk$: the ground ring
$\mathbb{D}^n$: the unit n-disc, see §I.4.1.1
$\mathbb{\Delta}^n$: the topological n-simplex, see §0.3, §II.1.3.4
pt: the one-point set (also denoted by $*$ when regarded as a terminal object)
Δ: the simplicial category, see §0.3, §II.1.3.2
Δ^n: the n-simplex object of the category of simplicial sets, see §0.3, §II.1.3.4

Generic categorical notation

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ...: general categories
$\mathcal{I}$, $\mathcal{J}$, ...: indexing categories, as well as the set of generating cofibrations and the set of generating acyclic cofibrations in a cofibrantly generated model category, see §II.4.1.3
$\mathcal{F}$, $\mathcal{G}$, ...: some classes of morphisms in a category
$\mathcal{M}$, $\mathcal{N}$, ...: (symmetric) monoidal categories, see §0.8
$\mathbb{1}$: the unit object of a (symmetric) monoidal category, see §0.8
eq: the equalizer of parallel arrows in a category
coeq: the coequalizer of parallel arrows in a category

Fundamental categories

$\mathcal{M}od$: the category of modules over the ground ring
$\mathcal{S}et$: the category of sets
$\mathcal{T}op$: the category of topological spaces, see §II.1.3
$s\mathcal{S}et$: the category of simplicial sets, see §0.3, §II.1.3
$\mathcal{G}rp$: the category of groups
$\mathcal{G}rd$: the category of groupoids, see §I.5.2.1
$\mathcal{C}at$: the category of small categories, see §I.5.2.1
$\mathcal{A}b$: the category of abelian groups

Categories of algebras and of coalgebras

$\mathcal{C}om$: the category of non-unitary commutative algebras
$\mathcal{A}s$: the category of non-unitary associative algebras
$\mathcal{L}ie$: the category of Lie algebras
$\mathcal{A}s_+$: the category of unitary associative algebras
$\mathcal{C}om_+$: the category of unitary commutative algebras, see §I.3.0.1
$\mathcal{C}om_+^c$: the category of counitary cocommutative coalgebras, see §I.3.0.4

$\mathcal{H}opf\mathcal{A}lg$: the category of Hopf algebras (defined as the category of bialgebras equipped with an antipode operation), see §I.7.1.8
$\mathcal{H}opf\mathcal{G}rd$: the category of Hopf groupoids (defined as the category of small categories equipped with an antipode operation), see §I.9.0.2

Functors and constructions for filtered objects

$\mathtt{F}_s$: the sth layer of a decreasing filtration
$\mathtt{E}^0_s$: the sth subquotient of a filtered object, see §I.7.3.6 (also used to denote the sth fiber of a tower of set maps in the context of homotopy spectral sequences, see §III.1.1.7)
$\mathtt{E}^0$: the weight graded object associated to a filtered object in a category (e.g. the weight graded module associated to a filtered module, see §I.7.3.6, the weight graded Lie algebra associated to a Malcev complete group, see §I.8.2.2, ...)
$(-)\widehat{\ }$: the completion functor on a category of objects equipped with a decreasing filtration, as well as the Malcev completion for groups and groupoids, see §I.7.3.4, §I.8.3 (also the rationalization functor on spaces, see the section about the constructions of homotopy theory in this glossary)

Functors and constructions on algebras and coalgebras

$\mathbb{S}$: the symmetric algebra functor (in any symmetric monoidal category), see §I.7.2.4
$\mathbb{T}$: the tensor algebra functor (in any symmetric monoidal category), see §I.7.2.4
$\mathbb{L}$: the free Lie algebra functor (in any $\mathbb{Q}$-additive symmetric monoidal category and in abelian groups), see §I.7.2.3
$\mathbb{U}$: the enveloping algebra functor (on the category of Lie algebras in any $\mathbb{Q}$-additive symmetric monoidal category), see §I.7.2.7
$\hat{\mathbb{S}}$, $\hat{\mathbb{T}}$, ...: the complete variants of the symmetric algebra functor, of the tensor algebra functor, ... in the context of a category of complete filtered modules, see §I.7.3.22
$\mathbb{G}$: the group-like element functor on coalgebras, see §I.7.1.14, and on complete Hopf coalgebras, see §I.8.1.2
$\mathbb{P}$: the primitive element functor on Hopf algebras, see §I.7.2.11
$\mathbb{I}(-)$: the augmentation ideal of Hopf algebras, see §I.8.1.1

Categorical prefixes

dg: prefix for a category of differential graded objects in a category (e.g. the category of dg-modules $dg\,\mathcal{M}od$, see §0.1, §II.5.0.1)
dg_*, dg^*: prefix for the chain graded and cochain graded variants of the categories of differential graded objects (e.g. the category of chain graded dg-modules $dg_*\,\mathcal{M}od$, see §II.5.0.1, the category of cochain graded dg-modules $dg^*\,\mathcal{M}od$, see §II.5.0.1, §II.5.1, and the category of unitary commutative cochain dg-algebras $dg^*\,\mathcal{C}om_+$, see §II.6.1.1, ...)
gr: prefix for a category of graded objects in a category when the grading underlies a differential graded structure (e.g. the category of graded modules $gr\,\mathcal{M}od$, see §0.1, see §II.5.0.2, ...)
s: prefix for a category of simplicial objects in a category (e.g. the category of simplicial modules $s\,\mathcal{M}od$, see §0.6, §II.5.0.4, the category of simplicial sets $s\mathcal{S}et$, see §0.3, §II.1.3, ...)

c: prefix for a category of cosimplicial objects in a category (e.g. the category of cosimplicial modules $c\,\mathcal{M}od$, see §0.6, §II.5.0.4, the category of cosimplicial unitary commutative algebras $c\,\mathcal{C}om_+$, see §II.6.1.3, ...)
f: prefix for a category of filtered objects in a category (e.g. the category of filtered modules $f\,\mathcal{M}od$, see §I.7.3.1)
$\hat{f}$: prefix for a category of complete filtered objects in a category (e.g. the category of complete filtered modules $\hat{f}\,\mathcal{M}od$, see §I.7.3.4, the category of Malcev complete groups $\hat{f}\,\mathcal{G}rp$, see §I.8.2). Note that the categories of complete Hopf algebras $\hat{f}\,\mathcal{H}opf\mathcal{A}lg$ and of complete Lie algebras $\hat{f}\,\mathcal{L}ie$ consist of Hopf algebras and Lie algebras in complete filtered modules that satisfy an extra connectedness requirement and a similar convention is made for the category of complete Hopf groupoids $\hat{f}\,\mathcal{H}opf\mathcal{G}rd$, see §I.7.3.15, §I.7.3.20, §I.9.1.2
w: prefix for a category of weight graded objects in a category (e.g. the category of weight graded modules $w\,\mathcal{M}od$, see §I.7.3.5)

