Homotopy of Operads and Grothendieck–Teichmüller Groups
Part 1: The Algebraic Theory and its Topological Background

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American Mathematical Society
Homotopy of Operads and Grothendieck–Teichmüller Groups

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Un souffle ouvre des brèches opéradiques

dans les cloisons, – brouille le pivotement
des toits rongés, – disperse les limites des
foyers, – éclipse les croisées.

Arthur Rimbaud
Nocturne vulgaire. Les Illuminations (1875)
Contents


References to chapters, sections, paragraphs, and statements of the book are given by §x.y.z when these cross references are done within a part (I, II, and III) and by §P.x.y.z where P = I, II, III otherwise. The cross references to the sections, paragraphs, and statements of the appendices are given by §P.x.y throughout the book, where §P = §A, §B, §C. The preliminary part of the first volume of this book also includes a Foundations and Conventions section, whose paragraphs, numbered §§0.1-0.16, give a summary of the main conventions used in this work.

Preliminaries

Preface

Mathematical Objectives

Foundations and Conventions

Reading Guide and Overview of this Volume

Part I. From Operads to Grothendieck–Teichmüller Groups

Part I(a). The General Theory of Operads

Chapter 1. The Basic Concepts of the Theory of Operads
  1.1. The notion of an operad and of an algebra over an operad
  1.2. Categorical constructions for operads
  1.3. Categorical constructions for algebras over operads
  1.4. Appendix: Filtered colimits and reflexive coequalizers
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>ix</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.2. The Malcev completion of operads in groupoids</td>
<td>328</td>
</tr>
<tr>
<td>9.3. Appendix: The local connectedness of complete Hopf groupoids</td>
<td>334</td>
</tr>
</tbody>
</table>

**Part I(d). The Operadic Definition of the Grothendieck–Teichmüller Group**

Chapter 10. The Malcev Completion of the Braid Operads and Drinfeld’s Associators
- 10.0. The Malcev completion of the pure braid groups and the Drinfeld–Kohno Lie algebras | 341 |
- 10.1. The Malcev completion of the braid operads and the Drinfeld–Kohno Lie algebra operad | 349 |
- 10.2. The operad of chord diagrams and Drinfeld’s associators | 355 |
- 10.3. The graded Grothendieck–Teichmüller group | 368 |
- 10.4. Tower decompositions, the graded Grothendieck–Teichmüller Lie algebra and the existence of rational Drinfeld’s associators | 385 |

Chapter 11. The Grothendieck–Teichmüller Group
- 11.1. The operadic definition of the Grothendieck–Teichmüller group | 400 |
- 11.2. The action on the set of Drinfeld’s associators | 408 |
- 11.3. Tower decompositions | 411 |
- 11.4. The graded Lie algebra of the Grothendieck–Teichmüller group | 414 |

Chapter 12. A Glimpse at the Grothendieck Program | 421 |

**Appendices**

Appendix A. Trees and the Construction of Free Operads
- A.1. Trees | 430 |
- A.2. Treewise tensor products and treewise composites | 442 |
- A.3. The construction of free operads | 453 |
- A.4. The construction of connected free operads | 462 |
- A.5. The construction of coproducts with free operads | 467 |

Appendix B. The Cotriple Resolution of Operads
- B.0. Tree morphisms | 478 |
- B.1. The definition of the cotriple resolution of operads | 485 |
- B.2. The monadic definition of operads | 498 |

Glossary of Notation | 503 |

Bibliography | 511 |

Index | 521 |
Preliminaries
Preface

The first purpose of this work is to give an overall reference, starting from scratch, on applications of methods of algebraic topology to the study of operads in topological spaces. Most definitions, notably fundamental concepts of the theory of operads and of homotopy theory, are reviewed in this book in order to make our account accessible to graduate students and to researchers coming from the various fields of mathematics related to our subject. Then our ultimate objective is to give a homotopical interpretation of a deep relationship between operads and Grothendieck–Teichmüller groups. This connection, which has emerged from research on the deformation quantization process in mathematical physics, gives a new approach to understanding internal symmetries of structures that occur in various constructions of algebra and topology.

We review the definition of an operad at the beginning of this monograph. Simply recall for the moment that an operad is a structure, formed by collections of abstract operations, which is used to define a category of algebras. In our study, we mainly consider the example of $E_n$-operads, $n = 1, 2, \ldots, \infty$, which are used to model a hierarchy of homotopy commutative structures, from fully homotopy associative but not commutative ($n = 1$), up to fully homotopy associative and commutative ($n = \infty$). Let us mention that the notion of an $E_1$-operad is synonymous to that of an $A_\infty$-operad, which is used in the literature when one deals only with purely homotopy associative structures.

The notion of an $E_n$-operad formally refers to a class of operads rather than to a singled-out object. This class consists, in the initial definition, of topological operads which are homotopically equivalent to a reference model, the Boardman–Vogt operad of little $n$-discs $D_n$. The operad of little $n$-cubes, which is a simple variant of the little $n$-discs operad, is also used in the literature to provide an equivalent definition of the class of $E_n$-operads. We provide a detailed account of the definition of these notions in this book. Nevertheless, as we soon explain, our ultimate purpose is not to study $E_n$-operads themselves, but homotopy automorphism groups attached to these structures.

Before explaining this goal, we survey some motivating applications of $E_n$-operads which are not our main subject matter (we only give short introductions to these topics) but illustrate our approach of the subject.

The operads of little $n$-discs $D_n$ were initially introduced to collect operations acting on iterated loop spaces. The first main application of these operads, which has motivated their definition, is the Boardman–Vogt and May recognition theorems of iterated loop spaces: any space $Y$ equipped with an action of the operad $D_n$ has the homotopy type of an $n$-fold loop space $\Omega^n X$ up to group-completion (see [27, 28] and [140]). Recall that the set of connected components of an $n$-fold loop space $\Omega^n X$ is identified with the $n$th homotopy group $\pi_n(X)$ of the space $X$. (Recall also
that this group is abelian as soon as \( n > 1 \).) The action of \( D_n \) on \( \Omega^n X \) includes a product operation \( \mu : \Omega^n X \times \Omega^n X \to \Omega^n X \) which, at the level of connected components, gives the composition operation of the group \( \pi_n(X) \) for any \( n > 0 \).

The operad \( D_n \) carries the homotopies that make this product associative (and commutative for \( n > 1 \)) and includes further operations, representing homotopy constraints, which we need to form a faithful picture of the structure of the \( n \)-fold loop space \( \Omega^n X \).

This outline gives the initial topological interpretation of \( E_n \)-operads. But this topological picture has also served as a guiding idea for a study of \( E_n \)-operads in other domains. Indeed, new applications of \( E_n \)-operads, which have initiated a complete renewal of the subject, have been discovered in the fields of algebra and mathematical physics, mostly after the proof of the Deligne conjecture asserting that the Hochschild cochain complex \( C^*(A, A) \) of an associative algebra \( A \) inherits an action of an \( E_2 \)-operad. In this context, we deal with a chain version of the previously considered topological little 2-discs operad \( D_2 \).

The cohomology of the Hochschild cochain complex \( C^*(A, A) \) is identified in degree 0 with the center \( Z(A) \) of the associative algebra \( A \). In a sense, the Hochschild cochain complex represents a derived version of this ordinary center \( Z(A) \). From this point of view, the construction of an \( E_2 \)-structure on \( C^*(A, A) \) determines, as in the study of iterated loop spaces, the level of homotopical commutativity of the derived center which lies beyond the apparent commutativity of the ordinary center. The first proofs of the Deligne conjecture have been given by Kontsevich-Soibelman \( 109 \) and McClure-Smith \( 142 \). The interpretation in terms of derived centers has been emphasized by Kontsevich \( 108 \) in order to formulate a natural extension of the conjecture for algebras over \( E_n \)-operads, where we now consider any \( n \geq 1 \) (we also refer to John Francis’s work \( 64 \) for a solution of this problem in the framework of \( \infty \)-category theory).

The verification of the Deligne conjecture has yielded a second generation of proofs, promoted by Tamarkin \( 173 \) and Kontsevich \( 108 \), of the Kontsevich formality theorem on Hochschild cochains. Recall that this result implies the existence of deformation quantizations of arbitrary Poisson manifolds (we also refer to \( 38, 148 \) for higher dimensional generalizations of the deformation quantization problem involving the categories of algebras associated to \( E_n \)-operads for all \( n \geq 1 \)). The new approaches of the Kontsevich formality theorem rely on the application of Drinfeld’s associators to transport the \( E_2 \)-structure yielded by the Deligne conjecture on the Hochschild cochain complex to the cohomology. In the final outcome, one obtains that each associator gives rise to a deformation quantization functor. This result has hinted the existence of a deep connection between the deformation quantization problem and the program, initiated in Grothendieck’s famous “esquisse” \( 83 \), which aims to understand Galois groups through geometric actions on curves. The Grothendieck–Teichmüller groups are devices, introduced in this program, which encode the information that can be captured through the actions considered by Grothendieck. The correspondence between associators and deformation quantizations imply that a rational pro-unipotent version of the Grothendieck–Teichmüller group \( GT(\mathbb{Q}) \) acts on the moduli space of deformation quantizations. The initial motivation of our work was the desire to understand this connection from a homotopical viewpoint, in terms of homotopical structures associated to \( E_2 \)-operads. The homotopy automorphisms of operads come into play at this point.
Recently, it has also been discovered that mapping spaces of $E_n$-operads can be used to compute the homotopy of the spaces of compactly supported embeddings of Euclidean spaces modulo immersions $\text{Emb}_c(R^m, R^n)$ (see notably the works of Dev Sinha [162], Lambrechts-Turchin-Volić [112], Arone-Turchin [9], Dwyer-Hess [59], and Boavida-Weiss [29] for various forms of this statement). We give more details on these developments in the concluding chapter of this book. In a related field, it has been observed that algebras over $E_n$-operads can be used to define multiplicative analogues of the classical singular homology theory for manifolds (different but equivalent constructions of such multiplicative homology theories are the “topological chiral homology”, studied by Jacob Lurie in [127, 128], and the “factorization homology”, studied by John Francis in [64] and by Costello-Gwilliam in [50]). These new developments give further motivations for the study of mapping spaces and homotopy automorphism spaces of $E_n$-operads.

Recall again that an operad is a structure which governs a category of algebras. The homotopy automorphisms of an operad $P$ are transformations, defined at the operad level, which encode natural homotopy equivalences on the category of algebras associated to $P$. In this interpretation, the group of homotopy automorphism classes of an $E_2$-operad, which we actually aim to determine, represents the internal symmetries of the first level of homotopy commutative structures which $E_2$-operads serve to encode. To obtain our result, we mainly work in the setting of rational homotopy theory and we consider a rational version of the notion of an $E_2$-operad in topological spaces. We precisely establish that the group of rational homotopy automorphism classes of $E_2$-operads is isomorphic to the pro-unipotent Grothendieck–Teichmüller group $\text{GT}(\mathbb{Q})$. This result is new and represents the main outcome of our work.

In the conclusion of this monograph, we will give an overview of a sequel of this research [67], where the author, Victor Turchin, and Thomas Willwacher tackle the computation of the homotopy of the spaces of rational homotopy automorphisms of $E_n$-operads in dimension $n \geq 2$ (see §III.6). The main outcome of this study is that these spaces of rational homotopy automorphisms of general $E_n$-operads can be described in terms of the homology of a complex of graphs which, according to an earlier work of Willwacher [184], reduces to the Grothendieck–Teichmüller group in the case $n = 2$. (We have a similar description of the mapping spaces of $E_n$-operads which occur in the study of embedding spaces.)

To reach all of these results, we have to set up a new rational homotopy theory for topological operads beforehand and to give a sense to the rationalization of operads in topological spaces. We actually define an analogue of the Sullivan model of the rational homotopy of spaces for operads. We then consider cooperads, the structures which are dual to operads in the categorical sense. We precisely show that the rational homotopy of an operad in topological spaces is determined by an associated cooperad in commutative dg-algebras (a Hopf dg-cooperad). We have a small model of the operad of little 2-discs which is given by the Chevalley–Eilenberg cochain complex of the Drinfeld–Kohno Lie algebras (the Lie algebras of infinitesimal braids). We use this model in our proof that the group of rational homotopy automorphism classes of $E_2$-operads reduces to the pro-unipotent Grothendieck–Teichmüller group. In the course of our study, we also define a cosimplicial analogue of the Sullivan model of operads. This cosimplicial model remains well defined in
the positive characteristic setting and gives, in this context, a model for the homotopy of the completion of topological operads at a prime.

The other main topics considered in our study include the application of homotopy spectral sequences and of Koszul duality techniques for the computation of mapping spaces attached to operads. We aim to give a detailed and comprehensive introduction to the applications of these methods for the study of operads from the point of view of homotopy theory. Besides, we thoroughly review the applications of Hopf algebras to the Malcev completion (the rationalization) of general groups. For the applications to operads, we actually consider an extension of the classical Malcev completion of groups to groupoids. Indeed, we will explain that the pro-unipotent Grothendieck–Teichmüller group can be defined as the group of automorphisms of the Malcev completion of a certain operad in groupoids which governs operations acting on braided monoidal categories. We use this observation and classical constructions of homotopy theory to define our correspondence between the Grothendieck–Teichmüller group and the space of homotopy automorphisms of $E_2$-operads. The previously mentioned homotopy spectral sequence techniques are used to check that this correspondence induces a bijection in homotopy. This first volume of this monograph is mainly devoted to the fundamental and algebraic aspects of our subject, from the definition of the notion of an operad to the definition of the Grothendieck–Teichmüller group. The applications of homotopy theory to operads and the proof of our isomorphism statement between the Grothendieck–Teichmüller group and the group of homotopy automorphisms classes of $E_2$-operads are explained in the second volume.

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Mathematical Objectives

The ultimate goal of this work, as we explain in the general introduction, is to prove that the Grothendieck–Teichmüller group represents, in the rational setting at least, the group of homotopy automorphism classes of $E_2$-operads. This objective can be taken as a motivation to read this book or as a guiding example of an application of our methods.

The definition of an operad is recalled with full details in the first part of this volume. In this introductory section, we only aim to give an idea of our main results. Let us simply recall that an operad $P$ basically consists of a collection $P(r)$, $r \in \mathbb{N}$, where each object $P(r)$ parameterizes operations with $r$ inputs $p = p(x_1, \ldots, x_r)$, together with a multiplicative structure which models the composition of such operations. We can define operads in any category equipped with a symmetric monoidal structure $\mathcal{M}$. We then assume $P(r) \in \mathcal{M}$, and we use the tensor product operation, given with this category $\mathcal{M}$, to define the composition structure attached to our operad. The operads in a base symmetric monoidal category form a category, which we denote by $\mathcal{MO}_p$, or more simply, by $\mathcal{O}_P = \mathcal{MO}_p$, when this ambient category $\mathcal{M}$ is fixed by the context. An operad morphism $f : P \to Q$ naturally consists of a collection of morphisms in the base category $f : P(r) \to Q(r)$, $r \in \mathbb{N}$, which preserve the composition structure of our operads.

For technical reasons, we have to consider operads $P_+$ equipped with a distinguished element $* \in P_+(0)$ (whenever the notion of an element makes sense), which represents an operation with zero input (a unitary operation in our terminology). In the context of sets, we moreover assume that $P_+(0)$ is a one-point set reduced to this element. In the module context, we assume that $P_+(0)$ is a one dimensional module over the ground ring. In a general setting, we assume that $P_+(0)$ is the unit object given with the symmetric monoidal structure of our base category. We then say that $P_+$ forms a unitary operad. We use the notation $\mathcal{O}_P_+$ to refer to the category of unitary operads. The subscript $*$ indicates the fixed arity zero component which we assign to the objects of this category of operads. We usually consider together both a non-unitary operad $P$, which has no term in arity 0, and an associated unitary operad $P_+$, where the arity zero term, spanned by the distinguished operation $* \in P_+(0)$, is added. We therefore follow the convention to use a subscript $+$, marking the addition of this term, for the notation of the unitary operad $P_+$. We also say that the operad $P_+$ arises as a unitary extension of the non-unitary operad $P$. We often perform constructions on the non-unitary operad $P$ first, and on the unitary operad $P_+$ afterwards, by assuming that the additional distinguished element (or unit term) of $P_+$ is preserved by the operations involved in our construction.

In topology, an $E_2$-operad is usually defined as an operad in the category of spaces which is equivalent to Boardman–Vogt’s operad of little 2-discs $D_2$ in the
homotopy category of operads. The spaces $\mathcal{D}_2(r)$ underlying this operad have a trivial homotopy in dimension $*=1$, and for $*=1$, we have $\pi_1 \mathcal{D}_2(r) = P_r$, where $P_r$ denotes the pure braid group on $r$ strands. Thus, the space $\mathcal{D}_2(r)$ is an Eilenberg-MacLane space $K(P_r, 1)$ associated to the pure braid group $P_r$. For our purpose, we consider a rationalization of the little 2-discs operad $\mathcal{D}_2$, for which we have $\pi_1 \mathcal{D}_2(2r) = \hat{P}_r$, where $\hat{P}_r$ denotes the Malcev completion of the group $P_r$. We give a general definition of the rationalization for operads in topological spaces in the second volume of this book. In the case of the little 2-discs operad $\mathcal{D}_2$, we have a simple model of the rationalization $\mathcal{D}_2$ which relies on the Eilenberg-MacLane space interpretation of the little 2-discs spaces. We give a brief outline of this approach soon.

Homotopy automorphisms can be defined in the general setting of model categories which provides a suitable axiomatic framework for the application of homotopy theory concepts to operads. In order to introduce our subject, we first explain a basic interpretation of the general definition of a homotopy automorphism in the context of topological operads.

We have a natural notion of homotopy $\sim$ for morphisms of operads in topological spaces. To a topological operad $Q$, we associate the collection of path spaces $Q^{\Delta^1}(r) = \text{Map}_{\mathcal{T}op}([0, 1], Q(r))$, which inherits an operad structure from $Q$ and defines a path-object associated to $Q$ in the category of topological operads. We explicitly define a homotopy between operad morphisms $f, g : \mathcal{P} \to Q$ as an operad morphism $h : \mathcal{P} \to Q^{\Delta^1}$ which satisfies $d_0 h = f$, $d_1 h = g$, where $d_0, d_1 : Q^{\Delta^1} \to Q$ are the natural structure morphisms (evaluation at the origin and at the end point) associated with our path-object $Q^{\Delta^1}$. This homotopy $h$ is intuitively equivalent to a continuous family of operad morphisms $h_t : \mathcal{P} \to Q$ going from $h_0 = f$ to $h_1 = g$.

In a first approximation, we take the sets of homotopy classes of operad morphisms as the morphism sets of the homotopy category $\mathcal{Ho}(\mathcal{T}op \mathcal{Op})$ which we associate to the category of topological operads $\mathcal{T}op \mathcal{Op}$. In principle, we have to deal with a suitable notion of cofibrant object in the category of operads and to replace any operad by a cofibrant resolution in order to use this definition of morphism set. But we will explain this issue later on. We focus on the basic definition of the morphism sets of the homotopy category for the moment.

The groups of homotopy automorphism classes, which we aim to determine, are the groups of automorphisms of the homotopy category $\mathcal{Ho}(\mathcal{T}op \mathcal{Op})$. The automorphism group $\text{Aut}_{\mathcal{Ho}(\mathcal{T}op \mathcal{Op})}(\mathcal{P})$ associated to a given operad $\mathcal{P} \in \mathcal{T}op \mathcal{Op}$ accordingly consists of homotopy classes of morphisms $f : \mathcal{P} \to \mathcal{P}$, which have a homotopy inverse $g : \mathcal{P} \to \mathcal{P}$ such that $fg \sim \text{id}$ and $gf \sim \text{id}$. We consider the operadic homotopy relation at each step of this definition.

