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Introduction

Integrable models in field theory and statistical mechanics constitute a research area that has made significant advances in the last few decades. Many of the deep results are intimately related in one way or another to the understanding of local operators of the model. The present book is aimed to expound upon the structure of the space of local operators.

We start this introduction by some historical remarks, which are by no means complete. Historical reviews at a more technical level will be given in the first chapters of both Volume I and Volume II.

The theory of quantum integrable models has a dual origin.

The first discovery in the field is due to Bethe [**Be**] who proposed a method for solving a one-dimensional model of isotropic magnet. This method is known now as the Bethe Ansatz. Then comes the famous solution of the two-dimensional Ising model by Onsager [**O1**]. Onsager also pointed out a deep relation between two-dimensional classical statistical mechanics and one-dimensional quantum mechanics [**O2**]. This observation set Bethe's and Onsager's work into a common perspective. The Bethe Ansatz technique was generalised and applied to various other models, including the Bose gas with a δ function interaction. Using this method, Yang and Yang [**YY**] investigated the thermodynamics. Onsager's work was further developed by Lieb [**Li**] and Sutherland [**Su**] who were able to solve the ice model. Finally a giant step was taken by Baxter who solved the eight-vertex and related models [**Ba1**], [**Ba2**], introducing novel machinery and thereby laying the foundation for all further progress in the field [**Ba3**].

The second development came from the theory of soliton equations. The seminal paper by Gardner, Greene, Kruskal, and Miura [**GGKM**] on the Korteweg-de Vries (KdV) equation opened up the whole new subject of integrable non-linear differential equations. Among important contributions are the work of Lax [**La**] who reformulated the equation in terms of Lax pair, and of Faddeev and Zakharov [**ZF**] who reinterpreted the KdV equation as a completely integrable Hamiltonian system. Faddeev and Takhtajan [**FT**] found an integrable model of relativistic classical field theory, known now as the sine-Gordon (sG) model. There arose naturally the issue of quantisation. Faddeev and Korepin [**FK**] successfully addressed the semi-classical quantisation, revealing a striking feature of the model: the spectrum includes solitons. These are excitations which correspond to non-trivial classical solutions and yet do not require the introduction of additional fields, contrary to the naïve particle-field correspondence. Coleman [**C**] and Mandelstam [**Ma**] explained that solitons can be interpreted as fermions in the massive Thirring model. This was the first occurrence of duality in Quantum Field Theory (QFT). Another unusual feature of the model, discovered in [**FK**] by semi-classical quantisation, is that the scattering matrices completely factorise. This property allowed the Zamolodchikov brothers [**ZaS**, **ZZ**] to compute the S-matrices exactly by a bootstrap approach.

The Quantum Inverse Scattering Method (QISM) proposed by Faddeev, Sklyanin, and Takhtajan [**STF**] synthesised these two major streams in the field of integrable models (see also [**KIB**] for a review). The main achievement of QISM is in the formulation of the Algebraic Bethe Ansatz (ABA). This method is based on the notion of R-matrix, a mathematical abstraction of a set of Boltzmann weights which plays a fundamental role in Baxter's work. As it turned out, certain R-matrices coincide with the factorised S-matrices of the Zamolodchikov brothers. The only explanation for this coincidence would be that there are not so many good mathematical objects around. The notion of R-matrix eventually gave birth to the theory of quantum groups. After the first example found by Kulish and Reshetikhin [**KR**], the general formulation was given by Drinfeld [**Dr**] and Jimbo [**Ji1**].

The most complicated part of the theory of integrable models concerns the computation of off-shell objects: form factors, correlation functions, etc. Achievements in this direction include the following. For relativistic models of QFT, Smirnov formulated a system of axioms for form factors, and solved the corresponding equations for a number of models [**Sm**]. Subsequently, correlation functions for the six-vertex model (or equivalently, static correlation functions for the XXZ model) were found by Jimbo, Miwa, and collaborators [**JM**]. The latter work relies heavily on the theory of quantum groups. Surprisingly, form factors in Conformal Field Theory (CFT) and correlation functions on the lattice are described by quite similar (actually dual) equations, though the variables involved have completely different meaning. This is yet another instance which indicates that good mathematical structures are limited in number. Later on, using ABA, Kitanine, Maillet, and Terras [**KMT**] generalised the results of [**JM**] to correlation functions for the XXZ model with a magnetic field. Göhmann, Klümper, and Seel [**GKS**] managed further generalisation to the case of finite temperature. The works [**KMT**, **GKS**] use the determinant formula found much earlier by Slavnov [**Sl**]. All these formulas for correlation functions are given in terms of multiple integrals which are hard to investigate. It is all the more so in the last two cases where the integrand involves certain functions which are not quite explicit.

