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Hopf Algebras and Root Systems

István Heckenberger
Hans-Jürgen Schneider

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Dedicated to Nicolás Andruskiewitsch

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Preface

This book is an introduction to Hopf algebras in braided monoidal categories with applications to Hopf algebras in the usual sense, that is, in the category of vector spaces. By now there exists a wide variety of deep results in this area, and we don't aim to provide a complete overview. We will discuss some of these topics in Chapter 17.

Our main goal is to present from scratch and with complete proofs the theory of Nichols algebras (or quantum symmetric algebras) and the surprising relationship between Nichols algebras and (generalized) root systems. Hopefully our book makes the vast literature in the area more accessible, and it is useful for future research.

Since its beginnings some 70 years ago, the theory of Hopf algebras has developed rapidly into various directions. Its origins came from algebraic topology, algebraic and formal groups, and operator algebras. The influential book of Sweedler from 1969 [**Swe69**] laid the foundations of a general theory of abstract (non-commutative and non-cocommutative) Hopf algebras. After the work of Drinfeld and Jimbo on quantum groups, and Drinfeld's report "Quantum groups" [**Dri87**] at the International Congress of Mathematicians 1986, the interest in the topic drastically increased.

Quantum groups are prominent examples of pointed Hopf algebras (their irreducible comodules are one-dimensional). Several years after their discovery, general classification results for pointed Hopf algebras were obtained ([**AS02**]; [**AS04**], [**AA08**], [**AS10**] depending on [**Ros98**], [**Kha99**], [**Hec06**], [**Hec08**]). In these papers, the classical theory of quantum groups and of the small quantum groups as developed in [**Lus93**] is applied.

Although quantum groups are intrinsically related to Lie theoretical structures, it is not at all obvious to which extent this is true for general pointed Hopf algebras. The lifting method introduced in [**AS98**] showed that the classification of Nichols algebras is an essential step in the classification theory of pointed Hopf algebras. And here, in the theory of Nichols algebras, the combinatorics of root systems and Weyl groups, or better Weyl groupoids, plays an important role. Weyl groupoids were introduced in [**Hec06**] for diagonal braidings using Kharchenko's PBW basis [**Kha99**] based on the theory of Lyndon words, and in [**AHS10**] in general.

Nichols algebras as a special class of braided pointed Hopf algebras are studied in great detail in this book. They appeared first in [**Nic78**], independently as braided algebras in [**Wor89**]. It follows from the work of Lusztig [**Lus93**] that $U_q^+(\mathfrak{g})$, $\mathfrak{g}$ symmetrizable Kac-Moody Lie algebra, q transcendental, is a Nichols algebra; see [**Ros98**] (where a dual description of Nichols algebras as quantum shuffle algebras is used), [**Gre97**], and [**Sch96**].

We emphasize categorical constructions and one-sided coideal subalgebras. The introduction of Nichols systems, which are generalizations of Nichols algebras together with a grading by a free abelian group, allows us to develop the theory in a very general setting. We do not use the theory of Lyndon words, and we do not assume results from quantum groups. Our theory can be applied to quantum groups, and some of our results on right coideal subalgebras are new also in the special case of quantum groups.

Prerequisites. The reader is expected to be familiar with linear algebra and algebra on the graduate level including tensor products of modules, basic noncommutative algebra, and the language of categories, functors, and natural transformations. For a better understanding, a course in semisimple Lie algebras would be helpful but is not strictly necessary.

We now describe the contents of the book in more detail.

(1) Foundations. We begin in Chapter 1 with a quick introduction to Nichols algebras. Our goal is to give a complete exposition of the basics of Nichols algebras which are scattered over various papers.

The most important example of a braided monoidal category in this book is the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over some Hopf algebra H with bijective antipode. If $H = \Bbbk G$ is the group algebra of a group G over a field $\Bbbk$, then an object in ${}^H_H\mathcal{YD}$ is a G-graded vector space $V = \bigoplus_{g\in G} V_g$ with a G-action such that for all $g, h \in G$, $g \cdot V_h = V_{ghg^{-1}}$. The braiding $c_{V,W}$ between objects $V, W \in {}^H_H\mathcal{YD}$ is given by

$$c_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto g \cdot w \otimes v, \quad v \in V_g,\ w \in W.$$