Note that a topological operad $\mathcal{P}$ gives rise to an operad object in the homotopy category of topological spaces $\mathcal{Ho}(\mathcal{T}op)$, and we could also study the automorphism group $\text{Aut}_{\mathcal{Ho}(\mathcal{T}op \mathcal{Op})}(\mathcal{P})$ formed in this naive category of homotopy operads $\mathcal{Ho}(\mathcal{T}op) \mathcal{Op}$. But these automorphism groups differ from our groups of homotopy automorphisms and do not give the appropriate structure for the homotopy version of usual constructions of group theory (like homotopy fixed points). Indeed, an automorphism of the operad $\mathcal{P}$ in the homotopy category of spaces $\mathcal{Ho}(\mathcal{T}op)$ is just a collection of homotopy classes of maps $f \in [\mathcal{P}(r), \mathcal{P}(r)]$, invertible in the homotopy category of spaces, and which preserve the operadic
structures up to homotopy, unlike our homotopy automorphisms that preserve operadic structures strictly. Moreover, actual operad morphisms \( f, g : P \to Q \) define the same morphism of operads in the homotopy category of spaces \( Ho(\mathcal{O}) \) as soon as we have a homotopy between the individual maps \( f : P(r) \to P(r) \) and \( g : P(r) \to P(r) \), for all \( r \in \mathbb{N} \) (regardless of operad structures). Thus, operad morphisms which are homotopic in the strong operadic sense determine the same morphism of operads in the homotopy category of spaces \( Ho(\mathcal{O}) \), but the converse implication does not hold. By associating the collection of homotopy classes of maps \( f : P(r) \to P(r) \) to a homotopy automorphism \( f \in Aut_{Ho(\mathcal{O})}(P) \), we obtain a mapping \( Aut_{Ho(\mathcal{O})}(P) \to Aut_{Ho(\mathcal{O})}(P) \), from the group of homotopy classes of homotopy automorphisms towards the group of automorphisms of the operad in the homotopy category of spaces, but this mapping is neither an injection nor a surjection in general.

To apply methods of algebraic topology, we associate to any operad \( P \) a whole simplicial set of homotopy automorphisms \( Aut^h_{\mathcal{O}}(P) \) rather than a single group of homotopy automorphism classes. To be more precise, this group \( Aut_{Ho(\mathcal{O})}(P) \), which we primarily aim to determine, is identified with the set of connected components of our homotopy automorphism space \( \pi_0(Aut^h_{\mathcal{O}}(P)) \). In the second volume of this work, we explain the definition of these homotopy automorphism spaces in the general context of simplicial model categories. For the moment, we simply give a short outline of the definition in the context of topological operads.

First, we extend the definition of our path object and we consider, for each \( n \in \mathbb{N} \), an operad \( P^{\Delta^n} \) which is defined by the collection of function spaces \( P^{\Delta^n}(r) = Map_{\mathcal{O}}(\Delta^n, P(r)) \) on the \( n \)-simplex \( \Delta^n \). This operad sequence \( P^{\Delta^n} \) inherits a simplicial structure from the topological simplices \( \Delta^n \). In particular, since we obviously have \( P = P^{\Delta^0} \), we have a morphism \( v^* : P^{\Delta^n} \to P \) associated to each vertex \( v \) of the \( n \)-simplex \( \Delta^n \). The simplicial set \( Aut^h_{\mathcal{O}}(P) \) precisely consists, in dimension \( n \in \mathbb{N} \), of the morphisms of topological operads \( f : P \to P^{\Delta^n} \) such that the composite \( v^*f \) defines a homotopy equivalence of the operad \( P = P^{\Delta^0} \), for each vertex \( v \in \Delta^n \). From this definition, we immediately see that the 0-simplices of the simplicial set \( Aut^h_{\mathcal{O}}(P) \) are the homotopy equivalences of the operad \( P = P^{\Delta^0} \), the 1-simplices are the operadic homotopies \( h : P \to P^{\Delta^1} \) between homotopy equivalences, and therefore, we have a formal identity \( Aut_{Ho(\mathcal{O})}(P) = \pi_0(Aut^h_{\mathcal{O}}(P)) \) between our group of homotopy automorphism classes \( Aut_{Ho(\mathcal{O})}(P) \) and the set of connected components of \( Aut^h_{\mathcal{O}}(P) \).

In what follows, we often consider simplicial sets as combinatorial models of topological spaces. In this situation, we adopt a common usage of homotopy theory to use the name ‘space’ for a simplicial set. Therefore we refer to the simplicial set \( Aut^h_{\mathcal{O}}(P) \) as the homotopy automorphism space associated to the operad \( P \).

Besides homotopy equivalences, we consider a class of morphisms, called weak-equivalences, which are included in the definition of a model structure on the category of operads. We adopt the standard notation of the theory of model categories \( \xrightarrow{\sim} \) to refer to this class of distinguished morphisms. The notion of a model category also includes the definition of a class of cofibrant objects, generalizing the cell complexes of topology, and which are well suited for the homotopy constructions we aim to address.
To be more specific, recall that a map of topological spaces \( f : X \to Y \) is a weak-equivalence when this map induces a bijection on connected components \( f_\ast : \pi_0(X) \cong \pi_0(Y) \) and an isomorphism on homotopy groups \( f_\ast : \pi_\ast(X, x) \cong \pi_\ast(Y, f(x)) \), for all \( \ast > 0 \) and for any choice of base point \( x \in X \). We define a weak-equivalence of operads as an operad morphism \( f : P \to Q \) of which underlying maps \( f : P(r) \to Q(r) \) are weak-equivalences of topological spaces. In the context of topological spaces, a classical result asserts that any weak-equivalence between cell complexes is homotopically invertible as a map of topological spaces. In the context of topological operads, we similarly obtain that any weak-equivalence between cofibrant operads \( f : P \cong \to Q \) is homotopically invertible as an operad morphism: we have an operad morphism \( g : Q \to P \) in the converse direction as our weak-equivalence \( f : P \cong \to Q \) such that \( fg \sim id \) and \( gf \sim id \), where we now consider the operadic homotopy relation (as in the definition of a homotopy automorphism for operads).

The proof of the model category axioms for operads includes the construction of a cofibrant resolution functor, which assigns a cofibrant operad \( R \) equipped with a weak-equivalence \( R \cong \to P \) to any given operad \( P \). The definition of the homotopy category of topological operads in terms of homotopy class sets of morphisms is actually the right one when we replace each operad \( P \) by such a cofibrant model \( R \cong \to P \). In particular, when we form the group of homotopy automorphism classes of an operad \( \text{Aut}_{\text{Ho}(\text{Top}_{\text{op}})}(P) \), we have to assume that \( P \) is cofibrant as an operad, otherwise we tacitly assume that we apply our construction to a cofibrant resolution of \( P \). The general theory of model categories ensures that the obtained group \( \text{Aut}_{\text{Ho}(\text{Top}_{\text{op}})}(P) \) does not depend, up to isomorphism, on the choice of this cofibrant resolution. We have similar results and we apply similar conventions for the homotopy automorphism spaces \( \text{Aut}_{\text{Top}_{\text{op}}}(P) \). To be precise, in the general context of the theory of model categories, we have a notion of fibrant object, which is dual to the notion of a cofibrant object, and we actually have to consider objects which are both cofibrant and fibrant when we use the above definition of the group of homotopy automorphism classes of an object. We have a similar observation for the definition of homotopy automorphism spaces. But we can neglect this issue for the moment, because all objects of our model category of topological operads are fibrant.

We go back to the case of the little 2-discs operad. We aim to determine the homotopy groups of the homotopy automorphism space \( \text{Aut}_{\text{op}_{\text{op}}}(D_{2+}) \) associated to the rationalization of \( D_2 \) and in the unitary operad context, which we mark by the addition of the subscript + in our notation. Recall that the connected components of this space \( \text{Aut}_{\text{op}_{\text{op}}}(D_{2+}) \) correspond to homotopy classes of operad homotopy equivalences \( f : \hat{R}_{2+} \cong \to \hat{R}_{2+} \), where \( \hat{R}_2 \) denotes a cofibrant model of the rationalized little 2-discs operad \( D_2^\wedge \). Our result reads:

**Theorem A.** The homotopy automorphism space of the rationalization of the little 2-discs operad \( D_{2+}^\wedge \) satisfies 

\[
\pi_\ast \text{Aut}_{\text{op}_{\text{op}}}(D_{2+}^\wedge) = \begin{cases} 
\text{GT}(\mathbb{Q}), & \text{for } \ast = 0, \\
\mathbb{Q}, & \text{for } \ast = 1, \\
0, & \text{otherwise,}
\end{cases}
\]
where $GT(Q)$ denotes the rational pro-unipotent version of the Grothendieck–Teichmüller group, such as defined by V. Drinfeld in \[57\].

The identity established in this theorem is a new result. The ultimate goal of this work precisely consists in proving this statement. In fact, we will more precisely associate a well-defined rational homotopy automorphism of the $E_2$-operad $R_{2+}$ to our element of the Grothendieck–Teichmüller group $\phi \in GT(Q)$. Our main theorem asserts that, when we work in the rational setting, this construction gives exactly all homotopy automorphism classes of $E_2$-operads.

We consider a pro-unipotent version of the Grothendieck–Teichmüller group in our theorem. We mostly study this group for the applications in the rational homotopy theory of operads, but we also have a pro-finite version of the
Grothendieck–Teichmüller group which is better suited for the purposes of the original Grothendieck’s program in Galois theory. In fact, a pro-finite analogue of our result, which relates this pro-finite Grothendieck–Teichmüller group to a pro-finite completion of $E_2$-operads, has been obtained by Horel during the writing of this monograph. We will give more explanations on this statement and on other generalizations of the result of Theorem in the concluding chapter of the second volume of this monograph (§ III.6).

The (rational) homology of the little 2-discs spaces $H_*(D_2(r)) = H_*(D_2(r), \mathbb{Q})$, $r \in \mathbb{N}$, forms an operad in graded modules $H_*(D_2)$. We will see that this homology operad $H_*(D_2)$ is identified with an operad, defined in terms of generating operations and relations, and which we can associate to the category of Gerstenhaber algebras (a graded version of the notion of a Poisson algebra). We therefore use the notation $\text{Gerst}_2$ for this operad such that $\text{Gerst}_2 = H_*(D_2)$. In what follows, we consider the (rational) cohomology of the little 2-discs spaces $H^*(D_2) = H^*(D_2, \mathbb{Q})$ rather than the homology $H_*(D_2)$. We use that the cohomology of a space inherits a unitary commutative algebra structure and that the collection of the cohomology algebras $H^*(D_2)$ associated to the little 2-discs spaces $D_2$ form a cooperad in the category of unitary commutative algebras in graded modules, where the name cooperad obviously refers to the structure dual to an operad in the categorical sense (we go back to this concept later on). We also use the phrase ‘(graded) Hopf cooperad’ to refer to this particular case of the structure defined by a cooperad in the category of unitary commutative algebras (in graded modules).

For our purpose, we actually need a counterpart, in the category of graded Hopf cooperads, of the category of unitary operads. We have to adapt the definition of a cooperad in this case, because the consideration of an arity zero term in the context of a cooperad creates convergence difficulties in the definition of cofree objects. We work out this problem by integrating this part of the composition structure of our cooperads into a diagram structure. We will be more precise later on. For the moment, we just use the notation $\text{gr Hopf Op}_c^*$, with our distinguishing * mark, for this category of (graded) Hopf cooperads which we associate to unitary operads. We will adopt another notation as soon as we will be able to make the definition of this category more precise.

We have a general approach to compute the homotopy of mapping spaces in the category of operads $\text{Map}_{\text{Top Op}}(P, Q)$. In short, the idea is to determine the homotopy of mapping spaces on free simplicial resolutions of our objects $P, Q$ in the category of operads in topological spaces $\text{Top Op}$. We then get a spectral sequence whose second page reduces to the cohomology of a deformation complex associated to the cohomology cooperads $H^*(P), H^*(Q) \in \text{gr Hopf Op}_c^*$. The definition of this deformation complex involves both the commutative algebra structure and the cooperad structure of these graded Hopf cooperads $H^*(P), H^*(Q) \in \text{gr Hopf Op}_c^*$. To be explicit, at the deformation complex level, we deal with a free resolution of the Hopf cooperad $H^*(Q)$ in the commutative algebra direction and with a cofree resolution of the object $H^*(P)$ in the cooperad direction. Then our deformation complex precisely consists of modules of biderivations associated to these resolutions in the category of graded Hopf cooperads.

In fact, we only use this general approach in a follow-up (see [68]) as we can use a simplification of the free commutative algebra resolution when we establish our result about the homotopy automorphism space of the operad of little 2-discs.
To be explicit, instead of a resolution in the category of commutative algebras, we use a counterpart, for operads, of the classical Postnikov decomposition of spaces. To ease the definition of this Postnikov decomposition for the rationalization of the operad of little 2-discs $D^*_2$, we actually consider a classifying space $B(CD^*)$ of an operad of chord diagrams $CD^*$ which is equivalent to the (Malcev completion of the) operad of parenthesized braids $PaB^+$.

To perform our computation, we moreover decompose the general homotopy spectral sequences of operadic mapping spaces in two intermediate spectral sequences, where we deal with the resolution in the operad direction in a first step and with the obstruction problems associated to the Postnikov decomposition of our target object in the second step. We will see that these spectral sequences vanish in degree $* > 1$, reduce to the module of rank one $Q$ in degree $* = 1$, and reduce to a graded Lie algebra $grt$ associated to the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ in degree $* = 0$. We use this correspondence to check that all classes of degree $* = 0$ in the second page of our spectral sequence are hit by homotopy automorphisms of our operad which come from the Grothendieck–Teichmüller group. We conclude from this result that our mapping from the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ to the space of homotopy automorphisms of the operad $D^*_2$ induces a bijection in homotopy.

We tackle this verification in the third part of this work, in the second volume of this monograph. We review Drinfeld’s definition of the pro-unipotent Grothendieck–Teichmüller group first. We explain that the pro-unipotent Grothendieck–Teichmüller group can be defined as a group of automorphisms associated to the Malcev completion of the parenthesized braid operad $PaB^+$. We then develop a new rational homotopy theory of operads before tackling the computation of the homotopy of the homotopy automorphism space of rational $E_2$-operads.

For this purpose, we notably define an analogue of the Sullivan model for the rational homotopy of operads in topological spaces. Briefly recall that the classical Sullivan model of a simplicial set $X$ is defined by a commutative cochain differential graded algebra $\Omega^*(X)$ (a cochain commutative dg-algebra for short), which consists of piecewise linear differential forms on $X$. We consider cooperads in commutative cochain dg-algebras to define the Sullivan model of operads in topological spaces. We use the notation $dg^*Hopf Op^+_c$ for this category of cooperads and, for short, we also call ‘Hopf cochain dg-cooperads’ the objects of this category. We also adopt the notation $dg^* Com_+$ for the category of unitary commutative cochain dg-algebras in what follows. We already mentioned that a cooperad is a structure dual to an operad in the categorical sense. Briefly say for the moment that a cooperad $C$ in a category $\mathcal{M}$ essentially consists of a collection of objects $C(r) \in \mathcal{M}$ together with a comultiplicative structure of a form opposite to the composition operations of an operad. We simply take $\mathcal{M} = dg^* Com_+$ in this general definition when we form our category of Hopf cochain dg-cooperads. Recall that we temporarily use the subscript mark $*$ in our notation of the category of Hopf cochain dg-cooperads $dg^* Hopf Op^*_c$ in order to indicate that we actually consider a counterpart, in the cooperad context, of our category of unitary operads.

The Sullivan dg-algebra $\Omega^*(X)$ does not preserve operad structures, but we will explain in the second volume of this monograph that we can define an operadic enhancement of the Sullivan functor in order to assign a well-defined Hopf cochain dg-cooperad $\Omega^*_+(P)$ to any operad in simplicial sets $P$. We will prove that the
commutative cochain dg-algebras $\Omega^*_x(P)(r)$ which define the components of this Hopf cochain dg-cooperad $\Omega^*_x(P)$ are weakly-equivalent (quasi-isomorphic) to the Sullivan dg-algebras $\Omega^*(P(r))$ associated to the spaces $P(r)$ which underlie our operad $P$. We use this observation to check that this Hopf dg-cooperad $\Omega^*_x(P)$ determines the operad in simplicial sets $\mathcal{P}$ up to rational equivalence.

We will also see that our functor $\Omega^* : \mathcal{P} \to \Omega^*(\mathcal{P})$, from the category of operads in simplicial sets to the category of Hopf cochain dg-cooperads, admits a left adjoint $\mathcal{G}_*(\mathcal{P}) : \mathcal{P} \to \mathcal{G}_*(\mathcal{P})$, which assigns an operad in simplicial sets $\mathcal{G}_*(\mathcal{P})$ to any Hopf cochain dg-cooperad $K$. We consider the image of the Hopf cochain dg-cooperad $K = \Omega^*_x(P)$ associated to an operad in simplicial sets $\mathcal{P}$ under a left derived functor of this left adjoint $L \mathcal{G}_* : K \to L \mathcal{G}_*(K)$. We will prove that this operad in simplicial sets $P^\sim = LG(\Omega^*_x(P))$ forms, under mild finiteness assumptions, a suitable model for the rationalization of the operad $P$ in the sense that the components of this operad $P^\sim(r)$ are equivalent, in the homotopy category of spaces, to the Sullivan rationalization $X^\sim = P(r)^\sim$ of the simplicial sets $X = P(r)$.

The Sullivan dg-algebra functor (and our operadic enhancement of this functor similarly) is defined on the category of simplicial sets $sSet$. But we can use the classical singular complex functor $\text{Sing}_*(\mathcal{P})$, from topological spaces to simplicial sets, and the geometric realization functor which goes the other way round, in order to prolong our constructions on operads in simplicial sets to operads in topological spaces.

The category $dg^* \mathcal{H}opf \mathcal{O}p_{p^*}$ inherits a model structure, like the category of topological operads, and we can therefore apply the general theory of model categories to associate homotopy automorphism groups $\text{Aut}_{ho}(dg^* \mathcal{H}opf \mathcal{O}p_{p^*})\mathcal{A}$, as well as homotopy automorphisms spaces $\text{Aut}_{ho}(dg^* \mathcal{H}opf \mathcal{O}p_{p^*})\mathcal{A}$, to any object of the category of Hopf cochain dg-cooperads $A \in dg^* \mathcal{H}opf \mathcal{O}p_{p^*}$. In the case of topological operads, we already mentioned that homotopy automorphisms spaces are well defined for cofibrant objects only. In the case of Hopf cochain dg-cooperads, we have to perform both cofibrant and fibrant resolutions before applying the homotopy automorphism construction.

The results obtained in our study of the rational homotopy of operads imply that the group of homotopy automorphisms attached to the model $\Omega^*_x(P)$ of an operad in spaces $\mathcal{P}$ is isomorphic to the group of homotopy automorphisms attached to the rational completion of this operad $P^\sim$. Theorem A is therefore equivalent to the following statement:

**Theorem B.** Let $E_2$ be a (cofibrant) model of $E_2$-operad in the category of topological space. Let $K_2 = \Omega^*_x(E_2)$ be the Hopf cochain dg-cooperad associated to this operad $E_2$. The homotopy automorphism space of this object in the category of Hopf cochain dg-cooperads $\text{Aut}_{ho}(dg^* \mathcal{H}opf \mathcal{O}p_{p^*}(K_2))$ has trivial homotopy groups

$$\pi_*(\text{Aut}_{ho}(dg^* \mathcal{H}opf \mathcal{O}p_{p^*}(K_2))) = 0$$

in dimension $* > 1$, the $\mathbb{Q}$-module of rank one as homotopy group

$$\pi_1(\text{Aut}_{ho}(dg^* \mathcal{H}opf \mathcal{O}p_{p^*}(K_2))) = \mathbb{Q}$$

in dimension $* = 1$, and we have an isomorphism of groups

$$GT(\mathbb{Q})^\sim \cong \pi_0(\text{Aut}_{ho}(dg^* \mathcal{H}opf \mathcal{O}p_{p^*}(K_2)))$$

in dimension $* = 0$. 

The assertions of this theorem have been foreseen by M. Kontsevich in [108]. First results in the direction of Theorem B also occur in articles of D. Tamarkin [173] and T. Willwacher [184]. But these authors deal with operads within the category of differential graded modules, after forgetting about commutative algebra structures, and their results actually give a stable version (in the sense of homotopy theory) of our statements. The definition of a setting, where we can combine a model for operadic structures and a commutative algebra model for the topology underlying our objects, is a new contribution of this monograph. The proof of Theorem B in this context is also a new outcome of our work, like the result of Theorem A.