We finish the historical part of Introduction by mentioning a series of papers by Bazhanov, Lukyanov, and Zamolodchikov [**BLZ1**, **BLZ2**, **BLZ3**]. The construction based on quantum groups proposed there was very important for us.

This monograph summarises results of our long effort on computing expectation values of local operators in integrable models [**BJMST1**, **BJMST2**, **JMS1**, **BJMS**, **JMS2**]. It is not easy to discuss here the details of this work, so we prefer rather to motivate the reader by demonstrating some examples of our results. We shall consider two best known examples, one on the lattice and the other in the continuum.

As an example of a lattice model, we choose the isotropic Heisenberg antiferromagnet. Formally the Hamiltonian is written as

$$H_{\mathrm{XXX}} = \sum_{j=-\infty}^{\infty} \sum_{a=1}^{3} \sigma_j^a \sigma_{j+1}^a \,.$$

To make sense of this formal definition, we start with a finite chain of $2L$ sites $j = -L+1, \cdots, L$. On the space $\left(\mathbb{C}^2\right)^{\otimes 2L}$, we define operators σ_j^a given by conventional Pauli matrices (see (1.2) below) acting non-trivially on the j-th copy of $\mathbb{C}^2$ and as identity elsewhere. We impose the periodic boundary condition and take the limit $L \to \infty$. Denote by $|\mathrm{vac}\rangle$ the ground state, and consider the correlation functions

$$\langle \sigma_1^3 \sigma_n^3 \rangle = \frac{\langle \mathrm{vac}|\sigma_1^3\sigma_n^3|\mathrm{vac}\rangle}{\langle \mathrm{vac}|\mathrm{vac}\rangle}\,.$$

Long ago, Takahashi [**T**] found a nice formula for $n = 3$:

$$\langle \sigma_1^3 \sigma_3^3 \rangle = \frac{1}{3} - \frac{16}{3}\log 2 + 3\zeta(3)\,,$$

where $\zeta(s)$ is the Riemann zeta function. The appearance of a ζ-value is quite remarkable. Later, Boos and Korepin [**BK**] managed to evaluate the multiple integral formulas, obtaining

$$\langle \sigma_1^3 \sigma_4^3 \rangle = \frac{1}{3} - 12\log 2 + \frac{74}{3}\zeta(3) - \frac{56}{3}\zeta(3)\log 2 - 6\zeta(3)^2 - \frac{125}{6}\zeta(5) + \frac{100}{3}\zeta(5)\log 2\,.$$

We shall see that this kind of formulas hold in general: $\langle \sigma_1^3 \sigma_n^3 \rangle$ is a polynomial of degree $\left[\frac{n}{2}\right]$ of odd ζ-values with argument up to $2n-3$ (and $\log 2$ as well) with rational coefficients. This was confirmed by [**SST**] where the correlation functions were computed up to $n = 8$. We compute them up to $n = 11$. For this value the exact correlation function is already quite close to Lukyanov's asymptotic prediction [**Lk1**, **LT**]. We are also able to compute the entanglement density matrices, i.e., the expectation values of all local operators, up to 10 sites. In the paper [**BGKS**] evidence was given that the multiple integral formulas factorise even in the case of finite temperature and magnetic field. This work gave an important impetus to our study.

As an example of QFT, we consider the sine-Gordon model. The Hamiltonian is

$$H_{\mathrm{sG}} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\partial_0\varphi(x)^2 + \partial_1\varphi(x)^2 + \mu\bigl(1-\cos(\beta\varphi(x))\bigr)\right]dx\,,$$

where ∂_μ are derivatives with respect to space-time coordinates; we shall also use $\partial_\pm$ for the light-cone derivatives. The commutator of $\varphi(x)$ and $\partial_0\varphi(x)$ is canonical.

