10.1090/surv/224/23

CHAPTER 23

Ideals in Graded Algebras

Let B be a graded C*-algebra with grading $\{B_g\}_{g\in G}$. If J is an ideal (always assumed to be closed and two-sided) in B, there might be no relationship between J and the grading of B. It is even possible that $J \cap B_g$ is trivial for every g in G.

For example, let $\mathscr{B}$ be the group bundle over $\mathbb{Z}$, so that $C^*(\mathscr{B})$ is isomorphic to $C(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle. Fixing $z_0 \in \mathbb{T}$, the ideal

$$J = \{f \in C(\mathbb{T}) : f(z_0) = 0\}$$

has trivial intersection with every homogeneous subspace $B_n = \mathbb{C}z^n$. This is because a nonzero element in B_n is invertible, and hence cannot belong to any proper ideal.

The purpose of this chapter is thus to study the relationship between ideals in graded algebras and the grading itself. For this we shall temporarily

► fix a graded C*-algebra B, with grading $\{B_g\}_{g\in G}$, and a closed two-sided ideal $J \trianglelefteq B$.

Proposition 23.1. *The closed two-sided ideal of B generated by $J \cap B_1$ coincides with the closure of $\bigoplus_{g\in G} J \cap B_g$.*

Proof. Given g and h in G, notice that

$$(J \cap B_g)B_h \subseteq (JB_h) \cap (B_gB_h) \subseteq J \cap B_{gh}.$$

Therefore we see that

$$K := \overline{\bigoplus_{g\in G} J \cap B_g}$$

is invariant under right multiplication by elements of B_h, and a similar reasoning shows invariance under left multiplication as well. Since K is closed by definition, we then deduce that K is a two-sided ideal. Consequently, noticing that $J \cap B_1$ is contained in K, we have that

$$\langle J \cap B_1\rangle \subseteq K,$$

where the angle brackets above indicate the closed two-sided ideal generated by $J \cap B_1$.

In order to prove the reverse inclusion, given g in G, and $x \in J \cap B_g$, notice that $x^*x \in J \cap B_1$. Using (15.3) (which is stated for Hilbert modules, and hence also holds for C*-algebras) we then have

$$x = \lim x(x^*x)^{1/n} \in \langle J \cap B_1\rangle.$$

We therefore conclude that

$$J \cap B_g \subseteq \langle J \cap B_1\rangle,$$

from where it follows that $K \subseteq \langle J \cap B_1\rangle$, as desired. □

This justifies the introduction of the following concept:

Definition 23.2. We shall say that J is an *induced ideal*[30] (sometimes also called a *graded ideal*) provided any one of the following equivalent conditions hold:

(a) J coincides with the ideal generated by $J \cap B_1$,

(b) $\bigoplus_{g\in G} J \cap B_g$ is dense in J.

For topologically graded algebras there is a lot more to be said, so we shall assume from now on that B is topologically graded. Recall from (19.6) that in this case B admits *Fourier coefficient operators*

$$F_g : B \to B_g, \quad \forall\, g \in G,$$

such that

$$F_g(b) = \delta_{g,h} b, \quad \forall\, g, h \in G, \quad \forall\, b \in B_h.$$

Given an arbitrary ideal $J \trianglelefteq B$, let us consider the following subsets of B:

$$\begin{aligned} J' &= \langle J \cap B_1 \rangle, \\ J'' &= \{x \in B : F_g(x) \in J,\ \forall g \in G\}, \\ J''' &= \{x \in B : F_1(x^*x) \in J\}, \end{aligned} \tag{23.3}$$

where the angle brackets in the definition of J' are again supposed to mean the closed two-sided ideal generated.

Proposition 23.4. *Given any ideal J in a topologically graded C*-algebra B, one has that the sets J', J'' and J''' defined above are closed two-sided ideals in B, and moreover*

$$J' \subseteq J'' = J'''.$$

Proof. It is evident that these are closed subspaces of B, and moreover that J' is an ideal.

In order to prove that J'' is an ideal, let $b \in B$, and $x \in J''$, and let us prove that $bx \in J''$. Since the B_g span a dense subspace of B, we may assume that $b \in B_h$, for some $h \in G$. Then

$$F_g(bx) \overset{(19.6)}{=} bF_{h^{-1}g}(x) \in J,$$

proving that $bx \in J''$, and hence that J'' is a left ideal. One similarly proves that J'' is a right ideal. Since we will eventually prove that $J''' = J''$, we skip the proof that J''' is an ideal for now.

Observing that $J \cap B_1$ is contained in J'', and since we now know that J'' is an ideal, the ideal generated by $J \cap B_1$, namely J', is also contained in J''.

