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Unitary Representations of Groups, Duals, and Characters

Bachir Bekka
Pierre de la Harpe



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Providence, Rhode Island

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Contents

Foreword	ix
Acknowledgments	xi
Introduction	1
Representation theory of finite groups	1
Representations of compact groups and abelian groups	2
Primitive dual and locally compact groups of type I	4
The normal quasi-dual of a locally compact group	5
Characters of a locally compact group	6
Thoma's dual space	7
Historical comments on dual spaces of groups	7
Overview	10
Chapter 1. Unitary dual and primitive dual	13
1.A. Definition of the unitary dual	13
1.B. Functions of positive type and GNS construction	19
1.C. Weak containment and Fell topology for representations of topological groups	27
1.D. Topological properties of the dual of a group	32
1.E. Primitive dual of a topological group	38
1.F. Induced representations, irreducibility, and equivalence	40
1.G. On decomposing representations into irreducible representations	52
1.H. On decomposable and diagonalisable operators	59
1.I. Direct integrals of von Neumann algebras	63
Chapter 2. Representations of locally compact abelian groups	67
2.A. Canonical representations of abelian groups	67
2.B. Canonical decomposition of representations of abelian groups	73
2.C. The SNAG Theorem	80
2.D. Containment and weak containment in terms of projection-valued measures	85
2.E. Canonical decomposition of projection-valued measures	88
2.F. Applying the SNAG Theorem to nonabelian groups	89
Chapter 3. Examples of irreducible representations	95
3.A. Infinite dihedral group	95
3.B. Two-step nilpotent groups	99
3.C. The affine group of a field	110
3.D. Solvable Baumslag–Solitar groups	114
3.E. Lamplighter group	118

3.F.	General linear groups	119
3.G.	Some nondiscrete examples	121
Chapter 4.	Finite-dimensional irreducible representations	123
4.A.	Finite-dimensional irreducible representations of some semi-direct products	123
4.B.	All finite-dimensional irreducible representations for some groups	126
4.C.	Classes of groups in terms of finite-dimensional representations	132
Chapter 5.	Describing all irreducible representations of some semi-direct products	147
5.A.	Constructing some irreducible representations	147
5.B.	Cocycles over measurable group actions	149
5.C.	Cocycles with values in the unitary group of a Hilbert space	151
5.D.	Constructing all irreducible representations	155
5.E.	Identifying induced representations	162
5.F.	On the existence of infinite nonatomic ergodic invariant measures	165
Chapter 6.	Types for representations, quasi-duals, groups of type I	169
6.A.	Comparing representations, quasi-equivalence	170
6.B.	Group representations and von Neumann algebras	176
6.C.	The quasi-dual of a topological group	192
6.D.	Groups of type I	196
6.E.	A class of groups of type I	200
Chapter 7.	Non type I groups	211
7.A.	A class of non type I groups	211
7.B.	Operator algebras, traces, and types	214
7.C.	Types of group representations	219
7.D.	Types of representations of discrete groups	221
7.E.	On types of representations of locally compact groups	227
7.F.	Non type I factor representations and irreducible representations	228
Chapter 8.	Representations of C^* -algebras and of LC groups, the Glimm Theorem	231
8.A.	Spectrum and primitive ideal space of a C^* -algebra	232
8.B.	C^* -algebras and representations of LC groups	242
8.C.	Functorial properties of group C^* -algebras	246
8.D.	Second-countable and σ -compact LC groups	254
8.E.	The central character of a representation, of a primitive ideal	256
8.F.	Characterization of type I groups: The Glimm Theorem	257
8.G.	The von Neumann algebra of a group representation	259
8.H.	Variants	260
Chapter 9.	Examples of primitive duals	263
9.A.	A weak containment result for induced representations	263
9.B.	Two-step nilpotent groups	265
9.C.	Affine groups of infinite fields	271
9.D.	Solvable Baumslag–Solitar groups	272
9.E.	Lamplighter group	275