The problem which we address here concerns the computation of one-point functions of local operators:

$$\langle \mathcal{O}(0)\rangle = \frac{\langle \mathrm{vac}|\mathcal{O}(0)|\mathrm{vac}\rangle}{\langle \mathrm{vac}|\mathrm{vac}\rangle}\,,$$

where $|\mathrm{vac}\rangle$ is again the ground state. Lukyanov and Zamolodchikov [**LZ**] computed the one-point functions for the operators $e^{ia\varphi(0)}$ (primary fields). Though this is a remarkable result, for application to perturbed CFT one needs to know one-point functions of descendants as well. The simplest non-trivial result was found by Fateev, Fradkin , Lukyanov, Zamolodchikov, and Zamolodchikov [**FFLZZ**]:

$$\langle (\partial_+\varphi)^2(\partial_-\varphi)^2 e^{ia\varphi}\rangle = -m^4 F_1(a)F_1(-a)\langle e^{ia\varphi}\rangle\,,$$

where we omit the argument 0 writing φ instead of $\varphi(0)$,

$$F_{2j-1}(a) = ((j-1)!)^2 \gamma\Big(\frac{2a-(2j-1)\beta}{2\eta}\Big)\gamma\Big(\frac{(2j-1)\beta^{-1}-2a}{2\eta}\Big)\,,$$

$\eta = \beta - \beta^{-1}$, and $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. The coupling constant μ has the dimension of mass to the power $2 - 2\beta^2$ because of the anomalous dimension of $\cos(\beta\varphi)$. The mass scale m is proportional to $\mu^{\frac{1}{2(1-\beta^2)}}$ with the proportionality coefficient known exactly due to Al. Zamolodchikov [**ZaA**]. In general, the levels (number of derivatives) for two light cone components (chiralities) must coincide due to Lorentz invariance. The simplest case above has level 2. The next non-trivial case is one on level 4. We shall see that there are two non-trivial expectation values. To describe them introduce

$$A_\epsilon = \frac{\eta^{-2}}{72}\Big\{\big(3-4a^2(\eta^2+2)\big)(\partial_\epsilon\varphi)^4 + 12(1-\eta^2)(\partial_\epsilon^2\varphi)^2\Big\}\,,$$
$$B_\epsilon = \frac{\eta^{-2}}{108}(\beta+\beta^{-1})\Big\{(3+4a^2)(\partial_\epsilon\varphi)^4 + 12(\partial_\epsilon^2\varphi)^2)\Big\}\,,$$

where $\epsilon = \pm$. Then

$$\langle (A_+ + aB_+)(A_- + aB_-)e^{ia\varphi}\rangle = m^8 F_1(a)F_3(-a)\langle e^{ia\varphi}\rangle\,,$$
$$\langle (A_+ - aB_+)(A_- - aB_-)e^{ia\varphi}\rangle = m^8 F_1(-a)F_3(a)\langle e^{ia\varphi}\rangle\,.$$

The second equation follows from the first one by C-symmetry.

We shall see that the general picture is as follows. Given a level L, we take two sets I^+, I^- of distinct odd positive integers, which are equal in size, such that the sum of elements of their union equals L. For such a datum, there is an operator of level L in both chiralities whose expectation value equals

$$m^{2L}\prod_{j\in I^+}F_j(a)\prod_{j\in I^-}F_j(-a)\,.$$

Thus we have a general method for exact computation of expectation values, both on the lattice and in the continuum. The attractive feature is that it is based on the same construction. Moreover our method allows a simple and straightforward generalisation to the case of non-zero temperature, or even to that of the generalised Gibbs ensemble.

Both the isotropic anti-ferromagnet and the sine-Gordon model can be considered as limits of the anisotropic XXZ magnet (inhomogeneous one in the second case). Due to Onsager's observation [**O2**], this magnet is closely related to the six-vertex model. The main tool which we shall use is the fermionic basis. In order to explain its meaning we invoke the analogy with CFT. In CFT, the space of all local operators is organised into a direct sum of irreducible representations of the Virasoro algebra. This structure is decisive for obtaining many exact results. Similarly, for spin chains we consider local observables, i.e., operators with finite supports. The linear span of these operators does not look a very interesting object. However, we manage to introduce on it the structure of a module created by one family of bosonic operators and two families of fermionic operators. The fermionic basis thus constructed has a great advantage that the expectation values for its elements are simple— basically given by determinants. The main goal of Volume I is to explain the construction and applications of the fermionic basis.

We now outline the contents of the present volume.

