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Kirillov's Seminar on Representation Theory
Kirillov's Seminar on Representation Theory

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Editor
ADVANCES IN THE MATHEMATICAL SCIENCES
EDITORIAL COMMITTEE

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Translation edited by A. B. Sossinsky

1991 Mathematics Subject Classification. Primary 05Exx, 17Bxx, 22E15;
Secondary 53C35.

ABSTRACT. The book is a collection of papers written by students of A. A. Kirillov and participants
of his seminar on Representation Theory at Moscow University. The papers deal with various
aspects of representation theory for Lie algebras and Lie groups and its relations to algebraic
combinatorics, theory of quantum groups, and geometry. The book is useful for researchers and
graduate students working in representation theory and its applications.

Library of Congress Card Number 91-640741
ISBN 0-8218-0669-6
ISSN 0065-9290

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Preface

The present volume was prepared for publication by students and friends of
Alexandr Alexandrovich Kirillov in connection with his 60th anniversary.

A. A. Kirillov’s numerous students (and not only his students) studied at his
seminar on representation theory at Moscow State University. This seminar func-
tioned for nearly 30 years, beginning in the early sixties when A. A. began teaching
at the chair of function theory and functional analysis of the Mechanics and Math-
ematics Department of MSU, and continuing until A. A. started working at the
University of Pennsylvania in Philadelphia. I first came to A. A.’s seminar in the
winter of 1964–65 as a freshman, so I am one of his first students and one of the
oldest participants of the seminar.

For many years Kirillov’s seminar was one of the best known and popular
Moscow mathematical seminars, and for me as well as for Kirillov’s other students,
the most customary and comforting one. It took place on Mondays, two hours
before the Gelfand seminar. On Thursdays, A. A. also conducted a seminar for
beginners (first and second year students), which was especially well attended.
Active students of the latter would eventually move on to the Monday seminar,
intended for older undergraduates, graduate students, and professional research
mathematicians.

The topics discussed at the seminar ranged quite widely, reflecting Kirillov’s
broad research interests. It included finite-dimensional representation theory;
unitary representations of reductive, solvable, and general Lie groups; representa-
tions of infinite-dimensional groups. Of course, the orbit method; the universal formula
for characters; symplectic geometry. The fractional fields of enveloping algebras
and other noncommutative rings; identities in noncommutative rings. Infinite-
groups. Mathematical physics . . . (I am afraid that I have missed many topics.)

The Kirillov seminar was neither primarily intended to inform on various top-
ics, with experts taking turns to lecture on them, nor was it a working group
concentrating on a specific cycle of papers, although to some extent it performed

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1When first talking to a new student, A. A. would usually ask what the latter would like to
study under him—algebra, geometry, or analysis.

2In general, the organization of the seminar did not involve any rigid planning: lecturers
and titles were not written out and displayed in advance, and everything seemed to take place
spontaneously.
both functions. Above all, it was a place where one learned to do mathematics “according to Kirillov”.

Participants of the seminar would usually assemble in advance, and while waiting for A. A. (who often came a bit late), they would conduct animated conversations. When Kirillov appeared at the doorstep, all the participants would rise. If no talk was planned and A. A. did not intend to lecture himself, he would conduct a poll: who had done something new? He would then call someone to the blackboard and ask to state the result “in five minutes”. In fact, few succeeded in complying with this sacramental time interval, but if the topic was interesting, A. A. would often forget this constraint, and “five minutes” could easily become a detailed account with a subsequent discussion.4

The atmosphere of the seminar was very free, relaxed, and informal. The lecturer was often interrupted by questions, and whenever A. A. felt that the listeners were losing track, he would explain the difficult parts in his own way or discuss improvised examples.

I am convinced that for us, just beginning to do mathematics, the main profit from participating in the seminar had to do with the impact of A. A. Kirillov’s personality, his manner of explaining things simply, his light irony concerning an overly “scientific” style of exposition, his sharp remarks, and strong dislike of artificial constructions. All this contributed to form a proper taste in mathematical style, such an important component of one’s mathematical education. And, of course, a crucial role was played by the problems that Kirillov systematically produced during the seminar. Some were prepared in advance, others arose spontaneously during discussions. A good result leading to new problems was particularly praised by Kirillov.5

One of the specific traits of A. A. Kirillov’s style as a teacher was that he never liked to impose research topics for kursovye (term papers), diplomnye (MS theses), or kandidatskie (PhD dissertations). It was assumed that each student must find a topic himself on the basis of problems set at the seminar. Of course, this was not an absolute rule, but to the students that he rated among the best, Kirillov always gave complete freedom in the choice of a research topic.