9.F. General and special linear groups	276
9.G. On the noninjectivity of the map from the dual to the primitive dual	288
9.H. Borel comparison for duals of groups	290
Chapter 10. Normal quasi-dual and characters	297
10.A. Traces and Hilbert algebras	297
10.B. The standard representation of a semi-finite von Neumann algebra	307
10.C. Group representations associated with traces	308
10.D. Normal factor representations and characters	312
Chapter 11. Finite characters and Thoma's dual	319
11.A. Factor representations of finite type and finite characters	319
11.B. GNS Construction for traces on groups	320
11.C. Thoma's dual	324
11.D. Characters and primitive duals	328
Chapter 12. Examples of Thoma's duals	331
12.A. Two-step nilpotent groups	331
12.B. Affine groups of infinite fields	336
12.C. Solvable Baumslag–Solitar groups	338
12.D. Lamplighter group	341
12.E. General linear groups	346
Chapter 13. The group measure space construction	353
13.A. Construction of factors of different types	353
13.B. Ergodic group actions without invariant measure	365
Chapter 14. Construction of factor representations for some semi-direct products	373
14.A. Crossed product von Neumann algebras for semi-direct products	373
14.B. Some factor representations of semi-direct products	376
14.C. Some normal factor representations	377
14.D. The von Neumann algebra of the regular representation as a crossed product	383
14.E. Examples of normal factor representations	387
Chapter 15. Separating families of finite type representations	397
15.A. Finite type representations and bi-invariant metrics	397
15.B. Groups for which finite type representations are separating	399
15.C. Locally compact groups with a finite von Neumann algebra	401
15.D. Connected and totally disconnected LC groups	403
15.E. Faithful traces on group C*-algebras	405
15.F. Traces and Invariant Random Subgroups	416
Appendix	421
A.1. Topology	421
A.2. Borel spaces	424
A.3. Measures on Borel spaces and σ -finiteness	425
A.4. Radon measures on locally compact spaces	429
A.5. Groups and actions	431

A.6. Locally compact groups	435
A.7. Locally compact abelian groups and duality	437
A.8. Hilbert spaces and operators	440
A.9. Projection-valued measures	441
A.10. C^* -algebras	442
A.11. Von Neumann algebras	443
Bibliography	447
Notation Index	463
Index	467

Foreword

This is an expository book on unitary representations of topological groups, and of several dual spaces, which are spaces of such representations up to some equivalence. The most important notions are defined for topological groups, but special attention is paid to the case of discrete groups.

The unitary dual of a group G is the space of equivalence classes of its irreducible unitary representations; it is both a topological space and a Borel space. It is useful in important situations, but much less useful in many other situations, for example because, for an infinite discrete group, it is usually not countably separated as a Borel space.

We discuss other candidates for being a dual space of a group. The two most important ones are the primitive dual, the space of weak equivalence classes of unitary irreducible representations, and the normal quasi-dual, the space of quasi-equivalence classes of traceable factor representations. The last space is parametrized by “characters”, which can be finite or infinite. The space of finite characters is an object particularly easy to describe: it is in natural bijection with the space of indecomposable central positive definite functions on the group, that we call the Thoma dual. We discuss other related spaces, such as the space of finite-dimensional irreducible unitary representations.

These spaces are much better behaved than the unitary dual. They deserve to be studied on their own, and related to each other. In contrast to the unitary dual, they can in principle be classified for every specific group. We illustrate the theory by giving an explicit description of these alternative dual spaces for a series of examples: Heisenberg groups over fields and over the integers, affine groups of infinite fields, solvable Baumslag–Solitar groups, lamplighter groups, and general (or special) linear groups over infinite fields.

Operator algebras play an important role in the discussion, in particular von Neumann algebras associated with unitary representations and C^* -algebras associated with locally compact groups.

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It is the occasion to celebrate Jacques Dixmier and the writing of his two books [Dixm–vN, Dixm–C] on operator algebras.*

*More than 50 years after publication,
they remain unrivalled references on the subject.*

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Notation Index

The following table contains a list of symbols that are frequently used throughout the book. The numbers indicate the pages of an appearance or a definition of each symbol. In a few cases, the same symbol may have different meanings, depending on the context.