In Chapter 1, we explain the formulation of the problem and introduce our notation. We consider the six vertex model on an infinitely long cylinder with finite circumference. We call the generatrix the *Space* direction and the circumference the *Matsubara* direction. The main object of our study is the expectation values of quasi-local operators. The latter are operators localised on a finite segment in the *Space* direction having a half-infinite tail of the form $q^{\alpha\sum_{j=-\infty}^{0}\sigma_j^3}$; see (1.9). We explain how the expectation values thus formulated are related in various limits to those at a finite temperature or at zero temperature on the infinite plane, as well as to density matrix and entanglement entroy. We then explain our strategy to evaluate the expectation values by introducing fermionic basis.

In Chapter 2, we prepare algebraic tools which will be extensively used in the later Chapters. We begin with the basics of algebraic Bethe ansatz in the *Matsubara* direction, algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and the universal R-matrix. We then proceed to the Bazhanov-Lukyanov-Zamolodchikov construction based on representations of the q-oscillator algebra *Osc*. Used originally to construct Q-operators, this method is indispensable for the construction of fermionic basis given in the next Chapter. We also touch upon the Destri-de Vega integral equation for the distribution of Bethe roots, which will become a convenient tool to handle the limit $\mathbf{n} \to \infty$ to infinite *Matsubara* chain.

Chapter 3 is devoted to the construction of fermionic annihilation and creation operators. Starting with the R-matrix whose auxiliary space is the tensor product of oscillator and two-dimensional representations, we introduce certain operator $\mathbf{k}(\zeta, \alpha)$ as a trace of monodromy matrix acting on operators on a finite interval. The fermionic annihilation operators $\mathbf{b}(\zeta, \alpha)$, $\mathbf{c}(\zeta, \alpha)$ are defined from the singular part of $\mathbf{k}(\zeta, \alpha)$. They are shown to anti-commute and are naturally extended to the whole space of quasi-local operators. The bosonic creation operators $\mathbf{t}^*(\zeta, \alpha)$ are simple to construct; they are basically the adjoint action by local integrals of motion of the XXZ chain. In contrast, the construction of fermionic creation operators $\mathbf{b}^*(\zeta, \alpha)$, $\mathbf{c}^*(\zeta, \alpha)$ is more elaborate. Their construction and derivation of their properties, including the (anti-)commutation relations with the annihilation operators, occupy Section 3.5 through Section 3.7. We give in particular the Russian doll construction in Section 3.6 which allows us to express the creation operators in the homogeneous chain compactly in terms of those of finite inhomogeneous chains. Applying the creation operators to the primary field we obtain the fermionic basis of quasi-local operators.

In Chapter 4, we combine the results of the preceding Chapters to prove the main theorem. As a mathematical tool we introduce deformed Abelian differentials and deformed Riemann bilinear identities. We prove the main theorem which states that the expectation values of fermionic basis can be expressed in terms of two functions, $\rho(\zeta|\alpha)$ and $\omega(\zeta, \alpha)$. We show also that the fermionic operators indeed create a basis of quasi-local operators on a homogeneous chain. We end this Chapter explaining the relation of our construction to cohomologies of affine Jacobi varieties.

In Chapter 5, we discuss various applications of the fermionic construction. We first rewrite the function $\omega(\zeta, \alpha)$ into a form suitable for taking the limit $\mathbf{n} \to \infty$. The main theorem allows us to compute expectation values of physically interesting operators, once we know their expression in the fermionic basis. We explain that the main theorem can again be used as a practical tool for finding the relevant coefficients. After examining the limit $\alpha \to 0$, we present numerical computation of the entanglement entropy in the XXZ model. We also consider the RSOS reduction corresponding to the minimal unitary series of CFT. We then present results for the entanglement entropy and correlation functions in the XXX model at zero temperature. Lastly we make a comment on the attempt to extend our approach to the completely anisotropic XYZ model.

In the Appendix we discuss the quasi-classical limit, and explain the meaning of our construction in the light of the theory of Abelian integrals. First we compute the quasi-classical limit of the canonical differential discussed in Chapter 4, expressing it in terms of differentials on a hyper-elliptic Riemann surface. We give a brief summary of basic facts about Riemann surfaces. In the quasi-classical limit, the algebra of local observables turn into the Poisson algebra generated by entries of the classical monodromy matrix. Via separation of variables, the Jacobi inversion formula allows us to express these matrix elements as meromorphic functions on the affine Jacobi varieties. We show that the fermionic creation operators have purely classical origin: on the top cohomology of the affine Jacobi variety, the operators $\mathbf{c}_j^*$ act by removing the first kind differentials and $\mathbf{b}_j^*$ by multiplying by the second kind differentials.