Now, when Kirillov’s seminar in Moscow no longer functions, while his students have dispersed all around the world and mostly communicate by e-mail,6 I piercingly realize how much I owe to the seminar. I have no doubts that similar feelings are experienced by Kirillov’s other students.

* * *

I shall briefly review the contents of the contributions.

1. In the paper “Screenings and a universal Lie–de Rham cocycle” by V. Ginzburg and V. Schechtman, a generalization of the classical Feigin–Fuchs construction

---

3Many Moscow seminars were, to some extent, something like clubs (this was especially true of I. M. Gelfand’s famous seminar) and the discussions before they formally began, as well as the positive influence of the late arrival of the seminar’s head, deserve special analysis.

4Such a poll would invariably take place at the first session after vacations. I vividly remember the feeling of frustration that arose if my turn to be polled was not reached.

5In assessing mathematical achievements, A. A. half-jokingly used “economic” terminology, distinguishing results that destroy “workplaces” for mathematicians from those that create them.

6Recently A. A. told me that he can invite people to his new seminar in Philadelphia from within a radius of $300 (that is the amount that can be allocated for travel expenses). Unfortunately, typical distances are now measured by larger amounts.
is presented. It provides canonical mappings from the homology of one-dimensional local systems on the configuration spaces appearing in conformal field theory to the $\text{Ext}$-spaces between modules of semi-infinite forms over the Virasoro algebra or Wakimoto modules over affine Lie algebras. An analog of this construction for finite-dimensional semisimple Lie algebras is given.

2. The paper “Interlacing measures” by S. Kerov deals with the asymptotic behavior of pairs of interlacing sequences,

$$x_1 < y_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < x_n.$$ 

A typical example of such pairs is provided by roots of polynomials of adjacent degrees in a family of orthogonal polynomials. The author introduces and studies a more general object, a pair of interlacing measures. As a matter of fact, to each pair of interlacing measures with difference $\tau$ there corresponds a unique probability distribution $\mu$ such that

$$\exp \int \ln \frac{1}{z - u} \tau(du) = \int \frac{\mu(du)}{z - u}, \quad \text{Im} \, z \neq 0.$$ 

This equation has a number of interesting applications, including

(1) the connection between additive and multiplicative integral representations of analytic functions of negative imaginary type;

(2) the Markov moment problem;

(3) distributions of mean values of Dirichlet random measures;

(4) the theory of spectral shift function in scattering theory;

(5) the Plancherel measure of the infinite symmetric group.

Apparently, the paper gives the first unified survey of all these topics. A special emphasis is on the combinatorial connections between the moments of the measures $\tau$ and $\mu$ in the above formula. One of the new results is an explicit formula for the multiplicative integral representation of the Gaussian measure on the real line.

3. The paper “Quasicommuting families of quantum Plücker coordinates” by B. Leclerc and A. Zelevinsky is devoted to the study of $q$-deformations of Plücker coordinates on the flag variety. The authors give a criterion for quasi-commutativity of two such coordinates and study their maximal quasi-commuting families (here “quasi-commutativity” means “commutativity up to a power of $q$”). The results have applications to the description of canonical bases for the quantum group $GL_n$, the geometry of Bott–Samelson desingularizations of Schubert varieties, and combinatorics of the “second Bruhat order” due to Manin and Schechtman.

