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Volume 1

Selected Applications of Geometry to Low-Dimensional Topology

Michael H. Freedman
Feng Luo

American Mathematical Society

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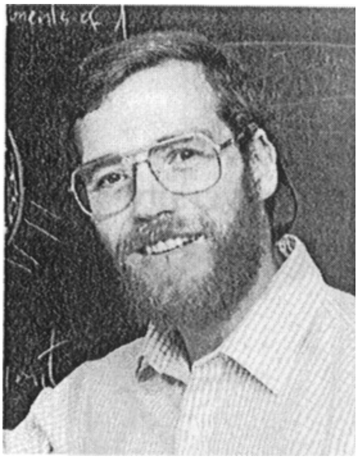
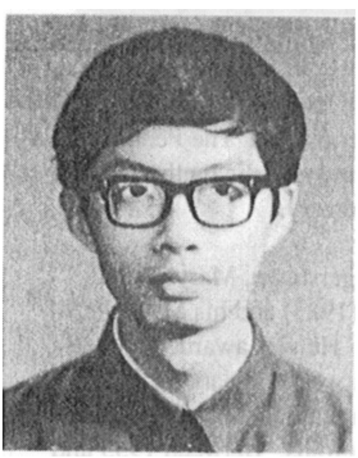


Michael H. Freedman
was born on April 21, 1951, in Los Angeles, California. He earned his Ph.D. at Princeton University in 1973 at the age of 21. He joined the University of California, San Diego (UCSD) as an assistant professor in 1976. He currently holds the Charles Lee Powell Chair in Mathematics at UCSD.

A 1986 winner of the Fields Medal, Professor Freedman was honored for his contributions to four-dimensional geometry. His work has led to a deeper understanding of geometry and topology of four-dimensional manifolds and is intimately connected with the properties of Yang-Mills equations of theoretical physics.



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Contents

Introduction

These lectures attempt to give a taste of the accomplishments of manifold topology over the last thirty years. The decades 1930–59 witnessed a massive introduction of algebra into topology and the importance of this current continues unabated. Our emphasis here is on the application of geometric ideas to the robust alliance of algebra and topology. The working definition of geometry will be psychological—the arguments should be amenable to rather sharp mental pictures and not too dependent on calculations. Geometry ranges at least from the point-set theory of decomposition spaces to the Yang-Mills equation.

We begin with the notions of manifold and smooth structures and a rather detailed discussion of the Gauss-Bonnet theorem—the prototypical link between geometry and topology. A certain amount of background and reference will be supplied as we move forward, but we do not attempt complete treatments (for example, we define "manifold" but leave "submanifold" to the imagination). Our goal is to provide a not-too-technical glimpse into several active areas in geometric topology. Much of the material can be appreciated with only a general background, but in places we refer to notions well within topology or analysis on manifolds.

From Gauss-Bonnet we proceed to a discussion of the topology and geometry of foliated 3-manifolds, a beautiful topic which might be viewed as a toy model for Thurston's work on 3-manifolds. Thurston proves some purely topological theorems by using topological hypotheses (e.g., M^3 is Haken) to create a geometric structure (a toral sum of homogeneous spaces) and then uses this new description to conclude topological information ($\pi_1(M^3)$ is residually finite—the details of this example were worked out by Hemple **[He2]**). The same pattern arises for foliated 3-manifolds although the results are less spectacular—most of the topological conclusions are available by other means.

The next chapter (4) explains in terms of general position why dimension-four is so special. Two-dimensional disks play a preferred role in topology—they mediate the algebra and geometry of double points, the simplest of all singularities. In dimension-four two-dimensional disks tend to cross themselves as do arcs on a surface. We follow Casson's ideas to resolve this difficulty and sketch the course of subsequent work in the topological category.

Chapter (5) on Donaldson's theory implicitly offers a different reason why dimension-four is special: $SO(4)$ is not simple, but rather $so(4) \cong so(3) \oplus so(3)$. This leads to the existence of the anti-self dual equation and all its consequences for the smooth category. We provide much of the elementary material on bundles, connections, and curvature needed to appreciate this idea.

A final chapter on exotic $\mathbf{R}^4$'s summarizes a few of the surprising consequences of having two competing theories of 4-dimensional manifolds—one topological and one smooth.

These notes adhere closely to lectures the first author gave at Pennsylvania State University in February 1987. However, background material has been added to clarify the discussion. The second author has assumed much of the task of rounding out the lectures. In addition, the solution to "Reeb's problem" presented in section 3 is his work. We would like to thank Richard Herman and Steve Armentrout for arranging the lectures and Mr. Marker for hosting them. We also thank Kathy Wong for her careful typing and Benedict Freedman for his comments on an early version. The second author would like to take this opportunity to thank his thesis advisor Michael H. Freedman.

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