10.1090/surv/224/04

CHAPTER 4

Inverse Semigroups

In the study of partial actions, the notion of inverse semigroups plays a predominant role. So let us briefly introduce this concept, referring the reader to [**79**] for proofs and further details.

Definition 4.1. An *inverse semigroup* is a nonempty set S equipped with a binary associative operation (i.e., S is a semigroup) such that, for every s in S, there exists a unique element s^* in S, such that

$$ss^*s = s, \quad \text{and} \quad s^*ss^* = s^*.$$

Given an inverse semigroup S, one may prove that the collection of idempotent elements in S, namely

$$E(S) = \{s \in S : s^2 = s\},$$

is a commutative sub-semigroup. Under the partial order relation defined in $E(S)$ by

$$e \leq f \;\Leftrightarrow\; ef = e, \quad \forall\, e, f \in E(S),$$

$E(S)$ becomes a *semilattice*, meaning that for every e and f in $E(S)$, there exists a largest element which is smaller than e and f, namely ef. One therefore often refers to $E(S)$ as the *idempotent semilattice* of S.

There is also a natural partial order relation defined on S itself by

$$s \leq t \;\Leftrightarrow\; ts^*s = s \;\Leftrightarrow\; ss^*t = s, \quad \forall\, s, t \in S. \tag{4.2}$$

This is compatible with the multiplication operation in the sense that

$$s \leq t,\; s' \leq t' \quad \Rightarrow \quad ss' \leq tt'. \tag{4.3}$$

One of the main examples of inverse semigroups is as follows: given a set X, two subsets $C, D \subseteq X$, and a bijective map

$$f : C \to D,$$

we will say that f is a *partial symmetry* of X. The set

$$\mathcal{I}(X) = \{f : f \text{ is a partial symmetry of } X\} \tag{4.4}$$

may be turned into a semigroup by equipping it with the operation of composition, where, as before, the composition of two partially defined maps is defined on the largest domain where it makes sense.

It may be easily proven that $\mathcal{I}(X)$ is an inverse semigroup, where, for every f in $\mathcal{I}(X)$, one has that f^* is the inverse of f.

The idempotent elements of $\mathcal{I}(X)$ are just the identity maps defined on subsets of X. The order among idempotents happens to be the same as the order of inclusion of their domains.

More generally, the natural order among general elements of $\mathcal{I}(X)$ is the order given by "extension", meaning that, for f and g in $\mathcal{I}(X)$, one has that $f \leq g$, if and only if g is an extension of f, in symbols

$$f \leq g \iff f \subseteq g, \quad \forall f, g \in \mathcal{I}(X).$$

The classical Wagner-Preston Theorem [**105**], [**92**] asserts that any inverse semigroup is isomorphic to a *-invariant sub-semigroup of $\mathcal{I}(X)$, for some X. This may be considered as the version for inverse semigroups of the well known Cayley Theorem for groups.

Given a partial action θ of a group G on a set X, notice that each θ_g is an element of $\mathcal{I}(X)$.

Proposition 4.5. *Let G be a group, X be a set, and*

$$\theta : G \to \mathcal{I}(X)$$

be a map. Then θ is a partial action of G on X if and only if, for every g and h in G, one has that

(i) θ_1 *is the identity map of* X,
(ii) $\theta_{g^{-1}} = (\theta_g)^*$,
(iii) $\theta_g \theta_h \theta_{h^{-1}} = \theta_{gh} \theta_{h^{-1}}$,
(iv) $\theta_{g^{-1}} \theta_g \theta_h = \theta_{g^{-1}} \theta_{gh}$.

PROOF. Assuming that θ is a partial action, one immediately checks (i) and (ii). As for (iii), observe that, following (2.2), the domain of $\theta_{gh}\theta_{h^{-1}}$ is

$$\theta_h(D_{h^{-1}} \cap D_{(gh)^{-1}}) \overset{(2.6)}{=} D_h \cap D_{g^{-1}},$$

which is clearly also the domain of $\theta_g \theta_h \theta_{h^{-1}}$. For any x in this common domain we have that

$$\theta_{h^{-1}}(x) \in \theta_{h^{-1}}(D_h \cap D_{g^{-1}}) = D_{h^{-1}} \cap D_{h^{-1}g^{-1}},$$

so (iii) follows immediately from (2.5.ii). Finally, (iv) follows from (iii) and (ii), in view of the fact that the star operation reverses multiplication.

Conversely, suppose that θ satisfies (i–iv) and denote the range of θ_g by D_g. Since $\theta_{g^{-1}}$ is the inverse of θ_g by (i), we have that the range of $\theta_{g^{-1}}$ is the domain of θ_g, meaning that θ_g is a map

$$\theta_g : D_{g^{-1}} \to D_g.$$

Since (2.1.i) is granted, let us verify (2.1.ii). For this, let us suppose we are given g and h in G, as well as an element x in the domain of $\theta_g \theta_h$. Then the element $y := \theta_h(x)$ clearly lies in the domain of $\theta_g \theta_h \theta_{h^{-1}}$, and hence (iii) implies that it is also in the domain of $\theta_{gh}\theta_{h^{-1}}$, which is to say that x is in the domain of θ_{gh}. Moreover

$$\theta_g\big(\theta_h(x)\big) = \theta_g\big(\theta_h(\theta_{h^{-1}}(y))\big) = \theta_{gh}\big(\theta_{h^{-1}}(y)\big) = \theta_{gh}(x),$$

proving that $\theta_g \theta_h \subseteq \theta_{gh}$. □

The language of inverse semigroups is especially well suited to the introduction of our next example. In order to describe it, let X be a set and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be any collection of partial symmetries of X.