Notation	Meaning	Page
\simeq	equivalence of representations	15
\simeq_{Γ}	equivalence of representations of Γ (occasionally)	291
\preceq	weak containment of representations	27
\sim	weak equivalence of representations	28
\approx	quasi-equivalence of representations	170
\cong	isomorphism of sets, i.e., bijection	2
\cong	isomorphism of other structures (Hilbert spaces, C^* -algebras, ...)	
$\mathbf{N} = \{0, 1, 2, 3, \dots\}$	nonnegative integers	
$\mathbf{N}^* = \{1, 2, 3, \dots\}$	positive integers	
$\overline{\mathbf{N}}^* = \{1, 2, \dots, \infty\}$	extended positive integers	73
\mathbf{Z}	rational integers	10
\mathbf{Q}	rational numbers	31
\mathbf{Z}_p	p -adic integers	35
\mathbf{Q}_p	p -adic numbers	35
\mathbf{R}	real numbers	17
$\mathbf{R}_+ = [0, \infty[$	nonnegative real numbers	112
$\overline{\mathbf{R}}_+^{\times} =]0, \infty[$	positive real numbers	112
$\mathbf{R}_+^{\times} =]0, \infty]$	extended positive real numbers	366
$\quad = \mathbf{R}_+^{\times} \cup \{\infty\}$		
$\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$	real projective line	369
$\quad = \mathbf{P}^1(\mathbf{R})$		
$\mathbf{Sol}_p = (\mathbf{Q}_p \times \mathbf{R})/\Delta$	p -adic solenoid	116
\mathbf{C}	complex numbers	14
$\mathbf{T} = \mathbf{R}/\mathbf{Z}$	circle group	17
$\quad = \{z \in \mathbf{C} \mid z = 1\}$		
\mathbf{K}	field	36
\mathbf{F}_p	finite field of order p	105

\mathcal{H} , or \mathcal{K}	Hilbert space	13
$\mathcal{U}(\mathcal{H})$	unitary group of \mathcal{H}	14
$\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$	bounded linear operators from \mathcal{H}_1 to \mathcal{H}_2	14
$\mathcal{L}(\mathcal{H})$	bounded linear operators on \mathcal{H}	14
$I = \text{Id}_{\mathcal{H}} \in \mathcal{L}(\mathcal{H})$	identity operator on \mathcal{H}	24
$\text{Proj}(\mathcal{H})$	set of projections in $\mathcal{L}(\mathcal{H})$	441
$\ell^2(T, \mathcal{K})$	Hilbert space	43
$L^2(X, \mu, \mathcal{K})$	Hilbert space	54
$L^2(\widehat{G}, \mu, \mathcal{K})$	Hilbert space	73
$L^p(X, \mu)$ or $L^p(X)$	Banach space	426
$\mathbf{1}_U$	characteristic function of U	430
G	topological group	14
e	unit element in a group	14
μ_G	left Haar measure on an LC group G	2
Δ_G	modular function of an LC group G	435
$G_{\text{ab}} = G/\overline{[G, G]}$	abelianization of a group G	31
G_{disc}	topological group G made discrete	26
$G = H \ltimes N$	semi-direct product	124
$\check{\varphi}$	$\check{\varphi}(g) = \check{\varphi}(g^{-1})$ for $\varphi : G \rightarrow \mathbf{C}$	19
φ^*	$\varphi^*(g) = \Delta(g^{-1})\overline{\varphi(g^{-1})}$ for $\varphi : G \rightarrow \mathbf{C}$	436
$\mathbf{1}_G$	unit representation of G on \mathbf{C}	14
π, ρ, σ	(unitary) representations of a topological group	14
λ_G	left regular representation of an LC group G	26
ρ_G	right regular representation of an LC group G	26
$\text{Hom}_G(\pi_1, \pi_2)$	space of intertwiner operators	14
$\mathbf{C}[G]$	group algebra of a group G	20
$L^1(G, \mu_G)$ or $L^1(G)$	group algebra of an LC group G	242
$C_\lambda^*(G)$	reduced C*-algebra of an LC group G	243
$C_{\text{max}}^*(G)$	maximal C*-algebra of an LC group G	239
$\mathcal{L}(\Gamma)$	von Neumann algebra of λ_Γ	218
\widehat{G}	(unitary) dual of a topological group G	18
\widehat{G}_{fd}	finite-dimensional part of \widehat{G}	31
$\text{Hom}(G, \mathbf{T})$	group of continuous homomorphisms $G \rightarrow \mathbf{T}$	18
$P(G), P_{\leq 1}(G), P_1(G)$	functions of positive type on G	19
$\text{Extr}(P_1(G))$	indecomposable elements in $P_1(G)$	23
$\text{Prim}(G)$	primitive dual of a topological group G	38
$\text{QD}(G)$	quasi-dual of a topological group G	192
$\text{QD}(G)_{\text{norm}}$	normal quasi-dual of an LC group G	314
$\text{QD}(G)_{\text{fin}}$	finite part of $\text{QD}(G)$	319