Morphisms, hom-objects, duals, and analogous constructions

`Mor`: the notation for the morphism sets of any category (e.g. $\mathtt{Mor}_{\mathcal{M}od}(-,-)$ for the morphism sets of the category of modules over the ground ring $\mathcal{M}od$)
`Aut`: the notation for the automorphism group of an object in a category
`Hom`: the notation for the hom-objects of an enriched category structure (not to be confused with the morphism sets), see §0.12
`D`: the duality functor for ordinary modules, dg-modules, simplicial modules and cosimplicial modules, see §II.5.0.13
$(-)^\vee$: the dual of individual objects, or of objects equipped with extra structures (algebras, operads, ...), see §II.5.0.13
`Der`: the modules of derivations (for algebras, operads, ...), see §III.2.1
`Map`, $\mathtt{Aut}^h$: see the section of this glossary about the constructions of homotopy theory

Constructions of homotopy theory

Fundamental constructions in model categories

$\mathtt{Ho}(-)$: the homotopy of a model category, see §II.1.2
$\mathtt{Aut}^h$: the notation for the homotopy automorphism space of an object in a model category, see §II.2.2
`Map`: the notation for the mapping spaces of a pair of objects in simplicial model categories and in general model categories, see §II.2.1, §II.3.2.11

Fundamental simplicial and cosimplicial constructions

`B`: the classifying space construction for groups, groupoids, categories, ..., see §I.5.2.3 (also the bar construction of algebras and of operads, see the relevant sections of this glossary)
$\mathtt{sk}_r$: the rth skeleton of a simplicial set, of a simplicial and of a cosimplicial object in a model category, see §II.1.3.8, §II.3.1.7, §II.3.1.17
`Tot`: the totalization of cosimplicial spaces, of cosimplicial objects in a model category, see §II.3.3.13

$|-|$: the geometric realization of simplicial sets, of simplicial objects in a model category, see §0.5, §II.1.3.5, §II.3.3.5
$\mathtt{Diag}$: the diagonal complex of a bisimplicial set, of a bisimplicial object and of a bicosimplicial object in a model category, see §II.3.3.19
$\mathtt{L}_r(X)$: the rth latching object of a simplicial object in a category, see §II.3.1.14
$\mathtt{M}_r(X)$: the rth matching object of a simplicial object in a category, see §II.3.1.15 (also the matching objects of Λ-sequences, see the section about operads and related structures of this glossary)
$\mathtt{L}^r(X)$, $\mathtt{M}^r(X)$: the cosimplicial variants of the matching and matching object constructions, see §II.3.1.3, §II.3.1.5

Differential graded constructions

$\mathbb{b}^m$, $\mathbb{e}^m$: notation for particular homogeneous elements (of upper degree m) notably used to define the generating (acyclic) cofibrations of the category of cochain graded dg-modules, see §II.5.1.2
$\mathbb{b}_m$, $\mathbb{e}_m$: same as $\mathbb{b}^m$ and $\mathbb{e}^m$ but in the chain graded context
$\mathbb{B}^m$: source objects of the generating cofibrations of the category of cochain graded dg-modules, see §II.5.1.2
$\mathbb{E}^m$: target objects of the generating (acyclic) cofibrations of the category of cochain graded dg-modules, see §II.5.1.2
$\mathbb{B}_m$, $\mathbb{E}_m$: dual objects of the dg-modules $\mathbb{B}^m$ and $\mathbb{E}^m$
$\mathbb{o}$: notation for particular homogeneous elements used in the definition of suspension functors on dg-modules, see §C.2.3
$\mathbb{p}_r$, $\mathbb{p}_r^{\mathrm{s}}$: notation for particular homogeneous elements used in the definition of the operadic suspension functor for operads in dg-modules, see §II.4.1.1
$\mathtt{Cyl}$: the standard cylinder object functor on the category of dg-modules, see §II.13.1.10
$\mathtt{B}$: the bar construction for algebras, see §II.6.3 (also the classifying space of groups, categories, and the bar construction of operads, see the relevant sections of this glossary)
τ_*: the right adjoint $\tau_* : dg\,\mathcal{M}od \to dg_*\,\mathcal{M}od$ of the embedding $\iota : dg_*\,\mathcal{M}od \hookrightarrow dg\,\mathcal{M}od$ of the category of chain graded dg-modules $dg_*\,\mathcal{M}od$ into the category of all dg-modules $dg\,\mathcal{M}od$, see §II.5.3.2
τ^*: the left adjoint $\tau^* : dg\,\mathcal{M}od \to dg^*\,\mathcal{M}od$ of the embedding $\iota : dg^*\,\mathcal{M}od \hookrightarrow dg\,\mathcal{M}od$ of the category of cochain graded dg-modules $dg^*\,\mathcal{M}od$ into the category of all dg-modules $dg\,\mathcal{M}od$, see §II.5.0.1
$(-)_\flat$: the forgetful functor from dg-modules to graded modules, see §0.1

The Dold–Kan correspondence

$\mathtt{N}_*$: the normalized chain complex functor on the category of simplicial modules, see §0.6, §II.5.0.5
$\mathtt{N}^*$: the conormalized cochain complex functor on the category of cosimplicial modules, see §II.5.0.9
$\Gamma_\bullet$: the Dold–Kan functor on the category of chain graded dg-modules, see §II.5.0.6
$\Gamma^\bullet$: the cosimplicial version of the Dold–Kan functor on the category of cochain graded dg-modules, see §II.5.0.9