The maps $c_{V,W}$ are G-graded and G-linear, where the monoidal structure is given by the usual grading and diagonal action on the tensor product $V \otimes W$. For any object $V \in {}^H_H\mathcal{YD}$, the Nichols algebra $\mathcal{B}(V)$ is defined as follows. We want an $\mathbb{N}_0$-graded Hopf algebra R in the braided category ${}^H_H\mathcal{YD}$ in which the elements of V are primitive and generators of the algebra. Moreover, R should be minimal in the sense that there are no other primitive elements than those in V. Of course, the tensor algebra $T(V)$ is an $\mathbb{N}_0$-graded Hopf algebra generated by V, where the elements of V are primitive. But in general there are more primitive elements in higher degrees. We define the **Nichols algebra** $\mathcal{B}(V)$ by

$$\mathcal{B}(V) = T(V)/I(V), \ \ I(V) \text{ the largest coideal in degree } \geq 2.$$

This is an $\mathbb{N}_0$-graded braided quotient Hopf algebra of the tensor algebra. Thus the Nichols algebra is defined by a universal property, which means that it is very often quite difficult to really compute $\mathcal{B}(V)$. In Corollary 1.9.7 we prove that the relations of the Nichols algebra can be described by the quantum symmetrizer maps defined by the action of the braid group. This is an important theoretical result. However, it does not immediately help, for example, to decide which Nichols algebras are finite-dimensional.

Let A be a Hopf algebra whose coradical $A_0 = H$ is a Hopf subalgebra, and let $\operatorname{gr} A$ be the associated $\mathbb{N}_0$-graded Hopf algebra with respect to the coradical filtration. Then the Nichols algebra over H appears naturally as a subalgebra of $\operatorname{gr} A$ (see Corollary 7.1.17). Hence Nichols algebras are essential for the classification problem of such Hopf algebras A.

Chapter 2 is a collection of fairly standard results in the theory of Hopf algebras which we will need later on or which motivate more general constructions later.

In Chapter 3 the theory of Hopf algebras in braided (strict) monoidal categories $\mathcal{C}$ is presented, partly with new proofs. To our knowledge, this theory didn't appear so far in a textbook. Sections 3.8 and 3.10 contain detailed proofs of the Radford-Majid-Bespalov theory of bosonization and Hopf algebras with a projection in braided categories. Theorem 3.10.6 on left and right coinvariant subobjects seems to be new; it is used to prove the existence of the Hopf algebra isomorphism T in Theorem 12.3.3, which in this book plays the role of the Lusztig automorphisms of quantum groups.

In Chapter 4 we specialize Chapter 3 to the braided category ${}^H_H\mathcal{YD}$. By Theorem 4.4.11, a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$ has bijective antipode and is a Frobenius algebra. This was shown in the pioneering paper [**LS69**] for usual Hopf algebras.

In Chapter 5 a fairly general theory of filtrations by abelian monoids is presented, which will be applied in particular to $\mathbb{N}_0^\theta$, $\theta \geq 2$, to obtain appropriate gradings of Nichols algebras. In addition we study the coradical filtration assuming standard results from the theory of the Jacobson radical of algebras.

Chapters 6 and 7 deal with general braided vector spaces and their Nichols algebras. They are rather independent of the remaining parts of the book. In Corollary 7.2.8 we establish the fundamental non-degenerate pairing between $B(V^*)$ and $B(V)$, where V is a finite-dimensional object in ${}^H_H\mathcal{YD}$.

In Chapter 8 we discuss quantized enveloping algebras and, more generally, linkings of Nichols algebras. We define Hopf algebras $U(\mathcal{D}, \lambda)$ which generalize the quantum groups $U_q(\mathfrak{g})$; they are given by the Serre relations in each connected component of the Dynkin diagram and linking relations such as the relations between the E_i and F_i for quantum groups (introduced in [**AS02**]).

(2) The main motivating problem. Lusztig in [**Lus93**] defines the positive part U_q^+ of the deformed universal enveloping algebra of a Kac-Moody Lie algebra by a universal property which is easily seen to be an alternative description of the Nichols algebra of the degree one part V of U_q^+. In this case V is a Yetter-Drinfeld module over the group algebra of a free abelian group G with basis $K_1, \dots, K_n$, and

$$V = \bigoplus_{i=1}^{n} \Bbbk E_i, \quad E_i \in V_{K_i}, \quad K_i \cdot E_j = q^{d_i a_{ij}} \text{ for all } i, j.$$

Here, q is not a root of unity, and $(d_i a_{ij})_{1 \leq i,j \leq n}$ is the symmetrized Cartan matrix. (In Lusztig's book, q is transcendental, and $\mathrm{char}(\Bbbk) = 0$.) The Nichols algebras of the summands $\Bbbk E_i$ are simply polynomial algebras in the variable E_i. Much later in his book, Lusztig shows that U_q^+ is explicitly given by the quantum Serre relations.

