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CHAPTER 29

Topologically Free Partial Actions

In chapter (23) we have studied induced and Fourier ideals in topologically graded algebras. These may be considered as the ideals which respect the given grading. In the present chapter we will study conditions which imply that all ideals somehow take the grading into account, such as having a nonzero intersection with the unit fiber algebra or, in the best-case scenario, are induced ideals.

The graded algebra at the center of the stage here will be the crossed product of an abelian C*-algebra A by a given group G, and the main conditions we will impose relate to the corresponding partial action of G on the spectrum of A, namely that there are not too many fixed points. As an application we will give conditions for a partial crossed product to be simple.

Definition 29.1. Let $\theta = \big(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big)$ be a topological partial action of a group G on a locally compact Hausdorff space X, and let $g \in G$.

(a) A *fixed point* for g is any element x in $D_{g^{-1}}$, such that $\theta_g(x) = x$.
(b) The set of all fixed points for g will be denoted by F_g.
(c) We say that θ is *free* if F_g is empty for every $g \neq 1$.
(d) We say that θ is *topologically free* if the *interior* of F_g is empty for every $g \neq 1$.

Observe that $F_g = F_{g^{-1}}$, for all g in G. Also notice that F_g is a closed subset of $D_{g^{-1}}$ (and hence also of D_g by the previous remark) in the relative topology, but it is not necessarily closed in X. Incidentally, here is a useful result of general topology:

Lemma 29.2. *Let X be a topological space, and suppose we are given subsets $F \subseteq D \subseteq X$, such that F is closed relative to D, and D is open in X. If the interior of F is empty, then the interior of $\bar{F}$ is also empty, where $\bar{F}$ denotes the closure of F relative to X.*

Proof. We should observe that, since D is open, the interior of F relative to D is the same as the interior of F relative to X.

It suffices to prove that, if V is an open subset of X, with $V \subseteq \bar{F}$, then $V = \emptyset$. Notice that for each such V, we have

$$V \cap D \subseteq \bar{F} \cap D = F,$$

the last equality being the expression that F is closed in D. Since F has no interior, we deduce that $V \cap D = \emptyset$.

It is well known that, when two open subsets are disjoint, each one is necessarily disjoint from the other's closure, whence

$$\emptyset = V \cap \bar{D} \supseteq V \cap \bar{F} = V.$$

□

Another topological fact we will need is the following baby version of Baire's category Theorem, which incidentally holds in any topological space:

Lemma 29.3. *Let X be a topological space and let $F_1, F_2, \ldots, F_n$ be nowhere dense*[37] *subsets of X. Then $F_1 \cup F_2 \cup \ldots \cup F_n$ is nowhere dense.*

PROOF. Replacing each F_i by its closure, we may clearly assume them to be closed, and hence so is their union.

In order to prove the statement we must show that every open set

$$V \subseteq F_1 \cup F_2 \cup \ldots \cup F_n$$

is necessarily empty. Given such a V, notice that,

$$W := V \setminus (F_1 \cup F_2 \cup \ldots \cup F_{n-1})$$

is an open set contained in F_n, whence W is empty by hypothesis. Thus $V \subseteq F_1 \cup F_2 \cup \ldots \cup F_{n-1}$, and the conclusion follows by induction. □

► From now on we will fix a group G, a locally compact Hausdorff space X and a topological partial action

$$\theta = \big(\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G}\big)$$

of G on X. Our attention will be focused on the partial action of G on $C_0(X)$ induced by θ via (11.6), which we will henceforth denote by α. More precisely

$$\alpha = \Big(\big\{C_0(D_g)\big\}_{g \in G}, \{\alpha_g\}_{g \in G}\Big),$$

where each α_g is given by

$$\alpha_g : f \in C_0(D_{g^{-1}}) \mapsto f \circ \theta_{g^{-1}} \in C_0(D_g).$$

Above all, we are interested in the reduced crossed product

$$C_0(X) \rtimes_{\text{red}} G,$$

so, given f in $C_0(D_g)$, we will always interpret the expression "$f\delta_g$" as an element of the above crossed product algebra.

