University LECTURE Series

Volume 7

Lectures on Differential Galois Theory

Andy R. Magid

American Mathematical Society

http://dx.doi.org/10.1090/ulect/007

Titles in This Series

Volume

7 **Andy R. Magid**
Lectures on differential Galois theory
1994

6 **Dusa McDuff and Dietmar Salamon**
J-holomorphic curves and quantum cohomology
1994

5 **V. I. Arnold**
Topological invariants of plane curves and caustics
1994

4 **David M. Goldschmidt**
Group characters. symmetric functions. and the Hecke algebra
1993

3 **A. N. Varchenko and P. I. Etingof**
Why the boundary of a round drop becomes a curve of order four
1992

2 **Fritz John**
Nonlinear wave equations. formation of singularities
1990

1 **Michael H. Freedman and Feng Luo**
Selected applications of geometry to low-dimensional topology
1989

This page intentionally left blank

University
LECTURE
Series

Volume 7

Lectures on Differential Galois Theory

Andy R. Magid

American Mathematical Society
Providence, Rhode Island

1991 *Mathematics Subject Classification.* Primary 12H05

Library of Congress Cataloging-in-Publication Data

Magid, Andy R.
Lectures on differential Galois theory/Andy R. Magid.
p. cm. — (University lecture series, ISSN 1047-3998; 7)
Includes bibliographical references.
ISBN 0-8218-7004-1
1. Galois theory. 2. Differential equations. I. Title. II. Series: University lecture series (Providence, R.I.): 7.
QA174.2.M34 1994
512′.3—dc20 94-10431
CIP

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
This volume was typeset by the author using $\mathcal{A}\!\mathcal{M}\!\mathcal{S}$-TEX,
the American Mathematical Society's TEX macro system.

10 9 8 7 6 5 4 3 2 01 00 99 98 97

Contents

Preface

Differential Galois theory is the theory of solutions of differential equations over a differential base field, or rather the nature of the differential field extension generated by the solutions, in much the same way that ordinary Galois theory is the theory of field extensions generated by solutions of (one variable) polynomial equations, with the additional feature that the corresponding differential Galois groups (of automorphisms of the extension fixing the base and commuting with the derivation) are algebraic groups.

This book, despite its title, deals only with the differential Galois theory of linear homogeneous differential equations, whose differential Galois groups are algebraic matrix groups. This branch of the theory is known as the Picard–Vessiot theory, after its founders C. Picard (1856–1941) and E. Vessiot (1865–1952) whose fundamental papers on the subject appeared in 1883 and 1892, respectively. (These historical notes are due to Phyllis Cassidy.)

Thus differential Galois theory has a history dating back to the 19th century. It was subsequently advanced by J. F. Ritt at Columbia. However, it was only put on its present solid footing in the postwar work of Ellis Kolchin, which saw publication beginning with his 1948 *Annals* paper.

Kolchin's 1973 *Differential algebra and algebraic groups*, New York, Academic Press **[Ko2]**, comprehensively codifies his work, although his orginal papers from the 40s and 50s remain equally accessible. Readers who find Kolchin's idiosyncratic language difficult to dip into briefly will appreciate Irving Kaplansky's *Introduction to differential algebra*, 2nd edition, Hermann, Paris, 1976 **[K]**, which treats the topic marvelously concisely (although for some reason Kaplansky chooses to omit the construction of the Picard–Vessiot extensions, which play the same role in differential Galois theory that Galois extensions play in ordinary Galois theory, referring the reader to Kolchin's original papers instead). There are also many other accounts of differential Galois theory in the literature, for example by Deligne **[D]**, Fahim **[F]**, Levelt **[L]**, Katz **[Ka]**, Pommaret **[P]**, and Takeuchi **[T]** to name some recent examples.