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Notation

Chapter 1

1.2

q parameter q in the six vertex model, (1.1)
P permutation matrix, (1.3)
$T^{6\mathrm{v}}_{j,\mathbf{M}}$ *Matsubara* monodromy matrix, (1.4)
$T^{6\mathrm{v}}_{\mathbf{S},\mathbf{M}}$ monodromy matrix with *Space* and *Matsubara* directions, (1.5)
$T^{6\mathrm{v},\kappa}_{\mathbf{M}}$ *Matsubara* transfer matrix with twist, (1.8)

1.3

$q^{2\alpha S(0)}\mathcal{O}$ quasi-local operator, (1.9)
$\mathcal{Z}^{6\mathrm{v},\kappa}_{L,\mathbf{n}}$ expectation-value functional (finite chain) (1.10)
$\mathcal{Z}^{6\mathrm{v},\kappa}_{\mathbf{n}}$ expectation-value functional (infinite chain) (1.11)

1.4

$L_{j,\mathbf{m}}$ L operator (1.12)
$R(\zeta)$ R-matrix of 6 vertex model, (1.13)
$T^{\kappa}_{\mathbf{M}}(\xi,\kappa)$ *Matsubara* transfer matrix with spectral parameter and twist, (1.18)

1.5

$\mathbf{A}(\xi_1,\dots,\xi_l,\alpha)$ reduced qKZ operator, (1.21), (1.22)
θ charge conjugation, (1.23)
$\mathbb{R}_{j,j+1}(\xi_j/\xi_{j+1})$ adjoint action of R-matrix ,(1.25)

1.6

$T_{\mathbf{S}}(\zeta)$ *Space* transfer matrix (1.29)
$\widetilde{R}_{j,\mathbf{m}}(\zeta)$ $\tilde{R}$ matrix (1.32)
$\check{R}_{j,\mathbf{m}}(\zeta)$ $\check{R}$ matrix (1.33)
$\mathbf{I}_p$ local Hamiltonians (1.37)

1.7

$\rho(n,T)$ density matrix, (1.49)
$s(n,T)$ entanglement entropy, (1.51)

Chapter 2

2.1

$A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$ entries of *Matsubara* monodromy matrix (2.1)
$a(\xi)$, $d(\xi)$ vacuum eigenvalues of $A(\xi)$, $D(\xi)$ (2.7)
$|\lambda_1^+,\dots,\lambda_k^+\rangle$ Bethe vector for *Matsubara* transfer matrix (2.8), (2.93)
$\langle\lambda_1^-,\dots\lambda_{\mathbf{n}-k}^-|$ Bethe covector for *Matsubara* transfer matrix, (2.13)
$Q^+(\xi,\kappa)$, $Q^-(\xi,\kappa)$ Q-functions (2.11),(2.15)
$M_{\mathbf{n}}(\lambda_1,\dots\lambda_{\mathbf{n}}|\tau_1,\dots,\tau_{\mathbf{n}})$ domain wall partition function, (2.18), (2.20)

2.2

$\mathcal{R}$ universal R-matrix, (2.21),(2.22),(2.23),(2.24)
E, F, K generators of $U_q(\mathfrak{sl}_2)$, (2.25)
C Casimir element of $U_q(\mathfrak{sl}_2)$, (2.27),(2.30)
V^Λ Verma module of $U_q(\mathfrak{sl}_2)$, (2.28)

2.3

e_i, f_i, t_i generators of $U_q(\widehat{\mathfrak{sl}}_2)$, (2.32),(2.33),(2.34),(2.35)
$U_q\mathfrak{b}^+$, $U_q\mathfrak{b}^-$ Borel subalgebras of $U_q(\widehat{\mathfrak{sl}}_2)$, p.37
ev_ζ evaluation homomorphism, (2.43)
π_ζ^Λ evaluation Verma representation of $U_q(\widehat{\mathfrak{sl}}_2)$, (2.45)
$\pi_\zeta^{(2s)}$ spin s representation of $U_q(\widehat{\mathfrak{sl}}_2)$, (2.45)
$R^{(ev,1)}(\zeta)$ evaluation R-matrix with spin 1/2, (2.46)
Osc q-oscillator algebra, (2.49)
$\mathbf{a}$, $\mathbf{a}^*$, $q^{\pm D}$ generators of Osc, (2.49)
$W^\pm$ representations of Osc, (2.50)
$R_A^+(\zeta)$, $R_A^-(\zeta)$ R-matrix with Osc as auxiliary space ,(2.53), (2.57)
$R_a(\zeta)$ R-matrix with $\pi^{(1)}$ as auxiliary space (2.55)
$\check{R}_{A,B}(\zeta)$ intertwiner for $W_{\zeta_1}^+\otimes W_{\zeta_2}^+$, (2.59),(2.61)
$\mathfrak{A}^{\text{left}}(\zeta)$, $\mathfrak{A}^{\text{right}}(\zeta)$ (2.66), (2.71)
$\mathfrak{A}(\zeta)$ quotient algebra of $U_q(\mathfrak{sl}_2)$, (2.67)
$F_{a,A}$ matrix of conjugation used for fused R-matrix (2.79)