Assume more generally that

$$V = \bigoplus_{i=1}^{\theta} M_i \in {}^H_H\mathcal{YD}$$

is a finite direct sum of finite-dimensional irreducible objects $M_i \in {}^H_H\mathcal{YD}$, where H is a Hopf algebra with bijective antipode. If H is the group algebra of a finite group, and if the characteristic of the field does not divide the order of the group, then any finite-dimensional object V in ${}^H_H\mathcal{YD}$ is semisimple. The Nichols algebra $\mathcal{B}(V)$ has an additional important structure. It is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. We denote the standard basis of $\mathbb{Z}^\theta$ by $\alpha_1, \ldots, \alpha_\theta$, and define the degree of M_i as α_i. Suppose we know the $\mathcal{B}(M_i)$. Which additional information is needed to understand $\mathcal{B}(V)$? For example, when is $\mathcal{B}(V)$ finite-dimensional? Is there an analog of Lusztig's PBW-basis depending on the longest element in the Weyl group of a semisimple Lie algebra?

Note that in our general situation no Cartan matrix is given a priori. The key to the missing information will be the root system and the Weyl groupoid of the tuple $M = (M_1, \ldots, M_\theta)$. We define the Nichols algebra of the tuple by $\mathcal{B}(M) = \mathcal{B}(V)$.

(3) The combinatorics of Cartan graphs and their Weyl groupoids. This is a generalization of the notion of a Cartan matrix and its Weyl group to a family of Cartan matrices. Right now there are several approaches to this theory. Nevertheless we restrict ourselves in Part 2 of the book to a presentation based on families of Cartan matrices, since this approach appears to be most useful to explain the combinatorics in the theory of Nichols algebras. Part 2 is independent of the theory of Nichols algebras.

Let $\theta \geq 1$ be a natural number, $\mathbb{I} = \{1, \ldots, \theta\}$, $\mathcal{X}$ a non-empty set, $(r_i)_{i \in \mathbb{I}}$ a family of maps $r_i : \mathcal{X} \to \mathcal{X}$, and $(A^X)_{X \in \mathcal{X}}$ a family of (generalized) Cartan matrices. The quadruple $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, (r_i), (A^X))$ is called a **semi-Cartan graph** if the following axioms hold.

(CG1) For all $i \in \mathbb{I}$, $r_i^2 = \mathrm{id}_{\mathcal{X}}$.

(CG2) For all $i \in \mathbb{I}$, $X \in \mathcal{X}$, A^X and $A^{r_i(X)}$ have the same i-th row.

For all $X \in \mathcal{X}$ and $i \in \mathbb{I}$ let $s_i^X \in \mathrm{Aut}(\mathbb{Z}^\theta)$ be the reflection map defined by $s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i$ for all $j \in I$. Let $\mathcal{W}(\mathcal{G})$ be the groupoid with objects $\mathcal{X}$ and morphisms generated by formal maps $s_i^X : X \to r_i(X)$. Composition of such morphism is given by multiplication in $\mathrm{Aut}(\mathbb{Z}^\theta)$. Note that $\mathcal{W}(\mathcal{G})$ is a groupoid (a category where every morphism is an isomorphism), since $s_i^{r_i(X)}$ is inverse to s_i^X. The **real roots** of X are the elements in $\mathbb{Z}^\theta$ which can be written as $w(\alpha_i)$ for some morphism $w : Y \to X$ and $i \in \mathbb{I}$ ($w(\alpha_i) = f(\alpha_i)$, where w is given by $f \in \mathrm{Aut}(\mathbb{Z}^\theta)$).