Kolchin's book ends with a theorem that describes the structure of a differential Galois extension as a twisted form of the function field of the differential Galois group (remember this is an algebraic group so it has a function field) with scalars the base differential field. In 1986, Michael Singer **[S]** gave a proof of this theorem for the Picard–Vessiot case based on differential Galois theory, and hence understandable, say, to a reader who has mastered Kaplansky's book (Kolchin's proof, of a more widely applicable theorem, uses cohomology). On the other hand, if one knows the Kolchin theorem, then it should be easy to establish the fundamental correspondence theorem of differential Galois theory: if $E = F(G)$, then it is not very surprising that subfield extensions K between F and E correspond to subgroups of G. (Here we use

$F(G)$ for the function field of the algebraic group with coordinate ring $F[G]$.) Thus if one could establish Kolchin's theorem in the Picard–Vessiot case from first principles (meaning differential algebra without the fundamental theorem of differential Galois theory), then one could deduce differential Galois theory in the Picard–Vessiot case as a consequence.

I had the opportunity to follow this approach to differential Galois theory in the spring of 1992, and the present volume of lecture notes is the result. These lectures were first presented in a course at the Hebrew University in Jerusalem from February through June of 1992, and then reprised in a seminar at the University of Oklahoma from September through November of 1992. I am grateful to my students in Jerusalem, particularly Assaf Wool and Yaa'cov Kapelovich, and my colleagues in Oklahoma, particularly Richard Resco, Brent Gordon, Ed Cline, Murad Ozaydin, and Leonid Dickey, for their comments and suggestions on these lectures. I am also grateful to Michael Singer for some helpful suggestions and references.

The prerequisite for the course, and hence for following these notes, is a background in algebra, especially field theory and commutative algebra, in affine algebraic geometry, and in affine algebraic groups, such as typically acquired by graduate students who have had a years course in algebra and a semesters exposure to algebraic groups. For the latter, Borel **[B]** is a complete reference.

Ellis Kolchin, whom I had known since my days as a J. F. Ritt Instructor at Columbia in 1969–72, passed away in the fall of 1991. I deeply regret not having had the opportunity to discuss this project with him, and hope it is worthy of his memory, to which I dedicated the course.

Outline of approach

This approach to differential Galois theory via the structure theory of Picard–Vessiot extensions can be regarded as a sequence of six theorems, which we now present. (This outline does not correspond exactly to the chapter–by–chapter summary of topics which follows.)

We fix the following notation:

F is a differential field with derivation D_F, denoted D when there is no confusion. (A derivation is an additive map $D : F \to F$ such that $D(ab) = D(a)b + aD(b)$ for elements a, b of F, and a differential field is a pair (F, D_F) consisting of a field and a designated derivation.)

We also sometimes denote $D^i(y)$ by $y^{(i)}$.

$L = Y^{(\ell)} + a_{\ell-1}Y^{(\ell-1)} + \cdots + a_0 Y^{(0)}$, where $a_i \in F$, is a linear homogeneous differential operator over F.

$E \supseteq F$ is a differential extension field of F; that is, a differential field containing F such that D_E restricted to F is D_F.

For y in E, $L(y) = D^\ell y + a_{\ell-1}D^{\ell-1} + \cdots + a_0 y$ and the solutions of $L = 0$ in E are $\{y \in E | L(y) = 0\}$.

The first proposition shows that, unlike the situation with polynomial equations, no matter what F is and no matter what L is there is always a proper extension E of F generated by solutions of $L = 0$.

PROPOSITION. *Given L and F there exists a differential extension field $E \supseteq F$ in which $L = 0$ has ℓ solutions algebraically independent over F.*

PROOF. Let $R = F[y_{1,0}, \ldots, y_{\ell,\ell-1}]$, a polynomial ring over F in ℓ^2 indeterminates. Define a derivation on R by

$$D_R(y_{ij}) = y_{i,j+1} \qquad \text{for} \qquad j < \ell - 1$$

and

$$D_R(y_{i,\ell-1}) = -\sum_0^{\ell-1} a_k y_{i,k}.$$

R is a differential integral domain over F and $y_{i,0}$, $1 \leq i \leq \ell$, are algebraically independent solutions of $L = 0$ in R. We let $E = Q(R)$, the fraction field of R. □

Consider the following example:

EXAMPLE. Let $\mathbb{C}$ denote the complexes, let $F = \mathbb{C}(e^x)$, let $D = \frac{d}{dx}$, and let $L = Y^{(1)} - Y^{(0)}$. Then (omitting the subscripts) the proposition yields $E = \mathbb{C}(e^x)(y)$ with $D(y) = y$.

