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J-holomorphic Curves and Quantum Cohomology

Dusa McDuff
Dietmar Salamon

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Preface

The theory of \( J \)-holomorphic curves has been of great importance to symplectic topologists ever since its inception in Gromov's paper [26] of 1985. Its applications include many key results in symplectic topology: see, for example, Gromov [26], McDuff [42, 45], Lalonde–McDuff [36], and the collection of articles in Audin–Lafontaine [5]. It was also one of the main inspirations for the creation of Floer homology [18, 19, 73], and recently has caught the attention of mathematical physicists through the theory of quantum cohomology: see Vafa [82] and Aspinwall–Morrison [2].

Because of this increased interest on the part of the wider mathematical community, it is a good time to write an expository account of the field, which explains the main technical steps in the theory. Although all the details are available in the literature in some form or other, they are rather scattered. Also, some improvements in exposition are now possible. Our account is not, of course, complete, but it is written with a fair amount of analytic detail, and should serve as a useful introduction to the subject. We develop the theory of the Gromov-Witten invariants as formulated by Ruan in [64] and give a detailed account of their applications to quantum cohomology. In particular, we give a new proof of Ruan–Tian’s theorem [67, 68] that the quantum cup-product is associative.

Many people have made useful comments which have added significantly to our understanding. In particular, we wish to thank Givental for explaining quantum cohomology, Ruan for several useful discussions and for pointing out to us the connection between associativity of quantum multiplication and the WDVV-equation, Taubes for his elegant contribution to Section 3.4, and especially Gang Liu for pointing out a significant gap in an earlier version of the gluing argument. We are also grateful to Lalonde for making helpful comments on a first draft of this manuscript. The first author wishes to acknowledge the hospitality of the University of California at Berkeley, and the grant GER-9350075 under the NSF Visiting Professorship for Women program which provided partial support during some of the work on this book.

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</tr>
<tr>
<td>$(a, b)$</td>
<td>108, 131</td>
</tr>
<tr>
<td>$(a \ast b)_A$</td>
<td>111</td>
</tr>
<tr>
<td>$\Psi_{A,B}(\alpha, \beta; \gamma, \delta)$</td>
<td>116</td>
</tr>
<tr>
<td>$\widehat{J}, \widehat{A}$</td>
<td>143</td>
</tr>
<tr>
<td>$W^{k,p}(\Omega), W_0^{k,p}(\Omega)$</td>
<td>181</td>
</tr>
</tbody>
</table>
J-Holomorphic Curves and Quantum Cohomology
Dusa McDuff and Dietmar Salamon

All in all it is rewarding to read this book, as many delicate points are first explained in easy-to-understand terms before the authors dive into the proofs ... this book will certainly remain a standard for background on quantum cohomology for many years to come.

—*Mathematical Reviews*

J-holomorphic curves revolutionized the study of symplectic geometry when Gromov first introduced them in 1985. Through quantum cohomology, these curves are now linked to many of the most exciting new ideas in mathematical physics. This book presents the first coherent and full account of the theory of J-holomorphic curves, the details of which are presently scattered in various research papers. The first half of the book is an expository account of the field, explaining the main technical aspects. McDuff and Salamon give complete proofs of Gromov’s compactness theorem for spheres and of the existence of the Gromov-Witten invariants. The second half of the book focuses on the definition of quantum cohomology. The authors establish that this multiplication exists, and give a new proof of the Ruan-Tian result that is associative on appropriate manifolds. They then describe the Givental-Kim calculation of the quantum cohomology of flag manifolds, leading to quantum Chern classes and Witten’s calculation for Grassmannians, which relates to the Verlinde algebra. The Dubrovin connection, Gromov-Witten potential on quantum cohomology, and curve counting formulas are also discussed. The book closes with an outline of connections to Floer theory.