The axioms of a semi-Cartan graph are not yet strong enough to be useful. For example, we want that the real roots are positive or negative, that is, in $\mathbb{N}_0^\theta$ or in $-\mathbb{N}_0^\theta$. We define in Definition 9.1.14 a **Cartan graph** by two additional axioms (CG3) and (CG4). If $\mathcal{G}$ is a Cartan graph, we call $\mathcal{W}(\mathcal{G})$ the **Weyl groupoid** of $\mathcal{G}$. The importance of the axioms of a Cartan graph $\mathcal{G}$ comes from Theorem 9.4.8, where we show that the Weyl groupoid of a Cartan graph $\mathcal{G}$ is a **Coxeter groupoid** (in a different language this is a result of [**HY08**]), that is, the Weyl groupoid has defining relations of the same type as Coxeter groups have.

Most of the results in Part 2 have been already published in [**HY08**], [**CH09b**], [**CH09a**], and [**CH12**]. However, in Section 9.2 we present new axioms (CG3') and (CG4') of a Cartan graph in terms of reduced sequences. These axioms are those appearing most naturally for semi-Cartan graphs of Nichols systems.

(4) The Cartan graph of a Nichols algebra. Let $M = (M_1, \ldots, M_\theta)$ as above. First we have to define reflection operators on tuples of Yetter-Drinfeld modules. For each $i \in \mathbb{I}$ let $R_i(M) = (M'_1, \dots, M'_\theta)$, where

$$M'_j = \begin{cases} M_i^* & \text{if } j = i, \\ (\operatorname{ad} M_i)^{-a_{ij}^M}(M_j) & \text{if } j \neq i, \end{cases}$$

and where we assume that $a_{ij}^M = -\max\{m \in \mathbb{N}_0 \mid (\operatorname{ad} M_i)^m(M_j) \neq 0\}$ exists. The i-th component is the dual Yetter-Drinfeld module M_i^*, and ad is the braided adjoint action in the Nichols algebra $\mathcal{B}(M) = \mathcal{B}\big(\bigoplus_{i=1}^\theta M_i\big)$. By Lemma 13.4.4, $(a_{ij}^M)_{i,j\in\mathbb{I}}$ is a (generalized) Cartan matrix, when we set $a_{ii}^M = 2$. By Corollary 13.4.3, the components of $R_i(M)$ are again irreducible. Note the formal similarity with Lusztig's isomorphisms T_i of quantum groups, where

$$T_i(E_j) = \begin{cases} -F_i K_i & \text{if } j = i, \\ (\operatorname{ad} E_i)^{(-a_{ij})}(E_j) & \text{if } j \neq i. \end{cases}$$

The set of points $\mathcal{X}$ of $\mathcal{G}(M)$ is the set of isomorphism classes of all $R_{i_n} \cdots R_{i_1}(M)$, $n \geq 0$, which we assume to exist. We have attached to each $X = [M] \in \mathcal{X}$ a Cartan matrix $A^X = (a_{ij}^M)_{i,j\in\mathbb{I}}$, and we have defined maps $r_i : \mathcal{X} \to \mathcal{X}$, $[M] \mapsto [R_i(M)]$ ($[M]$ denotes the isomorphism class of M). By Theorem 13.6.2, $\mathcal{G}(M)$ is a semi-Cartan graph. This result was first obtained in [**AHS10**] with a different proof.

In order to implement the remaining axioms of a Cartan graph, sequences of graded right coideal subalgebras of Nichols algebras and their compatibility with reflections are studied in Chapter 14. Important results in this respect are Theorem 14.1.4, and in particular Theorem 14.1.9. The latter relates sequences of right coideal subalgebras of Nichols algebras to reduced sequences in the semi-Cartan graph. In Section 14.2 we introduce the notion of an exact factorization of bialgebras and Nichols systems. With this tool we prove in Theorem 14.2.12 that the semi-Cartan graph of a Nichols algebra admitting all reflections is indeed a Cartan graph. This is a new result; it was first shown in [**HS10b**] for finite semi-Cartan graphs $\mathcal{G}(M)$. It is more general than what was shown in the existing approaches, where the root system of the Nichols algebra, usually based on the theory of Lyndon words, was assumed.