Constructions of rational homotopy theory

$(-)\widehat{\ }$: the rationalization functor on spaces, see §II.7.2.3, and on operads in simplicial sets, see §II.10.2, §II.12.2 (also the completion of filtered objects, see the section of this glossary about the background of our constructions)
Ω^*: the Sullivan cochain dg-algebra functor on simplicial sets, see §II.7.1
$\Omega^*_\sharp$: the operadic upgrade of the cochain dg-algebra functor on operads in simplicial sets, see §II.10.1, §II.12.1
$\mathtt{G}_\bullet$: the functor from cochain dg-algebras to simplicial sets, see §II.7.2
$\mathtt{MC}_\bullet$: the Maurer–Cartan spaces associated to (complete) Lie algebras, see §II.13.1.8

Operads and related structures

Indexing of operads

Σ_r: the symmetric group on r letters
Σ: the category of finite ordinals and permutations, see §I.2.2.3
Λ: the category of finite ordinals and injections, see §I.2.2.2
Λ^+: the category of finite ordinals and increasing injections, see §I.2.2.2
$\Sigma_{>0}$, $\Sigma_{>1}$, $\Lambda_{>0}$, $\Lambda_{>1}$, $\ldots$: the full subcategory of the category Σ, Λ, $\ldots$ generated by the ordinals of cardinal $r > 0$, $r > 1$, see §I.2.2.2, §I.2.4.1
$\mathcal{B}ij$: the category of finite sets and bijections, see §I.2.5.1
$\mathcal{I}nj$: the category of finite sets and injections, see §I.2.5.9
$\mathcal{B}ij_{>0}$, $\mathcal{B}ij_{>1}$, $\mathcal{I}nj_{>0}$, $\mathcal{I}nj_{>1}$, $\ldots$: the full subcategory of the categories $\mathcal{B}ij$, $\mathcal{I}nj$, $\ldots$ generated by the finite sets of cardinal $r > 0$, $r > 1$, see §I.2.5.9
$\underline{\mathsf{m}}$, $\underline{\mathsf{n}}$, $\ldots$, $\underline{\mathsf{r}}$, $\ldots$: generic notation for finite ordinals $\underline{\mathsf{r}} = \{1 < \cdots < r\}$ or for finite sets $\underline{\mathsf{r}} = \{i_1, \ldots, i_r\}$ used to index the terms of operads, symmetric sequences and Λ-sequences
$\underline{0}$, $\underline{1}$, $\underline{2}$, $\ldots$: the empty ordinal, the ordinal of cardinal one $\underline{1} = \{1\}$, of cardinal two $\underline{2} = \{1 < 2\}$, $\ldots$

Categories of operads and related

$\mathcal{O}p$: the category of (symmetric) operads, see §I.1.1.2
$\mathcal{O}p_\varnothing$: the category of non-unitary (symmetric) operads, see §I.1.1.20
$\mathcal{O}p_{\varnothing 1}$: the category of connected (symmetric) operads, see §I.1.1.21
$\mathcal{O}p^c_{\varnothing 1}$: the category of (symmetric) cooperads, see §II.9.1.8
$\Lambda\,\mathcal{O}p_\varnothing\,/\,\mathit{Com}$: the category of augmented non-unitary Λ-operads (the postfix expression $-/\,\mathit{Com}$ can be discarded when the augmentation is trivial), see §I.2.2.17
$\Lambda\,\mathcal{O}p_{\varnothing 1}\,/\,\mathit{Com}$: the category of augmented connected Λ-operads (the postfix expression $-/\,\mathit{Com}$ can be discarded when the augmentation is trivial), see §I.2.4
$\mathcal{S}eq$: the category of symmetric sequences, see §I.1.2
$\mathcal{S}eq_{>0}$: the category of non-unitary symmetric sequences, see §I.1.2.13
$\mathcal{S}eq_{>1}$: the category of connected symmetric sequences, see §I.1.2.13
$\mathcal{S}eq^c$, $\mathcal{S}eq^c_{>0}$, $\mathcal{S}eq^c_{>1}$: same as $\mathcal{S}eq$, $\mathcal{S}eq_{>0}$, $\mathcal{S}eq_{>1}$ but used instead of this notation in the context of cooperads
$\Lambda\,\mathcal{S}eq$: the category of Λ-sequences, see §I.2.3
$\Lambda\,\mathcal{S}eq_{>0}$: the category of non-unitary Λ-sequences, see §I.2.3
$\Lambda\,\mathcal{S}eq_{>1}$: the category of connected Λ-sequences, see §I.2.4.1
$\Lambda\,\mathcal{S}eq^c$: the category of covariant Λ-sequences

$\Lambda\,\mathcal{S}eq^c_{>0}$: the category of covariant non-unitary Λ-sequences
$\Lambda\,\mathcal{S}eq^c_{>1}$: the category of covariant connected Λ-sequences, see §II.11.1.7
$\mathcal{C}oll$: the category of (symmetric) collections, see §I.2.5.1
$\mathcal{C}oll_{>0}$: the category of non-unitary (symmetric) collections
$\mathcal{C}oll_{>1}$: the category of connected (symmetric) collections

Categories of Hopf operads and related

$\mathcal{H}opf\,\mathcal{O}p$: the category of Hopf operads (defined as the category of operads in counitary cocommutative coalgebras), see §I.3.2
$\mathcal{H}opf\,\mathcal{O}p_{\varnothing}$, $\mathcal{H}opf\,\mathcal{O}p_{\varnothing 1}$: the non-unitary and connected variants of the category of Hopf operads
$\mathcal{H}opf\,\Lambda\,\mathcal{O}p_{\varnothing}$, $\mathcal{H}opf\,\Lambda\,\mathcal{O}p_{\varnothing 1}$: the Λ-operad variants of the categories of non-unitary and connected Hopf operads, see §I.3.2.15
$\mathcal{H}opf\,\mathcal{S}eq$: the category of Hopf symmetric sequences (defined as the category of symmetric sequences in counitary cocommutative coalgebras), see §I.3.2.6
$\mathcal{H}opf\,\mathcal{S}eq_{>0}$, $\mathcal{H}opf\,\mathcal{S}eq_{>1}$: the non-unitary and connected variants of the category of Hopf symmetric sequences
$\mathcal{H}opf\,\Lambda\,\mathcal{S}eq_{>0}$, $\mathcal{H}opf\,\Lambda\,\mathcal{S}eq_{>1}$: the Λ-sequence variants of the categories of non-unitary and connected Hopf symmetric sequences
$\mathcal{H}opf\,\mathcal{O}p^c_{\varnothing 1}$: the category of Hopf cooperads (defined as the category of cooperads in unitary commutative algebras), see §II.9.3.1
$\mathcal{H}opf\,\Lambda\,\mathcal{O}p^c_{\varnothing 1}$: the category of Hopf Λ-cooperads (defined as the category of cooperads in unitary commutative algebras), see §II.11.4.1
$\mathcal{H}opf\,\mathcal{S}eq^c_{>1}$: the category of connected Hopf symmetric sequences underlying Hopf cooperads (defined as the category of symmetric sequences in unitary commutative algebras), see §II.9.3.1
$\mathcal{H}opf\,\Lambda\,\mathcal{S}eq^c_{>1}$: the category of connected Hopf Λ-sequences underlying Hopf Λ-cooperads (defined as the category of Λ-sequences in unitary commutative algebras), see §II.11.4.1