4. The paper “Factorial supersymmetric Schur functions and super Capelli identities” by A. Molev is devoted to super generalization of a remarkable class of combinatorial functions—the so-called factorial Schur polynomials. These polynomials, introduced by the mathematical physicists L. C. Biedenharn and J. D. Louck and further studied by I. G. Macdonald and other authors, are certain multidimensional inhomogeneous polynomials whose highest degree terms are ordinary Schur polynomials. They have numerous applications in algebraic combinatorics and representation theory. The author develops the super counterpart of the theory. The main applications of his results are “factorial” analogs of the Jacobi–Trudi and Sergeev–Pragacz formulas; construction of a distinguished linear basis in the center of the universal enveloping algebra of $gl(m|n)$; a super analog of the higher Capelli identities. Related topics are discussed in the papers by M. Nazarov and by A. Okounkov and G. Olshanski (see below).
5. In the paper “Yangians and Capelli identities” by M. Nazarov, the $R$-matrix formalism is applied to higher Capelli identities. Recall that the classical Capelli identity (which is discussed in H. Weyl’s famous book on classical groups) provides remarkable determinantal expressions for canonical generators of the center of the universal enveloping algebra $U(gl(n))$. The higher Capelli identities are stated for a much wider family of central elements, which form a distinguished linear basis in the center of $U(gl(n))$. Note that under the Harish–Chandra isomorphism, these basis elements turn into the factorial Schur polynomials mentioned above. The methods of the paper are inspired by quantum group theory. The author studies the image of the universal $R$-matrix for the Yangian $Y(gl(n))$ with respect to the evaluation homomorphism of $Y(gl(n))$ to $U(gl(n))$. The fusion procedure as defined by I. Cherednik is used. The higher Capelli identities are obtained as a corollary of this machinery. Although the Yangian techniques used in the paper may first seem rather sophisticated, the Yangians are actually a very natural and powerful tool for handling many problems concerning classical Lie algebras. Note that in another paper by the same author, the same approach is carried over the “true” super analog of $gl(n)$, the queer Lie superalgebra $q(n)$, and in the recent paper by A. Molev and M. Nazarov, the Yangian techniques are used to obtain new Capelli-type identities (for the orthogonal and symplectic Lie algebras). Different approaches to the higher Capelli identities for $gl(n)$ were developed by A. Okounkov.

6. The aim of the paper “Hinges and the Study–Semple–Satake–Furstenberg–De Concini–Procesi–Oshima boundary” by Yu. Neretin is to propose a unified elementary geometric description for various boundaries and completions of groups and symmetric spaces—the Satake–Furstenberg boundary, the Martin boundary, the Karpelevich boundary, complete symmetric varieties in the sense of De Concini and Procesi, compactifications of Bruhat–Tits buildings, etc. The key element of the author’s constructions is the new concept of a “hinge” (a finite collection of points of a Grassmann manifold subject to certain conditions).

7. The paper “Multiplicities and Newton polytopes” by A. Okounkov deals with Newton polytopes associated in the author’s recent paper (Invent. Math. 125 (1996), 405–411) to $G$-spaces $X$, where $G$ is a connected reductive group,

$$(*) \quad X \subset \mathbb{P}(V), \quad X \text{ is closed, irreducible and } G\text{-stable},$$

and $V$ is a finite-dimensional representation of $G$. The first result of the paper is the explicit computation of the polytope for the case when $G$ is the symplectic group, $G = Sp(2n)$, and $X$ is the flag variety. The polytope thus obtained coincides with the Gelfand–Zetlin-type polytope that appears in the well-known description (due to Zhelobenko) of weight multiplicities for the reduction scheme $Sp(2n) \downarrow Sp(2n - 2) \downarrow \cdots$. This gives yet another proof and a geometric interpretation of Zhelobenko’s theorem. The second result is that the polytopes corresponding to all the different $G$-equivariant embeddings $(*)$ of $X$ can be arranged into a convex cone. This gives a strengthening of the theorem from the author’s paper cited above: the semi-classical limit of weight multiplicities for the action $(*)$ is a log-concave function of both the weight and the $G$-linearized invertible sheaf that defines the embedding $(*)$.

8. The paper “Shifted Schur functions II. The binomial formula for characters of classical groups and its applications” by A. Okounkov and G. Olshanski continues the authors’ previous work (referred to as Part I) but can be read independently.
Note that the shifted Schur functions are a modification (of a special case) of factorial Schur polynomials. The results of Part I have a direct relationship to the groups $GL(n)$; the aim of Part II was to find their counterparts for the orthogonal and symplectic groups. The paper starts with the binomial formula, which is a kind of Taylor expansion for finite-dimensional characters. This is a simple result, which has a number of important consequences. For instance, it suggests the definition of a distinguished linear basis in $Z(\mathfrak{g})$, the center of the universal enveloping algebra $U(\mathfrak{g})$, where $\mathfrak{g}$ stands for an orthogonal or symplectic Lie algebra. The basis elements can then be characterized in several different ways. Note that their images under the Harish-Chandra isomorphism can be expressed through certain factorial Schur polynomials. A natural basis in $I(\mathfrak{g})$, the subalgebra of invariants in the symmetric algebra $S(\mathfrak{g})$, is also examined. Both bases turn out to be related via the “special symmetrization map” $S(\mathfrak{g}) \to U(\mathfrak{g})$, an equivariant linear isomorphism, which differs from the usual symmetrization map. More involved versions of the binomial formula and the combinatorics of “generalized symmetrization maps” are studied in subsequent works by the same authors (cited in Part II).

G. Olshanski

Moscow, 1997
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