Notice what happened: F already contained solutions of $L = 0$, for example e^x, and the construction adjoined an additional independent solution, namely y. We also note that if $z = \frac{e^x}{y}$, then $D(z) = 0$.

Elements of a differential field are called *constants* if they have derivative zero. In the example, the adjunction of a superfluous solution added a new constant. As we will see, this is generic. For the record, we note that the set of constants of a differential field is a subfield, and that the set of solutions in the extension E of $L = 0$ is a vector space over the field of constants.

It turns out that the appropriate extensions to consider for differential Galois theory, the *Picard–Vessiot* extensions, are those with no new constants.

DEFINITION. $E \supseteq F$ is a Picard–Vessiot extension of F for L if
(1) E is generated over F as a differential field by the solutions of $L = 0$ in E.
(2) The constants of E are the constants of F.
(3) $L = 0$ has ℓ solutions in E linearly independent over constants.

REMARK. Suppose that $E \supseteq F$ is a Picard–Vessiot extension of F for L and that $E \supseteq K \supseteq F$ is an intermediate differential field. Then it follows that E is also a Picard–Vessiot extension of K for L also.

For convenience, we refer to an extension $E \supseteq F$ satisfying only part (2) of the definition as a "no new constants" extension.

For the rest of this outline, we make the following conventions:

Let C denote the field of constants of F.

Assume that F is characteristic zero and that C is algebraically closed.

The following result gives a condition for no new constants:

THEOREM 1. *Suppose that $R \supseteq F$ is a differential integral domain. If $Q(R)$, the fraction field of R, has a new constant, then R contains a non-zero prime differential ideal.*

Combining Theorem 1 with the proposition above then yields:

COROLLARY. *Let $P \subseteq F[y_{i,j}] = R$ be a maximal prime differential ideal, where R is as in the proposition. Then $E = Q(R/P) \supseteq F$ (where E is the fraction field of R/P) satisfies (1) and (2) of the definition of a Picard–Vessiot extension.*

PROOF. Since R is a Noetherian ring, maximal prime differential ideals exist. Then R/P has no non–zero prime differential ideals, hence by the theorem $Q(R/P)$ has no new constants. Since R is differentially generated over F by solutions of $L = 0$, so are R/P and E. □

To guarantee the third condition for Picard–Vessiot extensions, we need to use Wronskian determinants:

DEFINITION. Let $y_1, \ldots, y_s$ be elements of the differential field E. Then

$$w = w(y_1, \ldots, y_s) = \begin{vmatrix} y_1^{(0)} & y_2^{(0)} & y_3^{(0)} & \cdots & y_s^{(0)} \\ y_1^{(1)} & y_2^{(1)} & y_3^{(1)} & \cdots & y_s^{(1)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} & \cdots & y_s^{(2)} \\ \vdots & \vdots & \vdots & & \vdots \\ y_1^{(s-1)} & y_2^{(s-1)} & y_3^{(s-1)} & \cdots & y_s^{(s-1)} \end{vmatrix}.$$

Just like the classical Wronskian determinant used in analysis, the Wronskian determines linear independence:

PROPOSITION. *The elements $y_1, \ldots, y_s$ of the differential field E are linearly independent over the constants of E if and only if $w(y_1, \ldots, y_s) \neq 0$.*

As in analysis, this proposition implies that the dimension of the space of solutions of $L = 0$ is limited:

COROLLARY. *$L = 0$ has at most ℓ solutions in E linearly independent over constants.*

PROOF. Suppose there are $s = \ell + 1$ elements y_i, $1 \leq i \leq s$, with $L(y_i) = 0$. Then in the determinant $w(y_1, \ldots, y_s)$ the last row is a linear combination of the preceding ones, and hence the determinant is 0. But then by the proposition the y_i are linearly dependent over the constants. □

As a consequence, we now have a constructive proof of the existence of Picard–Vessiot extensions:

THEOREM 2. *Let $P \subseteq F[y_{i.j}][w(y_{1.0} \cdots y_{\ell.0})^{-1}] = R[w^{-1}] = S$ be a maximal prime differential ideal. Then $E = Q(S/P) \supseteq F$ (E is the fraction field of S/P) is a Picard–Vessiot extension of F for L.*

PROOF. By Theorem 1, E has no new constants. Let y_i denote the image of $y_{i.0}$ in E. Then $w(y_1, \ldots, y_\ell) \neq 0$ since it is the image of w, which is a unit in S and hence in S/P. This means that the y_i are linearly independent over C. Also, E is generated over F as a differential field by the y_i by construction. □

Next, we note the following normality property of Picard–Vessiot extensions:

NORMALITY. Suppose $E_i \supseteq F$, $i = 1, 2$, are Picard–Vessiot extensions of F for L and that $E \supseteq F$ is a no new constants extension. Suppose that $\sigma_i : E_i \to E$ are differential embeddings which are the identity on F. Let V_i be the solutions of $L = 0$ in E_i, and let V be the solutions in E. Then $\sigma_i(V_i) \subseteq V$ and a dimension count based on condition (3) for Picard–Vessiot extensions and the corollary preceding Theorem 2 establishes that $\sigma_i(V_i) = V$. It then follows from condition (1) for Picard–Vessiot extensions that $\sigma_1(E_1) = \sigma_2(E_2)$.

Using the normality of Picard–Vessiot extensions, we can prove their uniqueness:

THEOREM 3. *Any two Picard–Vessiot extensions of F for L are isomorphic over F.*

PROOF. We can assume that one of the extensions is constructed as in Theorem 2, namely $E_1 = Q(S/P)$; let E_2 be the other one. The differential ring $T = (S/P) \otimes_F E_2$ is a finitely generated F–algebra, hence Noetherian. Let M be a maximal prime differential ideal in T. Then the differential field $E = Q(T/M)$ has no new constants (corollary to Theorem 1) and there are differential injections $\sigma_i : E_i \to E$ induced by the tensor inclusions $s \mapsto s \otimes 1$ and $e \mapsto 1 \otimes e$. Then by the above remark on normality $\sigma = \sigma_2\sigma_1^{-1}$ is an isomorphism from E_1 to E_2. □

In fact, the same method of proof used in Theorem 3 also proves a strong assertion about the automorphisms of a Picard–Vessiot extension:

THEOREM 4. *Let E be a Picard–Vessiot extension of F for L, and let x be an element of E not in F. Then there is a differential automorphism σ of E over F with $\sigma(x) \neq x$.*

PROOF. In the proof of Theorem 3, assume $E_1 = E_2 = Q(S/P)$. One checks that $z = x \otimes 1 - 1 \otimes x$ is not nilpotent in T, and we localize T at the multiplicatively closed set generated by z, calling the result T also. Then proceed as in Theorem 3. By construction, z has non-zero image in $Q(T/M)$ from which it follows that $\sigma_1(x) \neq \sigma_2(x)$ and hence that $x \neq \sigma(x)$. □

The set of differential automorphisms of E over F forms a group, which is denoted $G(E/F)$. The next result is that this is an algebraic group over the field C of constants of F.

THEOREM 5. *Let E be a Picard–Vessiot extension of F for L. Then $G(E/F)$ is a linear algebraic group over C. In fact, if $E = Q(S/P)$ as in Theorem 2, and $V = \sum Cy_{i,0}$, then $GL(V)$ acts naturally on S and $G(E/F) = \{\sigma \in GL(V) | P^\sigma = P\}$.*

Of course one needs to verify that the right–hand side of the equation in Theorem 5 is Zariski closed.

