Lie Groups and Symmetric Spaces
In Memory of F. I. Karpelevich

S. G. Gindikin
Editor
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Contents

Preface vii

Friedrich Karpelevich: His early years in mathematics
E. B. Dynkin xi

Asymptotic distribution of eigenvalues for certain elements of the group ring
of a compact Lie group
Dmitri N. Akhiezer 1

Minimal homogeneous submanifolds of symmetric spaces
Dmitri V. Alekseevsky and Antonio J. Di Scala 11

The heat kernel on noncompact symmetric spaces
Jean-Philippe Anker and Patrick Ostellari 27

On the uniqueness of Fourier Jacobi models for representations of \( U(2, 1) \)
Ehud Moshe Baruch, Ilya Piatetski-Shapiro, and Stephen Rallis 47

Notes on integral geometry for manifolds of curves
Joseph Bernstein and Simon Gindikin 57

Quantization of Alekseev-Meinrenken dynamical \( r \)-matrices
Benjamin Enriquez and Pavel Etingof 81

Analysis on the crown of a Riemannian symmetric space
Jacques Faraut 99

Quaternionic quasideterminants and determinants
Israel Gelfand, Vladimir Retakh, and Robert Lee Wilson 111

Product formula for \( c \)-function and inverse horospherical transform
Simon Gindikin 125

The dual horospherical Radon transform as a limit of spherical Radon transforms
J. Hilgert, A. Pasquale, and E. B. Vinberg 135

The Gindikin-Karpelevič formula and intertwining operators
A. W. Knapp 145

Multiplicity one theorem in the orbit method
Toshiyuki Kobayashi and Salma Nasrin 161
The $c$-function for non-compactly causal symmetric spaces and its relations
to harmonic analysis and representation theory
BERNHARD KRÖTZ AND GESTUR ÓLAFSSON 171

A formal identity for affine root systems
I. G. MACDONALD 195

Canonical representations and overgroups
V. F. MOLCHANOV 213

Pencils of geodesics in symmetric spaces, Karpelevich boundary, and
associahedron-like polyhedra
YURII A. NERETIN 225

Poisson formula for a family of non-commutative Lobachevsky spaces
M. A. OLSHANETSKY AND V.-B. K. ROGOV 257

Real semisimple Lie algebras and their representations
A. L. ONISHCHIK 273

A calculation of $c$-functions for semisimple symmetric spaces
TOSHIO OSHIMA 307

The Abel transform on symmetric spaces of noncompact type
P. SAWYER 331
Preface

This volume contains papers which friends and colleagues of Friedrich Karpelevich (1927–2000) dedicate to his memory.

Friedrich Karpelevich was born in Moscow, on October 2, 1927. His teenage years coincided with the difficult war time. For several years high schools were closed. Friedrich worked on a factory; as a result of an accident he lost a part of a finger. He was already 20 when he entered the university, having seriously considered the option of foregoing university education. From 1947 to 1952 he was a student at Moscow University. He was one of most brilliant students and started research in his first undergraduate years under the supervision of E. Dynkin. Friedrich participated in Dynkin’s seminar for high school students and continued to participate in Dynkin’s seminars for a substantial part of his mathematical life (see Dynkin’s contribution in this volume). He established himself as a serious mathematician with his paper about the characteristic roots of matrices with nonnegative entries, published in 1949. This paper contains a complete solution of a problem posed by A. Kolmogorov. Before Karpelevich, this problem had been considered under some restrictions by N. Dmitriev and E. Dynkin.

In the early fifties Karpelevich studied subalgebras of semisimple Lie algebras. He started by giving a description of non-semisimple maximal subalgebras of simple complex Lie algebras. The classification of such subalgebras had been obtained earlier by V. Morosov. For the next five years he studied semisimple subalgebras of real semisimple Lie algebras. Shortly before that Dynkin had given a description of the semisimple subalgebras of complex semisimple Lie algebras. The case of real Lie algebras is much more difficult. Here Karpelevich’s results include a general statement about a canonical embedding of a real semisimple Lie subalgebra, which became widely used. To obtain this result, Karpelevich applied the theory of symmetric spaces, which became his favorite mathematical subject. To complete the study for the case of classical Lie algebras he had to work on very complex problems of linear algebra. He got a remarkable formula for the inertia index of an invariant symmetric or Hermitian form on the space of irreducible representation of the real semisimple Lie algebra, which solves the problem completely. These results comprised Friedrich’s PhD thesis, and they were rewarded in 1956 by a very prestigious Moscow Mathematical Society Prize for Young Mathematicians. A. Onishchik prepared for this volume an expository paper about this work.