(5) Categorical tools, and the role of the Lusztig isomorphisms. The proofs of these results on the Cartan graph $\mathcal{G}(M)$ depend on Chapters 12 and 13. For all $i \in \mathbb{I}$, let $K_i^{\mathcal{B}(M)}$ be the set of right coinvariant elements of the canonical projection $\mathcal{B}(M) \to \mathcal{B}(M_i)$. By the braided version of the Theorem of Radford on projections of Hopf algebras, $K_i^{\mathcal{B}(M)}$ is a Hopf algebra in the braided category ${}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, where $\mathcal{C} = {}^H_H\mathcal{YD}$, and $\mathcal{B}(M)$ is isomorphic to the smash product Hopf algebra $K_i^{\mathcal{B}(M)} \# \mathcal{B}(M_i)$. In Theorem 12.3.2 (which first appeared in [**HS13b**] in an equivalent version and with a very different proof) we show that there is a braided isomorphism

$$(\Omega, \omega) : {}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}} \to {}^{\mathcal{B}(M_i^*)}_{\mathcal{B}(M_i^*)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}.$$

Hence $\Omega(K_i^{\mathcal{B}(M)})$ is a Hopf algebra in ${}^{\mathcal{B}(M_i^*)}_{\mathcal{B}(M_i^*)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, and we may consider its bosonization $\Omega(K_i^{\mathcal{B}(M)}) \# \mathcal{B}(M_i^*)$. By Theorem 13.4.9, this bosonization is isomorphic to $\mathcal{B}(R_i(M))$. The deeper results on $\mathcal{B}(R_i(M))$ depend on this isomorphism.

Theorem 12.3.3 is another categorical result on the isomorphism (Ω, ω). It implies a very close relationship between $\mathcal{B}(M)$ and $\mathcal{B}(R_i(M))$. There is an isomorphism of braided Hopf algebras

$$T_i^{\mathcal{B}(M)} : L_i^{\mathcal{B}(R_i(M))} \to K_i^{\mathcal{B}(M)}$$

between the left coinvariants $L_i^{\mathcal{B}(R_i(M))}$ of the projection

$$\mathcal{B}(R_i(M)) \cong \Omega(K_i^{\mathcal{B}(M)}) \# \mathcal{B}(M_i^*) \to \mathcal{B}(M_i^*)$$

and the right coinvariants $(K_i^{\mathcal{B}(M)})^{\mathrm{cop}}$ of $\mathcal{B}(M)$. To make sense, this Hopf algebra isomorphism has to be understood in the formulation of Theorem 12.3.3 which did not appear in print before.

The isomorphisms $T_i^{\mathcal{B}(M)}$ play the role of the Lusztig automorphisms to construct a PBW basis of U_q^+. Since the maps $T_i^{\mathcal{B}(M)}$ can be seen as isomorphisms of Hopf algebras, they can be used in Theorem 14.1.9 to construct right coideal subalgebras in $\mathcal{B}(M)$ stepwise (Lusztig's isomorphisms are maps of algebras not of coalgebras).

If the Cartan graph $\mathcal{G}(M)$ is finite, that is, there are only finitely many real roots, then we obtain by this procedure in Corollary 14.5.3 a tensor decomposition

$$\mathcal{B}(M_{\beta_m}) \otimes \cdots \otimes \mathcal{B}(M_{\beta_1}) \cong \mathcal{B}(M), \tag{0.0.1}$$

depending on the longest element in $\mathrm{Hom}(\mathcal{W}(M), [M])$, where $M_{\beta_m}, \ldots, M_{\beta_1}$ are irreducible subobjects of $\mathcal{B}(M)$ in ${}_H^H\mathcal{YD}$ which correspond to the higher root vectors of quantum groups, and $\deg(M_{\beta_i}) = \beta_i \in \mathbb{N}_0^\theta$ for all i. For all $1 \le l \le m$, the image of $\mathcal{B}(M_{\beta_l}) \otimes \cdots \otimes \mathcal{B}(M_{\beta_1})$ in $\mathcal{B}(M)$ is a right coideal subalgebra.

Assume that the components M_i of M are one-dimensional. Then the M_{β_l} in (0.0.1) are one-dimensional, the algebras $\mathcal{B}(M_{\beta_l})$ are polynomial rings or truncated polynomial rings. Thus we have constructed a PBW basis of $\mathcal{B}(M)$. In particular, we obtain Lusztig's PBW basis of $U_q^+(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, without any case by case considerations; see also Remark 16.2.6. The Levendorskii-Soibelman commutation relations are also shown in the general context of Nichols algebras over any field; see Theorem 14.1.12 and Theorem 16.3.16.

In Corollary 14.5.3 we prove that $\mathcal{G}(M)$ must be finite if $\mathcal{B}(M)$ is finite-dimensional.