Notation of operads

P, Q, ...: generic notation for operads (of any kind)
M, N, ...: generic notation for symmetric sequences, Λ-sequences, covariant Λ-sequences
C, D, ...: generic notation for cooperads (of any kind)
C_n: the operad of little n-cubes, see §I.4.1.3
D_n: the operad of little n-discs, see §I.4.1.7
As: the (non-unitary) associative operad, see §I.1.1.16, §I.1.2.6, §I.1.2.10
Com: the (non-unitary) commutative operad, see §I.1.1.16, §I.1.2.6, §§I.1.2.10-2.1.11
Lie: the Lie operad, see §I.1.2.10
Pois: the Poisson operad, see §I.1.2.12
Gerst_n: the n-Gerstenhaber operad (defined as a graded variant of the Poisson operad), see §I.4.2.13
Com^c: the commutative cooperad, see §II.9.1.3
CoS, PaS, CoB, PaB, ...: see the section about the applications of operads to the definition of Grothendieck–Teichmüller groups

Constructions on operads and on cooperads

τ: the truncation functors from non-unitary operads to connected operads and from augmented non-unitary Λ-operads to augmented connected Λ-operads, see §I.1.2.15, Proposition I.2.4.5
$\mathbb{O}$: the free operad functor, see §A.3
$\mathbb{O}^c$: the cofree cooperad functor, see §C.1
$\mathbb{O}_{\underline{\mathsf{T}}}(M)$: the treewise tensor product of a symmetric sequence M over a tree $\underline{\mathsf{T}}$ when regarded as a term of the free operad and of the cofree cooperad (same as the object denoted by $M(\underline{\mathsf{T}})$ in the section about trees), see §A.2
ΣF^r: the rth free symmetric sequence, see §II.8.1.2
ΛF^r: the rth free Λ-sequence, see §II.8.3.6
$\partial\Lambda F^r$: the boundary of the rth free Λ-sequence, see §II.8.3.7
$\partial'\Lambda F^r$: the boundary of the rth free Λ-sequence in the context of connected Λ-sequences, see §II.12.0.1
$\mathtt{Res}_\bullet$: the cotriple resolution functor on operads, see §B.1, §II.8.5
$\mathtt{B}$: the bar construction of operads, see §C.2 (also the classifying space of groups, categories, and the bar construction of algebras, see the relevant sections of this glossary)
$\mathtt{B}^c$: the cobar construction of cooperads, see §C.2
$\mathtt{K}$: the Koszul dual of operads, see §C.3
$\mathtt{M}(M)(r)$: the rth matching object of a Λ-sequence, see §II.8.3.1
$\mathtt{ar}_{\leq s}$: the sth layer of the arity filtration of a Λ-sequence, see Proof of Theorem II.8.3.20
$\mathtt{ar}^\sharp_{\leq s}$: the operadic upgrade of the arity filtration, see Proof of Theorem II.8.4.12
$\mathtt{cosk}^\Lambda_r$: the rth Λ-coskeleton of a Λ-sequence, see §II.8.3.3, of an augmented non-unitary Λ-operad, see Proof of Theorem II.8.4.12

Trees

$\mathcal{T}ree(\underline{\mathsf{r}})$: the category of $\underline{\mathsf{r}}$-trees (where $\underline{\mathsf{r}}$ is the indexing set of the inputs of the trees), see §A.1
$\mathcal{T}ree$: the operad of trees, see §A.1
$\widetilde{\mathcal{T}ree}(\underline{\mathsf{r}})$: the category of reduced $\underline{\mathsf{r}}$-trees (where $\underline{\mathsf{r}}$ is the indexing set of the inputs of the trees), see §A.1.12
$\widetilde{\mathcal{T}ree}$: the operad of reduced trees, see §A.1.12
$\mathcal{T}ree^\circ(\underline{\mathsf{r}})$: the category of planar $\underline{\mathsf{r}}$-trees (where $\underline{\mathsf{r}}$ is the indexing set of the inputs of the trees), see §A.3.16
$\mathcal{T}ree^\circ$: the operad of planar trees, see §A.3.16
$\underline{\mathsf{S}}$, $\underline{\mathsf{T}}$, ...: generic notation for trees
$\downarrow$: the unit tree (the tree with no vertex), see §A.1.4
$\underline{\mathsf{Y}}$: the notation of a corolla (a tree with a single vertex), see §A.1.4
$\underline{\Gamma}$: the notation of a tree with two vertices, see §A.2.3
$V(\underline{\mathsf{T}})$: the vertex set of a tree
$E(\underline{\mathsf{T}})$: the edge set of a tree
$\mathring{E}(\underline{\mathsf{T}})$: the set of inner edges of a tree
$\underline{\mathsf{r}}_v$: the set of ingoing edges of a vertex in a tree
$M(\underline{\mathsf{T}})$: the treewise tensor product of a symmetric sequence M over a tree $\underline{\mathsf{T}}$ (same as the object denoted by $\mathbb{O}_{\underline{\mathsf{T}}}(M)$ in the section about constructions on operads and on cooperads), see §A.2

$\lambda_{\underline{\mathsf{T}}}$: the treewise composition products associated to an operad, see §A.2.7
$\rho_{\underline{\mathsf{T}}}$: the treewise composition coproducts associated to a cooperad, see §C.1.5