2.4

$\mathfrak{H}_{\mathbf{M}}$ *Matsubara* space with arbitrary spin representations, (2.80), (2.81)
$R^{(ev,2s)}(\zeta)$ evaluation R-matrix with spin s, (2.82)
$R^{(+,2s)}(\zeta)$ R-matrix in oscillator and spin s representations, (2.84)
$\mathfrak{A}^{\text{left}}_{\text{loc}}(\zeta)$, $\mathfrak{A}^{\text{right}}_{\text{loc}}(\zeta)$ localisation of algebras $\mathfrak{A}^{\text{left}}(\zeta)$, $\mathfrak{A}^{\text{right}}(\zeta)$, Definition 2.7
$a(\xi)$,$d(\xi)$ functions $a(\xi)$, $d(\xi)$ for general spin, (2.93)

2.5

$Q^\pm_{\mathbf{M}}(\zeta,\kappa)$ Q-operators, (2.102)
$W(\xi)$ scalar function in the Wronskian identity, (2.109), (2.110)

2.6

$\mathfrak{a}(\zeta|\kappa)$ unknown function in DdV equation, (2.111)
$K_0(\zeta)$ kernel function in DdV equation, (2.113)
$R_0(\zeta)$ resolvent kernel in DdV equation, (2.115)

Chapter 3

$\mathcal{W}$, $\mathcal{W}_{\alpha,s}$ space of quasi-local operators, (3.1)
$\mathcal{W}_\alpha$ block α of $\mathcal{W}$, (3.4)

3.1

$R^+_{\{a,A\}}(\zeta)$ fusion of oscillator and spin 1/2 R-matrix, (3.5)
$R_{\{a,A\},\{b,B\}}(\zeta)$ R-matrix intertwining fused spaces, (3.6)
$\mathcal{F}_{\{a,A\},\{b,B\}}(\zeta)$ quasi R-matrix, (3.8)

3.2

$\mathbb{T}_{\mathrm{aux},J}(\zeta)$ adjoint monodromy matrix, (3.10)
$\mathbb{R}_{\mathrm{aux},j}(\zeta)$ adjoint R-matrix, (3.11)
$T_{\{a,A\},J}(\zeta)$ fused monodromy matrix, (3.13)
$\mathbb{T}_{\{a,A\},J}(\zeta,\alpha)$ fused adjoint monodromy matrix, (3.15)
$\mathbf{k}_J(\zeta,\alpha)$ operator $\mathbf{k}(\zeta,\alpha)$ on interval J, (3.16),(3.17)
Δ_ζ q-difference operator, (3.20)
$\mathbf{v}_J(\zeta,\alpha)$ operator $\mathbf{v}(\zeta,\alpha)$ on interval J, (3.22)
ϕ spin reversal operation on operators, Definition 3.6
$\psi(\zeta,\alpha)$ Cauchy kernel, (3.30)
$\mathbf{q}_J(\zeta,\alpha)$ adjoint Q-operator, (3.35)

3.3

$\overline{\mathbf{c}}_J(\zeta,\alpha)$ annihilation operator $\overline{\mathbf{c}}$ on J, (3.38)
$\mathbf{c}_J(\zeta,\alpha)$ annihilation operator $\mathbf{c}$ on J, (3.39)
$\mathbf{c}(\zeta,\alpha)$ annihilation operator $\mathbf{c}$, (3.42)
$\mathbf{b}(\zeta,\alpha)$ annihilation operator $\mathbf{b}$, (3.43)

3.4

$\mathbf{t}^*_J(\zeta,\alpha)$ creation operator $\mathbf{t}^*$ on J, (3.52)
$\mathbf{r}_{i,j}(\zeta^2)$ 'tail' of adjoint R-matrix, (3.54)
$\mathbf{t}^*(\zeta,\alpha)$ creation operator $\mathbf{t}^*$, (3.59)

