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CHAPTER 11

C*-Algebraic Partial Dynamical Systems

In this chapter we will adapt the construction of the partial crossed product to the category of C*-algebras. We begin with a quick review of basic concepts.

Definition 11.1. A *C*-algebra* is a *-algebra A over the field of complex numbers, equipped with a norm $\|\cdot\|$, with respect to which it is a Banach space, and such that for all a and b in A, one has that

(i) $\|ab\| \leq \|a\|\|b\|$,
(ii) $\|a^*\| = \|a\|$,
(iii) $\|a^*a\| = \|a\|^2$.

There are many references for the basic theory of C*-algebras where the interested reader will find the basic results, such as [**89**], [**9**], [**84**] and [**32**].

Of special relevance to us is Gelfand's Theorem [**84**, Theorem 2.1.10] which asserts that there is an equivalence between the category of locally compact Hausdorff (*LCH* for short) topological spaces, with proper[9] continuous maps, on the one hand, and the category of abelian C*-algebras, with *non-degenerate*[10] *-homomorphisms, on the other hand. This equivalence is implemented by the contravariant functor

$$X \rightsquigarrow C_0(X),$$

where $C_0(X)$ refers to the C*-algebra formed by all continuous complex valued functions f defined on X, vanishing[11] at ∞.

If X and Y are LCH spaces then any proper continuous map $h : X \to Y$ induces a non-degenerate *-homomorphism

$$\phi_h : f \in C_0(Y) \mapsto f \circ h \in C_0(X),$$

and conversely, any non-degenerate *-homomorphism from $C_0(Y)$ to $C_0(X)$ is induced, as above, by a unique proper continuous map from X to Y. Moreover, ϕ_h is an isomorphism if and only if h is a homeomorphism.

Ideals (always assumed to be norm-closed and two-sided) in C*-algebras are automatically self-adjoint. Every ideal in a C*-algebra is both non-degenerate and idempotent, and hence the conclusion of (7.9) holds for them.

If X is a LCH space then there is a one-to-one correspondence between open subsets of X and ideals of $C_0(X)$ given as follows: to an open set $U \subseteq X$ we attach

[9] A map between locally compact topological spaces is said to be *proper* if the inverse image of every compact set is compact.

[10] A *-homomorphism φ from a C*-algebra A to another C*-algebra B, or perhaps even into the multiplier algebra $\mathcal{M}(B)$, is said to be *non-degenerate* if $B = [\varphi(A)B]$ (brackets denoting closed linear span). By taking adjoints, this is the same as saying that $B = [B\varphi(A)]$.

[11] We say that a map f *vanishes at* ∞, if for every real number $\varepsilon > 0$, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact.

the ideal given by

$$C_0(U) := \{f \in C_0(X) : f = 0 \ \text{ on } \ X \setminus U\}.$$

The reader should be aware that any open set $U \subseteq X$ may also be seen as a LCH space in itself, so this notation has a potential risk of confusion since, besides the above meaning of $C_0(U)$, one could also think of $C_0(U)$ as the set of all continuous complex valued functions *defined on* U, and vanishing at ∞.

However the two meanings of $C_0(U)$ give rise to naturally isomorphic C*-algebras, the isomorphism taking any function defined on U to its extension to the whole of X, declared zero outside of U. The very slight distinction between the two interpretations of this notation will fortunately not cause us any problems.

Recall from (4.4) that $\mathcal{I}(X)$ denotes the inverse semigroup formed by all partial symmetries of a set X.

Definition 11.2. Given a C*-algebra A, we will say that a partial symmetry $\phi \in \mathcal{I}(A)$ is a *partial automorphism* of A, if the domain and range of ϕ are closed two-sided ideals of A, and ϕ is a *-isomorphism from its domain to its range. We will denote by $\text{pAut}(A)$ the collection of all partial automorphisms of A. It is evident that $\text{pAut}(A)$ is an inverse sub-semigroup of $\mathcal{I}(A)$.

Given any partial homeomorphism of a LCH space X, say $h : U \to V$, where U and V are open subsets of X, the map

$$\phi_h : C_0(V) \to C_0(U)$$

is a *-isomorphism between ideals of $C_0(X)$, and hence may be seen as a partial automorphism of $C_0(X)$. It follows from what was said above that the correspondence

$$h \in \text{pHomeo}(X) \ \mapsto \ \phi_{h^{-1}} \in \text{pAut}\big(C_0(X)\big) \tag{11.3}$$

is a semigroup isomorphism.

Definition 11.4. A *C*-algebraic partial action* of the group G on the C*-algebra A is a partial action $\theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big)$ on the underlying set A, such that each D_g is a *closed* two-sided ideal of A, and each θ_g is a *-isomorphism from $D_{g^{-1}}$ to D_g. By a *C*-algebraic partial dynamical system* we shall mean a partial dynamical system

$$\big(A,\ G,\ \{D_g\}_{g\in G},\ \{\theta_g\}_{g\in G}\big)$$

where A is a C*-algebra and $\big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big)$ is a C*-algebraic partial action of G on A.