The following technical Lemma is the key tool in the proof of our main result below. It is intended to *shut out* certain elements in the grading subspaces $C_0(D_g)\delta_g$, with $g \neq 1$, by *compressing* them away with positive elements lying in the unit fiber algebra.

Lemma 29.4. *Given*

(*a*) $g \in G$, *with* $g \neq 1$,
(*b*) $f \in C_0(D_g)$,
(*c*) $x_0 \in X \setminus F_g$, *and*
(*d*) $\varepsilon > 0$,

there exists h in $C_0(X)$ such that

(*i*) $h(x_0) = 1$,
(*ii*) $0 \leq h \leq 1$, *and*
(*iii*) $\|(h\delta_1)(f\delta_g)(h\delta_1)\| \leq \varepsilon$.

[37] A subset of a topological space is said to be *nowhere dense* when its closure has empty interior.

PROOF. We separate the proof in two cases, according to whether or not x_0 lies in D_g. If x_0 is not in D_g, let

$$K = \{x \in D_g : |f(x)| \geq \varepsilon\}.$$

Then K is a compact subset of D_g and, since x_0 is not in K, we may use Urysohn's Lemma obtaining a continuous, real valued function on X such that

$$0 \leq h \leq 1, \quad h(x_0) = 1, \quad \text{and} \quad h|_K = 0.$$

By temporarily working in the one-point compactification of X, and replacing K by $K \cup \{\infty\}$, we may also assume that h vanishes at infinity, meaning that h is in $C_0(X)$.

Since f is bounded by ε outside of K, it follows that $\|hf\| \leq \varepsilon$, from where (iii) easily follows, hence concluding the proof in the present case.

If x_0 is in D_g, then $\theta_{g^{-1}}(x_0)$ is defined and not equal to x_0. We may then take disjoint open sets V_1 and V_2 such that

$$x_0 \in V_1, \quad \text{and} \quad \theta_{g^{-1}}(x_0) \in V_2,$$

and we may clearly assume that $V_1 \subseteq D_g$, and $V_2 \subseteq D_{g^{-1}}$. We may further shrink V_1 by replacing it with

$$V_1' = V_1 \cap \theta_g(V_2),$$

(notice that x_0 is still in V_1'), so that $\theta_{g^{-1}}(V_1') \subseteq V_2$, and then

$$\theta_{g^{-1}}(V_1') \cap V_1' = \emptyset. \tag{29.4.1}$$

Using Urysohn's Lemma again, pick h in $C_0(X)$ such that

$$0 \leq h \leq 1, \; h(x_0) = 1, \quad \text{and} \quad h|_{X \setminus V_1'} = 0.$$

We then have

$$(h\delta_1)(f\delta_g)(h\delta_1) = \alpha_g\big(\alpha_{g^{-1}}(hf)h\big)\delta_g = 0,$$

because $\alpha_{g^{-1}}(hf)$ is supported $\theta_{g^{-1}}(V_1')$, while h is supported in V_1', and these are disjoint sets by (29.4.1). This verifies (iii) in the final case, and the proof is thus concluded. □

The following result, together with its consequences to be presented below, is gist of this chapter:

Theorem 29.5. *Let θ be a topologically free partial action of a group G on a locally compact Hausdorff space X. Then any nonzero closed two-sided ideal*

$$J \trianglelefteq C_0(X) \rtimes_{red} G$$

has a nonzero intersection with $C_0(X)$.

PROOF. We should notice that the last occurrence of $C_0(X)$, above, stands for its copy $C_0(X)\delta_1$ within $C_0(X) \rtimes_{\mathrm{red}} G$.

Denoting the conditional expectation provided by (17.8) by

$$E_1 : C_0(X) \rtimes_{\mathrm{red}} G \to C_0(X),$$

we claim that, for every z in $C_0(X) \rtimes_{\mathrm{red}} G$, and for every $\varepsilon > 0$, there exists $h \in C_0(X)$, such that

(i) $0 \leq h \leq 1$,
(ii) $\|hE_1(z)h\| \geq \|E_1(z)\| - \varepsilon$, and
(iii) $\|(h\delta_1)z(h\delta_1) - hE_1(z)h\delta_1\| \leq \varepsilon$.

