University
LECTURE
Series

Volume 3

Why the Boundary of a Round Drop Becomes a Curve of Order Four

A. N. Varchenko
P. I. Etingof

American Mathematical Society

http://dx.doi.org/10.1090/ulect/003

University
LECTURE
Series

Volume 3

Why the Boundary of a Round Drop Becomes a Curve of Order Four

A. N. Varchenko
P. I. Etingof

American Mathematical Society
Providence, Rhode Island

1991 *Mathematics Subject Classification.* Primary 30C99, 31A25, 58F07, 76S05.

Library of Congress Cataloging-in-Publication Data

Varchenko, A. N. (Aleksandr Nikolaevich)

[Pochemu granit͡sa krugloĭ kapli prevrashchaetsi͡a v inversnyĭ obraz ẹllipsa? English]

Why the boundary of a round drop becomes a curve of order four/A. N. Varchenko and P. I. Etingof.

p. cm. —(University lecture series, ISSN 1047-3998; 3)

Includes bibliographical references.

ISBN 0-8218-7002-5

1. Boundary value problems. 2. Curves, Elliptic. 3. Fluid dynamics–Mathematical models. I. Etingof, P. I. (Pavel I.), 1969– . II. Title. III. Series: University lecture series (Providence, R.I.).

QA379.V36 1992 92-20985

515′.35—dc20 CIP

The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability. ♾

This publication was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX, the American Mathematical Society's TeX macro system.

10 9 8 7 6 5 4 3 2 1 97 96 95 94 93 92

Contents

This page intentionally left blank

Preface

In the forties P. Ya. Polubarinova-Kochina and P. P. Kufarev studied the problem of evolution of a round oil spot surrounded by water when oil is extracted from a well inside the spot (Figure 1). It turned out that the boundary of the spot remains an algebraic curve of degree four in the course of evolution. This curve is the image of an ellipse under a reflection with respect to a circle. In 1950 Kufarev managed to generalize this property: if initially the oil spot is the image of the unit disk under a conformal map given by a rational function of the complex coordinate in the disk, then it retains this property in the course of evolution.

In 1972 S. Richardson found an infinite series of first integrals of motion of the spot. He proved that the integral of any harmonic function over the oil domain changes linearly in time. This allowed Richardson to give a new proof of the invariance of rationality and an effective method to construct explicit solutions.

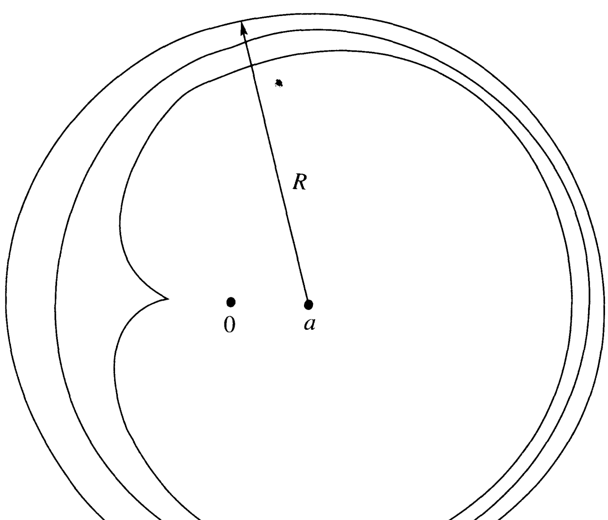

FIGURE 1

It was realized recently that this approach can be extended to multiply connected domains. In this case, the unit circle is replaced by a half of a complex algebraic curve with a specified M-structure, and the conformal map is given by a path integral of a meromorphic differential on the curve that has no zeros or poles on the chosen half.

Below we discuss these and other interesting mathematical subjects that arose recently in the theory of fluid flows with a moving boundary.

This text is an extended version of the first author's talk at a Moscow Mathematical Society meeting about the results of the second author. The authors gratefully acknowledge the crucial influence of Prof. V. M. Entov who introduced them to the circle of physical problems under consideration. The authors would also like to thank Prof. V. I. Arnold, D. Ya. Kleinbock, Prof. I. M. Krichever, and Dr. A. I. Shnirelman for very useful discussions, and Dr. Y. Peres for reading the English translation of the manuscript and making important remarks.