3.5

$\mathbf{f}_J(\zeta,\alpha)$ operator $\mathbf{f}^*$ on J, (3.67)
$\mathbf{f}_{\mathrm{rat},J}(\zeta,\alpha)$ rational version of $\mathbf{f}_J$, (3.69)
$\Delta_\zeta^{-1}\psi(\zeta,\alpha)$ q-primitive of $\psi(\zeta,\alpha)$, (3.72)
$\mathbf{b}^*_J(\zeta,\alpha)$ creation operator $\mathbf{b}^*$ on J, (3.74)
$\mathbf{g}_{c,L}(\zeta,\alpha)$ operator $\mathbf{g}$ on L, (3.75)
$\mathbf{b}^*(\zeta,\alpha)$ creation operator $\mathbf{b}^*$, (3.85)
$\mathbf{c}^*(\zeta,\alpha)$ creation operator $\mathbf{c}^*$, (3.87)

3.6

$\mathbf{g}_j^{\epsilon}(\xi_j,\alpha)$ Russian-doll operators, (3.90), (3.92)
$\mathbf{x}^{\epsilon *}(\zeta)$ combined notation for creation operators, (3.95)
$\mathbf{b}^*_{\mathrm{rat}}(\zeta)$ rational version of $\mathbf{b}^*$, (3.109)
$D_\zeta F(\zeta)$ q-Laplacian (operator version), (3.110)

3.8

$\mathbf{b}_p$, $\mathbf{c}_p$ Fourier components of annihilation operators (3.124)
$\mathbf{b}^*_p$, $\mathbf{c}^*_p$, $\mathbf{t}^*_p$ Fourier components of creation operators (3.125), (3.126), (3.127)

Chapter 4

4.2

$\varphi(\zeta)$ logarithmic q-primitive of $a(\zeta)/d(q\zeta)$, (4.12),(4.13)
$d\mu^{\pm}(\zeta)$ measures used for deformed Abelian differentials, (4.14)
$\rho(\zeta|\alpha)$ ratio of transfer matrix eigenvalues, (4.16)
δ^-_ζ q-difference operator, (4.17)
$E_\zeta\big(g^{\pm}(\zeta)\big)$ exact form, (4.19)
$r(\zeta,\xi)$, $r^+(\zeta,\xi)$, $r^-(\xi,\zeta)$ generating functions of deformed Abelian differentials, (4.22)
$\Omega^{\pm}_{\mathbf{m}}(\zeta)$, $\widetilde{\Omega}^{\pm}_{\mathbf{m}}(\zeta)$ deformed Abelian differentials, (4.27)
$\mathcal{A}^{\pm}_{\mathbf{i},\mathbf{j}}$, $\mathcal{B}^{\pm}_{\mathbf{i},\mathbf{j}}$ deformed period matrices, (4.28)
$\omega(\zeta,\xi|\alpha)$ function ω, (4.35)
$\omega_{\mathrm{sing}}(\zeta,\xi|\alpha)$ singular part of ω, (4.35)

4.3

$\mathbf{g}_{\mathrm{rat}}(\zeta,\alpha)$ rational version of $\mathbf{g}(\zeta,\alpha)$, (4.51)

4.4

$\bar{\mathbf{t}}^*(\zeta)$ dual version of $\mathbf{t}^*(\zeta)$, (4.70), (4.71)
$\bar{\mathbf{b}}^*(\zeta)$, $\bar{\mathbf{c}}^*(\zeta)$ dual version of $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$, (4.72)
$\mathbf{s}(\zeta,\alpha)$ spin 2 annihilation operator, (4.77), Remark 4.27
$B_n(\alpha)$ basic operator supported on $[1,n]$, (4.78), (4.79)
$\mathbf{s}_p$ Fourier components of $\mathbf{s}(\zeta,\alpha)$, (4.80)
$\mathbf{y}_m$ operators used to describe quasi-local operators of given length, (4.88)

Chapter 5

5.1

$K_\alpha(\zeta)$ generalised kernel function, (5.2)
$dm(\zeta)$ generalised measure, (5.3)
$(F\star G)(\zeta,\xi)$ convolution with respect to dm, (5.4)
R_{dress} dressed resolvent kernel, (5.5)
$f_{\mathrm{left}}(\zeta,\xi)$, $f_{\mathrm{right}}(\zeta,\xi)$ (5.6)
$G_{\mathrm{right}}(\zeta,\xi)$ right auxiliary function,(5.8)