Due to anti-Semitism, Karpelevich was not admitted to graduate school at Moscow University. He was preparing his thesis while teaching in a provincial technical school in Novocherkassk and, starting in 1953, in the Moscow Institute of
Transport Engineering. Karpelevich worked in this Institute up to last days of his life.

At the end of the fifties Friedrich’s interests shifted to geometry and analysis on homogeneous manifolds. Together with F. Berezin in 1958, he computed zonal spherical functions on Grassmannians in terms of special functions of one variable. If the space is of rank one, zonal functions are functions of one variable and can be expressed in terms of the Gauss hypergeometric function. For complex classical groups they were computed by Gelfand and Naimark. Numerous attempts to do it in other cases were unsuccessful, so the computation made by Berezin and Karpelevich remains unique up to this day.

The natural development of Friedrich’s interest in spherical functions was our collaboration on the computation of the c-function of Harish-Chandra in 1962, and later on the inverse horospherical transform. You can find more details about this work in my reminiscences in this volume.

One of the most important of Karpelevich’s results in the theory of symmetric spaces is his construction of the boundary of symmetric spaces of non-positive curvature in 1965. It is based on a detailed study of the asymptotic behavior of the geodesics. Karpelevich’s boundary has numerous applications in the theory of the eigenfunctions of the Laplace–Beltrami operator, which he studied for some time. Soon he changed his field and started to work in probability theory. Yu. Neretin wrote an expository paper on boundaries of symmetric spaces for this collection. Karpelevich’s works in probability are reflected in the memorial volume “Analytic methods in applied probability”, Yu. M. Suhov (ed.), American Mathematical Society, Providence, RI, 2002.

Friedrich Karpelevich was one of the deepest and most original mathematicians working in Lie groups and symmetric spaces in the second half of the 20th century. The contributors of this volume have different relationships with him. Some of them were happy to know Friedrich personally and to collaborate with him, some know him only through his works, but they all share the highest opinion about Karpelevich’s mathematical merits.

As we already mentioned, the volume includes Dynkin’s recollection on Karpelevich’s first steps in mathematics and two expository papers on Karpelevich’s results and their development: Onishchik’s paper on subalgebras of real semisimple Lie algebras and Neretin’s paper on boundaries of symmetric spaces. Several papers are connected with the product formula for the Harish-Chandra c-function. A. Knapp discusses its applications to intertwining operators. The papers by Oshima and by Krötz and Olafsson consider computations of $c$-functions for some pseudo-Riemannian symmetric spaces. I recall that Friedrich had a strong interest in geometry and analysis on pseudo-Riemannian symmetric spaces and undertook several attempts to work in this direction. I wrote down my reminiscences on our joint work on the $c$-function and the horospherical transform.

The paper of Hilgert, Pasquale and Vinberg considers an algebraic version of the inversion of the horospherical transform which is also connected with the $c$-function.

Several papers are dedicated to Karpelevich’s favorite area, symmetric spaces. Sawyer gave a survey of the Abel transform on noncompact Riemann symmetric spaces which has strong connections with the horospherical transform and its inversion. It also has connections with estimates of the heat kernel on such spaces. Anker and Ostellari prepared the survey on this area. In Faraut’s contribution,
some analytic problems of complex crowns of Riemann symmetric spaces are studied.

The volume contains several contributions on theory of representations and other aspects of Lie groups: Kobayashi and Nasrin’s paper on a new multiplicity one theorem, Molchanov’s paper on canonical representations on Hermitian symmetric spaces, and Baruch, Platetski-Shapiro and Rallis’ paper on representations of $U_{2,1}$ for local fields. Akhiezer investigates asymptotic problems for group rings of compact Lie groups.

Alekseevsky and Di Scala generalize Karpelevich’s 1953 result on totally geodesic orbits of reductive isometry groups on Riemann symmetric spaces. The volume also contains a paper by Macdonald on affine root systems, a paper by Gelfand, Retakh, and Wilson on quaternionic determinants and quasideterminants, a contribution by Olshanetsky and Rogov on non-commutative hyperbolic spaces, a paper by Enriquez and Etingof on quantization of dynamical $r$-matrices, and a paper by Bernstein and Gindikin on integral geometry for curves.

We can see that many of these contributions have a close relation with Karpelevich’s heritage and all of them, without a single exception, represent the areas of mathematics which he loved all his life.

