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Nilpotent Structures in Ergodic Theory

Bernard Host
Bryna Kra



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Nilpotent Structures in Ergodic Theory

**Bernard Host
Bryna Kra**

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To Bati
To Brian

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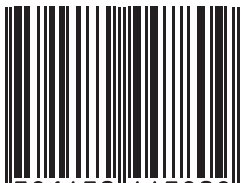
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