δ^{+}_{ζ} q-difference operator, (5.12)
$G_{\mathrm{left}}(\zeta,\xi)$ left auxiliary function, (5.17)
$\psi_{+}(\zeta,\alpha)$ modified Cauchy kernel, (5.23)
$dm_0(\zeta)$ generalised measure, (5.25)
$(F \circ G)(\zeta,\xi)$ convolution with respect to dm_0, (5.26)
$(F * G)(\zeta,\xi)$ convolution $*$, (5.27)
F_{left}, F_{right} (5.30)

5.2

$\omega_{\mathrm{rat}}(\zeta,\xi|\alpha)$ rational version of ω, (5.40)
$\overline{D}_{\zeta}$ q-Laplacian, (5.41)
$(\mathcal{A}f)(x_1,\cdots,x_l)$ operator A on symmetric polynomials, (5.45)
$(\mathcal{D}f)(x_1,\cdots,x_l)$ operator D on symmetric polynomials, (5.46)
$(\mathcal{B}f)(x_1,\cdots,x_l)$ operator B on symmetric polynomials, (5.47)
$(\mathcal{C}f)(x_1,\cdots,x_l)$ operator C on symmetric polynomials, (5.48)
$\mathcal{T}_{++}$ operator A on Young diagrams, (5.53)
$\mathcal{T}_{--}$ operator D on Young diagrams,(5.54)
$\mathcal{T}_{+-}$ operator B on Young diagrams,(5.55)
$\mathcal{T}_{-+}$ operator C on Young diagrams,(5.56)

5.3

Ω annihilation operator Ω, (5.65),(5.66)
$B^{\varepsilon}(\zeta)$ (5.67)
$\omega_0(\zeta_2,\zeta_1)$, $\omega_1(\zeta_2,\zeta_1)$ Taylor coefficients of ω_{rat} as α(5.71)
Ω_0, Ω_1 annihilation operator Ω as $\alpha \to 0$, (5.75),(5.76)
$\mathcal{Q}$ charge operator, (5.80)
$\tilde{X}_1(\zeta_1,\zeta_2)$, $\tilde{X}_0(\zeta_1,\zeta_2|\alpha)$ gauge transformed $X_1(\zeta_1,\zeta_2)$, $X_0(\zeta_1,\zeta_2|\alpha)$, (5.110)
$Y^{\vee}$ vector corresponding to operator Y, (5.116)
$\widetilde{O}(J)$ A-basis of invariant operators, (5.121)
$v_{k,J,m}$ vector corresponding to a Bratteli diagram J, (5.122)
$O(J)$ $3j$-basis of invariant operators, (5.125)

5.4

$\omega(x)$ function ω in the XXX case, (5.137)

5.5

$\varphi(\zeta,\nu)$ generating function for ω_0, ω_1 in XXZ case, (5.144)
J_a, J_{bc} structure constants of Sklyanin algebra, (5.148), (5.149)
K_0, K_2 Casimir elements of Sklyanin algebra, (5.153)
$\varphi(\mu,\nu,\tau)$ generating function for ω_1, ω_2, ω_3 in XYZ case, (5.169)

Appendix

A.1

g genus of the hyper-elliptic curve,

A.2

$\varkappa$ rescaled twist, finite in the quasi-classical limit, (A.4)
$T_{cl}(\zeta, \varkappa)$ quasi-classical limit of transfer matrix, (A.5)
ϱ differential of the third kind with poles at $0, \infty$ (A.10)
$dm^{\pm}(z)$ quasi-classical measure, (A.14), (A.32)

A.3

S hyper-elliptic curve, (A.23)
ω_i normalized differential of the first kind, (A.27)
$\omega(p_1, p_2)$ canonical differential of the second kind, (A.28)
ω_{q_1,q_2} normalized differential of the third kind, (A.29)

A.4

$\mathcal{t}(z)$ classical monodromy matrix, (A.33)
$\mathcal{a}(z)$, $\mathcal{b}(z)$, $\mathcal{c}(z)$, $\mathcal{d}(z)$ matrix elements of $\mathcal{t}(z)$, (A.33)
z_j, w_j separated variables, (A.36)
ϖ variable conjugate to $\varkappa$, (A.37)
$S(k)$ symmetric power of S, (A.40)
$A(p)$ Abel transformation, (A.42)
Θ Theta divisor, (A.45)

A.5

∇_j flat connection, (A.58)

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