Assume that $\mathcal{G}(M)$ is finite. In Corollary 14.6.8 we prove that the construction of right coideal subalgebras mentioned above defines a bijection

$$\mathrm{Hom}(\mathcal{W}(M), [M]) \to \mathcal{K}(\mathcal{B}(M))$$

between morphisms in the Weyl groupoid ending in $[M]$ and the set of all graded right coideal subalgebras of $\mathcal{B}(M)$. Kharchenko [**Kha11**] conjectured that the number of such right coideal subalgebras in $U_q^+(\mathfrak{g})$ (for simple Lie algebras) is equal to the order of the Weyl group. Our work on right coideal subalgebras in [**HS13a**] was motivated by this conjecture, which is now proved as a special case of Corollary 14.6.8. As a novelty, in Theorem 14.6.6 we generalize the correspondence in Corollary 14.6.8 to tuples with not necessarily finite Cartan graph.

The categorical results in Chapter 12 are very general. They can be applied to any Hopf algebra K in ${}_{\mathcal{B}(M_i)}^{\mathcal{B}(M_i)}\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, not just to $K_i^{\mathcal{B}(M)}$. This leads to a new and substantial extension of the theory of Nichols algebras in Section 13.5. There we introduce Nichols systems and define reflection operators for Nichols systems. The stepwise construction of right coideal subalgebras in Section 14.1 works for Nichols systems.

We use Nichols systems to establish criteria when a given pre-Nichols algebra is Nichols. By Theorem 14.5.4, any pre-Nichols system admitting all reflections and having a finite Cartan graph is in fact a Nichols algebra. Theorem 14.5.4 is fundamental for several proofs later on in the book. We would like to highlight Theorem 15.5.1 (finite-dimensional pre-Nichols algebras of diagonal type are Nichols), Theorem 16.2.5(2) (the positive part U_q^+ of a quantum group attached to a Cartan matrix of finite type, q not a root of 1, is a Nichols algebra), Theorem 16.4.23(2) (a pre-Nichols algebra with finite Gelfand-Kirillov dimension of a braided vector space of quasi-generic Cartan type is the Nichols algebra U_q^+), and Corollary 16.4.24 (a braided vector space of diagonal type with a Nichols algebra being a domain of finite Gelfand-Kirillov dimension is quasi-generic of finite Cartan type); see below for more details.

(6) Applications. After some basic observations on reflections of Yetter-Drinfeld modules of diagonal type in Section 15.1, we study root vector sequences in pre-Nichols systems. In the special case of usual quantum groups, the root vectors of Lusztig are shown later in Remark 16.2.6 to form root vector sequences. This has advantages for both approaches: Lusztig's root vectors satisfy integrality properties, and root vector sequences are defined by defining properties which can be used to develop new methods (such as braided commutators associated to Lyndon words) to construct them. Further important differences in the two approaches to quantum groups are that our root vectors are only unique up to scalar multiples, we don't use an analog of the braid relations for Lusztig's automorphisms, and we don't need to perform case by case analysis (except in Remark 16.2.6 to prove the correspondence). Note that root vector sequences, similarly to Lusztig's root vectors, are defined for any reduced decomposition of an element of the Weyl group(oid).

Using root vector sequences, Theorem 15.2.7 describes a basis of any right coideal subalgebra of a Nichols system attached to a reduced decomposition of an element of the Weyl groupoid.

Following [**HW15**], in Theorem 15.3.1 we classify two-dimensional braided vector spaces of diagonal type which have a finite Cartan graph, where the field $\Bbbk$ has characteristic 0. This classification uses explicitly the combinatorics of finite Cartan graphs of rank two from Section 10.3. The classification in [**Hec09**] of all finite-dimensional braided vector spaces of diagonal type and with finite Cartan graph is beyond the scope of this book.

Angiono in [**Ang15**] (using the results on right coideal subalgebras in Corollary 14.6.8) and [**Ang13**] found a celebrated presentation of the Nichols algebras appearing in [**Hec09**] in terms of generators and relations, where the ground field is algebraically closed of characteristic 0.

A conjecture in [**AS00a**] says that any finite-dimensional pointed Hopf algebra H over an algebraically closed field of characteristic 0 is generated as an algebra by group-like and skew-primitive elements. In Theorem 15.5.1 we prove that finite-dimensional pre-Nichols algebras of diagonal type over a field of characteristic 0 are Nichols algebras. This proves the conjecture when the group of group-like elements of H is abelian. This theorem was originally proved by I. Angiono in [**Ang13**] using his list of defining relations of the finite-dimensional Nichols algebras classified in [**Hec09**]. In contrast, our proof is based on the aforementioned Theorem 14.5.4 and some results in rank two and partially in rank three.

