

Mathematical
Surveys
and
Monographs

Volume 264

Diagrammatic Algebra

J. Scott Carter
Seiichi Kamada



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2020 *Mathematics Subject Classification*. Primary 18M15, 18M30, 18N25, 55N99, 57K10, 57K16, 57K45, 57R42.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-264

Library of Congress Cataloging-in-Publication Data

Names: Carter, J. Scott, author. | Kamada, Seiichi, 1964– author.

Title: Diagrammatic algebra / J. Scott Carter, Seiichi Kamada.

Description: Providence, Rhode Island : American Mathematical Society, [2021] | Series: Mathematical surveys and monographs, 0076-5376 ; Volume 264 | Includes bibliographical references and index.

Identifiers: LCCN 2021035872 | ISBN 9781470466718 (paperback) | ISBN 9781470468156 (ebook)

Subjects: LCSH: Algebraic topology. | Homology theory. | Intersection homology theory. | AMS: Algebraic topology – Homology and cohomology theories [See also 57Txx] – None of the above, but in this section. | Manifolds and cell complexes {For complex manifolds, see 32Qxx} – Differential topology {For foundational questions of differentiable manifolds, see 58Axx; for infinite-dimensional manifolds, see 58B}

Classification: LCC QA612 .C37 2021 | DDC 514/.2–dc23

LC record available at <https://lcn.loc.gov/2021035872>

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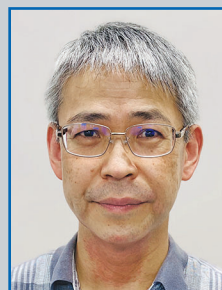
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