Assume first that z is a linear combination of the form

$$z = z_1\delta_1 + \sum_{g\in T} z_g\delta_g, \tag{29.5.1}$$

where T is a finite subset of G, with $1 \notin T$, in which case $E_1(z) = z_1$. Let

$$V = \{x \in X : |z_1(x)| > \|z_1\| - \varepsilon\},$$

which is clearly open and nonempty. By the topological freeness hypothesis and by (29.2), each F_g is nowhere dense in X. Furthermore, by (29.3) one has that $\bigcup_{g\in T} F_g$ is likewise nowhere dense, hence there exists some

$$x_0 \in V \setminus \Big(\bigcup_{g\in T} F_g\Big).$$

For each g in T we may then apply (29.4), obtaining an h_g in $C_0(X)$ satisfying

$$h_g(x_0) = 1, \quad 0 \le h_g \le 1, \quad \text{and} \quad \|(h_g\delta_1)(z_g\delta_g)(h_g\delta_1)\| \le \frac{\varepsilon}{|T|}.$$

Here we are tacitly assuming that $|T| > 0$, and we leave it for the reader to treat the trivial case in which $|T| = 0$.

We will now show that $h := \prod_{g\in T} h_g$, satisfies conditions (i–iii), above. Noticing that (i) is immediate, we prove (ii) by observing that x_0 is in V, so

$$\|hE_1(z)h\| = \|hz_1h\| \ge |z_1(x_0)| > \|z_1\| - \varepsilon = \|E_1(z)\| - \varepsilon.$$

As for (iii), we have

$$\begin{aligned}\|(h\delta_1)z(h\delta_1) - hE_1(z)h\delta_1\| &= \|(h\delta_1)\big(z - E_1(z)\delta_1\big)(h\delta_1)\| \\ &= \Big\|\sum_{g\in T}(h\delta_1)(z_g\delta_g)(h\delta_1)\Big\| \\ &\le \sum_{g\in T}\|(h\delta_1)(z_g\delta_g)(h\delta_1)\| \\ &\le \sum_{g\in T}\|(h_g\delta_1)(z_g\delta_g)(h_g\delta_1)\| \\ &\le \varepsilon.\end{aligned}$$

This proves (i–iii) under special case (29.5.1), but since the elements of that form are dense in $C_0(X) \rtimes_{\text{red}} G$, a standard approximation argument gives the general case.

The claim verified, let us now address the statement. Arguing by contradiction we suppose that J is a nonzero ideal such that

$$J \cap C_0(X) = \{0\}.$$

Pick a nonzero element y in J, and let $z = y^*y$. Using the claim, for each positive ε we choose h in $C_0(X)$, satisfying (i–iii) above. Let

$$q \;:\; C_0(X) \rtimes_{\text{red}} G \;\to\; \frac{C_0(X) \rtimes_{\text{red}} G}{J}$$

be the quotient map. Since z lies in J, we have that

$$q\big((h\delta_1)z(h\delta_1)\big) = 0,$$

whence

$$\begin{aligned}\big\|q\big(hE_1(z)h\delta_1\big)\big\| &= \big\|q\big((h\delta_1)z(h\delta_1) - hE_1(z)h\delta_1\big)\big\| \\ &\leq \big\|(h\delta_1)z(h\delta_1) - hE_1(z)h\delta_1\big\| \\ &\overset{\text{(iii)}}{\leq} \varepsilon.\end{aligned}$$

Since $J \cap C_0(X) = \{0\}$, we deduce that q is injective, hence isometric, on $C_0(X)$. So

$$\varepsilon \geq \|hE_1(z)h\| \overset{\text{(ii)}}{\geq} \|E_1(z)\| - \varepsilon,$$

from where we see that $\|E_1(z)\| \leq 2\varepsilon$, and since ε is arbitrary, we have

$$0 = E_1(z) = E_1(y^*y).$$

It then results from (17.13) that $y = 0$, a contradiction. This concludes the proof. □

A useful consequence to the representation theory of reduced crossed products is as follows:

Corollary 29.6. *Let θ be a topologically free partial action of a group G on a locally compact Hausdorff space X. Then a *-representation of $C_0(X) \rtimes_{red} G$ is faithful if and only if it is faithful on $C_0(X)$.*

PROOF. Apply (29.5) to the kernel of the given representation. □

Definition 29.7. We will say that θ is a *minimal partial action* if there are no θ-invariant closed subsets of X, other than X and the empty set.