$\text{Char}(G)$	equivalence classes of characters of an LC group G	314
$\text{Char}(G)_{\text{fin}}$	finite part of $\text{Char}(G)$	319
$\text{Tr}(G), \text{Tr}_{\leq 1}(G), \text{Tr}_1(G)$	traces on G	320
$E(G) = \text{Extr}(\text{Tr}_1(G))$	Thoma's dual of a topological group G	324
\widehat{G}	Pontryagin dual of an LCA group G	437
\mathcal{F}	Fourier transform	438
$\overline{\mathcal{F}}$	inverse Fourier transform	438
A	C*-algebra	214
A_+	positive cone in A	214
\widehat{A}	spectrum of A	233
$\text{Prim}(A)$	primitive ideal space of A	233
$A_+^*, A_{+, \leq 1}^*, A_{+, 1}^*$	positive linear forms on a C*-algebra A	234
$\text{P}(A) = \overline{\text{Extr}}(A_{+, 1}^*)$	set of pure states of A	235
$\mathcal{J}(A)$	set of closed two-sided ideals of A	238
$\mathcal{N}(A)$	C*-semi-norms on a C*-algebra A	238
$\mathcal{E}(A)$	extremal C*-semi-norms on a C*-algebra A	239
$t : A_+ \rightarrow [0, \infty]$	trace on A	214
\mathfrak{m}_t	ideal of definition of t	298
\mathfrak{n}_t	$\{x \in A \mid t(x^*x) < \infty\}$	297
\mathcal{M}	von Neumann algebra	218
\mathcal{M}_+	positive cone in \mathcal{M}	218
\mathcal{A}	Hilbert algebra	298
σ -compact	sigma-compact topological space	423
$C(X)$	space of continuous functions on X	423
$C^b(X)$	space of bounded continuous functions on X	423
$C^c(X)$	subspace of $C(X)$ of functions with compact support	423
$C^{c,K}(X)$	subspace of $C^c(X)$ of functions with support in K	423
$C^0(X)$	subspace of $C(X)$ of functions vanishing at infinity	423
$M^b(X)$	space of bounded complex measures on (X, \mathcal{B})	431

Index

- Absolute value of an operator on a Hilbert space, 441
- Accessible topological space, T_1 , 421
- Action
 - 2-transitive, 49
 - aperiodic, 434
 - ergodic, 353, 434
 - essentially free, 353
 - essentially transitive, 353
 - nonsingular, 353
 - type II, III_λ , 366
- Action of G on X , respectively $G \curvearrowright X$ and $X \curvearrowleft G$, 18, 433
- Affine group
 - $\text{Aff}(\mathbf{R})$, $\text{Aff}(\mathbf{R})_0$, 41, 113, 202, 228
 - $\text{Aff}(\mathbf{Q})$, nonuniqueness of direct integral decomposition in irreducible, 59
 - $\text{Aff}(\mathbf{Q})$, quasi-regular representation of type III, 225
 - $\text{Aff}(\mathbf{K})$ of a field \mathbf{K} , 110
- Affine group $\text{Aff}(\mathbf{K})$ of an infinite field
 - examples of irreducible representations, 110
 - finite-dimensional representations, 129
 - primitive dual, 271
 - examples of irreducible representations, 165
 - icc group, 432
 - Thoma dual, 336
- Algebras
 - continuous functions, $C(X)$, $C^b(X)$, $C^c(X)$, $C^0(X)$, 19, 215, 424
 - trigonometric polynomials on \widehat{G} , 375, 439
 - convolution algebras, $C^c(G)$, $C^0(G)$, $L^1(G)$, 242, 259, 305
 - Hilbert algebra, 298
 - C*-algebra, 442
 - von Neumann algebra, 443
- Almost periodic function, 135
- Amenable
 - group, 28
 - homogeneous space, 51
- Aperiodic action, 434
- Approximate identity, 443
- Atom
 - of a measure, 425
 - of a projection-valued measure, 442
- Baire space, 238, 422
- Banach involutive algebra, 442
- Baumslag–Solitar group $BS(1, p)$
 - examples of irreducible representations, 114, 165
 - finite-dimensional representations, 130
 - icc group, 432
 - normal factor representations, 388
 - primitive dual, 272
 - Thoma dual, 338
- Bi-invariant pseudo-metric on a group, 398
- Bihomomorphism, 99
- Bohr compactification, 134
- Borel equivalence relation, 290
 - Borel equivalent, 291
 - Borel reducible, 291
 - smooth, 290
- Borel map, 424
- Borel space, 424
 - countably generated, 424

