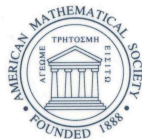


University
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Volume 3

Why the Boundary
of a Round Drop
Becomes a Curve
of Order Four

A. N. Varchenko
P. I. Etingof



American Mathematical Society

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Preface

In the forties P. Ya. Polubarinova-Kochina and P. P. Kufarev studied the problem of evolution of a round oil spot surrounded by water when oil is extracted from a well inside the spot (Figure 1). It turned out that the boundary of the spot remains an algebraic curve of degree four in the course of evolution. This curve is the image of an ellipse under a reflection with respect to a circle. In 1950 Kufarev managed to generalize this property: if initially the oil spot is the image of the unit disk under a conformal map given by a rational function of the complex coordinate in the disk, then it retains this property in the course of evolution.

In 1972 S. Richardson found an infinite series of first integrals of motion of the spot. He proved that the integral of any harmonic function over the oil domain changes linearly in time. This allowed Richardson to give a new proof of the invariance of rationality and an effective method to construct explicit solutions.

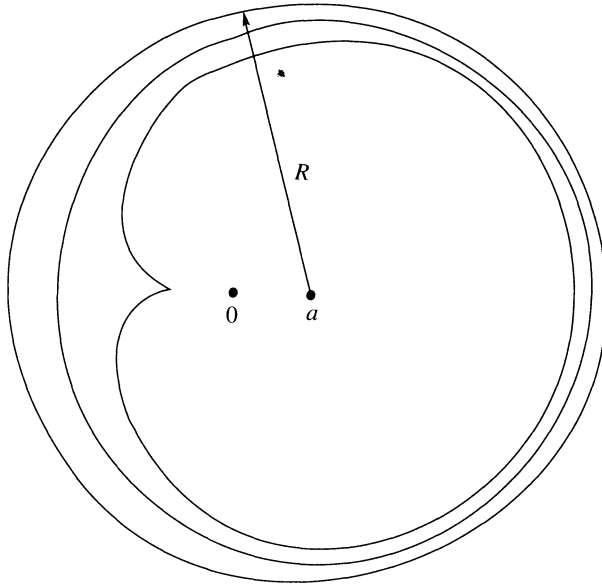


FIGURE 1

It was realized recently that this approach can be extended to multiply connected domains. In this case, the unit circle is replaced by a half of a complex algebraic curve with a specified M -structure, and the conformal map is given by a path integral of a meromorphic differential on the curve that has no zeros or poles on the chosen half.

Below we discuss these and other interesting mathematical subjects that arose recently in the theory of fluid flows with a moving boundary.

This text is an extended version of the first author's talk at a Moscow Mathematical Society meeting about the results of the second author. The authors gratefully acknowledge the crucial influence of Prof. V. M. Entov who introduced them to the circle of physical problems under consideration. The authors would also like to thank Prof. V. I. Arnold, D. Ya. Kleinbock, Prof. I. M. Krichever, and Dr. A. I. Shnirelman for very useful discussions, and Dr. Y. Peres for reading the English translation of the manuscript and making important remarks.

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