Recall that $E_2$-operads only give the second layer of a full sequence of homotopy structures, ranging from $E_1$, fully homotopy associative but non-commutative, up to $E_\infty$, fully homotopy associative and commutative. In a follow-up [67], the author, Victor Turchin and Thomas Willwacher give a computation, in terms of graph complexes, of the homotopy of the homotopy automorphism spaces of $E_n$-operads for $2 < n < \infty$. We give an overview of these results in the concluding chapter of the second volume of this monograph. Let us mention that the group of homotopy automorphism classes of $E_1$-operads can easily be determined, but the result is trivial in this case. The group of homotopy automorphisms of an $E_\infty$-operad is trivial too.

The proof of our result requires the elaboration of new theories, like the definition of a model for the rational homotopy of topological operads, and the first aim of this monograph is to work out such general problems. The purpose of this book is also to give a comprehensive introduction to our subject, heading to our main theorems as straight as possible and with minimal background, for mathematicians coming from other domains and for graduate students.

We heavily use the formalism of Quillen’s model categories [152], which we apply to operads in order to form our model for the rational homotopy of topological operads. We rely on the modern reference books by Hirschhorn [89] and by Hovey [91] for the subject of Quillen’s model categories. We also refer to the book [61] for a comprehensive introduction to the rational homotopy theory and to Bousfield-Gugenheim’s memoir [36] for an interpretation of the Sullivan model in the formalism of model categories. We also refer to Sullivan’s article [171] for the applications of rational homotopy theory to the study of homotopy automorphisms of spaces. We review these subjects thoroughly before tackling our own constructions.

We first explain the connections between little 2-discs operads and braided structures, as well as the definition of the Grothendieck–Teichmüller group in terms of automorphisms of operads in groupoids. We give a comprehensive account of these topics in the first volume of this monograph, after an introduction to the general theory of operads. We notably give an operadic formulation of the classical coherence theorems of monoidal categories, of braided monoidal categories, and of symmetric monoidal categories. We will explain that the previously considered operad of parenthesized braids $PaB$, which we define by using the fundamental groupoid of the little 2-discs operad, actually governs the structure of a braided monoidal category.

We also review the applications of the theory of Hopf algebras to the Malcev completion of groups with the aim of extending this completion process to
groupoids and to operads. We focus on the study of a pro-unipotent Grothendieck–Teichmüller group in this work and we actually rely on the operadic Malcev completion construction in our definition of the Grothendieck–Teichmüller group. In passing, we will explain an operadic interpretation of the notion of a Drinfeld associator which was used by Tamarkin in order to prove the rational formality of $E_2$-operads.

We address the definition of our model for the rational homotopy of operads in the second volume of this monograph, after a survey of the general theory of model categories and of the rational homotopy theory of spaces. We mentioned in the introduction of this work that the Chevalley–Eilenberg cochain complex of the Drinfeld–Kohno Lie algebras (the Lie algebras of infinitesimal braids) can be used to define a small model of a rational $E_2$-operad. We also explain this construction in the second volume of this monograph. We review the already alluded proof of the rational formality of $E_2$-operads in the second volume too. We actually explain the definition of small models of $E_n$-operads for all $n \geq 2$ by using a graded version of the Drinfeld–Kohno Lie algebras. We tackle the computation of the homotopy automorphism space of rational $E_2$-operads afterwards, in the concluding part of the second volume.
Foundations and Conventions

The reader is assumed to be familiar with the language of category theory and to have basic knowledge about fundamental concepts like adjoint and representable functors, colimits and limits, categorical duality, which we freely use throughout this work. The reader is also assumed to be aware of the applications of colimits and limits in basic examples of categories (including sets, topological spaces, and modules). Nonetheless, we will review some specialized topics, like reflexive coequalizers and filtered colimits, which are considered in applications of category theory to operads.

We use single script letters (like \( \mathcal{C}, \mathcal{M}, \ldots \)) as general notation for abstract categories. We use script expressions (like \( \mathcal{M}od, \mathcal{A}s, \mathcal{O}p, \ldots \)) for particular instances of categories which we consider in this work (like modules, associative algebras, operads, \ldots). We are going to explain that the formal definition of many algebraic structures remains the same in any instance of base category \( \mathcal{M} \) and essentially depends on a symmetric monoidal structure given with this category \( \mathcal{M} \). We usually assume that the category \( \mathcal{M} \), to which we assign the role of a base category, is equipped with enriched hom-bifunctors \( \mathbf{Hom}_\mathcal{M}(\cdot, \cdot) \). We give a more detailed reminder on this notion in \( \S \S \) 0.12-0.13.

In practice, we take our base category \( \mathcal{M} \) among the category of sets \( \text{Set} \), the category of simplicial sets \( s\text{Set} \), the category of topological spaces \( \text{Top} \), a category of \( k \)-modules \( \text{Mod} \) (where \( k \) refers to a fixed ground ring), or among a variant of these categories. To be precise, besides plain \( k \)-modules, we have to consider categories formed by differential graded modules \( dg\text{Mod} \) (we usually speak about \( dg\text{-modules} \) for short), graded modules \( gr\text{Mod} \), simplicial modules \( s\text{Mod} \), and cosimplicial modules \( c\text{Mod} \). The first purpose of this preliminary chapter is to quickly recall the definition of these categories (at least, in order to fix our conventions). By the way, we also recall the definition of the category of simplicial sets \( s\text{Set} \), which we use along with the familiar category of topological spaces \( \text{Top} \).

To complete our account, we recall the general definition of a symmetric monoidal category and we explain some general constructions which we associate to this notion. But we postpone our reminder on the definition of the monoidal structure of the category of \( dg \)-modules, simplicial modules and cosimplicial modules until the moment where we tackle the applications of these base categories.

In the module context, we assume that a ground ring \( k \) is given and is fixed once and for all. In certain constructions, we have to assume that this ground ring \( k \) is a field of characteristic 0.

0.1. Graded and differential graded modules. The category of differential graded modules \( dg\text{Mod} \) (\( dg\text{-modules} \) for short) consists of \( k \)-modules equipped with a decomposition \( C = \bigoplus_{n \in \mathbb{Z}} C_n \), which ranges over \( \mathbb{Z} \), and with a morphism \( \delta : C \to C \) (the differential) such that we have \( \delta^2 = 0 \) and \( \delta(C_n) \subset C_{n-1} \), for all \( n \in \mathbb{Z} \). We
obviously define a morphism of dg-modules as a morphism of $k$-modules $f : C \to D$ which intertwines differentials and which satisfies the relation $f(C_n) \subset D_n$, for all $n \in \mathbb{Z}$.

In textbooks of homological algebra (like [181]), authors mostly deal with an equivalent notion of chain complex, of which components are split off into a sequence of $k$-modules $C_n$ linked by the differentials $\delta : C_n \to C_{n-1}$ rather than being put together in a single object. The idea of a dg-module (used for instance in [129]) is more natural for our purpose and is also more widely used in homotopy theory. In what follows, we rather reserve the phrase ‘chain complex’ for certain specific constructions of dg-modules.

The category of graded modules $gr\text{Mod}$ consists of $k$-modules equipped with a decomposition $C = \bigoplus_{n \in \mathbb{Z}} C^*_n$, which ranges over $\mathbb{Z}$, but where we have no differential. We obviously define a morphism of graded modules as a morphism of $k$-modules $f : C \to D$ which satisfies the relation $f(C_n) \subset D_n$, for all $n \in \mathbb{Z}$.

We have an obvious functor $(-) : dg\text{Mod} \to gr\text{Mod}$ defined by retaining the underlying graded structure of a dg-module and by forgetting about the differential. We notably consider the underlying graded module of dg-modules, which this forgetful process formalizes, when we define the notion of a quasi-free object (in the category of commutative algebras, in the category of operads, . . .). The other way round, we can embed the category of graded modules $gr\text{Mod}$ into the category of dg-modules $dg\text{Mod}$ by viewing a graded module as a dg-module equipped with a trivial differential $\delta = 0$. We use this identification at various places.

Recall that the homology of a dg-module $C$ is defined by the quotient $k$-module $H_\ast(C) = \ker \delta / \text{im} \delta$ which inherits a natural grading from $C$. The homology defines a functor $H_\ast(-) : dg\text{Mod} \to gr\text{Mod}$. In most references of homological algebra, authors use the phrase ‘quasi-isomorphism’ for the class of morphisms of dg-modules which induce an isomorphism in homology. In what follows, we rather use the name ‘weak-equivalence’ which we borrow from the general formalism of model categories (see §II.1 for this notion).

We generally use the mark $\sim\to$ to refer to the class of weak-equivalences in a model category (see §II.1) and we naturally use the same notation in the dg-module context. We mostly use the notions introduced in this paragraph in the second part of this book and we review these definitions with full details in §II.5.

### 0.2. Degrees and signs of dg-algebra.

The component $C_n$ of a dg-module (respectively, graded module) $C$ defines the homogeneous component of degree $n$ of $C$. To specify the degree of a homogeneous element $x \in C_n$, we use the expression $\deg(x) = n$. We adopt the standard convention of dg-algebra to associate a sign $(-1)^{\deg(x)\deg(y)}$ to each transposition of homogeneous elements $(x, y)$. We do not specify such a sign in general and we simply use the notation $\pm$ to refer to it. We will see that the introduction of these signs is forced by the definition of the symmetry isomorphism of the tensor product of dg-modules (see §II.5).

We usually consider lower graded dg-modules, but we also have a standard notion of dg-module equipped with a decomposition in upper graded components $C = \bigoplus_{n \in \mathbb{Z}} C^*_n$ such that the differential satisfies $\delta(C^n) \subset C^{n+1}$. Certain constructions (like the duality of $k$-modules and the conormalized complex of cosimplicial spaces) naturally produce upper graded dg-modules. In what follows, we apply the relation $C_{-n} = C^n$ to identify an upper graded with a lower graded dg-module. We also review these concepts in §II.5.
0.3. Simplicial objects and cosimplicial objects in a category. The simplicial category $\Delta$, which models the structure of simplicial and cosimplicial objects in a category, is defined by the collection of finite ordinals $\underline{n} = \{0 < \cdots < n\}$ as objects together with the non-decreasing maps between finite ordinals $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$ as morphisms. We define a simplicial object $X$ in a category $\mathcal{C}$ as a contravariant functor $X : \Delta^{op} \to \mathcal{C}$ which assigns an object $X_n \in \mathcal{C}$ to each $n \in \mathbb{N}$ and a morphism $u^* : X_n \to X_m$ to each non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$. We similarly define a cosimplicial object in $\mathcal{C}$ as a covariant functor $X : \Delta \to \mathcal{C}$ which assigns an object $X^n \in \mathcal{C}$ to each $n \in \mathbb{N}$ and a morphism $u_* : X^n \to X^m$ to each non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$. Naturally, we define a morphism of simplicial objects $f : X \to Y$ (and a morphism of cosimplicial object similarly) as a sequence of morphisms $f : X_n \to Y_n$ in the ambient category $\mathcal{C}$ which intertwine the action of the simplicial operators $u^*$ on our objects $X$ and $Y$.

We use the notation $s \mathcal{C}$ for the category of simplicial objects in a given ambient category $\mathcal{C}$ and the notation $c \mathcal{C}$ for the category of cosimplicial objects. The category of simplicial sets $sSet$, for instance, formally consists of the simplicial objects in the category of sets $X : \Delta^{op} \to Set$.

The simplices $\Delta^n$, $n \in \mathbb{N}$, are the fundamental examples of simplicial sets which are given by the representable functors $\text{Mor}_\Delta(\cdot, u) : \Delta^{op} \to Set$, where we use the notation $\text{Mor}_\Delta(\cdot, u)$ to refer to the morphism sets of the simplicial category $\Delta$. The collection of the $n$-simplices $\Delta^n$, $n \in \mathbb{N}$, forms a cosimplicial object in the category of simplicial sets itself, with the covariant action of non-decreasing maps $u_* : \Delta^m \to \Delta^n$ defined by the composition on the target in these morphism sets $\Delta^n = \text{Mor}_\Delta(\cdot, u)$.

In the case of a simplicial set $X$, an element $\sigma \in X_n$ is called an $n$-dimensional simplex (or more simply an $n$-simplex) in $X$. The definition of the $n$-simplex $\Delta^n$ as a representable functor $\Delta^n = \text{Mor}_\Delta(\cdot, u)$ implies that we have the relation $\text{Mor}_{sSet}(\Delta^n, X) = X_n$, for any simplicial set $X \in sSet$, where we use the notation $\text{Mor}_{sSet}(\cdot, -)$ for the morphism set of the category $sSet$. To make this correspondence explicit, we consider the $n$-simplex, denoted by $t_n \in (\Delta^n)_n$, which corresponds to the identity of the object $\underline{n}$ in the simplicial category $\Delta$. The morphism $\sigma_* : \Delta^n \to X$, associated to any $n$-simplex $\sigma \in X_n$, is characterized by the relation $\sigma_*(t_n) = \sigma$.

The topological simplices $\Delta^n = \{(t_0, \ldots, t_n)|0 \leq t_i \leq 1, t_0 + \cdots + t_n = 1\}$ form another fundamental instance of a cosimplicial object which is defined in the category of topological spaces. The cosimplicial structure map $u_* : \Delta^m \to \Delta^n$ associated to a morphism of the simplicial category $u \in \text{Mor}_\Delta(m, u)$ sends any element $(s_0, \ldots, s_m) \in \Delta^m$ to the point $(t_0, \ldots, t_n) \in \Delta^n$ such that $t_i = \sum_{u(k) = i} s_k$.

We mainly use simplicial objects and cosimplicial objects in the second volume of this book and we go back to the definitions of this paragraph in §II.5. We also study simplicial and cosimplicial modules (simplicial and cosimplicial objects in module categories) in depth in §II.6.

0.4. Faces and degeneracies in a simplicial object. The maps $d^i : \{0 < \cdots < n - 1\} \to \{0 < \cdots < n\}$, $i = 0, \ldots, n$, such that

\[
d^i(x) = \begin{cases} x, & \text{for } x < i, \\ x + 1, & \text{for } x \geq i, \end{cases}
\]
and the maps \( s^j : \{0 < \cdots < n + 1\} \to \{0 < \cdots < n\}, j = 0, \ldots, n \), such that

\[
s^j(x) = \begin{cases} 
x, & \text{for } x \leq j, \\
x - 1, & \text{for } x > j,
\end{cases}
\]

generate the simplicial category in the sense that any non-decreasing map \( u : \{0 < \cdots < m\} \to \{0 < \cdots < n\} \) can be written as a composite of maps of that form. Moreover, any relation between these generating morphisms can be deduced from the following generating relations:

\[
d^i d^j = d^i d^{j-1}, \text{ for } i < j, \\
s^j d^i = \begin{cases} 
d^i s^{j-1}, & \text{for } i < j, \\
\text{id}, & \text{for } i = j, j + 1, \\
d^{i-1} s^j, & \text{for } i > j + 1,
\end{cases} \\
s^i s^j = s^i s^{j+1}, \text{ for } i \leq j.
\]

The structure of a cosimplicial object is, as a consequence, fully determined by a sequence of objects \( X^n \in \mathcal{C} \) together with morphisms \( d^i : X^{n-1} \to X^n, i = 0, \ldots, n \), and \( s^j : X^{n+1} \to X^n, j = 0, \ldots, n \), for which these relations (3) hold. The morphisms \( d^i : X^{n-1} \to X^n, i = 0, \ldots, n \), which represent the image of the maps \( d^i \) under the functor defined by \( X \), are the \textit{coface operators} of the cosimplicial object \( X \) (we may also speak about the \textit{cofaces} of \( X \) for short). The morphisms \( s^j : X^{n+1} \to X^n, j = 0, \ldots, n \), which represent the image of the maps \( s^j \) are the \textit{codegeneracy operators} of \( X \) (or, more simply, the \textit{codegeneracies} of \( X \)).

Dually, the structure of a simplicial object is fully determined by a sequence of objects \( X_n \in \mathcal{C} \) together with morphisms \( d_i : X_n \to X_{n-1}, i = 0, \ldots, n \), and \( s_j : X_n \to X_{n+1}, j = 0, \ldots, n \), for which relations

\[
d_i d_j = d_{j-1} d_i, \text{ for } i < j, \\
d_i s_j = \begin{cases} 
s_{j-1} d_i, & \text{for } i < j, \\
\text{id}, & \text{for } i = j, j + 1, \\
s_j d_{i-1}, & \text{for } i > j + 1,
\end{cases} \\
s_i s_j = s_{j+1} s_i, \text{ for } i \leq j,
\]

opposite to (3), hold. The morphisms \( d_i : X_n \to X_{n-1}, i = 0, \ldots, n \), which represent the image of the maps \( d^i \) under the contravariant functor defined by \( X \), are the \textit{face operators} of the simplicial object \( X \), and the morphisms \( s_j : X_n \to X_{n+1}, j = 0, \ldots, n \), which represent the image of the maps \( s^j \), are the \textit{degeneracy operators} of \( X \). We also recall the definition of these operators in the course of our study (in §§II.13).

0.5. \textit{Geometric realization of simplicial sets and singular complex of topological spaces}. Recall that a topological space \(|K|\), traditionally called the geometric realization of \( K \), is naturally associated to each simplicial set \( K \in s\textit{Set} \). This space is defined by the coend

\[
|K| = \int_{n \in \Delta} K_n \times \Delta^n.
\]

where each set \( K_n \) is viewed as a discrete space and we consider the topological \( n \)-simplices \( \Delta^n \) (of which definition is recalled in §[0.3]). The coend which we consider in
this construction is equivalent to the quotient object of the coproduct \( \coprod_n K_n \times \Delta^n = \coprod_n \{ \coprod_{\sigma \in K_n} \{ \sigma \} \times \Delta^n \} \) under the relations

\[
(u^*(\sigma), (t_0, \ldots, t_m)) \equiv (\sigma, u_*(t_0, \ldots, t_m)),
\]

for \( u \in \text{Mor}_\Delta(m, n), \sigma \in K_n, \) and \((t_0, \ldots, t_m) \in \Delta^m\). The definition of the map \( u_* : \Delta^m \to \Delta^n \) associated to each \( u \in \text{Mor}_\Delta(m, n) \) involves the cosimplicial structure of the topological \( n \)-simplices \( \Delta^n \). One easily checks that the realization of the \( n \)-simplex \( \Delta^n = \text{Mor}_\Delta(-, n) \) is identified with the topological \( n \)-simplex \( \Delta^n \).

In the converse direction, we can use the singular complex construction to associate a simplicial set \( \text{Sing}_* (X) \) to any topological space \( X \). This simplicial set \( \text{Sing}_* (X) \) consists in dimension \( n \) of the set of continuous maps \( \sigma : \Delta^n \to X \) going from the topological \( n \)-simplex \( \Delta^n \) to \( X \). The composition of simplices \( \sigma : \Delta^n \to X \) with the cosimplicial operator \( u_* : \Delta^m \to \Delta^n \) associated to any \( u \in \text{Mor}_\Delta(m, n) \) yields a map \( u^* : \text{Sing}_* (X) \to \text{Sing}_* (X) \) so that the collection of sets \( \text{Sing}_n (X) = \text{Mor}_\text{op}(\Delta^n, X), n \in \mathbb{N} \), inherits a natural simplicial structure.

The geometric realization obviously gives a functor \( | - | : s\text{Set} \to \text{Top} \). The singular complex construction gives a functor in the converse direction \( \text{Sing}_* : \text{Top} \to s\text{Set} \), which is actually a right adjoint of the geometric realization functor (see [79, §1.2]). We study generalizations of these constructions in §II.8

0.6. Simplicial modules, cosimplicial modules, and homology. The category of simplicial modules \( s\text{Mod} \) is the category of simplicial objects in the category of \( k \)-modules \( \text{Mod} \). Thus, a simplicial module \( K \) can be defined either as a covariant functor from the simplicial category \( \Delta \) to the category of \( k \)-modules \( \text{Mod} \), or, equivalently, as a collection of \( k \)-modules \( K_n, n \in \mathbb{N} \) equipped with face operators \( d_i : K_n \to K_{n-1}, i = 0, \ldots, n \), and degeneracy operators \( s_j : K_n \to K_{n+1}, j = 0, \ldots, n \), which satisfy the simplicial relations.

The category of cosimplicial modules \( c\text{Mod} \) similarly consists of the cosimplicial objects in the category of \( k \)-modules.