Letting $\mathbb{F} = \mathbb{F}(\Lambda)$ be the free group on the index set Λ, our plan is to construct a partial action of $\mathbb{F}$ on X in the form of a map

$$\theta : \mathbb{F} \to \mathcal{I}(X).$$

As a first step, let us define

$$\theta_\lambda := f_\lambda, \quad \text{and} \quad \theta_{\lambda^{-1}} := f_\lambda^{-1}, \quad \forall\, \lambda \in \Lambda.$$

Given any element $g \in \mathbb{F}$, write

$$g = x_1 x_2 \ldots x_n,$$

in *reduced form*, meaning that each $x_j \in \Lambda \cup \Lambda^{-1}$, and $x_{j+1} \neq x_j^{-1}$. It is well known that a unique such decomposition of g exists. We then put

$$\theta_g := \theta_{x_1}\theta_{x_2}\ldots\theta_{x_n}, \tag{4.6}$$

with the convention that, if $g = 1$, then its reduced form is "empty", and θ_1 is the identity function on X.

Proposition 4.7. *The map θ defined in* (4.6)*, above, is a partial action of $\mathbb{F}$ on X.*

Proof. We leave it for the reader to prove (2.1.i). In order to verify (2.1.ii), pick g and h in $\mathbb{F}$, and write

$$g = x_n x_{n-1} \ldots x_2 x_1, \quad \text{and} \quad h = y_1 y_2 \ldots y_{m-1} y_m,$$

in reduced form, as above (for reasons which will soon become clear, we have chosen to reverse the indices in the reduced form of g).

Let p be the number of cancellations occurring when performing the multiplication gh, meaning that

$$x_i = y_i^{-1}, \quad \forall\, i = 1, \ldots, p,$$

and that p is maximal with this property. Define g', h', and k in $\mathbb{F}$, as follows

$$g = \overbrace{x_n \ldots x_{p+1}}^{g'}\ \overbrace{x_p \ldots x_1}^{k}, \quad \text{and} \quad h = \overbrace{y_1 \ldots y_p}^{k^{-1}}\ \overbrace{y_{p+1} \ldots y_m}^{h'},$$

so that

$$gh = g'h' = x_n \ldots x_{p+1}\ y_{p+1} \ldots y_m$$

is in reduced form. Denoting the identity map on X by id_X, we then clearly have that

$$\theta_k \theta_k^{-1} \subseteq id_X,$$

so

$$\theta_g\theta_h = \theta_{g'}\theta_k\theta_k^{-1}\theta'_h \overset{(4.3)}{\subseteq} \theta_{g'}\, id_X\, \theta_{h'} = \theta_{g'}\theta_{h'} = \theta_{g'h'} = \theta_{gh}, \tag{4.7.1}$$

concluding the proof. □

Observe that in (4.7.1) we were allowed to use that

$$\theta_{g'}\theta_{h'} = \theta_{g'h'}$$

because the juxtaposition of the reduced forms of g' and h' turned out to be the reduced form of $g'h'$. Equivalently, $|g'h'| = |g'| + |h'|$, where $|\cdot|$ refers to *word length*.

Definition 4.8. A *length function* on a group G is a function $\ell : G \to \mathbb{R}_+$ such that

(i) $\ell(1) = 0$, and
(ii) $\ell(gh) \leq \ell(g) + \ell(h)$, for all g and h in G.

Evidently $|\cdot|$ is a length function for the free group.

Definition 4.9. Let G be a group equipped with a length function ℓ. A partial action θ of G is said to be *semi-saturated* (with respect to the given length function ℓ) if

$$\ell(gh) = \ell(g) + \ell(h) \quad \Longrightarrow \quad \theta_g\theta_h = \theta_{gh}, \quad \forall\, g, h \in G.$$

In the free group, observe that the condition that $|gh| = |g| + |h|$ means that the juxtaposition of the reduced forms of g and h is precisely the reduced form of gh. So the partial action θ defined in (4.6) is easily seen to be semi-saturated.

Summarizing we have:

Proposition 4.10. *Let X be a set and let $\{f_\lambda\}_{\lambda\in\Lambda}$ be any collection of partial symmetries of X. Then there exists a unique semi-saturated partial action θ of $\mathbb{F}(\Lambda)$ on X such that*

$$\theta_\lambda = f_\lambda, \quad \forall\, \lambda \in \Lambda.$$

PROOF. The existence was proved above and we leave the proof of uniqueness as an easy exercise. □

Notes and remarks. Semi-saturated partial actions are related to Quigg and Raeburn's notion of *multiplicativity*, as defined in [**94**, Definition 5.1]. The relationship between partial actions and inverse semigroups is further discussed in [**101**], [**50**], [**55**], [**61**], [**22**], [**23**] and [**24**].