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Preface

This book aims to treat all currently known examples of amenable groups and techniques behind proving amenability. We do not elaborate on any examples of non-amenable groups in detail (except free groups); rather, we list all known examples of non-amenable groups which do not contain free groups in the Introduction.

The subject of amenability has its roots in measure theory. In 1904, Lebesgue asked whether a finitely additive translation-invariant measure defined on all Lebesgue measurable sets is necessarily the Lebesgue measure. This question was answered negatively by Banach. The core of his proof is the Hahn-Banach theorem and a connection between finitely additive measures and means. This essentially led to a first example of an amenable group: the group of real numbers.

The next step in the development of amenability was the Hausdorff paradox. It states that the unit sphere can be decomposed into finitely many pieces in a way that rotating these pieces around one can obtain two spheres. Hausdorff showed that the non-abelian free group on two generators (which is now considered a canonical example of a non-amenable group) is a subgroup of the group of rotations. It is easy to make a paradoxical decomposition of the free group itself. Having this decomposition in hand, Hausdorff pushed it to the action of the rotation group on the sphere, which implied his famous paradox.

Considering a unit ball and projecting each point of the ball (except the center) onto the sphere, one obtains a paradoxical decomposition of the ball without a center. Notably, the entire ball admits a paradoxical decomposition if one allows use of rotations and translations in the rearranging of the sets. This counter-intuitive paradox was proved by Banach and Tarski in 1929. A strong version of the Banach-Tarski paradox claims that any bounded set with non-empty interior can be decomposed and rearranged to any other bounded set with non-empty interior. This is striking, as it implies that a tiny pea that can be decomposed to the sun.

In 1929, von Neumann extracted an essential property of the group that forbids paradoxical decomposition: the existence of an invariant mean. He coined a German name "meßbar", which translates to English as "measurable". Studying the properties of meßbar groups, he showed that this class is closed under taking natural group operations: subgroups, extensions, quotients, and direct limits. Moreover, it contains all finite and abelian groups. This immediately explained why the Banach-Tarski paradox is impossible in the two dimensional case, where the rotation group is solvable.

In 1957, Day gave an English term "amenable" as analog of von Neumann's "meßbar" groups. He defined the class of elementary amenable groups as the smallest class of groups containing all finite groups and abelian groups, and closed under taking subgroups, extensions, quotients, and direct limits. No substantial progress in understanding this class was made until the '80s, when Chou showed that all elementary amenable groups have either polynomial or exponential growth, and Grigorchuk gave an example of a group with intermediate growth. Grigorchuk's group served as a starting point in developing the theory of groups with intermediate growth, all of them being non-elementary amenable (it is easy to show that all non-amenable groups have exponential growth).

The class of non-elementary amenable groups of exponential growth challenged many mathematicians. One of the primary difficulties is that the amenability for these groups does not come for free, as it does for the subexponential growth groups. Additionally, proving that these groups are not elementary amenable can be algebraically difficult for a given group. An important class of groups containing many non-elementary amenable groups of exponential growth is the class of automata groups. The first example of a non-elementary amenable group with exponential growth is the Basilica group introduced by Grigorchuk and Zuk. Furthermore, amenability of the Basilica group is a technically difficult problem solved by Bartholdi and Virág in 2005. In 2009, Bartholdi, Nekrashevych, and Kaimanovich demonstrated the amenability of the group of bounded automata of finite state, which as a consequence gives another proof of amenability of the Basilica group. In 2010, this result was extended to automata of linear activity by Amir, Angel, and Virág. In 2012, Monod and the author showed that the topological full group of a Cantor minimal system is amenable. By results of Matui in 2006, these groups have a simple and finitely generated commutator subgroup; in particular, it is not elementary amenable, giving the first examples of simple finitely generated infinite and amenable groups. In 2013, Nekrashevych, de la Salle, and the author produced a unified proof of amenability of all known examples of amenable groups. This approach further covers amenability of automata groups of quadratic activity. In 2016, Nekrashevych [**117**] found subgroups of topological full groups which give examples of simple Burnside groups of intermediate growth.

* * *

The purpose of this monograph is to give an introduction to amenability of discrete groups and provide a self-contained presentation of all currently known amenable groups. The techniques that are used for proving amenability in this book are largely based on the recent work of my collaborators and myself, which is mainly a combination of analytic and probabilistic tools with geometric group theory. In this book, I do not cover all available material related to topological groups; instead, I concentrate on treating examples of finitely generated amenable groups and their properties.

The monograph was developed from a sequence of lectures on amenability that I gave on different occasions in 2013–2017. This includes a winter school in CIRM at Luminy; three graduate courses at Northwestern University in Evanston; a winter school in Santiago de Chili; a summer school at Bernoulli Center in Lausanne; a winter school in RIMS, Kyoto; a YMCA summer school; and an Israeli Women in Mathematics meeting. Many open problems in the end of the book were discussed during a workshop on amenability at American Institute of Mathematics in 2016.

I am grateful to many people who provided me with numerous comments on the early drafts. I am thankful to Narutaka Ozawa and Stefaan Vaes for allowing me to reproduce their unpublished proofs: lemma on recurrency implies extensive amenability, and a group-theoretical reproof of the main result of [**88**]. I am very grateful to Yves de Cornulier, Max Chorniy, Tsachik Gelander, Yair Glasner, Pierre de la Harpe, Rostislav Grigorchuk, Rostyslav Kravchenko, Nicolas Monod, Vasco Schiavo, Emmanuel Rauzy, Anatoliy Vershik, and Tom Ward for providing useful remarks on all stages of the book. Nico Matte Bon, Nicolas Monod, and Mikael de la Salle had numerous discussions on the topic of extensive amenability with me, and I am truly grateful for opportunity to learn from them and their contribution to my understanding of the subject A very elegant proof of the growth dichotomy for elementary amenable groups was kindly provided by Phillip Wesolek and Pierre-Emmanuel Caprace. I thank to them for the opportunity to include it in the book. I thank to Volodia Nekrashevych for numerous discussions on the structure of the book, and support during the writing process.

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