To any simplicial module \( K \), we associate a \( dg \)-module \( \mathbb{N}_*(K) \), called the normalized complex of \( K \), and defined by the quotient \( \mathbb{N}_n(K) = K_n/s_0K_{n-1} + \cdots + s_{n-1}K_{n-1} \) in degree \( n \), together with the differential \( \delta : \mathbb{N}_n(K) \to \mathbb{N}_{n-1}(K) \) such that \( \delta = \sum_{i=0}^n (-1)^i d_i \). This normalized chain complex construction naturally gives a functor \( \mathbb{N}_* : s\text{Mod} \to dg \text{Mod} \). The homology of a simplicial module \( K \) is defined as the homology of the associated normalized complex \( \mathbb{N}_*(K) \). For simplicity, we use the same notation for the homology functor on simplicial modules and on \( dg \)-modules. Hence, we set \( \mathbb{H}_*(K) = \mathbb{H}_*(\mathbb{N}_*(K)) \), for any \( K \in s\text{Mod} \). We study simplicial and cosimplicial modules in depth in §II.8 and we recall the definition of the normalized complex construction at this moment.

0.7. Normalized complex and homology of simplicial sets. We will consider the functor \( k[-] : s\text{Set} \to s\text{Mod} \) which maps a simplicial set \( X \) to the simplicial module \( k[X] \) generated by the set \( X_n \) in dimension \( n \), for any \( n \in \mathbb{N} \), and which inherits an obvious simplicial structure. We also have a contravariant functor \( A : s\text{Set}^{op} \to c\text{Mod} \) which maps a simplicial set \( X \) to the cosimplicial module \( A(X) = k^X \), dual to \( k[X] \), and defined in dimension \( n \) by the \( k \)-module of functions \( u : X_n \to k \) on the set \( X_n \in s\text{Set} \).

We use the notation \( \mathbb{N}_*(X) \) for the normalized complex of the simplicial \( k \)-module \( k[X] \) associated to a simplicial set \( X \). We retrieve the classical homology of simplicial sets by considering the homology of these simplicial modules. We also
use the notation \( H_\ast(-) \) for the homology functor on simplicial sets. We accordingly have the formula \( H_\ast(X) = H_\ast(N_\ast(X)) \), for any \( X \in sSet \).

The normalized complexes of the simplices \( \Delta^n, \, n \in \mathbb{N} \), naturally form a simplicial object in the category of dg-modules \( N_\ast(\Delta^\bullet) \). For a given simplicial module \( K \), we have a coend formula

\[
N_\ast(K) = \int^{n \in \Delta} K_n \otimes N_\ast(\Delta^n),
\]

and the normalized complex construction of \( \Sect\ ) can be regarded as a dg-module version of the geometric realization of simplicial sets (we explain this idea in \( \Sect\ ).

**0.8. Symmetric monoidal categories and the structure of base categories.** In the introduction of this chapter, we briefly mentioned that our base categories, let \( \mathcal{M} = \Set, \Top, \Mod, \ldots \), are all instances of symmetric monoidal categories.

By definition, a symmetric monoidal category is a category \( \mathcal{M} \) equipped with a tensor product \( \otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) that satisfies natural unit, associativity and symmetry relations which we express as follows:

(a) We have a unit object \( 1 \in \mathcal{M} \) such that we have a natural isomorphism \( A \otimes 1 \simeq A \simeq 1 \otimes A \), for \( A \in \mathcal{M} \).

(b) We have a natural isomorphism \( (A \otimes B) \otimes C \simeq A \otimes (B \otimes C) \), for every triple of objects \( A, B, C \in \mathcal{M} \), which satisfies a pentagonal coherence relation (Mac Lane’s pentagon relation) when we put together the associativity isomorphisms associated to a 4-fold tensor product, and additional triangular coherence relations with respect to the unit isomorphism when we assume that one of our objects is the unit object of our category (we refer to [130, \Sect\] for the expression of these constraints).

(c) We have a natural symmetry isomorphism \( A \otimes B \simeq B \otimes A \), associated to every pair of objects \( A, B \in \mathcal{M} \), which satisfies hexagonal coherence relations (Drinfeld’s hexagon relation) when we apply the symmetry isomorphism to a 3-fold tensor product, and additional triangular coherence relations with respect to the unit isomorphism when we assume that one of our objects is the unit object (see again [130, \Sect\] for details).

In the case of \( k \)-modules \( \Mod \), the monoidal structure is given by the usual tensor product of \( k \)-modules, taken over the ground ring, together with the ground ring itself as unit object. The definition of the tensor product of dg-modules, simplicial modules, cosimplicial modules is reviewed later on, when we tackle applications of these base categories. In the category of sets \( \Set \) (respectively, topological spaces \( \Top \), simplicial sets \( sSet \)), the tensor product is simply given by the cartesian product \( \otimes = \times \) together with the one-point set \( 1 = pt \) as unit object. In what follows, we also use the general notation \( * \) for the terminal object of a category, and we may write \( pt = * \) when we want to stress that the one point-set actually represents the terminal object of the category of sets (respectively, topological spaces, simplicial sets).

The unit object and the isomorphisms that come with the unit, associativity and commutativity relations of a symmetric monoidal category are part of the structure. Therefore, these morphisms have, in principle, to be given with the definition. But, in our examples, we can assume that the unit and associativity relations are identities, and in general, we just make explicit the definition of the symmetry isomorphism \( c = c(A, B) : A \otimes B \xrightarrow{\cong} B \otimes A \).
We make explicit the coherence constraints for the unit, associativity, and symmetry isomorphisms of symmetric monoidal categories in §I.6 (we use a braided analogue of the structure of a symmetric monoidal category in our definition of the Grothendieck–Teichmüller group). We also review the definition of several notions of structure preserving functors between symmetric monoidal categories in §I.3.

0.9. Tensor products and colimits. In many constructions, we consider symmetric monoidal categories $\mathcal{M}$ equipped with colimits and limits and whose the tensor product distributes over colimits in the sense that:

(a) The canonical morphism $\text{colim}_{\alpha \in \mathcal{I}}(A_\alpha \otimes B) \rightarrow (\text{colim}_{\alpha \in \mathcal{J}} A_\alpha) \otimes B$ associated to a diagram $A_\alpha \in \mathcal{M}$, $\alpha \in \mathcal{I}$, is an isomorphism for all $B \in \mathcal{M}$, and similarly as regards the canonical morphism $\text{colim}_{\beta \in \mathcal{J}} (A \otimes B_\beta) \rightarrow A \otimes (\text{colim}_{\beta \in \mathcal{J}} B_\beta)$ associated to a diagram $B_\beta \in \mathcal{M}$, $\beta \in \mathcal{J}$, where we now fix the object $A \in \mathcal{M}$.

This requirement is fulfilled by all categories which we take as base symmetric monoidal categories in $\mathcal{M}$ equipped with colimits and limits and whose the tensor product distributes over colimits in the sense that:

- One simple example is given by taking the direct sum $\oplus$ of symmetric monoidal categories which do not satisfy this distribution relation.
- In many constructions, we consider symmetric monoidal categories $\mathcal{M}$ equipped with colimits and limits and whose the tensor product distributes over colimits in the sense that:

0.10. Symmetric groups and tensor permutations. We use the notation $\Sigma_r$ for the group of permutations of \{1, \ldots, r\}. Depending on the context, we regard a permutation $s \in \Sigma_r$ as a bijection $s : \{1, \ldots, r\} \rightarrow \{1, \ldots, r\}$ or as a sequence $s = (s(1), \ldots, s(r))$ equivalent to an ordering of the set \{1, \ldots, r\}. In any case, we will use the notation $id = id_r$ for the identity permutation in $\Sigma_r$. We omit the subscript $r$ which indicates the cardinal of our permutation when we do not need to specify this information.

In a symmetric monoidal category equipped with a strictly associative tensor product, we can form $r$-fold tensor products $T = X_1 \otimes \cdots \otimes X_r$ without care and omit unnecessary bracketing. Then we also have a natural isomorphism

$$X_1 \otimes \cdots \otimes X_r \overset{s^*}{\rightarrow} X_{s(1)} \otimes \cdots \otimes X_{s(r)},$$

associated to each permutation $s \in \Sigma_r$, and such that the standard unit and associativity relations $id^* = id$ and $t^* s^* = (st)^*$ hold. To construct this action, we use the classical presentation of $\Sigma_r$ with the transpositions $t_i = (i \ i + 1)$ as generating elements and the identities

1. $t_i^2 = id$, for $i = 1, \ldots, r - 1$,
2. $t_i t_j = t_j t_i$, for $i, j = 1, \ldots, r - 1$, with $|i - j| \geq 2$,
3. $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$, for $i = 1, \ldots, r - 2$,

as generating relations. We assign the morphism

$$X_1 \otimes \cdots \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_r \overset{c_{X_i,X_{i+1}}}{\rightarrow} X_1 \otimes \cdots \otimes X_{i+1} \otimes X_i \otimes \cdots \otimes X_r,$$

induced by the symmetry isomorphism $c(X_i, X_{i+1}) : X_i \otimes X_{i+1} \overset{\cong}{\rightarrow} X_{i+1} \otimes X_i$, to the transposition $t_i = (i \ i + 1)$. The axioms of symmetric monoidal categories imply that these morphisms satisfy the relations (I.3) attached to the elementary
transpositions in $\Sigma_r$. Hence, we can use the presentation of $\Sigma_r$ to coherently extend the action of the transpositions $t_i \in \Sigma_r$ on tensor powers to the whole symmetric group.

0.11. Tensor products over arbitrary finite sets. In our constructions, we often deal with tensor products $\bigotimes_{i \in I} X_i$ that range over an arbitrary set $I = \{i_1, \ldots, i_r\}$ (not necessarily equipped with a canonical ordering). In fact, we effectively realize such a tensor product $\bigotimes_{i \in I} X_i$ as an ordered tensor product $X_{u(1)} \otimes \cdots \otimes X_{u(r)}$, which we associate to the choice of a bijection $u : \{1 < \cdots < r\} \xrightarrow{\sim} I$. The tensor products associated to different bijection choices $u, v : \{1 < \cdots < r\} \xrightarrow{\sim} I$ differ by a canonical isomorphism $s^* : X_{u(1)} \otimes \cdots \otimes X_{u(r)} \xrightarrow{\sim} X_{v(1)} \otimes \cdots \otimes X_{v(r)}$ which we determine from the permutation $s \in \Sigma_r$ such that $v = us$ by using the just defined action of symmetric group on tensors.

In principle, the tensor product $\bigotimes_{i \in I} X_i$ is only defined up to these canonical isomorphisms. However, we can adapt the general Kan extension process to make this construction more rigid. Formally, we define the unordered tensor product as the colimit $\bigotimes_{i \in I} X_i = \text{colim}_{u : \{1 < \cdots < r\} \xrightarrow{\sim} I} X_{u(1)} \otimes \cdots \otimes X_{u(r)}$ that ranges over the category formed by the bijections $u : \{1 < \cdots < r\} \xrightarrow{\sim} I$ as objects and the permutations $s \in \Sigma_r$ such that $v = us$ as morphisms. The colimit process automatically performs the identification of the tensors associated to different bijection choices.

This construction can be regarded as an instance of a Kan extension process which we will apply to structures, called symmetric sequences, underlying operads (see §2.5).

0.12. Enriched category structure of base categories. The morphism sets of a category $\mathcal{C}$ will always be denoted by $\text{Mor}_{\mathcal{C}}(A, B)$. But many categories which we consider come equipped with a hom-bifunctor $\text{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{M}$ with values in one of our base symmetric monoidal categories $\mathcal{M} = \text{Set}, \text{Top}, \text{Mod}, \ldots$, and which provides $\mathcal{C}$ with an enriched category structure.

The structure of an enriched category includes operations that extend the classical composition operations attached to the morphism sets of ordinary categories. In the usual setting, the units of the composition are given by identity morphisms $\text{id}_A \in \text{Mor}_{\mathcal{C}}(A, A)$ associated to all objects $A \in \mathcal{C}$. In the case of an enriched category, the units of the composition are morphisms

$$\text{id}_A : 1 \to \text{Hom}_{\mathcal{C}}(A, A),$$

given for all objects $A \in \mathcal{C}$, and defined on the tensor unit of the base category $1$. The composition products are morphisms

$$\circ : \text{Hom}_{\mathcal{C}}(B, C) \otimes \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{C}}(A, C),$$

given for all $A, B, C \in \mathcal{C}$, and where we consider the tensor product of hom-objects in the base category instead of the cartesian product of morphism sets. These composition products are assumed to satisfy obvious analogues, now expressed in terms of commutative diagrams, of the unit and associativity relations of the composition in ordinary categories. Each of our base categories $\mathcal{M} = \text{Set}, \text{Top}, \text{Mod}, \ldots$ is enriched over itself. In the case of sets $\text{Set}$, we trivially take $\text{Hom}_{\text{Set}}(-, -) = \text{Mor}_{\text{Set}}(-, -)$. In the case of topological spaces $\text{Top}$, the hom-objects $\text{Hom}_{\text{Top}}(A, B)$ are given by the morphism sets $\text{Mor}_{\text{Top}}(A, B)$ equipped with the usual compact-open topology. In the case of modules $\text{Mod}$, the hom-objects $\text{Hom}_{\text{Mod}}(A, B)$ are similarly given by the morphism sets of the category $\text{Hom}_{\text{Mod}}(A, B) = \text{Mor}_{\text{Mod}}(A, B)$, which
come naturally equipped with a module structure (the usual one). In our remaining fundamental examples $\mathcal{M} = s\text{Set}, \text{dg Mod}, \ldots$, the hom-objects $\text{Hom}_\mathcal{M}(A, B)$ consist of maps which are given by an extension of the definition of the morphisms of our category $\mathcal{M}$. (We give the explicit definition of these hom-objects later on, when we begin to use these categories.)

In all these examples, we actually take hom-objects which fit an adjunction relation with respect to the symmetric monoidal structure (authors say that our base categories are instances of closed monoidal categories). We review this connection in a next paragraph.

0.13. The general notion of an enriched category, morphisms and homomorphisms. In what follows, we actually use enriched categories both as a natural framework to perform constructions on objects and as examples of structured objects. The base categories $\mathcal{M} = \text{Set, Mod, sSet, dg Mod,} \ldots$ correspond to the first usage of enriched category structures, while the Hopf categories, which we consider in our definition of the Malcev completion of groupoids (see §I.9.0), correspond to the second form of applications of enriched categories.

In the first case, an enriched category structure is often given as an extra structure associated with an ordinary category $\mathcal{C}$. Then we deal with both morphism sets $\text{Mor}_\mathcal{C}(-, -)$ and with hom-objects $\text{Hom}_\mathcal{C}(-, -)$ with values in a given symmetric monoidal category $\mathcal{M}$ (not necessarily a base category). We say that our category $\mathcal{C}$ is enriched over $\mathcal{M}$ when we need to specify this category where our hom-objects are defined. We assume that the hom-objects are equipped with unit and composition morphisms §0.12(1-2) formed within our symmetric monoidal category $\mathcal{M}$.

In this context, where enriched categories arise as extra-structures associated with an underlying ordinary category $\mathcal{C}$, we also naturally assume that the hom-objects form a bifunctor $\text{Hom}_\mathcal{C}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$ so that we have morphisms

\begin{align*}
(1) & \quad f_* : \text{Hom}_\mathcal{C}(-, A) \to \text{Hom}_\mathcal{C}(-, B) \quad \text{and} \quad f^* : \text{Hom}_\mathcal{C}(B, -) \to \text{Hom}_\mathcal{C}(A, -),
\end{align*}

for every $f \in \text{Mor}_\mathcal{C}(A, B)$. The unit morphisms and the composition operations §0.12(1-2) have to be invariant under these actions of morphisms on hom-objects.

In our basic examples, where hom-objects are made from point-sets, we can identify the actual morphisms of the category $f \in \text{Mor}_\mathcal{C}(A, B)$ with particular elements of the hom-objects $\text{Hom}_\mathcal{C}(A, B)$. The general elements $u \in \text{Hom}_\mathcal{C}(A, B)$ are conversely identified with maps $u : A \to B$ which satisfy some mild requirements, and these hom-objects $\text{Hom}_\mathcal{C}(A, B)$ are generally given by an extension of the morphism sets of our category $\text{Mor}_\mathcal{C}(A, B)$. In this setting, we use the name ‘homomorphism’ to refer to the general elements of the hom-objects $\text{Hom}_\mathcal{C}(A, B)$ as opposed to the ‘morphisms’, which refer to the elements of the morphism sets $\text{Mor}_\mathcal{C}(A, B)$. We may however use the arrow notation $u : A \to B$ when we want to regard such a homomorphism $u \in \text{Hom}_\mathcal{C}(A, B)$ as a map. In this case, the belonging category of the arrow $u$ is specified by the context. The composition on hom-objects also usually extends the composition on morphisms, and the morphisms (1), which make the hom-objects into a bifunctor, are generally identified with the left (respectively, right) composition with the homomorphism which we associate to any morphism $f \in \text{Mor}_\mathcal{C}(A, B)$.

In a general setting, we can define a correspondence between morphisms and homomorphisms by using a natural transformation

\begin{align*}
(2) & \quad \iota^\#: \text{1[Mor}_\mathcal{C}(A, B)] \to \text{Hom}_\mathcal{C}(A, B),
\end{align*}
where the expression $\mathbb{1}[\text{Mor}_C(A, B)]$ denotes the coproduct, ranging over the set of morphisms $f \in \text{Mor}_C(A, B)$, of copies of the unit object $\mathbb{1}$.

0.14. Closed symmetric monoidal categories. In the case of our base categories $\mathcal{M} = \text{Set}, \mathcal{Top}, \text{Mod}, \ldots$, we actually take hom-bifunctors that fit in an adjunction relation $\text{Mor}_\mathcal{M}(A \otimes B, C) \simeq \text{Mor}_\mathcal{M}(A, \text{Hom}_\mathcal{M}(B, C))$ with respect to the symmetric monoidal structure of the category $\mathcal{M}$. The bijection which gives this adjunction relation is also assumed to be natural in $A, B, C \in \mathcal{M}$.

We generally say that a symmetric monoidal category $\mathcal{M}$ is closed when the tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ has a right adjoint $\text{Hom}_\mathcal{M}(-, -) : \mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{M}$ so that we have an isomorphism

\[ \text{Mor}_\mathcal{M}(A \otimes B, C) \simeq \text{Mor}_\mathcal{M}(A, \text{Hom}_\mathcal{M}(B, C)), \]

for any $A, B, C \in \mathcal{M}$. Note that the existence of this adjoint implies that our tensor product distributes over colimits as we require in \([0.9]\). The other way round, the hom-object bifunctor $\text{Hom}_\mathcal{M}(-, -)$, which we define by an adjunction relation of this form \((1)\), automatically satisfies the same distribution relations with respect to colimits and limits as the general morphism set bifunctor of our base category. Namely, we have the identity $\text{Hom}_\mathcal{M}(\text{colim}_\alpha A_\alpha, B) = \lim_\alpha \text{Hom}_\mathcal{M}(A_\alpha, B)$, when we take a colimit on the source of our hom-object $A = \text{colim}_\alpha A_\alpha$, and the identity $\text{Hom}_\mathcal{M}(A, \lim_\beta B_\beta) = \lim_\beta \text{Hom}_\mathcal{M}(A, B_\beta)$, when we consider a limit on the target $B = \lim_\beta B_\beta$.

The hom-objects $\text{Hom}_\mathcal{M}(A, B)$ defined by an internal hom-functor naturally inherit an evaluation morphism

\[ \epsilon : \text{Hom}_\mathcal{M}(A, B) \otimes A \to B \]

which represents the augmentation of the adjunction \((1)\) and which generalizes the usual evaluation of maps in the category of sets. The unit of our adjunction is given by a morphism

\[ \iota : A \to \text{Hom}_\mathcal{M}(B, A \otimes B) \]

associated to each pair of objects in our category $A, B \in \mathcal{M}$.