- countably separated, 424
- standard, 424
- Bounded element of a Hilbert algebra, 298
- Bruhat decomposition
 - of $GL_n(\mathbf{K})$, 277
 - of $SL_n(\mathbf{K})$, 285
- C^* -algebra, 442
 - commutative C^* -algebra, 424
 - cartesian product, 245
 - restricted product, 245
- C^* -kernel, 244
- C^* -maximally almost periodic group, or C^* -MAP group, 408
- C^* -residually finite group, 412
- C^* -semi-norm, 238
- Canonical representation π_μ of an LCA group G on $L^2(\widehat{G}, \mu)$, 68
- Cartesian product of C^* -algebras, 245
- CCR
 - representation, 202, 265
 - group, 202
 - C^* -algebra, 257
- Central character, 17, 257
- Central direct integral
 - decomposition, 196
- Central function = class function, 1
- Central support of an element in a von Neumann algebra, 445
- Character
 - unit 1_G , 14
 - unitary character, 17
 - central, 17, 257
 - of an LC group, 314
 - finite, 319, 325
 - of a normal factor representation of an LC group, 315
- Class function = central function, 1
- Cocycle, 42, 149
 - Radon–Nikodym, 148, 365
 - cohomologous, 149, 152
 - irreducible, 152
- Commensurate subgroups, 51, 432
- Commensurator, 48, 432
- Commutant, 14, 443
- Commutation Theorem for Hilbert algebras, 299
- Commutator $[a, b] := a^{-1}b^{-1}ab$, 31
- Commutator subgroup = derived group, 31
- Conditional expectation, 359
- Conjugate representation, 18, 44
- Connes embedding problem, 406
- Continuity of induction, 44
- Convolution
 - of two functions on an LC group, 436
 - of two measures on an LC group, 435
- Crossed product, 357, 387
- Cyclic representation, 20
- Decomposable operator over a measure space, 54, 59
- Decomposable von Neumann algebra, 64
- Decomposition of representations
 - into irreducible representations for type I groups, 199
 - into irreducible representations, 57
 - into factor representations, 195
- Derived group = commutator subgroup, 31
- Diagonal matrix coefficient, 20
- Diagonalisable operator, 55
- Dihedral group D_∞
 - dual, 95
 - Thoma dual, 326
- Dimension
 - of a Hilbert space, 440
- Direct integral
 - of Hilbert spaces, 53
 - of representations, 55
 - of von Neumann algebras, 64
- Direct sum of representations, 15
- Disjoint
 - measures, 426
 - representations, 15, 170
- Dual
 - short for unitary dual, 18
 - surjection on primitive dual, 38
 - finite-dimensional part, 31
 - injection in quasi-dual, 192

- Enveloping C*-algebra of measure algebra, 248
- Equivalent
 - representations, 15
 - cyclic representations, 22
 - measures, 426
 - projection-valued measures, 82
- Ergodic
 - action, 353, 434
 - IRS, 416
 - measure, 434
- Essential range
 - of a cocycle, 366
 - of a function, 426
- Essential supremum, 426
- Essentially free action, 353
- Essentially transitive action, 353
- Extremal semi-norm, 239

- Factor, 89, 444
- Factor representation, 89, 173, 179
- Faithful trace, 214
- FC-centre $FC(\Gamma)$ of a group Γ , 431
- FC-group, 431
- Fell topology, 29, 245
- Field of Hilbert spaces over a measure space, 53
- Finite part of the quasi-dual, 319
- Finite trace, 214
- Finite type, for a Polish group, 398
- Finite-dimensional part of the dual, 31
- First return time, 390
- Fourier transform, 438
- Fourier–Stieltjes transform, 438
- Free group
 - C*-MAP group, 410
 - C*-residually finite group, 413
 - icc, 432
 - inside $SL_2(\mathbf{Q})$ and $SL_2(\mathbf{F}_p(T))$, 282
 - nonnormal irreducible
 - infinite-dimensional representation, 313
 - nonuniqueness of direct integral decomposition in irreducible, 58
- Full Hilbert algebra, 300
- Function of positive type, 19
 - associated with a representation, 20
- Fundamental family of measurable vector fields over a Borel space, 53