References

1. R. Courant, *Dirichlet Principle, Conformal Mapping, and Minimal Surfaces*, Interscience, New York, 1950.
2. Stanley Richardson, *Some Hele-Shaw flows with time-dependent free boundaries*, J. Fluid Mech. **102** (1981), 263–278.
3. P. S. Novikov, *On the uniqueness for the inverse problem of potential theory*, Dokl. Akad. Nauk SSSR **18** (1938), 165–168. (Russian)
4. Makoto Sakai, *A moment problem on Jordan domains*, Proc. Amer. Math. Soc. **70** (1978), 35–38.
5. P. I. Etingof, *Integrability of the problem of filtration with a moving boundary*, Dokl. Akad. Nauk SSSR **313** (1990), 42–47; English transl. in Soviet Phys. Dokl. **35** (1990).
6. B. Gustafsson, *Quadrature identities and the Schottky double*, Acta Appl. Math. **1** (1983), 209–240.
7. Stanley Richardson, *Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel*, J. Fluid Mech. **56** (1972), 609–618.
8. A. I. Markushevich, *Theory of functions of a complex variable*, vol. 1, GITTL, Moscow, 1950; English transl., Prentice Hall, Englewood Cliffs, NJ, 1967 .
9. L. A. Galin, *Unsteady seepage with a free surface*, Dokl. Akad. Nauk SSSR **47** (1945), 250–253. (Russian)
10. B. Gustafsson, *On the differential equation arising in a Hele-Shaw flow moving boundary problem*, Ark. Mat. **22** (1984), 251–268.
11. P. Ya. Polubarinova-Kochina, *On the motion of the oil contour*, Dokl. Akad. Nauk SSSR **47** (1945), 254–257. (Russian)
12. P. P. Kufarev, *The oil contour problem for the circle with any number of wells*, Dokl. Akad. Nauk SSSR **75** (1950), 507–510 (Russian)
13. Makoto Sakai, *Quadrature domains*, Lecture Notes in Math., vol. 934, Springer-Verlag, Berlin and New York, 1982.
14. V. M. Entov and P. I. Etingof, *Bubble contraction in Hele-Shaw cells*, Quart. J. Mech. Appl. Math. **44** (1991), 507–535.
15. Murray H. Protter and Hans F. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, Berlin and New York, 1984.
16. J. W. Milnor, *Singular points of complex hypersurfaces*, Princeton University Press, Princeton, NJ, 1968.
17. Makoto Sakai, *Null quadrature domains*, J. Analyse Math. **40** (1981), 144–154.
18. S. D. Howison, *Bubble growth in porous media and Hele-Shaw cells*, Proc. Royal Soc. of Edinburgh Sect. A **102** (1986), 141–148.
19. A. I. Markushevich, *Theory of functions of a complex variable*, vol. 2, GITTL, Moscow, 1950; English transl., Prentice Hall, Englewood Cliffs, NJ, 1967 .
20. H. Cartan, *Théorie elémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1961; English transl., Hermann, Paris, and Addison-Wesley, Reading, MA, 1963 .
21. B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern geometry: methods and applications*, 2nd ed., "Nauka", Moscow, 1986; English transl. of 1st ed., Springer-Verlag, Berlin and New York, 1984 .

22. L. Zalcman, *Some inverse problems of potential theory*, Contemp. Math. **63** (1987), 337–350.
23. P. I. Etingof, *Inverse problems of potential theory and flows in porous media with time-dependent free boundary*, Comp. Math. Appl. **22** (1991), 93–99.
24. I. M. Krichever and S. P. Novikov, *Algebras of Virasoro type, Riemann surfaces, and structures of the theory of solitons*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 2, 46–63; English transl., Functional Anal. Appl. **21** (1987), no. 2, 126–142.
25. V. Isakov, *On the uniqueness of a solution in the inverse problem of potential theory*, Dokl. Akad. Nauk SSSR **245** (1979), 1045–1047; English transl. in Soviet Math. Dokl., **20** (1979).
26. D. Aharonov and H. S. Shapiro, *Domains on which analytic functions satisfy quadrature identities*, J. Analyse Math. **30** (1976), 39–73.
27. C. Ullemar, *A uniqueness theorem for domains satisfying quadrature identity for analytic functions*, Preprint TRITA-MAT-1980-37, Mathematics, Royal Inst. of Technology, S-100-44, Stockholm, Sweden.
28. V. N. Strakhov and M. A. Brodsky, *On the uniqueness of the solution of the inverse logarithmic potential problem*, SIAM J. Appl. Math. **46** (1986), 324–344.
29. L. Karp, *Construction of quadrature domains in* $\mathbf{R}^n$ *from quadrature domains in* $\mathbf{R}^2$, Complex Variables: Theory Appl. **17** (1992), 179–188.

ISBN 0-8218-7002-5

9 780821 870020