The hom-objects of a closed symmetric monoidal category automatically inherit composition units $\text{id}_A : \mathbb{1} \to \text{Hom}_\mathcal{C}(A, A)$, given by the right adjoint of the unit isomorphisms $\mathbb{1} \otimes A \cong A$ of the symmetric monoidal structure, as well as composition operations $\circ : \text{Hom}_\mathcal{C}(B, C) \otimes \text{Hom}_\mathcal{C}(A, B) \to \text{Hom}_\mathcal{C}(A, C)$, given by the right adjoint of the composite evaluation morphisms $\text{Hom}_\mathcal{M}(B, C) \otimes \text{Hom}_\mathcal{M}(A, B) \to \text{Hom}_\mathcal{M}(B, C) \otimes B \xrightarrow{\epsilon} C$. Thus, any closed symmetric monoidal category is automatically enriched in the sense of \([0.12]\).

Besides, we have tensor product operations $\text{Hom}_\mathcal{M}(A, B) \otimes \text{Hom}_\mathcal{M}(C, D) \xrightarrow{\otimes} \text{Hom}_\mathcal{M}(A \otimes C, B \otimes D)$, which are given by the right adjoint of the composites $\text{Hom}_\mathcal{M}(A, B) \otimes \text{Hom}_\mathcal{M}(C, D) \otimes A \otimes C \simeq \text{Hom}_\mathcal{M}(A, B) \otimes A \otimes \text{Hom}_\mathcal{M}(C, D) \otimes C \xrightarrow{\otimes \epsilon} B \otimes D$ where we apply the symmetry operator of $\mathcal{M}$ and we form the tensor product of the evaluation morphisms associated to the hom-objects. This tensor product operation gives an extension, at the level of enriched hom-objects, of the tensor product of morphisms and satisfies the same unit, associativity, and symmetry relations.

\[ \text{In the case } \mathcal{M} = \mathcal{Top}, \text{we just need to restrict ourselves to a good category of topological spaces, such as the usual category of compactly generated spaces, in order to ensure that such an adjunction relation holds (see for instance [130], §VII.8) for an overview of this subject).} \]
The existence of the hom-object bifunctor $\text{Hom}_M(\cdot, \cdot)$ is notably useful for the study of algebras over operads and we give a short account of these applications in the first chapter of this book §I.1. In this book, we mostly study operad themselves and we mainly deal with the usual internal hom-objects associated to our fundamental examples of base model categories. To be specific, we use simplicial hom-objects, which we deduce from the standard internal hom-objects of the category of simplicial sets, when we study mapping spaces of operads in the second part of this work and we use hom-objects of dg-modules as an auxiliary device to compute the homotopy of these mapping spaces on the category of operads.

0.15. Concrete (symmetric monoidal) categories. Recall that a category $C$ is concrete when we have a faithful functor $U : C \to \text{Set}$ from this category $C$ to the category of sets $\text{Set}$. Most usual categories, which are defined in terms of point sets equipped with extra structures, are naturally equipped with the structure of a concrete category.

In what follows, we will say that a base symmetric monoidal category $M$ is concrete (as a symmetric monoidal category) when such a faithful functor $U : M \to \text{Set}$ is given by the representable functor $U = \text{Mor}_M(1, \cdot)$ associated to the unit object of our category $1 \in M$. The category of sets $M = \text{Set}$, the category of topological spaces $M = \text{Top}$, the category of modules $M = \text{Mod}$, ... are examples of concrete symmetric monoidal categories. In this situation, we regard the morphism set $U(X) = \text{Mor}_M(1, X)$ as a set of points, which we faithfully associate to any object of our category $X \in C$, and we have a natural pointwise tensor product operation $x \otimes y \in U(X \otimes Y)$, which we define by the obvious composition operation $1 \xrightarrow{\sim} 1 \otimes 1 \xrightarrow{x \otimes y} X \otimes Y$, for any $x \in U(X)$ and $y \in U(Y)$, where we consider the unit isomorphism $1 \simeq 1 \otimes 1$ of our category $M$. We just have $x \otimes y = (x, y) \in X \times Y$ in the case $M = \text{Set}, \text{Top}$, and we retrieve the standard notion of a tensor product of elements in the case $M = \text{Mod}$.

We mainly use the concept of a concrete symmetric monoidal category informally, in order to give a sense to set-theoretic tensor products which we define to illustrate some constructions of the theory. Let us mention that we can still form such set-theoretic tensor products (with some restriction) in the category of dg-modules, in the category of graded modules, in the category of simplicial modules and in the category of cosimplicial modules though these categories do not form concrete symmetric monoidal categories in the sense of our definition.

0.16. The notation of colimits, limits and universal objects. We adopt the following conventions for the notation of colimits, limits, and universal objects in categories. We generally use the unbased set notation $\emptyset$ for the initial object of a base category, the notation $\amalg$ for coproducts, and the notation $\ast$ for the terminal object. In certain situations, we use the empty set notation and we write $A = \emptyset$ to assert that an object $A$ is undefined.

We use additive category notation when we deal with additive structures, or when our base category consists of modules. We then write $0$ for the initial object of the category (the zero object). We also use $\oplus$ as a generic notation for the coproduct in the additive case.

When we deal with a category of objects equipped with a multiplicative structure (algebras, operads, ...), we generally adopt the base set notation $\vee$ for the coproduct, but we do not have any general convention for the notation of the initial object in this setting. In fact, we usually keep the notation of a particular object
of the base category which we use to effectively realize the initial object of our categories of structured objects. We can use a similar convention for coproducts when we can deduce the definition of this categorical operation from structure operations of our base category. For instance, we generally use the tensor product notation to refer to a coproduct of unitary commutative algebras, because we will observe in §1.3.0.3 that the coproduct is realized by the tensor product in this case.
Reading Guide and Overview of this Volume


These parts are widely independent from each others. Each part of this book is also divided into subparts which, by themselves, form self-contained groupings of chapters, devoted to specific topics, and organized according to an internal progression of the level of the chapters each. There is a progression in the level of the parts of the book too, but the chapters are written so that a reader with a minimal background could tackle any of these subparts straight away in order to get a self-contained reference and an overview of the literature on each of the subjects addressed in this monograph.

This volume comprises the first named part of the book, “From Operads to Grothendieck-Teichmüller Groups”, and two appendices, “Trees and the Construction of Free Operads” and “The Cotriple Resolution of Operads”, where we revisit with full details the definition of operads in terms of composition operations shaped on trees and we explain the applications of trees to the definition of universal objects in the category of operads.

The following overview is not intended for a linear reading but should serve as a guide each time the reader tackles new parts of this volume.

Part I. From Operads to Grothendieck–Teichmüller Groups. The first part of this book includes: an introduction to the fundamental concepts of the theory of operads; a survey on the definition of the little discs operads and of $E_n$-operads together with a detailed study of the connections between the little 2-disc operad and braided category structures; an introduction to the theory of Hopf algebras together with a study of the applications of Hopf algebras to the Malcev completion of groups, groupoids and operads; and a detailed account of the definition of the Grothendieck-Teichmüller group from the viewpoint of the theory of algebraic operads.

Part I(a). The General Theory of Operads. We give a detailed survey of the general definitions of the theory of operads in this part. The first chapter §1 is introductory and does not contain any original idea. We mainly explain the relationship between operads and algebras. In the second chapter §2 we explain our working definition of the notion of an operad and we give a new approach
to handle unitary operads (the operads equipped with a distinguished arity zero operation which can be used to model categories of algebras with unit). In the third chapter §3 we study the applications of general concepts of the theory of monoidal categories to operads.

Chapter 1. The Basic Concepts of the Theory of Operads. In this first chapter, we explain May’s definition of the notion of an operad as an object which governs the structure defined by collections of operations \( p = p(x_1, \ldots, x_r) \), where \( r \in \mathbb{N} \) (see §1.1). We examine the applications of usual categorical constructions to operads and we study the categories of algebras associated to operads afterwards (see §§1.2-1.3). We also recall the definition of particular instances of colimits (filtered colimits and reflexive coequalizers) which we heavily use in the theory of operads in an appendix section of this chapter (§1.4).

Chapter 2. The Definition of Operadic Composition Structures Revisited. The definition of an operad depends on composition schemes which we associate to operations \( p = p(x_1, \ldots, x_r) \). In May’s definition, which we recall in §1, we consider composition products where we can plug operations \( q_i = q_i(x_1, \ldots, x_{n_i}) \) in all inputs \( i = 1, \ldots, r \) of a given operation \( p = p(x_1, \ldots, x_r) \). This definition is perfectly suited for an introduction of the subject and for the study of algebras associated to operads. However, to work with operads themselves, we need to revisit the definition of our objects in order to get more insights into the structure of the composition products. We devote this second chapter to this subject. In a first step (§2.1), we check that the composition products of an operad, are, according to an observation of Martin Markl, fully determined by composition products on two factors (these operations are also called the partial composition products in the operad literature). In a second step (§§2.2-2.4), we explain a new approach to handle unitary operads. In short, we will see that the compositions with an extra operation of arity zero of a unitary operad can be encoded in a diagram structure associated to our object. We crucially use this observation in our study of the (rational) homotopy of operads in the second part of this book.

In general, we assume that an operad consists of a sequence of terms \( P(r) \), indexed by non-negative integers \( r \in \mathbb{N} \), and whose elements intuitively represent operations with \( r \) inputs indexed by the ordinal \( r = \{1 < \cdots < r\} \). To complete the account of this chapter, we explain an extension of the definition of an operad where terms \( P(r) \) indexed by arbitrary finite sets \( r = \{i_1, \ldots, i_r\} \) are considered (§2.5). In general, we can use bijections \( \{1 < \cdots < r\} \xrightarrow{\sim} \{i_1, \ldots, i_r\} \) to make the indexing by an arbitrary finite set \( \{i_1, \ldots, i_r\} \) equivalent to an indexing by an ordinal \( \{1 < \cdots < r\} \). Nevertheless, certain constructions on operads produce operations with no canonical input numbering and the extension of the input indexing to arbitrary finite sets becomes useful in this case. (The construction of the free operad in §A gives a motivating application of this concept.) These ideas go back to Joyal’s theory of species [98] and were first applied to operads in Getzler–Jones’s paper [77] and in Ginzburg–Kapranov’s paper [78].

Chapter 3. Symmetric Monoidal Categories and Operads. The third chapter of this part is devoted to applications of the theory of symmetric monoidal categories to the study of operads. The ideas of this chapter are not original, apart when we tackle the applications of our constructions to the new model of unitary operads which we introduced in the previous chapter.
We devote a preliminary section of the chapter (§3.0) to a survey of the definition of unitary commutative algebras and of counitary cocommutative coalgebras in symmetric monoidal categories. We examine the definition of operads in general symmetric monoidal categories afterwards (§3.1). We notably study the image of operads under functors between symmetric monoidal categories. We also survey the definition of the category of Hopf operads, which we define as the category of operads in the symmetric monoidal category of counitary cocommutative coalgebras (§3.2). The Hopf cooperads, which we consider in the summary of our mathematical objectives, are the dual structures of these Hopf operads.

We have various notions of functors associated to symmetric monoidal categories. We devote an appendix section of this chapter to a survey of this subject (§3.3).

Part I(b). Braids and $E_2$-operads. The main purpose of this part is to recall the general definition of an $E_n$-operad (by using the model of the little $n$-discs operads) and to study the connections between $E_n$-operads, Artin’s braid groups, and braided monoidal categories in the case $n = 2$.

Chapter 4. The Little Discs Model of $E_n$-operads. We recall the definition of the little $n$-discs operad and we make explicit the definition of an $E_n$-operad in the first section of this chapter (§4.1). We devote the next section of the chapter (§4.2) to a survey on the computation of the cohomology and of the homology of the little $n$-discs operad. We then give an overview of several variants of the little discs operads in an outlook section (§4.3). This chapter is mostly a survey of the literature and does not contain any original result.

We will see that the homology of an operad in topological spaces forms an operad in graded modules. In the second part of this book, we will also use that the cohomology of a topological operad with coefficients in a field inherits the dual structure of a cooperad in graded modules (when our objects satisfy mild finiteness assumptions). We also have a natural unitary commutative algebra structure on the cohomology of a space. We actually get that the cohomology of a topological operad with coefficients in a field forms a cooperad in unitary commutative algebras in graded modules (a graded Hopf cooperad) when we put this unitary commutative algebra structure and the cooperad structure together. We just make explicit the definition of the homology operad associated to the operad of little $n$-discs and the commutative algebra structure of the spaces of little $n$-discs in the second section of this chapter. (We will see that the homology of the operad of little $n$-discs is identified with an operad governing graded Poisson algebra structures.)

We use that the category of graded modules inherits a symmetric monoidal structure to give a sense to these notions of commutative algebras and of operads in graded modules. We devote an appendix section of this chapter to an account of our conventions on graded modules and to a survey of the definition of this symmetric monoidal structure on the category of graded modules (§4.4).

Chapter 5. Braids and the Recognition of $E_2$-operads. We tackle the study of the relationship between $E_2$-operads and braids in this chapter. We recall the definition of the Artin braid groups and some conventions on braids in a preliminary section of the chapter (§5.0).

We then give an account of Fiedorowicz’s definition of models of $E_2$-operads from contractible operads endowed with an action of braid groups (§5.1). We use this approach to prove that the classifying spaces of a certain operad in groupoids,
the colored braid operad, define an $E_2$-operad (§5.2). This operad of colored braids is closely related to the operad of parenthesized braids which we consider in the summary of our mathematical objectives. For the moment, simply say that the operad of colored braids is formed by groupoids whose morphism sets are identified with cosets of the pure braid groups inside the full Artin braid groups.

We prove in a second part of the chapter that the operad of colored braids is equivalent to the operad in groupoids formed defined by the fundamental groupoids of the underlying spaces of the little 2-discs operad (§5.3). We then use the adjunction between the fundamental groupoid and the classifying space construction in homotopy theory to give a second proof that the classifying spaces of the colored braid operad define an operad which is weakly-equivalent to the operad of little 2-discs, and hence, define a model of $E_2$-operad.

We can regard Fiedorowicz construction as a recognition criterion for the class of $E_2$-operads in topological spaces. We give an overview of more general recognition methods, which address the problem of giving an intrinsic definition of $E_n$-operads for all $n \geq 1$, in the concluding section of this chapter (§5.4).

**Chapter 6. The Magma and Parenthesized Braid Operads.** In the introductory chapter (§1), we recalled a general correspondence between operads and categories of algebras. In the case of an operad in the category of small categories (or groupoids), like the operad of colored braids considered in §5 the algebras are objects of the category of categories, and our operad therefore governs a class of monoidal structures which can be associated to a category. The operad of colored braids of §5 actually encodes the structure of a strict braided monoidal category, where we have a tensor product which is associative in the strict sense. The main purpose of this chapter is to explain this correspondence with full details and to give the definition of a variant of the colored braid operad, the operad of parenthesized braids, which we associate to braided monoidal categories with general associativity isomorphisms.

We first give a definition of an operad governing general monoidal categories (where the tensor product is just associative up to coherently defined isomorphisms) by elaborating on the classical Mac Lane Coherence Theorem of which we give an operadic interpretation (§6.1). We explain the definition of the operad of parenthesized braids afterwards (§6.2). To complete the account of this chapter, we also explain the definition of an operad of parenthesized symmetries which is an analogue for symmetric monoidal categories of the operad of parenthesized braids (§6.3).

**Part 1(c). Hopf Algebras and the Malcev Completion.** In this part, we revisit the fundamental results of the theory of Hopf algebras and we study the applications of Hopf algebras to the definition of a rationalization process, the Malcev completion, which extends the classical rationalization of abelian groups to general (possibly non-abelian) groups. We then check that the Malcev completion process applies to groupoids and to operads in groupoids. We use the Malcev completion of the parenthesized braid operad in our definition of the Grothendieck–Teichmüller group (in the next part).

**Chapter 7. Hopf Algebras.** We review the foundations of the theory of Hopf algebras first and we devote this first chapter of the part to this subject. We explain the definition of a Hopf algebra in the general context of additive symmetric monoidal categories enriched in $\mathbb{Q}$-modules and we check that the main results of the theory remain valid in this framework. In what follows, we mainly apply
our constructions to Hopf algebras in a category of modules over a characteristic zero field and to Hopf algebras in complete filtered modules, but our framework covers other examples of categories where Hopf algebras are usually defined in the literature (for instance, the categories of motives).

We explain the general definition of a Hopf algebra in the first section of the chapter (§7.1). We review the relationship between Hopf algebras and Lie algebras in the second section of the chapter (§7.2). We recall the definition of the enveloping algebra of a Lie algebra in the course of this study and we revisit the proof of the classical structure theorems of the theory of Hopf algebras, namely the Poincaré-Birkhoff-Witt Theorem and the Milnor-Moore Theorem. We devote the third section of chapter (§7.3) to a thorough study of the structure of Hopf algebras in complete filtered modules. We then use the phrase ‘complete Hopf algebras’ to refer to a subcategory of the category of Hopf algebras in complete filtered modules formed by objects which satisfy a natural connectedness condition. We notably consider this subcategory of complete Hopf algebras when we define our Malcev completion functor on groups.

Chapter 8. The Malcev Completion for Groups. We just examine the applications of Hopf algebras to the Malcev completion of groups in this chapter.

We first explain the definition of a general completion process on Hopf algebras. We apply this completion process to group algebras in order to get a completed group algebra functor from the category of groups to the category of complete Hopf algebras. We have a natural (complete) group-like element functor which goes the other way round, from complete Hopf algebras to groups. We precisely define the Malcev completion of a group as the group of complete group-like elements in the completed group algebra of our group. We devote the first section of the chapter to the definition of these functors (§8.1).

We also say that a group is Malcev complete when this group occurs as the image of a complete Hopf algebra under the group-like element functor. We study the category of Malcev complete groups and the properties of the Malcev completion process in the second and third sections of the chapter (§§8.2-8.3). We will notably explain that the elements of a Malcev complete group can be represented as the exponential of elements of a complete Lie algebra associated to our group. We use this correspondence to check that, in a Malcev complete group, we can define power operations $g^a$ with exponents in an arbitrary field of coefficients $a \in k$.

We devote the rest of the chapter to the study of the Malcev completion of free groups and of semi-direct products (§§8.4-8.5). In the course of this study, we also recall the definition of a counterpart, for Hopf algebras and for complete Hopf algebras, of the classical semi-direct product of groups.

Chapter 9. The Malcev Completion for Groupoids and Operads. The (complete) Hopf algebras of §§7-8 can be identified with group objects in the category of (complete) counitary cocommutative algebras. In this chapter, we introduce a generalization of this notion, which we call (complete) Hopf groupoids (see §§9.0-9.1), in order to extend the Malcev completion process of the previous chapter from groups to groupoids (§9.1). Then we check that this Malcev completion functor on groupoids preserves symmetric monoidal category structures, and as a consequence, gives rise to a Malcev completion functor on the category of operads in groupoids (§9.2). By the way, we explain the definition of an operadic version
of the classical notion of a local coefficient system. We naturally get such structures, which we call ‘local coefficient system operads’, when we study the tower decomposition of the Malcev completion of operads in groupoids.

We devote an appendix section of the chapter (§5.5) to the study of the existence of group-like elements in the complete counitary cocommutative algebras underlying a complete Hopf groupoid. We use the results of this appendix to formulate connectedness hypothesis which we naturally need for our study of complete Hopf groupoids.

Most of the statements explained in this chapter are new, though the completion of Hopf groupoids was already considered for the definition of motivic fundamental groupoids by Deligne and Deligne–Goncharov (see [52, 54]), and this chapter could also serve as a basic reference for the algebraic background of this subject.

Part I(d). The Operadic Definition of the Grothendieck–Teichmüller Group. The main goal of this part is to explain the definition of the pro-unipotent Grothendieck–Teichmüller group as a group of automorphisms associated to a Malcev completion of the parenthesized braid operad. We devote the first chapter of the part to a preliminary study of this Malcev complete operad of parenthesized braids. By the way, we explain the definition of a related operad, the operad of chord diagrams, and we give an operadic interpretation of the notion of a Drinfeld associator, which we use as equivalences of operads in Malcev complete groupoids between the Malcev completion of the parenthesized braid operad and the chord diagram operad. We also explain the definition of a graded version of the Grothendieck–Teichmüller group as a group of automorphisms associated to a parenthesized version of the chord diagram operad. We tackle the definition of the pro-unipotent Grothendieck–Teichmüller group itself in the second chapter of the part.