Simon Gindikin

March 2003

Papers of F. Karpelevich on Lie Groups and Symmetric Spaces

2. _______, *Classification of simple subgroups of the real forms of the group of complex unimodular matrices*, Doklady Akad. Nauk SSSR 85 (1952), 1205–1208. (Russian)
3. _______, *Subgroups of real Lie groups*, Uspekhi Mat. Nauk 7 (1952), no. 5, 203–204. (Russian)
7. _______, *On semisimple subgroups of semisimple Lie groups*, Uspekhi Mat. Nauk 10 (1955), no. 1, 196. (Russian)


Friedrich Karpelevich: His Early Years in Mathematics
E. B. Dynkin

We met first in 1946 when Karpelevich was a senior in high school, and I was a graduate student at Moscow University (MGU). He came to a mathematics circle at MGU, to a section which I was running for the second year. Among about 20 participants of the section were Agranovich, Berezin, Chentsov, Minlos, Uspenski, and Yushkevich, all of whom later contributed significantly to various areas of mathematics. Very soon Karpelevich distinguished himself by solving some challenging problems.

In 1989 he remembers:

“I solved a nice special case of the four color problem: if the number of every country’s borders is a multiple of three, then a regular coloring is possible. I used the familiar fact that a coloring of countries can be reduced to a coloring of boundaries, and I proved a crucial lemma by induction on the number of countries inside a contour. I remember that you claimed all attempts to use induction in this context failed, and I was proud when I demonstrated that sometimes it works.”

[Here and later I cite a tape recording of our conversations is made during my visit to Moscow in 1989 on an exchange program between the Academy of Sciences of the USSR and the National Academy of Sciences of the USA.]

I asked him:

“When you graduated from the high school, you told me that you are not going to apply for the university but rather go to a work in a factory. I was disappointed, and asked why.”

“You know,” he responded, “I grew up in a very poor family. My mother had three children and also her mother to care for. My parents divorced when I was 7 and my sister was 1. Practically, the father has not supported us. My mother worked as a sorter at a factory, earning 70 rubles per month. My brother came back from the army disabled after the war, and he studied at a medical school. The sister was in school.

“During the war my family stayed in Moscow. For two academic years, 1941–42 and 1942–43, schools in Moscow did not function. I was 14 and I worked as a milling-machine operator for two years. When regular school resumed, I returned there.”

In 1947 several active participants in the circle (including Fred) were admitted to the Department of Mechanics and Mathematics at MGU and the circle itself was
transformed into the seminar “Selected problems of contemporary mathematics” for freshmen. The seminar was continued under various names for a number of years. In 1955 it was divided into two daughter seminars: for algebra and for probability theory. Participants of these seminars were encouraged not only to solve problems but also to prepare their solutions for publication. Over a period of time, the first publications of Dobrushin, Karpelevich, Kirillov, Margulis, Sinai, Uspenskii, and others appeared this way. Karpelevich recalls:

“My first publication was on pseudo-norms of integers. I was requested to review papers of Mahler and to improve his presentation. My note was rewritten at least four times and each time you requested a new revision. I believe that this drill was an excellent training.”

As a sophomore, Friedrich made his first serious contribution to mathematics. It is related to a problem posed by Kolmogorov in 1945: to describe the set of all eigenvalues of $n \times n$ stochastic matrices. Dmitriev and Dynkin gave a partial solution to this problem. A final solution was given by Karpelevich. His paper was published in Izvestiya Akademii Nauk SSSR and translated into English by the American Mathematical Society.

“When I was a fourth year student, you asked me to review a paper of Gantmakher devoted to Cartan’s classification of real forms of complex semisimple Lie algebras. I simplified the presentation by using simple roots. In particular, I proved a theorem on canonical embedding which is referred to in recent textbooks. This was the contents of my diploma work.”

Fred was one of the brightest students in his class. He was regarded highly by Kolmogorov and by Petrovskii (Karpelevich worked successfully at his seminar). However in 1952 — the peak of Stalin’s anti-Semitism — it was impossible for Karpelevich, a Jew, to be admitted to a graduate school. He was sent to teach at a provincial technical school in Novocherkassk. As a result of a serious illness he was permitted to return to Moscow in 1953.

He continued to work on subalgebras of semisimple Lie algebras. He obtained a number of remarkable results on this subject (see Onishchik’s survey in this volume) which were the core of his Ph.D. thesis, published in 1955. He was awarded a prestigious prize by the Moscow Mathematical Society in 1956.

Karpelevich was one of the active participants in the seminar on Lie groups which I ran in 1957–1962 at Moscow University. His well-known work on a geodesic approach to the boundary problem started at this seminar. [More on the seminar can be found in “Lie Groups and Lie Algebras: E. B. Dynkin Seminar”, S. G. Gindikin, E. B. Vinberg (eds.), American Mathematical Society, Providence, RI, 1995.]
The book contains survey and research articles devoted mainly to geometry and harmonic analysis of symmetric spaces and to corresponding aspects of group representation theory. The volume is dedicated to the memory of Russian mathematician F. I. Karpelevich (1927–2000).