From operads to Grothendieck–Teichmüller groups

Permutations, braids, and related objects

Σ_r: the symmetric group on r letters
B_r: the Artin braid group on r strands, see §I.5.0
P_r: the pure braid group on r strands, see §I.5.0
$\mathfrak{p}(r)$: the rth Drinfeld–Kohno Lie algebras (the Lie algebra of infinitesimal braids on r strands), see §I.10.0.2
$\hat{\mathfrak{p}}(r)$: the complete Drinfeld–Kohno Lie algebra, see §I.10.0.6
$\mathfrak{p}_n(r)$: the graded variants of the Drinfeld–Kohno Lie algebras (with $\mathfrak{p}(r) = \mathfrak{p}_2(r)$), see §II.14.1.1
$\mathfrak{p}$: the Drinfeld–Kohno Lie algebra operad, see §I.10.1.1
$\mathfrak{p}_n$: the graded variants of the Drinfeld–Kohno Lie algebra operad (with $\mathfrak{p} = \mathfrak{p}_2$), see §II.14.1.1
$\hat{\mathfrak{p}}$: the complete Drinfeld–Kohno Lie algebra operad, see §I.10.2.2
CoS: the operad of colored symmetries, see §I.6.3
PaS: the operad of parenthesized symmetries, see §I.6.3
CoB: the operad of colored braids, see §§I.5.2.8-5.2.11, §I.6.2.7
PaB: the operad of parenthesized braids, see §I.6.2
$\mathsf{CoB}\widehat{\ }$, $\mathsf{PaB}\widehat{\ }$: the Malcev completion of the colored and parenthesized braid operads, see §I.10.1
$\mathsf{CD}\widehat{\ }$: the operad of chord diagrams, see §I.10.2.4
$\mathsf{PaCD}\widehat{\ }$: the operad of parenthesized chord diagrams, see §I.10.3.2

Grothendieck–Teichmüller groups and related objects

$Ass(\Bbbk)$: the set of Drinfeld's associators, see §I.10.2.11
$GT(\Bbbk)$: the pro-unipotent Grothendieck–Teichmüller group, see §I.11.1
GRT: the graded Grothendieck–Teichmüller group, see §I.10.3
$GT\widehat{\ }$: the profinite Grothendieck–Teichmüller group
$\mathfrak{grt}$: the graded Grothendieck–Teichmüller Lie algebra, see §I.10.4.6, §I.11.4

Bibliography

[1] Eiichi Abe. *Hopf algebras*, volume 74 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge-New York, 1980. Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka.

[2] J. Adámek, J. Rosický, and E. M. Vitale. What are sifted colimits? *Theory Appl. Categ.*, 23:No. 13, 251–260, 2010.

[3] John Frank Adams. *Infinite loop spaces*, volume 90 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[4] Marcelo Aguiar and Swapneel Mahajan. *Coxeter groups and Hopf algebras*, volume 23 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2006. With a foreword by Nantel Bergeron.

[5] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown and Stephen Chase and André Joyal.

[6] Anton Alekseev and Charles Torossian. Kontsevich deformation quantization and flat connections. *Comm. Math. Phys.*, 300(1):47–64, 2010.

[7] Yves André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2004.

[8] V. I. Arnol′d. The cohomology ring of the group of dyed braids. *Mat. Zametki*, 5:227–231, 1969.

[9] Gregory Arone and Victor Turchin. On the rational homology of high-dimensional analogues of spaces of long knots. *Geom. Topol.*, 18(3):1261–1322, 2014.

[10] E. Artin. Theorie der Zöpfe. *Hamb. Abh.*, 4:47–72, 1925.

[11] E. Artin. Theory of braids. *Ann. of Math. (2)*, 48:101–126, 1947.

[12] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. II. *J. Differential Geom.*, 39(1):173–213, 1994.

[13] Joseph Ayoub. L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, I. *J. Reine Angew. Math.*, 693:1–149, 2014.

[14] Joseph Ayoub. L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, II. *J. Reine Angew. Math.*, 693:151–226, 2014.

[15] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, and R. Vogt. Iterated monoidal categories. *Adv. Math.*, 176(2):277–349, 2003.

[16] Dror Bar-Natan. On associators and the Grothendieck-Teichmuller group. I. *Selecta Math. (N.S.)*, 4(2):183–212, 1998.

[17] M. G. Barratt and Peter J. Eccles. Γ^+-structures. I. A free group functor for stable homotopy theory. *Topology*, 13:25–45, 1974.

[18] M. A. Batanin. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Adv. Math.*, 136(1):39–103, 1998.

[19] M. A. Batanin. Symmetrisation of n-operads and compactification of real configuration spaces. *Adv. Math.*, 211(2):684–725, 2007.

[20] M. A. Batanin. The Eckmann-Hilton argument and higher operads. *Adv. Math.*, 217(1):334–385, 2008.

[21] Michael A. Batanin. Locally constant n-operads as higher braided operads. *J. Noncommut. Geom.*, 4(2):237–263, 2010.

[22] G. V. Belyĭ. Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):267–276, 479, 1979.

[23] Clemens Berger. Opérades cellulaires et espaces de lacets itérés. *Ann. Inst. Fourier (Grenoble)*, 46(4):1125–1157, 1996.

[24] Clemens Berger. Combinatorial models for real configuration spaces and E_n-operads. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, volume 202 of *Contemp. Math.*, pages 37–52. Amer. Math. Soc., Providence, RI, 1997.

[25] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.

[26] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.

[27] J. M. Boardman and R. M. Vogt. Homotopy-everything H-spaces. *Bull. Amer. Math. Soc.*, 74:1117–1122, 1968.

[28] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[29] Pedro Boavida de Brito and Michael Weiss. Spaces of smooth embeddings and configuration categories. Preprint `arXiv:1502.01640`, 2015.

[30] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

[31] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[32] Francis Borceux and George Janelidze. *Galois theories*, volume 72 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2001.

[33] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[34] N. Bourbaki. *Éléments de mathématique. Groupes et algèbres de Lie. Chapitres 2 et 3*. Springer-Verlag, Berlin, 2006. Reprint of the 1972 original [Hermann, Paris; MR0573068].