The complement of an invariant set is invariant too, so minimality is equivalent to the absence of nontrivial *open* invariant subsets.

Corollary 29.8. *If, in addition to the conditions of* (29.5)*, θ is minimal, then $C_0(X) \rtimes_{red} G$ is a simple*[38] *C*-algebra.*

PROOF. Let J be a nonzero, closed two-sided ideal of $C_0(X) \rtimes_{\text{red}} G$. Employing (29.5), we have that

$$K := J \cap C_0(X) \neq \{0\}. \tag{29.8.1}$$

Since K is an ideal in $C_0(X)$, there is an open subset $U \subseteq X$, such that $K = C_0(U)$. Using (23.11) we have that K is α-invariant, and it is easy to see that this implies U to be θ-invariant. By minimality of θ, one has that either $U = \emptyset$, or $U = X$, but under the first case we would have $K = \{0\}$, contradicting (29.8.1). So $U = X$, and hence $K = C_0(X)$, meaning that $C_0(X) \subseteq J$.

By (16.27), which is stated for the full cross-sectional algebra, but which evidently also holds for the reduced one, we have that $C_0(X)$ (the unit fiber algebra in the semi-direct product bundle) generates $C_0(X) \rtimes_{\text{red}} G$, as an ideal, whence $J = C_0(X) \rtimes_{\text{red}} G$, and the proof is concluded. □

The upshot of this result is that when only two θ-invariant open subsets exist in X, namely $\emptyset$ and the whole space, then only two closed two-sided ideals exist in $C_0(X) \rtimes_{\text{red}} G$, namely $\{0\}$ and the whole algebra.

This raises the question as to whether a correspondence may still be found between invariant open subsets and closed two-sided ideals, in case these exist in

[38] A C*-algebra is said to be simple if there are no nontrivial closed two-sided ideals.

greater numbers. In order to find such a correspondence we need a bit more than topological freeness.

Theorem 29.9. *Let θ be a topological partial action of a group G on a locally compact Hausdorff space X, such that either:*

(i) G is an exact group, or
(ii) the semi-direct product bundle associated to θ satisfies the approximation property.

In addition we suppose that θ, as well as the restriction of θ to any closed θ-invariant subset of X, is topologically free. Then there is a one-to-one correspondence between θ-invariant open subsets $U \subseteq X$, and closed two-sided ideals in $C_0(X) \rtimes_{red} G$, given by

$$U \mapsto C_0(U) \rtimes_{red} G.$$

PROOF. Given a θ-invariant open subset $U \subseteq X$, it is easy to see that $C_0(U)$ is an α-invariant ideal in $C_0(X)$. By (22.9), one then has that $C_0(U) \rtimes_{\text{red}} G$ is an ideal in $C_0(X) \rtimes_{\text{red}} G$.

By first checking on elements of the algebraic crossed product, it is easy to see that the conditional expectation E_1 of (17.8) satisfies

$$E_1\big(C_0(U) \rtimes_{\text{red}} G\big) = C_0(U)\delta_1.$$

From this it follows that our correspondence is injective, and we are then left with the task of proving that any ideal

$$J \trianglelefteq C_0(X) \rtimes_{\text{red}} G$$

is of the above form. Given J, let $U \subseteq X$ be the unique open subset of X such that

$$J \cap C_0(X) = C_0(U).$$

By (23.11) we have that $C_0(U)$ is α-invariant, whence U is θ-invariant and hence so is