- GCR
 - group, 200
 - C*-algebra, 257
- Gel'fand pair, 187
- Gel'fand–Naimark–Segal
 - representation, 20, 234, 321
 - with $\varphi \in P(G)$, 20
 - with $\varphi \in A_+^*$, 234
 - with $\varphi \in \text{Tr}(G)$, 321
- Gel'fand–Raikov Theorem, 27
- General linear group $GL_n(\mathbf{K})$
 - Bruhat decomposition, 277
 - examples of irreducible
 - representations, 119
 - finite-dimensional irreducible representations, 145
 - not MAP, 137
 - primitive dual when $GL_n(\mathbf{K})$ is amenable, 276
 - principal series, 277
 - quotient $PGL_n(\mathbf{K})$ icc, 432
 - Thoma dual, 346
- Glimm's Theorem, 258
- GNS, short for
 - Gel'fand–Naimark–Segal, 20
- Group measure space construction, see crossed product, 357

- Haar measure, 435
- Hausdorff topological space, T_2 , 421
- Heisenberg group
 - $H(R)$ over a commutative ring R , 102
 - $H(\mathbf{K})$ over a field \mathbf{K} , 105
 - $H(\mathbf{Z})$, 106
 - $H(\mathbf{R})$, 109, 246
 - $H_\infty(\mathbf{F}_p)$, 227
- Heisenberg group $H(\mathbf{K})$
 - examples of irreducible representations, 105
 - finite-dimensional representations, 128
 - primitive dual, 268

- Thoma dual, 334
- Heisenberg group $H(\mathbf{Z})$
 - factor representations of type I_∞ , II_1 , II_∞ , and III, 376
 - examples of irreducible representations, 106
 - finite-dimensional representations, 129
 - no normal representation of infinite type, 329, 381
 - primitive dual, 270
 - representations involving cocycles, 164
 - Thoma dual, 335
- Hilbert algebra, 298
 - bounded element, 298
 - full, 300
 - natural trace, 301
- Homogeneous representation, 80
- Icc group, 211, 432
- Ideal of definition of a trace, 297
- Indecomposable
 - for $\varphi \in P_{\leq 1}(G)$, 23
 - for $\varphi \in A_{+, \leq 1}^*$, 235
 - for $\varphi \in \text{Tr}_{\leq 1}(G)$, 324
- Induced representation, 40
- Induced transformation, 391
- Induced von Neumann algebra, 445
- Integral, 425
- Intertwiner
 - between representations of a group, 14
 - between cocycles on a group, 152
- Invariant measure, 433
- Invariant random subgroup, or IRS, 416
- Involutive algebra, 442
- Involutive algebra $\mathcal{L}(\mathcal{H})$, 14
- Irreducible representation, 16
- Jacobson topology, 233
- Kazhdan Property (T), 38
- Kolmogorov topological space, T_0 , 421
- Koopman representation, 356
- Lamplighter group
 - examples of irreducible representations, 118, 165
 - finite-dimensional representations, 131
 - icc group, 432
 - normal factor representations, 389, 393
 - primitive dual, 275
 - Thoma dual, 341
- Large compact subgroup, 203
- LC for “locally compact”, 36
- LCA for “locally compact abelian”, 67
- Left regular representation, 26, 212, 224, 243
- Liminal
 - locally compact group, 202
 - representation, 202
 - C^* -algebra, 257
- Locally quasi-compact topological space, 32
- Lower semi-continuous trace, 214
- Mackey machine, 92
- Mackey–Borel structure, 193, 255
- Maharam extension, 368
- Matrix coefficient, 20
- Mautner group $M := \mathbf{C}^2 \rtimes \mathbf{R}$, 228
- Maximal C^* -algebra of an LC group, 243
- Maximally almost periodic group, or MAP group, 133, 400
- Measurable
 - space, 424
 - map, 424
 - action $G \curvearrowright (X, \mathcal{B})$, 433
- Measurable field
 - measurable vector field over a Borel space, 53
 - measurable field of Hilbert spaces over a Borel space, 53
 - measurable field of operators over a measure space, 54
 - measurable field of representations, 55
 - measurable field of von Neumann algebras, 63
- Measure
 - σ -finite, 425