When it is understood that we are working in the category of C*-algebras and there is no chance for confusion we will drop the adjective *C*-algebraic* and simply say *partial action* or *partial dynamical system.*

As an immediate consequence of (4.5) we have:

Proposition 11.5. *Let G be a group, A be a C*-algebra, and*

$$\theta : G \to \text{pAut}(A)$$

be a map. Then θ is a C-algebraic partial action of G on A if and only if conditions (i–iv) of* (4.5) *are fulfilled.*

Putting together (5.3), (11.5) and (11.3), one concludes:

Corollary 11.6. *If G is a group and X is a LCH space, then* (11.3) *induces a natural equivalence between topological partial actions of G on X and C*-algebraic partial actions of G on $C_0(X)$.*

► We now fix, for the time being, a C*-algebraic partial action

$$\theta = \big(\{D_g\}_{g\in G},\ \{\theta_g\}_{g\in G}\big)$$

of a group G on a C*-algebra A.

Since θ is in particular a *-algebraic partial action, we may apply the construction of the crossed product described in (8.3) to θ. However, the resulting algebra, which we will temporarily denote by

$$A \rtimes_{\text{alg}} G,$$

will most certainly not be a C*-algebra, so we will modify the construction a bit in order to stay in the category of C*-algebras. Meanwhile we observe that $A \rtimes_{\text{alg}} G$ is an associative algebra by (8.7), as well as a *-algebra by (8.9).

Definition 11.7. A *C*-seminorm* on a complex *-algebra B is a seminorm $p : B \to \mathbb{R}_+$, such that, for all $a, b \in B$, one has that

(i) $p(ab) \leq p(a)p(b)$,
(ii) $p(a^*) = p(a)$,
(iii) $p(a^*a) = p(a)^2$.

If B is a C*-algebra and p is a C*-seminorm on B, it is well known that

$$p(b) \leq \|b\|, \tag{11.8}$$

for all $b \in B$.

Proposition 11.9. *Let p be a C*-seminorm on $A \rtimes_{alg} G$. Then, for every $a = \sum_{g\in G} a_g\delta_g$ in $A \rtimes_{alg} G$, one has that*

$$p(a) \leq \sum_{g\in G} \|a_g\|.$$

Proof. Notice that $A\delta_1$ is isomorphic to the C*-algebra A, so by what was said above we have that $p(a\delta_1) \leq \|a\|$, for all $a \in A$. We then have that

$$p(a_g\delta_g)^2 = p\big((a_g\delta_g)(a_g\delta_g)^*\big) \overset{(8.14)}{=} p(a_g a_g^* \delta_1) \leq \|a_g a_g^*\| = \|a_g\|^2,$$

so the statement follows from the triangle inequality. □

Let us therefore define a seminorm on $A \rtimes_{\text{alg}} G$, by

$$\|a\|_{\max} = \sup\big\{p(a) : p \text{ is a C*-seminorm on } A \rtimes_{\text{alg}} G\big\}. \tag{11.10}$$

By (11.9) we see that $\|a\|_{\max}$ is always finite and it is not hard to see that $\|\cdot\|_{\max}$ is a C*-seminorm on $A \rtimes_{\text{alg}} G$ (we will later prove that it is in fact a norm).

Definition 11.11. The *C*-algebraic crossed product* of a C*-algebra A by a group G under a C*-algebraic partial action $\theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big)$ is the C*-algebra $A \rtimes G$ obtained by completing $A \rtimes_{\text{alg}} G$ relative to the seminorm $\|\cdot\|_{\max}$ defined above.

The process of completing a semi-normed space involves first modding out the subspace formed by all vectors of zero length. However, as already mentioned, $\|\cdot\|_{\max}$ will be shown to be a norm on $A \rtimes_{\rm alg} G$, so the modding out part will be seen to be unnecessary.

Definition 11.12. From now on, for any $a \in D_g$, we will let $a\delta_g^{alg}$ denote the element of $A \rtimes_{\rm alg} G$ we have so far been denoting by $a\delta_g$, while we will reserve the notation $a\delta_g$ for the canonical image of $a\delta_g^{alg}$ in $A \rtimes G$.

As we will be mostly working with the C*-algebraic crossed product, the notation $a\delta_g^{alg}$ will only rarely be used in the sequel.

Definition 11.13. We will denote by

$$\iota : A \to A \rtimes G,$$

the mapping defined by $\iota(a) = a\delta_1$, for every $a \in A$.

As already mentioned, we will later prove that the natural map from $A \rtimes_{\rm alg} G$ to $A \rtimes G$ is injective and consequently ι will be seen to be injective as well.

The following is a useful device in producing *-homomorphisms defined on crossed product algebras:

Proposition 11.14. *Let B be a C*-algebra and let*

$$\varphi_0 : A \rtimes_{alg} G \to B$$

*be a *-homomorphism. Then there exists a unique *-homomorphism φ from $A \rtimes G$ to B, such that the diagram*

$$\begin{array}{ccc} A \rtimes_{alg} G & \xrightarrow{\varphi_0} & B \\ \downarrow & \nearrow_{\varphi} & \\ A \rtimes G & & \end{array}$$

commutes, where the vertical arrow is the canonical mapping arising from the completion process.

Proof. It is enough to notice that $p(x) := \|\varphi_0(x)\|$ defines a C*-seminorm on $A \rtimes_{\rm alg} G$, which is therefore bounded by $\|\cdot\|_{\max}$. Thus φ_0 is continuous for the latter, and hence extends to the completion. □

Notes and remarks. As already mentioned, partial actions on C*-algebras were introduced in [**45**] for the case of the group of integers, and in [**81**] for general groups. Although not covered by this book, the notion of *continuous* partial actions of *topological groups* on C*-algebras, *twisted* by a *cocycle* or otherwise, has also been considered [**48**].