$$F := X \setminus U.$$

It is a well known fact that the map

$$q : f \in C_0(X) \mapsto f|_F \in C_0(F)$$

passes to the quotient modulo $C_0(U)$, leading up to a *-isomorphism

$$C_0(X)/C_0(U) \simeq C_0(F).$$

Observe that $C_0(X)/C_0(U)$ carries a quotient partial action of G as proved in (22.7), while the restriction of θ to F induces a partial action of G on $C_0(F)$, via (11.6). The isomorphism mentioned above may then be easily shown to be G equivariant. So, employing (22.9) we get the following exact sequence of C*-algebras and *-homomorphisms

$$0 \;\to\; C_0(U) \rtimes_{\text{red}} G \;\xrightarrow{\iota_{\text{red}}}\; C_0(X) \rtimes_{\text{red}} G \;\xrightarrow{q_{\text{red}}}\; C_0(F) \rtimes_{\text{red}} G \;\to\; 0.$$

By (16.27) notice that $C_0(U) \rtimes_{\text{red}} G$ coincides with the ideal generated by $C_0(U)$ within $C_0(X) \rtimes_{\text{red}} G$. Since $C_0(U) \subseteq J$, we then deduce that

$$C_0(U) \rtimes_{\text{red}} G \subseteq J. \tag{29.9.1}$$

We next claim that

$$q_{\text{red}}(J) \cap C_0(F) = \{0\}. \tag{29.9.2}$$

In order to see this, let z be in $q_{\rm red}(J) \cap C_0(F)$, and choose y in J such that $q_{\rm red}(y) = z$. Since $C_0(F) = q_{\rm red}\big(C_0(X)\big)$, we may also choose f in $C_0(X)$ such that $q_{\rm red}(f) = z$. Therefore $q_{\rm red}(f - y) = 0$, hence we see that

$$f - y \in \operatorname{Ker}(q_{\rm red}) = C_0(U) \rtimes_{\rm red} G \subseteq J,$$

from where it follows that

$$f \in J \cap C_0(X) = C_0(U),$$

so

$$0 = q_{\rm red}(f) = z.$$

This proves (29.9.2), and since the restriction of θ to F is topologically free by hypothesis, we may use (29.5) to deduce that $q_{\rm red}(J) = \{0\}$, which is to say that

$$J \subseteq \operatorname{Ker}(q_{\rm red}) = C_0(U) \rtimes_{\rm red} G$$

so the inclusion in (29.9.1) is in fact an equality of sets. This concludes the proof. □

As a consequence we have:

Corollary 29.10. *Under the hypotheses of Theorem* (29.9)*, every ideal of* $C_0(X) \rtimes_{red} G$ *is induced (in the sense of Definition* (23.2)*), and hence also a Fourier ideal.*

Proof. In view of (29.9) it is enough to prove that the ideal

$$J = C_0(U) \rtimes_{\rm red} G$$

is induced for every θ-invariant open subset $U \subseteq X$. Given any such U, we need to prove that J is generated by $J \cap C_0(X)\delta_1$. However, since it is clear that

$$C_0(U)\delta_1 \subseteq J \cap C_0(X)\delta_1,$$

we deduce that

$$J = C_0(U) \rtimes_{\rm red} G \overset{(16.27)}{=} \langle C_0(U)\delta_1 \rangle \subseteq \langle J \cap C_0(X)\delta_1 \rangle \subseteq J,$$

so J is indeed an induced ideal. By (23.9) we then also have that J is a Fourier ideal. □

Notes and remarks. Theorem (29.5) first appeared in [**59**, Theorem 2.6]. It is a direct generalization of [**74**, Theorem 4.1] for partial actions. See also [**56**, Theorem 4.4].

A groupoid version of Theorem (29.9) is stated in [**96**, Proposition 4.5], although there is a missing hypothesis in it, without which one might have the bad behavior discussed in [**97**, Remark 4.10]. The correct statement applies for amenable groupoids and is to be found in [**97**, Corollary 4.9].

Since a notion of *exact groupoid* does not seem to exist, Theorem (29.9) under hypothesis (i) does not appear to have a groupoid counterpart.

Many authors have studied results similar to the ones we proved above for global actions of discrete groups on non-abelian C*-algebras. See, for example, [**43**], [**74**] and [**8**].