- absolutely continuous, 426
- algebra, 436
- complex, 429
- disjoint, 426
- equivalent, 426
- ergodic, 434
- invariant, 433
- nonatomic, 425
- quasi-invariant, 434
- Radon, 430
- singular, 426
- total variation, 429
- Measure algebra, 436
- Minimally almost periodic group, or
m.a.p. group, 143
- Modular function of a locally
compact group, 435
- Monomial representation, 46
- Moore group, 132
- Motion group, 187
- Multiplicity of a representation, 184
- Multiplicity-free representation, 75,
175, 181, 185
- Natural trace of a Hilbert algebra,
301
- Nondegenerate representation of an
involutive algebra, 232
- Nonsingular action, 353
- Normal factor representation, 312
- Normal positive linear map from a
von Neumann algebra to
another, 443
- Normal quasi-dual, 314
- Normal trace, 218
- Orbit equivalence of group actions,
165
- Partial isometry, 441
 - initial space, 441
 - final space, 441
- Polar decomposition of an operator,
441
- Polish space, Polish topology, 422
- Pontryagin dual, 437
- Positive definite function, see
functions of positive type, 19
- Positive element in a C^* -algebra
 - square root, 442
- Positive element in a C^* -algebra,
440, 442
 - square root, 441
- Positive type
 - for a function, 19
 - for a Radon measure, 309
- Postliminal
 - C^* -algebra, 257
- Prime ideal, 239
- Primitive dual
 - of a topological group, 38, 244
- Primitive ideal, 233
- Primitive ideal space
 - of a C^* -algebra, 233
- Principal series
 - of $GL_n(\mathbf{K})$, 120, 277
 - of $SL_n(\mathbf{K})$, 285
- Profinite completion, 138
- Profinite group, 137
- Projection in a C^* -algebra, 442
- Projection-valued measure, 441
 - atom, 442
 - equivalent, 82
 - regular, 441
 - support, 442
- Projective kernel, 256
- Projective limit of finite groups, 137
- Properly infinite
 - von Neumann algebra, 218
 - representation, 220
- Property
 - \mathcal{P} -by- \mathcal{Q} , 431
- Pure state on a C^* -algebra, 235
- Purely infinite
 - von Neumann algebra, 218
 - representation, 220
 - C^* -algebra, 314
- Quasi-compact topological space,
422
- Quasi-dual, 192
 - type I part, 192
 - normal, 314
 - finite part, 319
- Quasi-equivalent
 - representations, 170, 177
 - trace representations of an LC
group, 308

- finite type representations of a topological group, 322
- Quasi-invariant measure, 434
- Quasi-orbit, quasi-orbit space, 273, 422
- Quasi-regular representation, 43, 49, 190, 191, 225
- Radon measure, 430
 - complex, 431
- Radon–Nikodym
 - cocycle, 148, 365
 - derivative, 427
- Reduced C^* -algebra of an LC group, 243
- Reduced Enveloping C^* -algebra of measure algebra, 248
- Regular projection-valued measure, 441
- Representation
 - C^* -kernel, 244
 - canonical (for an LCA group), 68
 - CCR, 202, 265
 - conjugate, 18, 44
 - cyclic, 20
 - direct sum, 15
 - disjoint, 15, 170
 - equivalent \simeq , 15
 - factor, 89, 173, 179
 - finite type, 220, 319
 - homogeneous, 80
 - induced, 40
 - irreducible, 16, 232
 - Koopman, 356
 - left regular, 26
 - monomial, 46
 - multiplicity of a representation, 184
 - multiplicity-free, 175, 181, 185
 - nondegenerate, 232
 - normal, 312
 - of $C^*_{\max}(G)$ associated to a representation of G , 244
 - of $L^1(G)$ associated to a representation of G , 243
 - of an involutive algebra, of a C^* -algebra, 232
 - quasi-equivalent \approx , 170, 177, 308, 322
 - quasi-regular, 43, 49, 190, 191, 225
 - right regular, 26
 - standard, 308
 - subordinate, 170
 - trace, 308
 - traceable, 309
 - trivial (for a C^* -algebra), 232
 - type I, 175, 181
 - type II, II_1 , II_∞ , III, 220
 - unit, 14
 - weakly contained \preceq , 27, 244
 - weakly equivalent \sim , 28, 244
 - with vanishing coefficients, 346
- Representation = unitary group representation, 14
- Representation of finite type, 320
- Residually finite group, 136
- Residually finite-dimensional C^* -algebra, 408
- Restricted product of C^* -algebras, 245
- Right regular representation, 26
- Schröder–Bernstein property for representations, 171
- Schur’s lemma, 17
- Second-countable topological space, 423
- Semi-direct product, 124
- Semi-finite trace, 214
- Separable topological space, 423
- Separating family
 - of Borel subsets of a space, 424
 - of continuous functions on a space, 439
 - of representations of a group, 26
 - of representations of an involutive algebra, 406
- Separation axioms T_0 , T_1 , T_2 , 421
- Sigma-compact topological space, 423
- SIN group, 136, 399
- Solenoid \mathbf{Sol}_p , 116
- Special linear group $\text{SL}_n(\mathbf{K})$
 - Bruhat decomposition, 285
 - minimally almost periodic, 144
 - principal series, 285
 - $(\text{SL}_n(\mathbf{R}), \text{SO}(n))$ is a Gelfand pair, 189