Given any $x \in B$, notice that by Parseval's identity (17.15), we have that

$$\sum_{g\in G} F_g(z)^* F_g(z) = F_1(z^*z).$$

In fact (17.15) refers to the E_g, but since $F_g = E_g \circ \psi$ (see the proof of (19.6)), our identity follows easily from (17.15). Since ideals are hereditary, we then have that

$$F_1(z^*z) \in J \;\Leftrightarrow\; F_g(z)^* F_g(z) \in J,\ \forall g \in G,$$

and we notice that the condition in the right-hand side above is also equivalent to $F_g(z) \in J$. This proves that $J'' = J'''$. □

[30] This terminology is inspired by [**85**, 1.3].

Having seen how the ideals defined in (23.3) relate to each other, let us also discuss how do they relate to J, itself. It is elementary to see that J always contains J', but the relationship between J and J'' is not straightforward. We will see below that $J' = J''$ under certain conditions, in which case it will follow that $J'' \subseteq J$. However there are examples in which J'' is not a subset of J, and in fact it may occur that, on the contrary, J is a proper subset of J''.

This is the case, for example, if B is the full group C*-algebra of a non-amenable group G, and $J = \{0\}$. One may then prove that J'' is the kernel of the regular representation, hence J'' is strictly larger than J.

One might suspect that the culprit for this anomaly is the failure of faithfulness of the standard conditional expectation on $C^*(G)$, but examples may also be found in topologically graded C*-algebras with faithful conditional expectations. Take, for example, a group G and a C*-algebra B. One may then prove that $B \otimes_{\min} C^*_{\rm red}(G)$ is isomorphic to the reduced crossed product of B by G under the trivial action, so the former is a topologically G-graded C*-algebra with faithful conditional expectation by (17.13).

Assuming that G is a non-exact group, one may find a short exact sequence of C*-algebras

$$0 \to J \xrightarrow{i} B \xrightarrow{\pi} A \to 0,$$

for which

$$0 \to J \underset{\min}{\otimes} C^*_{\rm red}(G) \xrightarrow{i\otimes 1} B \underset{\min}{\otimes} C^*_{\rm red}(G) \xrightarrow{\pi\otimes 1} A \underset{\min}{\otimes} C^*_{\rm red}(G) \to 0,$$

is not exact. It is well known that the only place where exactness may fail is at the mid point of this sequence, meaning that the range of $i \otimes 1$ is properly contained in the kernel of $\pi \otimes 1$. Letting J be the range of $i \otimes 1$, one may prove that J'' is the kernel of $\pi \otimes 1$, whence J is properly contained in J'', as claimed. Consequently we have that

$$J' \subseteq J \subsetneq J'', \tag{23.5}$$

so this also produces an example in which J'' is strictly larger than J'.

Proposition 23.6. *Let B be a topologically graded C*-algebra with grading $\{B_g\}_{g\in G}$, and assume that the associated Fell bundle has the approximation property. Then, for every ideal $J \trianglelefteq B$, one has that the ideals J' and J'' defined in* (23.3) *are equal.*

Proof. It clearly suffices to prove that $J'' \subseteq J'$. Given $x \in J''$, we have by definition that each $F_g(x) \in J$, and we claim that $F_g(x) \in J'$. In order to see this, notice that

$$F_g(x)^* F_g(x) \in J \cap B_1 \subseteq J',$$

so we have that $F_g(x)^* F_g(x) \equiv 0 \pmod{J'}$. Since B/J' is a C*-algebra, we have that $F_g(x) \equiv 0 \pmod{J'}$, as well, meaning that $F_g(x) \in J'$, thus proving our claim.

Let $\{a_i\}_i$ be a Cesaro net for $\mathscr{B}$, and let $\{S_i\}_i$ be the net of summation processes provided by (20.10). A glimpse at the formula defining S_i is enough to convince ourselves that $S_i(x)$ is also in J', hence also

$$x = \lim_i S_i(x) \in J'.$$

□

There is another situation in which we may guarantee the coincidence of the ideals J' and J''.

Theorem 23.7. *Let G be a discrete group and let B be a topologically G-graded C^*-algebra. Suppose that G is exact and that the standard conditional expectation $F : B \to B_1$ is faithful. Then for every ideal J of B one has that the ideals J' and J'' defined in* (23.3) *coincide.*

Proof. Denote by $\mathscr{B} = \{B_g\}_{g\in G}$ the underlying Fell bundle and note that B is isomorphic to $C^*_{\mathrm{red}}(\mathscr{B})$ by (19.8). For each g in G, let $J_g = J \cap B_g$ so that $\mathcal{J} := \{J_g\}_{g\in G}$ is an ideal in $\mathscr{B}$. Employing (21.18) we have that the sequence

$$0 \to C^*_{\mathrm{red}}(\mathcal{J}) \xrightarrow{\iota_{\mathrm{red}}} B \xrightarrow{q_{\mathrm{red}}} C^*_{\mathrm{red}}(\mathscr{B}/\mathcal{J}) \to 0 \tag{23.7.1}$$

is exact. We next claim that

$$J' = \iota_{\mathrm{red}}\big(C^*_{\mathrm{red}}(\mathcal{J})\big), \quad \text{and} \quad J'' = \mathrm{Ker}(q_{\mathrm{red}}). \tag{23.7.2}$$

Given g in G, notice that

$$J_g^* J_g \subseteq B_1 \cap J \subseteq J',$$

so by (16.12) one has that

$$J_g = [J_g J_g^* J_g] \subseteq [J_g J'] \subseteq J',$$

so $\iota_{\mathrm{red}}\big(C^*_{\mathrm{red}}(\mathcal{J})\big) \subseteq J'$. Since the reverse inclusion is evident, we have proven the first identity in (23.7.2).