The Grothendieck–Teichmüller group has a pro-finite version too, which we do not really use in this work, but which is the version to be considered for the applications to the Grothendieck proposal, where the goal is to study of the absolute Galois group through geometric actions on curves. We just give an overview of this program in the concluding chapter of this part.

Most of the ideas used in this part are known to experts, but only partial references on the operadic interpretation of the pro-unipotent Grothendieck–Teichmüller group and on the Drinfeld associators were available in the literature so far.

Chapter 10. The Malcev Completion of the Braid Operads and Drinfeld’s Associators. The operad of parenthesized braids is an operad in groupoids whose morphism sets consist (like the morphism sets of the colored braid operad of §5) of cosets of the pure braid groups inside the full Artin braid groups. We therefore study the Malcev completion of the pure braid groups in the first section of this chapter (§10.0) before studying the Malcev completion of the parenthesized braid operad (and of the colored braid operad). We notably recall the definition of Lie algebra counterparts of the braid groups, which we call the Drinfeld–Kohno Lie algebras and which we denote by \( \mathfrak{p}(r) \) (some authors call these Lie algebras the ‘Lie algebras of infinitesimal braids’). We will see that, by results of Drinfeld and Kohno, the Malcev completion \( \hat{P}_r \) of the pure braid group on \( r \)-strands \( P_r \) is isomorphic to the group of exponential elements which we associate to a completion \( \hat{\mathfrak{p}}(r) \) of the Drinfeld–Kohno Lie algebra \( \mathfrak{p}(r) \).
We study the Malcev completion of the operad of parenthesized braids (and of the colored braid operad) afterwards (in §10.1). We then explain the definition of the chord diagram operad and we revisit the definition of a Drinfeld associator (§10.2). In short, we will see that the Drinfeld–Kohno Lie algebras form an operad and the chord diagram operad is an operad in Malcev complete groups which we associate to this operad in Lie algebras.

We devote the rest of the chapter to the definition of a graded version of the Grothendieck–Teichmüller group (§10.3) and to the applications of this object for the study of natural tower decompositions of the set of Drinfeld’s associators (§10.4). In the course of this study, we explain the definition of the graded Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$ which we associate to this graded Grothendieck–Teichmüller group $GRT(k)$. We use this analysis of the tower decomposition of the set of Drinfeld’s associators when we examine the outcome of the homotopy spectral sequence associated to the space of homotopy automorphisms of $E_2$-operads in the third part of this book.

Chapter 11. The Grothendieck–Teichmüller Group. We explain the operadic definition of the pro-unipotent Grothendieck–Teichmüller group in the first section of this chapter (§11.1). We precisely check that this group $GT(k)$, such as defined by Drinfeld, can be identified with a group of operad automorphisms associated to the Malcev completion of the operad of parenthesized braids. We study a natural action of the group $GT(k)$ on the set of Drinfeld’s associators afterwards (in §11.2). We use this action to check, after Drinfeld, that the pro-unipotent Grothendieck–Teichmüller group $GT(k)$ is isomorphic to the graded Grothendieck–Teichmüller group studied in the previous chapter $GRT(k)$.

Then we explain the definition of a tower decomposition of the Grothendieck–Teichmüller group $GT(k) = \lim_m GT_{(m)}(k)$ (§11.3). This tower decomposition is associated to a filtration of the group $GT(k)$ by normal subgroups $F_m GT(k)$. We explicitly have $GT_{(m)}(k) = GT(k)/F_m GT(k)$ for all $m \geq 0$. We actually have a pro-unipotent structure on the first layer of this filtration $GT^1(k) = F_1 GT(k)$ (not on the whole group $GT(k)$), which we can also identify with the kernel of a natural character map $\lambda : GT(k) \to k^\times$ on the group $GT(k)$. To complete the account of this chapter, we check that the subquotients $E_m^0 GT(k) = F_m GT(k)/F_{m+1}(k)$, $m \geq 1$, form a weight graded Lie algebra which is isomorphic to the graded Grothendieck–Teichmüller Lie algebra of the previous chapter $\mathfrak{grt}$. (We address this subject in §11.4)

Chapter 12. A Glimpse at the Grothendieck Program. This chapter serves as a conclusion for this volume. We provide a brief introduction to the Grothendieck program in Galois theory and we give an overview of the literature about the connections between Grothendieck–Teichmüller groups, motivic Galois groups, and multizetas.

Appendix A. Trees and the Construction of Free Operads. In this appendix, we explain the applications of trees to the definition of universal constructions in the category of operads. We make our definition of a tree precise in a preliminary section (§A.1). We explain the definition of general treewise composition operations associated to operads afterwards (in §A.2). We then explain the applications of trees to the definition of free objects in the category of operads (§§A.3A.4) and to the definition of coproducts with free objects (§A.5).
Most of the ideas which we use in this appendix are not original, as the applications of trees for the study of operads go back to Stasheff’s work on the recognition of loop spaces \([167]\) and to Boardman–Vogt’s work on homotopy invariant structures \([28]\). The definition of free operads in terms of trees, in particular, is due to Ginzburg–Kapranov \([78]\). We just give a new definition of reduced free objects in the context of unitary operads.

Appendix \(B\) The Cotriple Resolution of Operads. In this appendix, we explain the definition of a simplicial resolution functor on the category of operads, the cotriple resolution, which we use in our study of the homotopy of operads in the next parts of this book. We restrict our analysis to the case of connected operads for simplicity. In short, we prove that the cotriple resolution of an operad has an explicit description which we obtain by inserting extra structures, modelled by chains of tree morphisms, in our previous construction of free operads. We explain the definition of our notion of a tree morphism in the first section of this appendix (§B.0) and we tackle the applications to the cotriple construction afterwards (§B.1).

The free operad functor inherits a natural composition operation which makes this object a monad on the category of symmetric sequences (underlying the category of operads). We recall the definition of the concept of a monad in §B.2 and we prove that the structure of an operad can also be defined in terms of an action of this free operad monad on a symmetric sequence. In the language of category theory, this result asserts that the category of operads is monadic.

Most of the results explained in this appendix are known to experts (like the constructions of the previous appendix). We still just give a new definition of a reduced version of the cotriple resolution for unitary operads.
Glossary of Notation

Background

Fundamental objects

$k$: the ground ring
$D^n$: the unit $n$-disc, see §I.4.1.1
$\Delta^n$: the topological $n$-simplex, see §0.3 §II.1.3.4
$pt$: the one-point set (also denoted by $*$ when regarded as a terminal object)
$\Delta$: the simplicial category, see §0.3 §II.1.3.2
$\Delta^n$: the $n$-simplex object of the category of simplicial sets, see §0.3 §II.1.3.4

Generic categorical notation

$A, B, C, \ldots$: general categories
$I, J, \ldots$: indexing categories, as well as the set of generating cofibrations and the set of generating acyclic cofibrations in a cofibrantly generated model category, see §II.4.1.3
$F, G, \ldots$: some classes of morphisms in a category
$M, N, \ldots$: (symmetric) monoidal categories, see §0.8
$1$: the unit object of a (symmetric) monoidal category, see §0.8
$eq$: the equalizer of parallel arrows in a category
$coeq$: the coequalizer of parallel arrows in a category

Fundamental categories

$\text{Mod}$: the category of modules over the ground ring
$\text{Set}$: the category of sets
$\text{Top}$: the category of topological spaces, see §II.1.3
$\text{sSet}$: the category of simplicial sets, see §0.3 §II.1.3
$\text{Grp}$: the category of groups
$\text{Grd}$: the category of groupoids, see §I.5.2.1
$\text{Cat}$: the category of small categories, see §I.5.2.1
$\text{Ab}$: the category of abelian groups

Categories of algebras and of coalgebras

$\text{Com}$: the category of non-unitary commutative algebras
$\text{As}$: the category of non-unitary associative algebras
$\text{Lie}$: the category of Lie algebras
$\text{As_+}$: the category of unitary associative algebras
$\text{Com_+}$: the category of unitary commutative algebras, see §I.3.0.1
$\text{Com}_{c+}$: the category of counitary cocommutative coalgebras, see §I.3.0.4
**HopfAlg**: the category of Hopf algebras (defined as the category of bialgebras equipped with an antipode operation), see §I.7.1.8

**HopfGpd**: the category of Hopf groupoids (defined as the category of small categories equipped with an antipode operation), see §I.9.0.2

### Functors and constructions for filtered objects

- **Fs**: the sth layer of a decreasing filtration
- **Es**: the sth subquotient of a filtered object, see §I.7.3.6 (also used to denote the sth fiber of a tower of set maps in the context of homotopy spectral sequences, see §III.11.7)
- **E**: the weight graded object associated to a filtered object in a category (e.g. the weight graded module associated to a filtered module, see §I.7.3.6 the weight graded Lie algebra associated to a Malcev complete group, see §I.8.2.2, ...)
- **(−)♯**: the completion functor on a category of objects equipped with a decreasing filtration, as well as the Malcev completion for groups and groupoids, see §I.7.3.4, §I.8.3 (also the rationalization functor on spaces, see the section about the constructions of homotopy theory in this glossary)

### Functors and constructions on algebras and coalgebras

- **S**: the symmetric algebra functor (in any symmetric monoidal category), see §I.7.2.4
- **T**: the tensor algebra functor (in any symmetric monoidal category), see §I.7.2.4
- **L**: the free Lie algebra functor (in any Q-additive symmetric monoidal category and in abelian groups), see §I.7.2.3
- **U**: the enveloping algebra functor (on the category of Lie algebras in any Q-additive symmetric monoidal category), see §I.7.2.7
- **S, T, ...**: the complete variants of the symmetric algebra functor, of the tensor algebra functor, ... in the context of a category of complete filtered modules, see §I.7.3.22
- **G**: the group-like element functor on coalgebras, see §I.7.1.14 and on complete Hopf coalgebras, see §I.8.1.2
- **P**: the primitive element functor on Hopf algebras, see §I.7.2.11
- **I(−)**: the augmentation ideal of Hopf algebras, see §I.8.1.1

### Categorical prefixes

- **dg**: prefix for a category of differential graded objects in a category (e.g. the category of dg-modules \(dg\)Mod, see §I11.5.0.1)
- **dg**, **dg***: prefix for the chain graded and cochain graded variants of the categories of differential graded objects (e.g. the category of chain graded dg-modules \(dg_\ast\)Mod, see §II.5.0.1, the category of cochain graded dg-modules \(dg^\ast\)Mod, see §II.5.0.1, §II.5.1, and the category of unitary commutative cochain dg-algebras \(dg^\ast\)Com+, see §II.6.1.1, ...)
- **gr**: prefix for a category of graded objects in a category when the grading underlies a differential graded structure (e.g. the category of graded modules \(gr\)Mod, see §I.11.5.0.2, ...)
- **s**: prefix for a category of simplicial objects in a category (e.g. the category of simplicial modules \(s\)Mod, see §I.3, §II.5.0.4, the category of simplicial sets \(s\)Set, see §II.5.3, §II.13, ...)
c: prefix for a category of cosimplicial objects in a category (e.g. the category of cosimplicial modules \(c\text{Mod}\), see §0.6, §II.5.0.4 the category of cosimplicial unitary commutative algebras \(c\text{Com}^+\), see §II.6.1.3 ...)

\(f\): prefix for a category of filtered objects in a category (e.g. the category of filtered modules \(f\text{Mod}\), see §I.7.3.1)

\(\hat{f}\): prefix for a category of complete filtered objects in a category (e.g. the category of complete filtered groups \(\hat{f}\text{Grp}\), see §I.7.2). Note that the categories of complete Hopf algebras \(\hat{f}\text{HopfAlg}\) and of complete Lie algebras \(\hat{f}\text{Lie}\) consist of Hopf algebras and Lie algebras in complete filtered modules that satisfy an extra connectedness requirement and a similar convention is made for the category of complete Hopf groupoids \(\hat{f}\text{HopfGrpd}\), see §I.7.3.15, §I.7.3.20, §I.9.1.2

\(w\): prefix for a category of weight graded objects in a category (e.g. the category of weight graded modules \(w\text{Mod}\), see §I.7.3.5)

Morphisms, hom-objects, duals, and analogous constructions

\(\text{Mor}\): the notation for the morphism sets of any category (e.g. \(\text{Mor}_{\text{Mod}}(-,-)\) for the morphism sets of the category of modules over the ground ring \(\text{Mod}\))

\(\text{Aut}\): the notation for the automorphism group of an object in a category

\(\text{Hom}\): the notation for the hom-objects of an enriched category structure (not to be confused with the morphism sets), see §0.12

\(\text{D}\): the duality functor for ordinary modules, dg-modules, simplicial modules and cosimplicial modules, see §II.5.0.13

\((-)^\vee\): the dual of individual objects, or of objects equipped with extra structures (algebras, operads, ...), see §II.5.0.13

\(\text{Der}\): the modules of derivations (for algebras, operads, ...), see §III.2.1

\(\text{Map}, \text{Aut}^h\): see the section of this glossary about the constructions of homotopy theory

Constructions of homotopy theory

Fundamental constructions in model categories

\(\text{Ho}(-)\): the homotopy of a model category, see §II.1.2

\(\text{Aut}^h\): the notation for the homotopy automorphism space of an object in a model category, see §II.2.2

\(\text{Map}\): the notation for the mapping spaces of a pair of objects in simplicial model categories and in general model categories, see §II.2.1, §II.3.2.11

Fundamental simplicial and cosimplicial constructions

\(B\): the classifying space construction for groups, groupoids, categories, ..., see §I.5.2.3 (also the bar construction of algebras and of operads, see the relevant sections of this glossary)

\(\text{sk}_r\): the \(r\)th skeleton of a simplicial set, of a simplicial and of a cosimplicial object in a model category, see §III.3.8, §II.8.1.7, §II.8.1.17

\(\text{Tot}\): the totalization of cosimplicial spaces, of cosimplicial objects in a model category, see §II.5.3.14
| − |: the geometric realization of simplicial sets, of simplicial objects in a model category, see §[0.5] II[1.3.5] II[3.3.5]

\**Diag:** the diagonal complex of a bisimplicial set, of a bisimplicial object and of a bicosimplicial object in a model category, see §II[3.3.19]

\**L_\* (X):** the \( r \)th latching object of a simplicial object in a category, see §II[3.1.14]

\**M_\* (X):** the \( r \)th matching object of a simplicial object in a category, see §II[3.1.15]

(also the matching objects of \( \Lambda \)-sequences, see the section about operads and related structures of this glossary)

\**\( \Lambda^r (X), M^r (X):** the cosimplicial variants of the matching and matching object constructions, see §II[3.1.3] §II[3.1.5]

**Differential graded constructions**

\( b^m, e^m \): notation for particular homogeneous elements (of upper degree \( m \)) notably used to define the generating (acyclic) cofibrations of the category of cochain graded dg-modules, see §II[5.1.2]

\( B^m, E^m \): as \( b^m, e^m \) but in the chain graded context

\( B^m \): source objects of the generating cofibrations of the category of cochain graded dg-modules, see §II[5.1.2]

\( E^m \): target objects of the generating (acyclic) cofibrations of the category of cochain graded dg-modules, see §II[5.1.2]

\( \sigma_m, \rho_m \): dual objects of the dg-modules \( B^m \) and \( E^m \)

\( \sigma \): notation for particular homogeneous elements used in the definition of suspension functors on dg-modules, see §II[3.3]

\( \rho_r, \rho_s \): notation for particular homogeneous elements used in the definition of the operadic suspension functor for operads in dg-modules, see §II[4.1.1]

\**Cyl:** the standard cylinder object functor on the category of dg-modules, see §II[13.1.10]

\**B:** the bar construction for algebras, see §II[6.3] (also the classifying space of groups, categories, and the bar construction of operads, see the relevant sections of this glossary)

\( \tau_* : dgMod \to dg_* Mod \) of the embedding \( \iota : dg_* Mod \hookrightarrow dgMod \) of the category of chain graded dg-modules \( dg_* Mod \) into the category of all dg-modules \( dgMod \), see §II[5.3.2]

\( \tau^* : dgMod \to dg^* Mod \) of the embedding \( \iota : dg^* Mod \hookrightarrow dgMod \) of the category of cochain graded dg-modules \( dg^* Mod \) into the category of all dg-modules \( dgMod \), see §II[5.0.1]

\( (-)_{\flat} : \) the forgetful functor from dg-modules to graded modules, see §II[0.1]

**The Dold–Kan correspondence**

\**N_\* :** the normalized chain complex functor on the category of simplicial modules, see §II[0.6] §II[5.0.5]

\**N^* :** the conormalized cochain complex functor on the category of cosimplicial modules, see §II[5.0.9]

\**\( \Gamma_\* : \)** the Dold–Kan functor on the category of chain graded dg-modules, see §II[5.0.6]

\**\( \Gamma^* : \)** the cosimplicial version of the Dold–Kan functor on the category of cochain graded dg-modules, see §II[5.0.9]
Constructions of rational homotopy theory

$(-)^\wedge$: the rationalization functor on spaces, see §II.10.2, §II.12.2 (also the completion of filtered objects, see the section of this glossary about the background of our constructions)

$\Omega^\bullet$: the Sullivan cochain dg-algebra functor on simplicial sets, see §II.7.1

$\Omega^\bullet\#$: the operadic upgrade of the cochain dg-algebra functor on operads in simplicial sets, see §II.10.1, §II.12.1

$G^\bullet$: the functor from cochain dg-algebras to simplicial sets, see §II.7.2

$MC^\bullet$: the Maurer–Cartan spaces associated to (complete) Lie algebras, see §II.13.1.8

Operads and related structures

Indexing of operads

$\Sigma_r$: the symmetric group on $r$ letters

$\Sigma$: the category of finite ordinals and permutations, see §I.2.2.3

$\Lambda$: the category of finite ordinals and injections, see §I.2.2.2

$\Lambda^+$: the category of finite ordinals and increasing injections, see §I.2.2.2

$\Sigma_{>0}, \Sigma_{>1}, \Lambda_{>0}, \Lambda_{>1}, \ldots$: the full subcategory of the category $\Sigma, \Lambda, \ldots$ generated by the ordinals of cardinal $r > 0$, $r > 1$, see §I.2.2.2, §I.2.4.1

$B_{ij}$: the category of finite sets and bijections, see §I.2.5.1

$B_{ij}, B_{ij}, I_{nj}, I_{nj}, \ldots$: the full subcategory of the categories $B_{ij}, I_{nj}, \ldots$ generated by the finite sets of cardinal $r > 0$, $r > 1$, see §I.2.5.9

$m, n, \ldots, r, \ldots$: generic notation for finite ordinals $r = \{1 < \cdots < r\}$ or for finite sets $r = \{i_1, \ldots, i_r\}$ used to index the terms of operads, symmetric sequences and $\Lambda$-sequences

$0, 1, 2, \ldots$: the empty ordinal, the ordinal of cardinal one $1 = \{1\}$, of cardinal two $2 = \{1 < 2\}$, ...