[35] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3*. Springer-Verlag, Berlin, 2007. Reprint of the 1970 original [Hermann, Paris; MR0274237].

[36] A. K. Bousfield and V. K. A. M. Gugenheim. On PL de Rham theory and rational homotopy type. *Mem. Amer. Math. Soc.*, 8(179):ix+94, 1976.

[37] Francis Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math. (2)*, 175(2):949–976, 2012.

[38] Damien Calaque, Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted Poisson structures and deformation quantization. Preprint `arXiv:1506.03699`, 2015.

[39] Pierre Cartier. A primer of Hopf algebras. Prépublication Inst. Hautes Études Sci., 2006.

[40] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1994.

[41] Kuo Tsai Chen. Iterated path integrals. *Bull. Amer. Math. Soc.*, 83(5):831–879, 1977.

[42] S. Chmutov, S. Duzhin, and J. Mostovoy. *Introduction to Vassiliev knot invariants*. Cambridge University Press, Cambridge, 2012.

[43] Utsav Choudhury and Martin Gallauer Alves de Souza. An isomorphism of motivic galois groups. 2014. Preprint `arXiv:1410.6104`.

[44] F. R. Cohen, J. P. May, and L. R. Taylor. Splitting of certain spaces CX. *Math. Proc. Cambridge Philos. Soc.*, 84(3):465–496, 1978.

[45] Frederick R. Cohen. The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$. In *The homology of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 533, pages 207–351. Springer-Verlag, Berlin-New York, 1976.

[46] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The homology of iterated loop spaces*. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[47] Benjamin Collas and Sylvain Maugeais. On galois action on stack inertia of moduli spaces of curves. 2014. Preprint `arXiv:1412.4644`.

[48] Benjamin Collas and Sylvain Maugeais. Composantes irréductibles de lieux spéciaux d'espaces de modules de courbes, action galoisienne en genre quelconque. *Ann. Inst. Fourier (Grenoble)*, 65(1):245–276, 2015.

[49] Alain Connes and Dirk Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Comm. Math. Phys.*, 199(1):203–242, 1998.

[50] Kevin Costello and Owen Gwilliam. Factorization algebras in quantum field theory. Volume 31 of *New Mathematical Monographs*. Cambridge Univ. Press, Cambridge, 2016.

[51] Edward B. Curtis. Simplicial homotopy theory. *Advances in Math.*, 6:107–209 (1971), 1971.

[52] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.

[53] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[54] Pierre Deligne and Alexander B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.

[55] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice ıt Corps de classes local par Michiel Hazewinkel.

[56] V. G. Drinfel′d. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 798–820.

Amer. Math. Soc., Providence, RI, 1987.

[57] V. G. Drinfel′d. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Algebra i Analiz*, 2(4):149–181, 1990.

[58] Gabriel C. Drummond-Cole. Homotopically trivializing the circle in the framed little disks. *J. Topol.*, 7(3):641–676, 2014.

[59] William Dwyer and Kathryn Hess. Long knots and maps between operads. *Geom. Topol.*, 16(2):919–955, 2012.

[60] Pavel I. Etingof, Igor B. Frenkel, and Alexander A. Kirillov, Jr. *Lectures on representation theory and Knizhnik-Zamolodchikov equations*, volume 58 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[61] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[62] Z. Fiedorowicz. The symmetric bar construction. Preprint, 1996.

[63] R. Fox and L. Neuwirth. The braid groups. *Math. Scand.*, 10:119–126, 1962.

[64] John Francis. The tangent complex and Hochschild cohomology of $\mathcal{E}_n$-rings. *Compos. Math.*, 149(3):430–480, 2013.

[65] Benoit Fresse. Koszul duality of operads and homology of partition posets. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 115–215. Amer. Math. Soc., Providence, RI, 2004.

[66] Benoit Fresse. *Modules over operads and functors*, volume 1967 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[67] Benoit Fresse, Victor Turchin, and Thomas Willwacher. Mapping spaces of the E_n operads. In preparation, 2015.

[68] Benoit Fresse and Thomas Willwacher. The intrinsic formality of E_n-operads. Preprint `arXiv:1503.08699`, 2015.

[69] William Fulton and Robert MacPherson. A compactification of configuration spaces. *Ann. of Math. (2)*, 139(1):183–225, 1994.

[70] Hidekazu Furusho. Pentagon and hexagon equations. *Ann. of Math. (2)*, 171(1):545–556, 2010.

[71] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*. Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York, 1971.

[72] Giovanni Gaiffi. Models for real subspace arrangements and stratified manifolds. *Int. Math. Res. Not.*, (12):627–656, 2003.

[73] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1994 edition.

[74] Murray Gerstenhaber. The cohomology structure of an associative ring. *Ann. of Math. (2)*, 78:267–288, 1963.

[75] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Comm. Math. Phys.*, 159(2):265–285, 1994.

[76] E. Getzler. Operads and moduli spaces of genus 0 Riemann surfaces. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 199–230. Birkhäuser Boston, Boston, MA, 1995.

[77] Ezra Getzler and John D.S. Jones. Operads, homotopy algebra and iterated integrals for double loop spaces. Preprint `arXiv:hep-th/9403055`, 1994.

[78] Victor Ginzburg and Mikhail Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.

[79] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612].

[80] A. B. Goncharov. Galois symmetries of fundamental groupoids and noncommutative geometry. *Duke Math. J.*, 128(2):209–284, 2005.

[81] Pierre-Paul Grivel. Une histoire du théorème de Poincaré-Birkhoff-Witt. *Expo. Math.*, 22(2):145–184, 2004.

[82] Alexander Grothendieck. *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5*, volume 1960/61 of *Séminaire de Géométrie Algébrique.* Institut des Hautes Études Scientifiques, Paris, 1963.

[83] Alexandre Grothendieck. Esquisse d'un programme. In *Geometric Galois actions, 1*, volume 242 of *London Math. Soc. Lecture Note Ser.*, pages 5–48. Cambridge Univ. Press, Cambridge, 1997. With an English translation on pp. 243–283.

[84] P. Hall. A Contribution to the Theory of Groups of Prime-Power Order. *Proc. London Math. Soc.*, S2-36(1):29, 1933.

[85] Stephen Halperin and James Stasheff. Obstructions to homotopy equivalences. *Adv. in Math.*, 32(3):233–279, 1979.