On the other hand, denoting by E the faithful standard conditional expectation of $C^*_{\mathrm{red}}(\mathscr{B}/\mathcal{J})$, it is easy to see that $E \circ q_{\mathrm{red}} = q_{\mathrm{red}} \circ F$, so for any b in B we have that

$$q_{\mathrm{red}}(b) = 0 \;\Leftrightarrow\; E\big(q_{\mathrm{red}}(b^*b)\big) = 0 \;\Leftrightarrow\; q_{\mathrm{red}}\big(F(b^*b)\big) = 0 \;\Leftrightarrow\; F(b^*b) \in J_1,$$

where the last step is a consequence of the fact that $F(b^*b)$ is in B_1, and that the behavior of q on B_1 is that given by (21.3). This shows that $\mathrm{Ker}(q_{\mathrm{red}}) = J''$, concluding the verification of (23.7.2).

Since the sequence (23.7.1) is exact, the proof follows. ☐

Definition 23.8. Let B be a topologically graded C*-algebra with grading $\{B_g\}_{g\in G}$ and Fourier coefficient operators F_g. We will say that a closed, two-sided ideal $J \trianglelefteq B$ is a *Fourier ideal*, if $F_g(J) \subseteq J$, for every g in G.

Thus J is a Fourier ideal if and only if $J \subseteq J''$, while J is induced if and only if $J = J'$. We may thus reinterpret (23.4), (23.6) and (23.7) as follows.

Proposition 23.9. *Let B be a topologically graded C^*-algebra. Then every induced ideal of B is a Fourier ideal. Moreover, the converse holds if either*

(i) the associated Fell bundle has the approximation property, or
(ii) G is exact and the standard conditional expectation of B onto B_1 is faithful.

For an example of a Fourier ideal which is not induced, see (23.5).

The next result is stated for Fourier ideals but, because of the reasoning above, it also holds for induced ones.

Proposition 23.10. *Let B be a topologically graded C^*-algebra with grading $\{B_g\}_{g\in G}$, and let J be a Fourier ideal of B. Then B/J is topologically graded by the spaces $q(B_g)$, where q is the quotient map.*

PROOF. Since J is invariant under each Fourier coefficient operator F_g, we have that F_g passes to the quotient giving a well defined bounded map on B/J, namely

$$\tilde{F}_g(x+J) = F_g(x) + J, \quad \forall\, x \in B.$$

Notice that

$$q(B_g) = \tilde{F}_g(B/J) = \operatorname{Ker}(id - \tilde{F}_g),$$

the last step holding thanks to the fact that $\tilde{F}_g$ is idempotent. As a consequence we deduce that $q(B_g)$ is a closed subspace of B/J.

It is now immediate to verify that the collection $\{q(B_g)\}_{g\in G}$ satisfies (19.1.i)–(19.1.iii), and that $\tilde{F}_1$ fills in the rest of the hypothesis there to allow us to conclude that this is in fact a topological grading for B/J. □

All of the above results have their versions in the setting of partial crossed product algebras, since these are graded algebras. The next simple result, which we will use later, has no counterpart for graded algebras since its conclusion explicitly mentions the partial action.

Proposition 23.11. *Let* $\theta = \big(A, G, \{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big)$ *be a C*-algebraic partial dynamical system. Given an ideal* J *of either* $A \rtimes G$ *or* $A \rtimes_{red} G$*, let* $K = J \cap A$*. Then* K *is a* θ*-invariant ideal of* A*.*

PROOF. Of course we are identifying A with its copy $A\delta_1$ in the crossed product algebra. Regardless of whether we are working with the full or reduced crossed product, the proof is the same: given a in $K \cap D_{g^{-1}}$, choose an approximate identity $\{v_i\}_i$ for D_g. Then

$$J \ni (v_i\delta_g)(a\delta_1)(v_i\delta_g)^* \overset{(8.14)}{=} v_i\theta_g(a)v_i^*\delta_1 \overset{i\to\infty}{\longrightarrow} \theta_g(a)\delta_1,$$

so $\theta_g(a)$ is in K, proving the statement. □

Notes and remarks. The motivation for this chapter comes from Nica's work on induced ideals of algebras of Wiener-Hopf operators [**85**, Section 6], which in turn is inspired by Strătilă and Voiculescu's work on AF-algebras [**104**]. Propositions (23.4) and (23.6) have been proven in [**49**], while Theorem (23.7) is from [**52**, Theorem 5.1][31].

[31] Please note that [**52**] is the preprint version of [**53**].