Categories of operads and related

$Op$: the category of (symmetric) operads, see §I.1.1.2

$Op_{\Lambda}$: the category of non-unitary (symmetric) operads, see §I.1.1.20

$Op_{\Lambda1}$: the category of connected (symmetric) operads, see §I.1.1.21

$Op_{\Lambda1}$: the category of (symmetric) cooperads, see §II.9.1.8

$\Lambda/Op_{\Lambda} / Com$: the category of augmented non-unitary $\Lambda$-operads (the postfix expression $- / Com$ can be discarded when the augmentation is trivial), see §II.2.2.17

$\Lambda/Op_{\Lambda1} / Com$: the category of augmented connected $\Lambda$-operads (the postfix expression $- / Com$ can be discarded when the augmentation is trivial), see §II.2.2.17

$Seq$: the category of symmetric sequences, see §I.2.2.2

$Seq_{>0}$: the category of non-unitary symmetric sequences, see §II.12.13

$Seq_{>1}$: the category of connected symmetric sequences, see §II.12.13

$Seq^c, Seq_{>0}, Seq_{>1}$: same as $Seq, Seq_{>0}, Seq_{>1}$ but used instead of this notation in the context of cooperads

$\Lambda Seq$: the category of $\Lambda$-sequences, see §I.2.3

$\Lambda Seq_{>0}$: the category of non-unitary $\Lambda$-sequences, see §I.2.3

$\Lambda Seq_{>1}$: the category of connected $\Lambda$-sequences, see §II.2.4.1

$\Lambda Seq^c$: the category of covariant $\Lambda$-sequences
$\Lambda Seq^c_{>0}$: the category of covariant non-unitary $\Lambda$-sequences
$\Lambda Seq^c_{>1}$: the category of covariant connected $\Lambda$-sequences, see §II.11.7
$\mathfrak{Coll}$: the category of (symmetric) collections, see §I.2.5.1
$\mathfrak{Coll}_{>0}$: the category of non-unitary (symmetric) collections
$\mathfrak{Coll}_{>1}$: the category of connected (symmetric) collections

**Categories of Hopf operads and related**

$\mathcal{H}opf Op$: the category of Hopf operads (defined as the category of operads in counitary cocommutative coalgebras), see §I.3.2
$\mathcal{H}opf Op_{\varnothing}, \mathcal{H}opf Op_{\varnothing 1}$: the non-unitary and connected variants of the category of Hopf operads
$\mathcal{H}opf \Lambda Op_{\varnothing}, \mathcal{H}opf \Lambda Op_{\varnothing 1}$: the $\Lambda$-operad variants of the categories of non-unitary and connected Hopf operads
$\mathcal{H}opf Seq$: the category of Hopf symmetric sequences (defined as the category of symmetric sequences in counitary cocommutative coalgebras), see §III.2.16
$\mathcal{H}opf Seq^c_{>0}$, $\mathcal{H}opf Seq^c_{>1}$: the non-unitary and connected variants of the category of Hopf symmetric sequences
$\mathcal{H}opf \Lambda Seq^c_{>0}, \mathcal{H}opf \Lambda Seq^c_{>1}$: the $\Lambda$-sequence variants of the categories of non-unitary and connected Hopf symmetric sequences
$\mathcal{H}opf Op_{\varnothing 1}$: the category of Hopf cooperads (defined as the category of cooperads in unitary commutative algebras), see §II.9.3.1
$\mathcal{H}opf \Lambda Op_{\varnothing 1}$: the category of Hopf $\Lambda$-cooperads (defined as the category of cooperads in unitary commutative algebras), see §II.11.4.1
$\mathcal{H}opf Seq^c_{>1}$: the category of connected Hopf symmetric sequences underlying Hopf cooperads (defined as the category of symmetric sequences in unitary commutative algebras), see §II.9.3.1
$\mathcal{H}opf \Lambda Seq^c_{>1}$: the category of connected Hopf $\Lambda$-sequences underlying Hopf $\Lambda$-cooperads (defined as the category of $\Lambda$-sequences in unitary commutative algebras), see §II.11.4.1

**Notation of operads**

$P, Q, \ldots$: generic notation for operads (of any kind)
$M, N, \ldots$: generic notation for symmetric sequences, $\Lambda$-sequences, covariant $\Lambda$-sequences
$C, D, \ldots$: generic notation for cooperads (of any kind)
$C_n$: the operad of little $n$-cubes, see §I.1.1.3
$D_n$: the operad of little $n$-discs, see §I.1.1.7
$As$: the (non-unitary) associative operad, see §II.11.16, §II.12.6, §II.12.10
$Com$: the (non-unitary) commutative operad, see §II.11.16, §II.12.6, §II.12.10, §II.2.11
$Lie$: the Lie operad, see §II.12.10
$Pois$: the Poisson operad, see §I.1.2.12
$Gerst_n$: the $n$-Gerstenhaber operad (defined as a graded variant of the Poisson operad), see §I.12.13
$Com^c$: the commutative cooperad, see §II.9.1.3
$CoS, PaS, CoB, PaB, \ldots$: see the section about the applications of operads to the definition of Grothendieck–Teichmüller groups
Constructions on operads and on cooperads

\( \tau \): the truncation functors from non-unitary operads to connected operads and from augmented non-unitary \( \Lambda \)-operads to augmented connected \( \Lambda \)-operads, see §I.2.15, Proposition I.2.4.5

\( \Theta \): the free operad functor, see §A.3

\( \Theta^c \): the cofree cooperad functor, see §C.1

\( \Theta(M) \): the treewise tensor product of a symmetric sequence \( M \) over a tree \( T \) when regarded as a term of the free operad and of the cofree cooperad (same as the object denoted by \( M(T) \) in the section about trees), see §A.2

\( \Sigma F^r \): the \( r \)th free symmetric sequence, see §II.8.1.2

\( \Lambda F^r \): the \( r \)th free \( \Lambda \)-sequence, see §II.8.3.6

\( \partial \Lambda F^r \): the boundary of the \( r \)th free \( \Lambda \)-sequence, see §II.3.7

\( \partial' \Lambda F^r \): the boundary of the \( r \)th free \( \Lambda \)-sequence in the context of connected \( \Lambda \)-sequences, see §II.12.0.1

\( \text{Res} \): the cotriple resolution functor on operads, see §B.1, §II.8.5

\( B \): the bar construction of operads, see §C.2 (also the classifying space of groups, categories, and the bar construction of algebras, see the relevant sections of this glossary)

\( B^c \): the cobar construction of cooperads, see §C.2

\( K \): the Koszul dual of operads, see §C.3

\( M(r) \): the \( r \)th matching object of a \( \Lambda \)-sequence, see §II.8.3.11

\( \text{ar} \leq s \): the \( s \)th layer of the arity filtration of a \( \Lambda \)-sequence, see Proof of Theorem II.8.3.20

\( \text{ar}^\Lambda \leq s \): the operadic upgrade of the arity filtration, see Proof of Theorem II.8.4.12

\( \text{cosk}_\Lambda \): the \( r \)th \( \Lambda \)-coskeleton of a \( \Lambda \)-sequence, see §II.8.3.3 of an augmented non-unitary \( \Lambda \)-operad, see Proof of Theorem II.8.4.12

Trees

\( \text{Tree}(r) \): the category of \( r \)-trees (where \( r \) is the indexing set of the inputs of the trees), see §A.1

\( \overline{\text{Tree}}(r) \): the category of reduced \( r \)-trees (where \( r \) is the indexing set of the inputs of the trees), see §A.1.12

\( \overline{\text{Tree}} \): the operad of reduced trees, see §A.1.12

\( \text{Tree}^o(r) \): the category of planar \( r \)-trees (where \( r \) is the indexing set of the inputs of the trees), see §A.3.16

\( \text{Tree}^o \): the operad of planar trees, see §A.3.16

\( S, T, \ldots \): generic notation for trees

\( \downarrow \): the unit tree (the tree with no vertex), see §A.1.4

\( \overline{Y} \): the notation of a corolla (a tree with a single vertex), see §A.1.4

\( \overline{\Gamma} \): the notation of a tree with two vertices, see §A.2.3

\( V(T) \): the vertex set of a tree

\( E(T) \): the edge set of a tree

\( \hat{E}(T) \): the set of inner edges of a tree

\( L \): the set of ingoing edges of a vertex in a tree

\( M(T) \): the treewise tensor product of a symmetric sequence \( M \) over a tree \( T \) (same as the object denoted by \( \bigodot_T(M) \) in the section about constructions on operads and on cooperads), see §A.2
\( \lambda_T \): the treewise composition products associated to an operad, see §A.2.7
\( \rho_T \): the treewise composition coproducts associated to a cooperad, see §C.1.5

**From operads to Grothendieck–Teichmüller groups**

*Permutations, braids, and related objects*

- \( \Sigma_r \): the symmetric group on \( r \) letters
- \( B_r \): the Artin braid group on \( r \) strands, see §I.5.0
- \( P_r \): the pure braid group on \( r \) strands, see §I.5.0
- \( p(r) \): the \( r \)th Drinfeld–Kohno Lie algebras (the Lie algebra of infinitesimal braids on \( r \) strands), see §I.10.0.2
- \( \hat{p}(r) \): the complete Drinfeld–Kohno Lie algebra, see §I.10.0.6
- \( p_n(r) \): the graded variants of the Drinfeld–Kohno Lie algebras (with \( p(r) = p_2(r) \)), see §II.14.1.1
- \( p \): the Drinfeld–Kohno Lie algebra operad, see §II.10.1.1
- \( p_n \): the graded variants of the Drinfeld–Kohno Lie algebra operad (with \( p = p_2 \)), see §II.14.1.1
- \( \hat{p} \): the complete Drinfeld–Kohno Lie algebra operad, see §II.10.2.2
- \( \text{CoS} \): the operad of colored symmetries, see §I.6.3
- \( \text{PaS} \): the operad of parenthesized symmetries, see §I.6.3
- \( \text{CoB} \): the operad of colored braids, see §§I.5.2.8, I.5.2.11, §I.6.2.7
- \( \text{PaB} \): the operad of parenthesized braids, see §I.6.2
- \( \text{CoB} \hat{}, \text{PaB} \hat{} \): the Malcev completion of the colored and parenthesized braid operads, see §I.10.1
- \( \text{CD} \hat{} \): the operad of chord diagrams, see §I.10.2.4
- \( \text{PaCD} \hat{} \): the operad of parenthesized chord diagrams, see §I.10.3.2

*Grothendieck–Teichmüller groups and related objects*

- \( \text{Ass}(k) \): the set of Drinfeld’s associators, see §I.10.2.11
- \( \text{GT}(k) \): the pro-unipotent Grothendieck–Teichmüller group, see §I.11.1
- \( \text{GRT} \): the graded Grothendieck–Teichmüller group, see §II.10.3
- \( \text{GT} \hat{} \): the profinite Grothendieck–Teichmüller group
- \( \text{grt} \): the graded Grothendieck–Teichmüller Lie algebra, see §II.10.4.6, §II.11.4
Bibliography


and Pierre Lochak.


Index

$E_1$-operads

- and the associative operad, 140
- recognition of, 140

$E_2$-operads, see also the entry on

- $E_n$-operads for the general definition
  - in terms of the little 2-discs operad
  - and the symmetrization of braided operads, 176
- recognition of, 176

$E_n$-operads, 139

$E_\infty$-operads

- and the commutative operad, 140
- recognition of, 140

$\Lambda$-operads, 70, 73

- augmented non-unitary, see also
  - augmented non-unitary $\Lambda$-operads
- non-unitary, see also non-unitary
  - $\Lambda$-operads

$\Lambda$-sequences, 74

- augmented connected, see also
  - augmented connected $\Lambda$-sequences
- augmented non-unitary, see also
  - augmented non-unitary $\Lambda$-sequences,
    see also augmented non-unitary $\Lambda$-sequences
- connected, see also connected
  - $\Lambda$-sequences
- non-unitary, see also non-unitary
  - $\Lambda$-sequences

$n$-Gerstenhaber operad, 147, 148

- and the homology of the little $n$-discs operad, 148

$n$-Poisson operad, see also $n$-Gerstenhaber operad

additive operads, 891

- algebras over an operad, 17, 18, 38
- colimits of, 111
- extension of structure of, 111
- filtered colimits of, 111
- free, 89, 111, 19, 29
- limits of, 111
- reflexive coequalizers of, 111
- restriction of structure of, 111

antipodes, 232, 235

- in Hopf groupoids, 312
- left, 232
- right, 232

arity, 6

aritywise tensor products

- of Hopf operads, 115
- of operads, 114
- of symmetric sequences, 116

associahedra, 204

associative operad, 13, 14, 30, 32, 33

- algebras over the, 38
- and $E_1$-operads, 140
- and the little 1-discs operad, 134, 147
- and the permutation operad, 150, 30, 58
- as a Hopf operad, 119
- presentation of the, 30, 92
- presentation of the unitary, 89

associativity isomorphism

- of the parenthesized braid operad, 211
- of the parenthesized chord diagram operad, 372
- of the parenthesized permutation operad, 202
- of the parenthesized symmetry operad, 221

augmentation ideal of a connected operad, 85, 129

augmentation ideal of an augmented connected $\Lambda$-operad, 82

augmentation morphisms on operads, 58

- and free operads, 498
- and the cotriple resolution, 496
- and tree morphisms, 496
- and trees, 498
- and treewise composition products, 451
- and treewise tensor products, 450
- treewise representation of the, 60, 150
- treewise representation of the connected, 60, 150

augmented connected $\Lambda$-operads

- and colimits, 57
- and free operads, 82, 85, 100
category of finite ordinals and injections, 59, 60, 81
and composition products, 66, 69
category of finite sets and bijections, 90
and composition products, 93, 94
chord diagram
and the Malcev completion of the
colored braided operad, 219
thesis, see also parentheses braided
chord diagram operad
tower decomposition of the, 358
chord diagrams
algebras of, 358

Baker-Campbell-Hausdorff formula, 291
bialgebras, 231, 232
braid diagrams, 161
braid groups, 161–167
and the homotopy of configuration
spaces, 161–167
generators of the, 165
presentation of the, 165
pure, 165, 167
braid isotopies, 163
braid operad, 169
braided operads, 167
and the recognition of $E_2$-operads, 176
and the universal coverings of the little
2-discs operad, 176
restriction operators on, 176
symmetrization of, 176
braiding isomorphism
of the parentheses braided operad, 211
braids, 161
block, 161, 167, 169
concatenation of, 169
direct sums of, 169

generating, 165

generating pure, 341

identity, 163
isotopies of, 163
categories
closed symmetric monoidal, 331
concrete, 331
concrete symmetric monoidal, 331
enriched, 331
monadic, 341
symmetric monoidal, 341

complete enveloping algebras, 273
image of — in weight graded modules, 275
complete filtered modules, 261
and weight graded modules, 261
complete colimits of, 261
direct sums of, 261
inclusions of, 261
quotients of, 261
symmetric monoidal category of, 266
tensor products of, 266
tower decomposition of, 261
complete free Lie algebras, 273
and the Malcev completion of free groups, 296
complete group algebras, 279
adjunction between — and the groups of group-like elements of complete Hopf algebras, 279
and cartesian products, 283
and semi-direct products, 303
of abelian groups, 281
of the Malcev completion of free groups, 290
complete Hopf algebras, 267–268, 278
and the Milnor-Moore Theorem, 275
and the Poincaré-Birkhoff-Witt Theorem, 275
and the Structure Theorem of Hopf algebras, 274
exponential correspondence in, 280–281
group-like elements in, 279
morphisms of the groups of group-like elements in, 281
complete Hopf categories
symmetric monoidal category of, 329
tensor products of, 329
complete Hopf groupoids, 316–319
adjunction between — and the groupoids of group-like elements in complete Hopf groupoids, 321
associated to groupoids, 321
and cartesian products, 329
equivalences of, 317
morphisms of the groupoids of group-like elements in, 322
symmetric monoidal category of, 329
tensor products of, 329
complete Lie algebras, 270–271
complete symmetric algebras, 273
group-like elements of, 281
image of — in weight graded modules, 275
complete tensor algebras, 273
and the Malcev completion of free groups, 296–299
image of — in weight graded modules, 275
complete unitary associative algebras, 271
complete unitary commutative algebras, 271
composition products of operads, 10
and augmentations, 60
and edge contractions, 448
and restriction operators, 69
and trees, 438–439
associativity relations, 55
equivariance relations, 199
full, 71–72
partial, 152
treewise, 224
treewise representation of the, 50
unit relations, 52–53
composition products of trees, 438
and tree morphisms, 439
and treewise tensor products, 454
concrete symmetric monoidal categories, 330
configuration spaces, 141
and the little n-discs operad, 142
cohomology of the, 142–144
homotopy of, 160
homotopy of — and braid groups, 161–165
restriction operators on, 142–144
connected A-sequences, 81
augmented, see also augmented connected A-sequences
connected Hopf A-operads, 124
connected Hopf operads, 121
connected operads, 23
connected operads, 37
and colimits, 37
and free operads, 38
as a monadic category, 50
augmentation ideal of, 35
coproducts of, 474
cotriple resolution of, see also cotriple resolution of operads
free, 36–363
free, 36–363
free, 36
free, 36
free, 36
free, 36
free, 36
free, 36
corollas, 433
treewise composition products of, 160–160
connected symmetric sequences, 39
and free operads, 38
connected symmetric sequences, 39
and free operads, 38
connected operads, 23
and free operads, 36
free, 36
connected truncation of augmented non-unitary A-operads, 86
connected truncation of non-unitary operads, 87
connected unitary operads, 22
and free operads, 85
connected unitary operads, 22
and free operads, 85
cosimplicial modules, 330
cosimplicial objects
cotriple resolution of operads, 486, 501
and augmented connected Λ-operad structures, 496
and augmented non-unitary Λ-operad structures, 496
and subtree decompositions, 430
and tree morphisms, 482
extra-degeneracies on the, 493
for general (non-augmented) operads, 497
latching objects of the, 495
latching objects of the, 495–496
restriction operators on the, 496
counitary cocommutative coalgebras, 102, 103, 228, 299
and lax symmetric comonoidal functors, 113
cartesian products of, 103, 249
group-like elements in, 105
in a symmetric monoidal category, 102
operads in, see also Hopf operads
symmetric monoidal category of, 103, 229
tensor products of, 103, 229
dg-modules, xxi
disc center mapping, 141, 161
dodecagon relation, 217
Drinfeld’s associators, 363
action of the graded
Grothendieck–Teichmüller group on, 383, 385
action of the Grothendieck–Teichmüller group on, 408
existence of, 364
existence of rational, 394
fibers of the tower decomposition of the set of, 396, 397
hexagon relations of the, 360, 363
involution relation of, 360, 363
tower decomposition of the set of, 387
unit relations of, 360, 363
Drinfeld’s hexagon, see also hexagon relations
Drinfeld–Kohno Lie algebra operad, 350, 351
unitary version of the, 351
weight graded, 350
Drinfeld–Kohno Lie algebras, 342
and the Malcev completion of the pure braid groups, 444, 446
complete, 445
conjugation actions on the, 445
presentation of the, 442
edge contractions in a tree
and composition products of operads, 496, 499
and tree morphisms, 431
edge set of a tree, 431
edges
contractions of — and composition products of operads, 448
endomorphism operads, 15, 18
enriched categories, xxviii
enveloping algebras, 242
and semi-direct products, 307
and the Poincaré-Birkhoff-Witt Theorem, 255
and the representations of Lie algebras, 243
complete, see also complete enveloping algebras
Hopf algebra structure on, 244
in complete filtered modules, see also complete enveloping algebras
primitive elements of, 250
Eulerian idempotents, 250
exponential correspondence
and the Baker-Campbell-Hausdorff formula, 291
exponential correspondence in complete
Hopf algebras, 280, 281
extension of structure, 42
filtered colimits, 13
and multifunctors, 13
of algebras over an operad, 11
of augmented non-unitary Λ-operads, 79
of operads, 26
filtered modules, 259
and towers, 260
and weight graded modules, 263
complete, see also complete filtered modules
completion of, 261
inclusions of, 259
quotients of, 259
symmetric monoidal category of, 260
tensor products of, 265
framed little $n$-discs operad, 156
free algebras over an operad, 139, 140, 143, 73
free Lie algebras, 40, 243
and tensor algebras, 243, 246
completion of, 272
in complete filtered modules, see also complete free Lie algebras
in weight graded modules, 269
weight decomposition of, 263
free module functor, 105
free non-symmetric operads, 161
free operads, 23, 26, 337, 463
adjunction augmentation of, 457, 464
and augmented connected Λ-operads, 82
and augmented non-unitary Λ-operads, 79, 78, 458
and composition products of trees, 441, 455
and connected symmetric sequences, 96
and Hopf operads, 117, 118
and treewise composition products, 455
and treewise tensor products, 453, 460, 461, 463
augmentation morphisms on, 75–78, 82–85, 458
composition products of, 454–455
coproducts with, 469–474
morphism on, 455–457
restriction operators on, 75, 78, 82, 85
universal property of, 75–78, 82–85, 458
weight decomposition of, 460