- ($SL_n(\mathbf{Q}_p)$, $SL_n(\mathbf{Z}_p)$) is a Gelfand pair, 189
- $SL_2(\mathbf{R})$ is a CCR group, 204
- Thoma duals of $SL_n(\mathbf{K})$ and $PSL_n(\mathbf{K})$, 351, 420
- Spectrum of a C*-algebra, 233
- Square root of a positive operator, 441
- Square-integrable vector field over a measure space, 53
- Standard
 - representation of (\mathcal{M}, τ) , 308
 - Borel space, 424
 - measure on a Borel space, 428
- State on a C*-algebra, 234
- Strong topology on $\mathcal{L}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$, 14, 440
- Subordinate representation, 170
- Subrepresentation, 14
- Support of a projection-valued measure, 442
- Support of an operator, 444

- Tame group, 259
- Thoma's dual, 324
- Topological space
 - σ -compact, 423
 - accessible, 421
 - Baire, 238, 422
 - compact, 422
 - Kolmogorov, 421
 - locally compact, 422
 - quasi-compact, 422
 - second-countable, 423
 - separable, 423
 - separation axioms T_0 , T_1 , T_2 , 421
- Trace
 - on a C*-algebra, 214, 297
 - domination, 214
 - on a von Neumann algebra, 218
 - ideal of definition, 297
 - natural trace on the von Neumann algebra associated with a Hilbert algebra, 301
 - on a topological group, 320
- Trace representation, 308
 - quasi-equivalent, 308, 322
- Traceable representation, 309
- Transversal T , and $G = \bigsqcup_{t \in T} tH$, 40

- Trigonometric polynomials on \widehat{G} , 375, 439
- Trivial extension of a function defined on a subgroup, 21
- Trivial representation, 232
- Two-step nilpotent groups
 - every normal factor representation of $H(\mathbf{K})$ is of finite type, 329
 - examples of irreducible representations, 99
 - primitive dual, 265
 - Thoma dual, 331
- Type I
 - representation, 175, 181
 - part of the quasi-dual, 192
 - group, 196, 258
 - von Neumann algebra, 218
- Type II, III
 - von Neumann algebra, 218
 - representation, 220
 - action, 366
 - III_λ , 366, 387

- Ultra-weak topology on $\mathcal{L}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$, 440
- Unimodular group, 435
- Unit representation or unit character 1_G , 14
- Unitary character, 17
- Unitary dual, 18
- Unitary representation = representation, 14
- Universal Kolmogorov quotient, 422

- Vanishing coefficients (representation), 346
- von Neumann algebra, 443
 - factor, 89, 444
 - type I, 218
 - continuous, 218
 - finite, 218
 - semi-finite, 218
 - type II_1 , 218
 - type II_∞ , 218
 - type III, 218
 - properly infinite, 218
 - purely infinite, 218
 - crossed product, 357
 - enveloping, 260

- group measure space construction,
357
- von Neumann algebra $\mathcal{L}(\mathbf{G})$ of an
LC group G , 212
- von Neumann kernel of a topological
group, 135
- Weak topology on $\mathcal{L}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$,
14, 440
- Weakly
contained (representation), 27, 244
equivalent (representation), 28,
244

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