[86] Masaki Hanamura. Mixed motives and algebraic cycles. I. *Math. Res. Lett.*, 2(6):811–821, 1995.

[87] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[88] Allen Hatcher, Pierre Lochak, and Leila Schneps. On the Teichmüller tower of mapping class groups. *J. Reine Angew. Math.*, 521:1–24, 2000.

[89] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2003.

[90] Goeffroy Horel. Profinite completion of operads and the Grothendieck-Teichmüller group. Preprint `arXiv:1504.01605`, 2005.

[91] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1999.

[92] Annette Huber and Stefan Müller-Stach. Periods and Nori motives. Volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer International Publishing, Switzerland, 2017.

[93] Yasutaka Ihara. The Galois representation arising from $\mathbf{P}^1 - \{0, 1, \infty\}$ and Tate twists of even degree. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 299–313. Springer, New York, 1989.

[94] Yasutaka Ihara. Braids, Galois groups, and some arithmetic functions. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 99–120. Math. Soc. Japan, Tokyo, 1991.

[95] Yasutaka Ihara. On the embedding of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\widehat{\mathrm{GT}}$. In *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*, volume 200 of *London Math. Soc. Lecture Note Ser.*, pages 289–321. Cambridge Univ. Press, Cambridge, 1994. With an appendix: the action of the absolute Galois group on the moduli space of spheres with four marked points by Michel Emsalem

and Pierre Lochak.

[96] I. M. James. Reduced product spaces. *Ann. of Math. (2)*, 62:170–197, 1955.

[97] P. T. Johnstone. Adjoint lifting theorems for categories of algebras. *Bull. London Math. Soc.*, 7(3):294–297, 1975.

[98] André Joyal. Foncteurs analytiques et espèces de structures. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, volume 1234 of *Lecture Notes in Math.*, pages 126–159. Springer, Berlin, 1986.

[99] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[100] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[101] Christian Kassel and Vladimir Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics*. Springer, New York, 2008. With the graphical assistance of Olivier Dodane.

[102] Sean Keel. Intersection theory of moduli space of stable n-pointed curves of genus zero. *Trans. Amer. Math. Soc.*, 330(2):545–574, 1992.

[103] G. M. Kelly. On the operads of J. P. May. *Repr. Theory Appl. Categ.*, (13):1–13, 2005.

[104] V. G. Knizhnik and A. B. Zamolodchikov. Current algebra and Wess-Zumino model in two dimensions. *Nuclear Phys. B*, 247(1):83–103, 1984.

[105] Finn F. Knudsen. The projectivity of the moduli space of stable curves. II and III. *Math. Scand.*, 52(2):161–212, 1983.

[106] Toshitake Kohno. Série de Poincaré-Koszul associée aux groupes de tresses pures. *Invent. Math.*, 82(1):57–75, 1985.

[107] Toshitake Kohno. Bar complex, configuration spaces and finite type invariants for braids. *Topology Appl.*, 157(1):2–9, 2010.

[108] Maxim Kontsevich. Operads and motives in deformation quantization. *Lett. Math. Phys.*, 48(1):35–72, 1999. Moshé Flato (1937–1998).

[109] Maxim Kontsevich and Yan Soibelman. Deformations of algebras over operads and the Deligne conjecture. In *Conférence Moshé Flato 1999, Vol. I (Dijon)*, volume 21 of *Math. Phys. Stud.*, pages 255–307. Kluwer Acad. Publ., Dordrecht, 2000.

[110] Maxim Kontsevich and Don Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[111] C. Lair. Sur le profil d'esquissabilité des catégories modelables (accessibles) possédant les noyaux. *Diagrammes*, 38:19–78, 1997.

[112] Pascal Lambrechts, Victor Turchin, and Ismar Volić. The rational homology of spaces of long knots in codimension > 2. *Geom. Topol.*, 14(4):2151–2187, 2010.

[113] Michel Lazard. Sur les groupes nilpotents et les anneaux de Lie. *Ann. Sci. Ecole Norm. Sup. (3)*, 71:101–190, 1954.

[114] Michel Lazard. Lois de groupes et analyseurs. *Ann. Sci. Ecole Norm. Sup. (3)*, 72:299–400, 1955.

[115] Thang Tu Quoc Le and Jun Murakami. Kontsevich's integral for the Kauffman polynomial. *Nagoya Math. J.*, 142:39–65, 1996.

[116] Carl W. Lee. The associahedron and triangulations of the n-gon. *European J. Combin.*, 10(6):551–560, 1989.

[117] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.

[118] Marc Levine. Tate motives and the vanishing conjectures for algebraic K-theory. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 167–188. Kluwer Acad. Publ., Dordrecht, 1993.

[119] Marc Levine. *Mixed motives*, volume 57 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[120] Marc Levine. Mixed motives. In *Handbook of K-theory. Vol. 1, 2*, pages 429–521. Springer, Berlin, 2005.

[121] Muriel Livernet. Koszul duality of the category of trees and bar constructions for operads. In *Operads and universal algebra*, volume 9 of *Nankai Ser. Pure Appl. Math. Theoret. Phys.*, pages 107–138. World Sci. Publ., Hackensack, NJ, 2012.

[122] Pierre Lochak, Hiroaki Nakamura, and Leila Schneps. On a new version of the Grothendieck-Teichmüller group. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(1):11–16, 1997.

[123] Pierre Lochak and Leila Schneps. A cohomological interpretation of the Grothendieck-Teichmüller group. *Invent. Math.*, 127(3):571–600, 1997. With an appendix by C. Scheiderer.

[124] Jean-Louis Loday. Série de Hausdorff, idempotents eulériens et algèbres de Hopf. *Exposition. Math.*, 12(2):165–178, 1994.

[125] Jean-Louis Loday. Realization of the Stasheff polytope. *Arch. Math. (Basel)*, 83(3):267–278, 2004.

[126] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.

[127] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[128] Jacob Lurie. Higher algebra. Book project, 2014.

[129] Saunders Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[130] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[131] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory*. Dover Publications, Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.

[132] A. I. Mal′cev. Nilpotent torsion-free groups. *Izvestiya Akad. Nauk. SSSR. Ser. Mat.*, 13:201–212, 1949.