Fulton–MacPherson operad, 151–154

fundamental groupoids, 188
and the adjunction with the classifying space functor, 192
of operads, 188, 189
see also the entry on the little 2-discs operad for the particular case of this operad

geometric realization of simplicial sets,
and operads, 108
Gerstenhaber operad, see also n-Gerstenhaber operad
graded Grothendieck–Teichmüller group,
action of the — on Drinfeld’s associators, 383
composition operation of the, 382
Drinfeld’s definition of the, 382
filtration of the, 384, 396
hexagon relations of the, 380
involutive condition of the, 381
involutions of the — with the pro-unipotent Grothendieck–Teichmüller group, 380
isomorphism of the pro-unipotent — with the graded Grothendieck–Teichmüller group, 409
pentagon relation of the, 380
pro-unipotent, see also Grothendieck–Teichmüller group
semi-direct product decomposition of the, 390
tower decomposition of the, 390

unit relations of the, 380

graded Grothendieck–Teichmüller Lie algebra, 390
and the Grothendieck–Teichmüller group, 420
hexagon relation of the, 390
involutive condition of the, 390
pentagon relation of the, 390
semi-classical hexagon equation of the, 392
semi-classical hexagon relation of the, 390
unit relations of the, 390
weight decomposition of the, 392

graded Hopf operads, 129, 141, 147
graded modules, 256, 155
hom-objects of, 155
Hopf operads in, see also graded Hopf operads
operads in, see also graded operads
symmetric monoidal category of, 157
tensor products of, 157

graded operads, 141
Grothendieck–Teichmüller group, 401, 408
action of the — on Drinfeld’s associators, 408
and the graded Grothendieck–Teichmüller Lie algebra, 420
composition operation of the, 407
Drinfeld’s definition of the, 407
filtration of the, 413, 417
filtration subquotients of the, 417
graded, see also graded Grothendieck–Teichmüller group
hexagon relations of the, 403
involutive condition of the, 403
isomorphism of the pro-unipotent — with the graded Grothendieck–Teichmüller group, 409
pentagon relation of the, 404
pro-unipotent, see also Grothendieck–Teichmüller group
semi-direct product decomposition of the, 410
tower decomposition of the, 410
unit relations of the, 410

group algebras, 238
adjunction between — and the groups of group-like elements of Hopf algebras, 239
complete, see also complete group algebras
group filtrations, 254
and the exponential correspondence, 254
and weight graded Lie algebras, 254
in Malcev complete groups, 285
in the Grothendieck–Teichmüller group, 115, 117
in the Malcev completion of free groups, 298
in the Malcev completion of the pure braid groups, 348

group-like elements
adjunction between the groups of — of complete Hopf algebras and complete group algebras, 279
adjunction between the groups of — of Hopf algebras and group algebras, 236
and tensor products, 283
and the exponential correspondence, 280
in a complete coalgebra, 279, 318
in a complete Hopf algebra, 279
in a complete Hopf groupoid, 318, 321
adjunction between the groupoid of — and complete Hopf groupoids, 321
in a Hopf algebra, 235, 236
in a Hopf groupoid, 314
adjunction between the groupoid of — and Hopf groupoids, 314
in complete Hopf groupoids, 355
and tensor products, 329
in counitary cocommutative coalgebras, 105
in the truncation of complete Hopf algebras, 280
of complete symmetric algebras, 281
of the complete group algebra of abelian groups, 281
groupoids, 178, 181
connected, 314
fundamental, see also fundamental groupoids
Malcev completion of, 328
operads in, see also operads in groupoids
hexagon relations, 213
for Drinfeld’s associators, 300, 363
of the graded Grothendieck–Teichmüller group, 380
of the graded Grothendieck–Teichmüller Lie algebra, 390
of the Grothendieck–Teichmüller group, 403
hom-objects, 333
homomorphisms, 333
Hopf A-operads, 124
Hopf algebras, 232
and the Milnor-Moore Theorem, 256
complete, see also complete Hopf algebras
completion of, 278

group-like elements in complete filtered modules, see also complete Hopf algebras
in weight graded modules, see also weight graded Hopf algebras
locally conilpotent, 247
of groups, 235
semi-direct products of, 303
smash products of, see also Hopf algebras, semi-direct products of
Structure Theorem of, 248
symmetric monoidal category of, 257
tensor products of, 257
weight graded, see also weight graded Hopf algebras
Hopf categories, 312
in complete filtered modules, 315
symmetric monoidal category of, 329
tensor products of, 329
Hopf groupoids, 312
adjunction between — and the groupoids of group-like elements in Hopf groupoids, 314
associated to groupoids, 315
complete, see also complete Hopf groupoids
completion of, 351
and the completion of Hopf algebras, 320
geometrically connected, 314
globally connected, 314
in complete filtered modules, see also complete Hopf groupoids, 315
locally connected, 314
symmetric monoidal category of, 329
tensor products of, 329
Hopf operads, 112, 115
algebras over, 121
and free operads, 117, 118
aritywise tensor products of, 115
connected, 121
free, 118
in graded modules, see also graded Hopf operads
presentation of — by generators and relations, 119
Hopf symmetric sequences, 116
infinitesimal braiding, 372
of the parenthesized chord diagram operad, 372
ingoing edges of a tree, 431
of a vertex in a tree, 433
inner edges of a tree, 432
iterated loop spaces, 136
Recognition Theorem of, 137
Knizhnik–Zamolodchikov associator, 363
126
and multizeta values, 426
Knizhnik–Zamolodchikov connection, 346
and the Knizhnik–Zamolodchikov
associated, 364
Kontsevich operad, 154
lax symmetric comonoidal functor, 123
unit-preserving, 123
lax symmetric comonoidal functors, 123
and counitary cocommutative coalgebras,
104
lax symmetric monoidal functors, 123
and operads, 107, 109–112
and unitary commutative algebras, 104
unit-preserving, 123
and augmented non-unitary Λ-operads,
107, 109–112
and unitary operads, 107, 109–112
Lie algebras, 237
and torsion phenomena, 238
colimits of, 256
complete, see also complete Lie algebras
direct sums of, 256
enveloping algebra of, 242
in a Q-additive symmetric monoidal
category, 247
in complete filtered modules, see also
complete Lie algebras
in weight graded modules, see also
weight graded Lie algebras
limits of, 256
representations of, 243
semi-direct products of, 207
weight graded, see also weight graded
Lie algebras
Lie operad, 32, 33, 238
algebras over the, 35, 238
presentation of the, 32
little 2-discs operad, see also the entry on
the little n-discs operad for the general
definition, 159, 160
braided operads and the universal
coverings of the, 172, 176
fundamental groupoids of the — and the
colored braid operad, 189, 192
fundamental groupoids of the — and the
parenthesized braid operad, 209, 211
little n-cubes operad, 135
little n-discs operad, 131, 132
and configuration spaces, 132
and iterated loop spaces, 132
composition products of the, 132
framed, 156
homology of the, 147, 150
unit of the, 147
unitary version of the, 135
local coefficient system operads, 332, 351
local coefficient systems, 236
locally conilpotent Hopf algebras, 217
and the Milnor-Moore Theorem, 260
and the Structure Theorem of Hopf
algebras, 248
and weight graded Hopf algebras, 268
Mac Lane Coherence Theorem, 203
Mac Lane’s pentagon, see also pentagon
relation, see also Stasheff’s
associahedra
magma operad, 195, 196
and parenthesized words, 198
unitary version of the, 198
Malcev complete groupoids, 322
equivalences of, 322
fibers of the tower decompositions of, 326
isomorphisms of, 327
local coefficient systems on, 326
operads in, see also operads in Malcev
complete groupoids
tower decomposition of, 323, 326
Malcev complete groups, 283
and complete enveloping algebras, 285
and weight graded Lie algebras, 285
group filtrations on, 285
isomorphisms of, 287
tower decomposition of, 285, 288, 289
Malcev completion, 293
and presentations of groups by
generators and relations, 301
and weight graded Lie algebras, 293, 295
idempotence of the — on free groups,
300, 301
idempotence of the — on the pure braid
groups, 343
of abelian groups, 293
of free groups, 290, 301
of groupoids, 328
and cartesian products, 330
and operads, see also Malcev
completion, of operads in groupoids
and the Malcev completion of groups,
328
universal property of the, 328
of groups, see also Malcev completion
universal property of the, 298
of nilpotent groups, 295
of operads in groupoids, see also the
entry corresponding to the name of the
operad for each particular example of
application of the Malcev completion
construction on operads, 334
universal property of the, 334
of semi-direct products, 334
of the pure braid groups, 312
and the Drinfeld–Kohno Lie algebras,
311, 310
Milnor-Moore Theorem, 250
Milnor-Moore Theorem, 256
locally conilpotent Hopf algebras, 217
and the Milnor-Moore Theorem, 260
and the Structure Theorem of Hopf
algebras, 248
and weight graded Hopf algebras, 268
Mac Lane Coherence Theorem, 203
Mac Lane’s pentagon, see also pentagon
relation, see also Stasheff’s
associahedra
magma operad, 195, 196
and parenthesized words, 198
unitary version of the, 198
Malcev complete groupoids, 322
equivalences of, 322
fibers of the tower decompositions of, 326
isomorphisms of, 327
local coefficient systems on, 326
operads in, see also operads in Malcev
complete groupoids
tower decomposition of, 323, 326
Malcev complete groups, 283
and complete enveloping algebras, 285
and weight graded Lie algebras, 285
group filtrations on, 285
isomorphisms of, 287
tower decomposition of, 285, 288, 289
Malcev completion, 293
and presentations of groups by
generators and relations, 301
and weight graded Lie algebras, 293, 295
idempotence of the — on free groups,
300, 301
idempotence of the — on the pure braid
groups, 343
of abelian groups, 293
of free groups, 290, 301
of groupoids, 328
and cartesian products, 330
and operads, see also Malcev
completion, of operads in groupoids
and the Malcev completion of groups,
328
universal property of the, 328
of groups, see also Malcev completion
universal property of the, 298
of nilpotent groups, 295
of operads in groupoids, see also the
entry corresponding to the name of the
operad for each particular example of
application of the Malcev completion
construction on operads, 334
universal property of the, 334
of semi-direct products, 334
of the pure braid groups, 312
and the Drinfeld–Kohno Lie algebras,
311, 310
Milnor-Moore Theorem, 250
and locally conilpotent Hopf algebras, 250
for complete Hopf algebras, 245
for weight graded Hopf algebras, 209
moduli spaces of curves
Deligne–Mumford–Knudsen compactification of the, 155
monadic categories, 477
and operads, 500
monads, 177 185 208
and adjunctions, 499
morphisms, 336

non-symmetric operads, 8
free, 461
non-unitary A-operads, 70
augmented, see also augmented non-unitary A-operads
non-unitary A-sequences, 61 74
augmented, see also augmented non-unitary A-sequences
non-unitary Hopf A-operads, 221
non-unitary operads, 8 231 319 374 574
and colimits, 34
as a monadic category, 500
connected truncation of, 37
unitary extensions of, 22 537 340
non-unitary symmetric sequences, 35
normalized complex
of a simplicial module, 335
one-point set operad, 15 20
and commutative monoids, 20
and the commutative operad, 60
operads, see also the entry corresponding to the name of the operad for each specific example of an operad, 56
algebras over, 17 18
and lax symmetric monoidal functors, 107 109 112
and lifting of adjoint functors, 100 112
and monads, 14
and symmetric monoidal adjunctions, 108
and symmetric monoidal functors, 107 108
aritywise tensor products of, 114
as a monadic category, 500
augmentation morphisms on
and composition products, 69
braided, see also braided operads
colimits of, 29
composition products of, 428
and augmentations, 409
and restriction operators, 63
associativity relations, 55 58 148
equivariance relations, 40 55
treewise representation of the, 50 52 444

unit relations, 52 55
connected, see also connected operads
coproducts of, 23 269 178
cotriple resolution of, see also cotriple resolution of operads
endomorphism, 15 17
filtered colimits of, 26
free, see also free operads
full composition products of, 7 111 192
associativity relations, 8 10 14
equivariance relations, 7 10 14
treewise representation of the, 111
unit relations, 8 14
ideals of, 32
in counitary cocommutative coalgebras, see also Hopf operads
in graded modules, see also graded operads
in groupoids, see also operads in groupoids
in Malcev complete groupoids, see also operads in Malcev complete groupoids
in small categories, see also operads in small categories
limits of, 20
May’s definition of, 7
non-symmetric, 8
free, 461
non-unitary, see also non-unitary operads
partial composition products of, see also composition products of operads, 10
18 152
associativity relations, 53 55 148
equivariance relations, 10 55
treewise representation of the, 50 52
unit relations, 52 55
presentation of — by generators and relations, 20 82 85
reflection operators on
and composition products, 69
simplicial, see also simplicial operads
symmetric, 8
topological, see also topological operads
treewise representation of, 111 112 30 32
unit morphism of, 7
unit of, 10
unitary, see also unitary operads
with terms indexed by finite sets, 95
operads in groupoids, see also the entry corresponding to the name of the operad each specific example of an operad in groupoids, 178.
categorical equivalences of, see also
operads in small categories, categorical
equivalences of, 178
Malcev completion of, 334
pullbacks of, 201
operads in Malcev complete groupoids, 330
categorical equivalences of, 330
fibers of the tower decomposition of, 332
tower decomposition of, 331–334
operads in small categories, see also the
two corresponding to the name of the
operad for each specific example of an
operad in small categories, 178
and classifying spaces, 179
fibers of the tower decomposition of the,
175
universal property of, 175
operads in Malcev complete groupoids,
330
operads in small categories,
see also the
entry corresponding to the name of the
operad for each specific example of an
operad in small categories, 178
and classifying spaces, 179–180
categorical equivalences of, 178
permutations
block, 18
composition products of, 18
direct sums of, 18
full composition products of, 18
partial composition products of, 18
planar
trees, 161
planar binary trees, 161
Poincaré-Birkhoff-Witt Theorem, 254
for complete Hopf algebras, 275
for weight graded Hopf algebras, 269
Poisson operad, 34
as a Hopf operad, 120
of degree $n - 1$, see also $n$-Gerstenhaber
operad
presentation of the, 34
presentation of the unitary, 89
presentation of Hopf operads by generators
and relations, 119
presentation of operads by generators and
relations, 20
and algebras, 20
presentation of unitary operads by
generators and relations, 88
primitive elements
and the exponential correspondence, 280
in coaugmented coalgebras, 215
in complete Hopf algebras, 225
and weight graded modules, 276
in Hopf algebras, 245
Lie algebra structure on, 245
of semi-direct products of Hopf algebras,
307
of symmetric algebras, 246
of tensor algebras, 246
pro-unipotent Grothendieck–Teichmüller
group, see also
Grothendieck–Teichmüller group
profinite Grothendieck–Teichmüller group,
125
pullbacks of operads in groupoids, 201
pure braid groups, 162
and the homotopy of configuration
spaces, 101
generators of the, 341
Malcev completion of the, 342, 343
and the Drinfeld–Kohno Lie algebras, 344, 346
presentation of the, 341
quasi-free operads, 494
in simplicial modules, 494
in simplicial sets, 494
reduced trees, 433, 441
isomorphisms of, 461
morphisms of, see also tree morphisms
operad of, 441
reflexive coequalizers, 45
and multifunctors, 45
of algebras over an operad, 41
of augmented non-unitary Λ-operads, 79
of operads, 26
restriction of structure, 41
restriction operators on operads, 55
and free operads, 158
and reduced trees, 441
and the cotriple resolution, 496
of augmented non-unitary Λ-operads, 41
and trees, 439, 441
and reduced trees, 439
and tree morphisms, 496
and trees, 439, 441
and tree morphisms, 496
and reduced trees, 439
and treewise composition products, 451
and treewise tensor products, 450, 465–466, 475
associativity relations, 60
equivariance relations, 60
and symmetric collections, 90
treewise representation of the, 63, 66
semi-alternate two-colored trees, 467
treewise tensor products over, 467–469, 473
semi-classical hexagon relation, 372
of the graded Grothendieck–Teichmüller group, 380
of the graded Grothendieck–Teichmüller Lie algebra, 390, 392
semi-direct products of Hopf algebras, 303
in weight graded modules, 308
semi-direct products of Lie algebras, 307
in weight graded modules, 308
sifted colimits, 46
simplices, xxxii
topological, xxxii
simplicial modules, xxxv
simplicial objects
in a category, xxxiii
simplicial operads, 5
homology of, see also topological operads, homology of weak-equivalences of, 180
simplicial sets, xxxiii
green realization, xxiv 120
and operads 108
smash products of Hopf algebras, see also semi-direct products of Hopf algebras, 303
Stasheff’s associahedra, see also associahedra, 205
Structure Theorem of Hopf algebras, 248
for complete Hopf algebras, 241
for weight graded Hopf algebras, 269
subtrees, 434
and tree morphisms, 481
edge set of, 435
vertex set of, 435
symmetric algebras, 10, 207, 211
completion of, 272
Hopf algebra structure on, 241
in complete filtered modules, see also complete symmetric algebras
in weight graded modules, 269
primitive elements of, 246
weight decomposition of, 239
symmetric collections, 90
and symmetric sequences, 91, 92
treewise representation of, 92, 93
symmetric comonoidal transformation, 123
symmetric monoidal adjunctions, 129
and operads, 108
symmetric monoidal categories, xxxvi
Q-additive, 206
and counitary cocommutative coalgebras, 102, 104
and Lie algebras, 237
and unitary commutative algebras, 100, 101
closed, xxxi
concrete, xxxi
symmetric monoidal functors, 123
and augmented non-unitary A-operads, 107
and counitary cocommutative coalgebras, 104
and Lie algebras, 237
and unitary commutative algebras, 100, 101
lax, see also lax symmetric monoidal functors
symmetric monoidal transformation, 123
symmetric operads, see also operads
symmetric sequences, 23
and symmetric collections, 111, 122
aritywise tensor products of, 116
connected, see also connected symmetric sequences
non-unitary, see also non-unitary symmetric sequences
treewise tensor products of, 143
symmetry isomorphism
of the parenthesized chord diagram operad, 572
INDEX 531

of the parenthesized symmetry operad, 221

Teichmüller tower, 422

tensor algebras, 40, 239–241

completion of, 242

Hopf algebra structure on, 241

in complete filtered modules, see also

complete tensor algebras

in weight graded modules, 269

primitive elements of, 246

weight decomposition of, 239

tensor products

aritywise — of Hopf operads, 115

aritywise — of operads, 114

aritywise — of symmetric sequences, 116

distribution over colimits, xxvii

of complete filtered modules, 265–266

of counitary cocommutative coalgebras, 103

of filtered modules, 265

of graded modules, 152

of Hopf algebras, 267

of unitary associative algebras, 239

of unitary commutative algebras, 102

of weight graded modules, 269

over finite sets, xxviii

treewise, 24, 443

topological operads, 8

homology of, 144–147

weak-equivalences of, 129

tower decomposition

of complete filtered modules, 261

of Malcev complete groupoids, 323–326

fibers of the, 326

of Malcev complete groups, 285

of operads in Malcev complete groupoids, 331–334

fibers of the, 332

of the chord diagram operad, 385–387

of the Grothendieck–Teichmüller group, 388

of the Grothendieck–Teichmüller group,

112–113

of the parentheses braid operad, 412

of the set of Drinfeld’s associators, 387

392

393

390

396

398–399

378

433

413

422

458

377

451

453

433

450

439

453

436

432

431

402

401

400

401

400

400

401

400

401

400

401
and lax symmetric monoidal functors, complete, see also complete unitary commutative algebras coproducts of, in a symmetric monoidal category, in complete filtered modules, see also complete unitary commutative algebras symmetric monoidal category of, tensor products of, unitary connected operads and colimits, unitary extensions of operads, unitary operads, and augmented non-unitary Λ-operads, and colimits, and free operads, and lifting of adjoint functors, and symmetric monoidal functors, and unit-preserving lax symmetric monoidal functors, and unitary extensions of operads, colimits of, connected, see also connected unitary operads free, limits of, presentation of — by generators and relations, with terms indexed by finite sets, vertex set of a tree, weight graded Hopf algebras, and the Milnor-Moore Theorem, and the Poincaré-Birkhoff-Witt Theorem, and the Structure Theorem of Hopf algebras, semi-direct products of, weight graded Lie algebras, and group filtrations, and the Malcev completion of semi-direct products of groups, semi-direct products of, weight graded modules, and complete filtered modules, and filtered modules, symmetric monoidal category of, tensor products of, Yang–Baxter relations,
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This volume gives a comprehensive survey on the algebraic aspects of this subject. The book explains the definition of an operad in a general context, reviews the definition of the little discs operads, and explains the definition of the Grothendieck–Teichmüller group from the viewpoint of the theory of operads. In the course of this study, the relationship between the little discs operads and the definition of universal operations associated to braided monoidal category structures is explained. Also provided is a comprehensive and self-contained survey of the applications of Hopf algebras to the definition of a rationalization process, the Malcev completion, for groups and groupoids.

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