In Chapter 16, especially in Theorems 16.2.5 and 16.3.17, we recover the results of Angiono on generators and relations for Nichols algebras of finite Cartan type (which include the algebras studied by Lusztig when the Cartan matrix is of finite type) except for a few cases with parameters of small order. In the discussed cases the Nichols algebras are presented by the quantum Serre relations and by root vector relations. The proof of Theorem 16.2.5, where the braiding matrix is quasi-generic, is a more or less direct application of Theorem 14.5.4. A proof of Theorem 16.3.17 along the same line, where the entries of the braiding matrix are roots of unity, appears to be problematic since the root vector relations depend on the choice of a presentation of the longest element of the Weyl group. Instead, we provide first in Theorem 16.3.14 a basis of the Hopf algebra $U_{\boldsymbol{q}}^+$ defined by the quantum Serre relations by analyzing root vector sequences. This together with an easy dimension argument yields the claim.

It is known that for the excluded exceptional cases additional defining relations are needed.

In Section 16.4 we study Nichols algebras of diagonal type, which are domains of finite Gelfand-Kirillov dimension. By Corollary 16.4.24, these are the Nichols algebras of finite Cartan type, where the diagonal entries of the braiding are 1 (only in characteristic 0) or not roots of 1.

In Theorem 16.5.10 we show that the pointed Hopf algebras with abelian coradical, generic infinitesimal braiding, and finite Gelfand-Kirillov dimension are exactly the Hopf algebras $U(\mathcal{D}, \lambda)$ defined in Section 8.3 generalizing the quantum groups $U_q(\mathfrak{g})$. This was shown in [**AS04**] for positive braidings using [**Ros98**], and extended in [**AA08**] to the general case using [**Hec06**].

In Chapter 17 Nichols algebras over non-abelian groups are studied. Among others we prove in Corollary 17.1.5 (partly following [**HS10b**]) that the Nichols algebra of a non-zero non-simple Yetter-Drinfeld module over a finite simple group is necessarily infinite-dimensional. A similar result for the symmetric groups $\mathbb{S}_n$ with $n \geq 3$ is shown in Corollary 17.1.8.

The theory of reflections does not give direct information about Nichols algebras of irreducible Yetter-Drinfeld modules over groups. However, it can be helpful to prove that a given Nichols algebra of an irreducible Yetter-Drinfeld module is infinite-dimensional by finding a braided subspace which can be realized over some other group with decomposable Yetter-Drinfeld module and which has infinite-dimensional Nichols algebra. This is demonstrated in Corollary 17.1.11 which led to the definition of racks of type D. The rack theoretical formulation of Corollary 17.1.11 (finite racks of type D collapse) was used for example in [**AF$^+$11a**] to show that any finite-dimensional pointed Hopf algebra H over $\mathbb{C}$ with group $G(H) \cong \mathbb{A}_n$, $n \geq 5$, is isomorphic to the group algebra $\mathbb{C}\mathbb{A}_n$ of the alternating group. (Racks of type D were not used for $\mathbb{A}_5$.)

We collect the known finite-dimensional examples of Nichols algebras of irreducible Yetter-Drinfeld modules over groups in characteristic 0 in Section 17.2 without proofs. Finally, in Section 17.3 the finite-dimensional Nichols algebras of direct sums of two simple Yetter-Drinfeld modules are listed without proof; this classification uses the finiteness of the corresponding Cartan graph by Corollary 14.5.3. For references, see Chapter 17.

In the notes in the end of each chapter we refer to the relevant literature. We do this to the best of our knowledge, and we apologize to all authors whose work we have unintentionally not mentioned appropriately.

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In general, Nichols algebras are not classified by Cartan graphs and their root systems. However, extending partial results in the literature, the authors were able to associate a Cartan graph to a large class of Nichols algebras. This allows them to determine the structure of right coideal subalgebras of Nichols systems which generalize Nichols algebras. As applications of these results, the book contains a classification of right coideal subalgebras of quantum groups and of the small quantum groups, and a proof of the existence of PBW-bases that does not involve case by case considerations. The authors also include short chapter summaries at the beginning of each chapter and historical notes at the end of each chapter.

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