[133] Yuri I. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces*, volume 47 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1999.

[134] Martin Markl. Distributive laws and Koszulness. *Ann. Inst. Fourier (Grenoble)*, 46(2):307–323, 1996.

[135] Martin Markl. Models for operads. *Comm. Algebra*, 24(4):1471–1500, 1996.

[136] Martin Markl. Operads and PROPs. In *Handbook of algebra. Vol. 5*, volume 5 of *Handb. Algebr.*, pages 87–140. Elsevier/North-Holland, Amsterdam, 2008.

[137] Martin Markl. *Deformation theory of algebras and their diagrams*, volume 116 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2012.

[138] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[139] William S. Massey. *A basic course in algebraic topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[140] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271.

[141] J. Peter May. *Simplicial objects in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. Reprint of the 1967 original.

[142] James E. McClure and Jeffrey H. Smith. A solution of Deligne's Hochschild cohomology conjecture. In *Recent progress in homotopy theory (Baltimore, MD, 2000)*, volume 293 of *Contemp. Math.*, pages 153–193. Amer. Math. Soc., Providence, RI, 2002.

[143] James E. McClure and Jeffrey H. Smith. Multivariable cochain operations and little n-cubes. *J. Amer. Math. Soc.*, 16(3):681–704, 2003.

[144] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[145] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[146] Shigeyuki Morita. *Geometry of differential forms*, volume 201 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the two-volume Japanese original (1997, 1998) by Teruko Nagase and Katsumi Nomizu, Iwanami Series in Modern Mathematics.

[147] Hiroaki Nakamura and Leila Schneps. On a subgroup of the Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups. *Invent. Math.*, 141(3):503–560, 2000.

[148] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publ. Math. Inst. Hautes Études Sci.*, 117:271–328, 2013.

[149] Frédéric Patras. *Homothéties simpliciales*. Thèse de doctorat, Université Paris 7, 1992.

[150] Frédéric Patras. La décomposition en poids des algèbres de Hopf. *Ann. Inst. Fourier (Grenoble)*, 43(4):1067–1087, 1993.

[151] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.

[152] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[153] Daniel G. Quillen. On the associated graded ring of a group ring. *J. Algebra*, 10:411–418, 1968.

[154] Christophe Reutenauer. Theorem of Poincaré-Birkhoff-Witt, logarithm and symmetric group representations of degrees equal to Stirling numbers. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, volume 1234 of *Lecture Notes in Math.*, pages 267–284. Springer, Berlin, 1986.

[155] Christophe Reutenauer. *Free Lie algebras*, volume 7 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[156] Charles Rezk. *Spaces of algebra structures and cohomology of operads*. PhD thesis, Massachusetts Institute of Technology, 1996.

[157] Paolo Salvatore. Configuration spaces with summable labels. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 375–395. Birkhäuser, Basel, 2001.

[158] Paolo Salvatore and Nathalie Wahl. Framed discs operads and Batalin-Vilkovisky algebras. *Q. J. Math.*, 54(2):213–231, 2003.

[159] Pavol Ševera and Thomas Willwacher. Equivalence of formalities of the little discs operad. *Duke Math. J.*, 160(1):175–206, 2011.

[160] Steven Shnider and Shlomo Sternberg. *Quantum groups*. Graduate Texts in Mathematical Physics, II. International Press, Cambridge, MA, 1993. From coalgebras to Drinfel′d algebras, A guided tour.

[161] Dev P. Sinha. Manifold-theoretic compactifications of configuration spaces. *Selecta Math. (N.S.)*, 10(3):391–428, 2004.

[162] Dev P. Sinha. Operads and knot spaces. *J. Amer. Math. Soc.*, 19(2):461–486, 2006.

[163] Dev P. Sinha. The (non-equivariant) homology of the little disks operad. In *OPERADS 2009*, volume 26 of *Sémin. Congr.*, pages 253–279. Soc. Math. France, Paris, 2013.

[164] V. A. Smirnov. On the chain complex of an iterated loop space. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(5):1108–1119, 1135–1136, 1989.

[165] Jeffrey Henderson Smith. Simplicial group models for $\Omega^n S^n X$. *Israel J. Math.*, 66(1-3):330–350, 1989.

[166] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York-Berlin, 1981. Corrected reprint.

[167] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[168] N. E. Steenrod. *Cohomology operations*. Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50. Princeton University Press, Princeton, N.J., 1962.

[169] Richard Steiner. A canonical operad pair. *Math. Proc. Cambridge Philos. Soc.*, 86(3):443–449, 1979.

[170] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, 47:269–331 (1978), 1977.

[171] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[172] Tamás Szamuely. *Galois groups and fundamental groups*, volume 117 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2009.

[173] Dmitry E. Tamarkin. Another proof of M. Kontsevich formality theorem. Preprint `arXiv:math.QA/9803025`, 1998.

[174] Dmitry E. Tamarkin. Action of the Grothendieck-Teichmüller group on the operad of Gerstenhaber algebras. Preprint `arXiv:math/0202039`, 2002.

[175] Dmitry E. Tamarkin. Formality of chain operad of little discs. *Lett. Math. Phys.*, 66(1-2):65–72, 2003.

[176] Tomohide Terasoma. Mixed Tate motives and multiple zeta values. *Invent. Math.*, 149(2):339–369, 2002.

[177] Tomohide Terasoma. Geometry of multiple zeta values. In *International Congress of Mathematicians. Vol. II*, pages 627–635. Eur. Math. Soc., Zürich, 2006.

[178] Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[179] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[180] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.

[181] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[182] George W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978.

[183] David Wigner. An identity in the free Lie algebra. *Proc. Amer. Math. Soc.*, 106(3):639–640, 1989.

[184] Thomas Willwacher. M. Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra. *Invent. Math.*, 200(3):671–760, 2015.

[185] Miguel A. Xicoténcatl. The Lie algebra of the pure braid group. *Bol. Soc. Mat. Mexicana (3)*, 6(1):55–62, 2000.

[186] Donald Yau. *Colored operads*, volume 170 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016.

[187] Donald Yau and Mark W. Johnson. *A foundation for PROPs, algebras, and modules*, volume 203 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[188] Wenbin Zhang. Group operads and homotopy theory. Preprint `arXiv:1111.7090`, 2011.

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