Theorem 4 is really half the Galois correspondence theorem. The other half will be provided by the following theorem, which is the version of Kolchin's theorem for which we have been aiming:

THEOREM 6. *Let E be a Picard–Vessiot extension of F for L. Let $G \leq G(E/F)$ be a Zariski closed subgroup. Let T be the set of all f in E which satisfy a linear homogeneous differential equation over E^G. Then T is a finitely generated G–stable E^G–algebra with fraction field E, and if $\overline{F}$ denotes the algebraic closure of F, then there is a G–isomorphism $\overline{F} \otimes_F T \to \overline{F} \otimes_C C[G]$. (Here $C[G]$ is the coordinate ring of the algebraic group G.) In fact T consists of all the elements of E such that the C–span of their G–orbit is finite dimensional.*

PROOF. In the proof, we can assume that $E^G = F$ (see the remark after the definition of Picard–Vessiot extensions). By Theorem 3 we can assume that $E = Q(S/P)$ as in Theorem 2. Let $\mathcal{S}$ denote the set of G–semi–invariants in E. Then one shows that $T = S[\mathcal{S}]$; from this description one shows first that T has no G–stable ideals, then uses this to see that $\overline{F} \otimes_F T$ has no G–stable ideals, and then uses this fact to conclude the desired isomorphism with $\overline{F} \otimes_C C[G]$. □

Theorem 6 shows how the extension $E \supseteq E^G$ determines the group G (or rather its scalar extension to $\overline{F}$). As already noted, Theorem 4 shows how the group $G(E/F)$ determines the base field F from E. We combine these to get the fundamental theorem of differential Galois theory:

FUNDAMENTAL THEOREM OF DIFFERENTIAL GALOIS THEORY. *Let E be a Picard–Vessiot extension of F for L. Then there is a lattice inverting bijective correspondence between*

$$\{E \subseteq K \subseteq F | K \text{ is a differential subfield (with the same derivation)}\}$$

and

$$\{G \leq G(E/F) | G \text{ is a Zariski closed subgroup}\}$$

given by

$$K \mapsto G(E/K) \qquad \text{and} \qquad G \mapsto E^G.$$

Picard–Vessiot intermediate field extensions correspond to normal subgroups.

PROOF. As remarked above, E is also a Picard–Vessiot extension of any intermediate field K. Then one can deduce from Theorem 5 that $G(E/K)$ is Zariski closed in $G(E/F)$, and Theorem 4 implies that $E^{G(E/K)} = K$. If G is a Zariski closed subgroup of $G(E/F)$, then $G \leq G(E/E^G)$ and $E^G = E^{G(E/E^G)}$. Let K denote this latter field. In Theorem 6, the ring T depends only on K, and hence from the isomorphism of that theorem we conclude that $\overline{F} \otimes_C C[G]$ equals $\overline{F} \otimes_C C[G(E/E^G)]$, and thus that $G = G(E/E^G)$. □

In addition to providing a convenient path to the Fundamental Theorem, Theorem 6 also points the way to an approach to the inverse problem of differential Galois theory. First we state the inverse problem:

INVERSE PROBLEM. Given a differential field F and a linear algebraic group G over the constant field C of F, find a Picard–Vessiot extension E of F with $G(E/F) = G$.

We know from Theorem 6 that if the inverse problem is solved, then E is the fraction field of a domain T with $\overline{F} \otimes_F T = \overline{F} \otimes_C C[G]$ (we denote this latter $\overline{F}[G_{\overline{F}}]$). At least in the case that G is connected, so that $C[G]$ is an integral domain, this can be shown to be equivalent to finding an appropriate derivation on $\overline{F}[G_{\overline{F}}]$, namely one commuting with G and having no proper differential ideals. It of course suffices to find such a derivation on $F[G_F]$ itself.

We conclude this outline with an historical curiosity. In 1895, on the occasion of the 100th birthday of l'École Normale Supérieure, Lie gave an address about Galois, in which he talked about the developing Galois theory of differential equations. With a little creative anachronism, one can read into those comments of nearly a century ago the main point of view of the present work:

"De là par example est née une théorie générales d'intégration pour les systèmes d'équations différentielles dont la solution la plus générale s'exprime en function particulière par les formules qui définissent un groupe fini et continu; cette théorie a une analogie frappante avec celle de Galois."

Sophus Lie, *Influence de Galois sur le développement des mathématiques*, Gesam. Abh., Volume 6 (quoted by Uta Mertzbach in *Lie's Galois theory of differential equations* I: *Historical background* (preliminary report); Special Session on History of Mathematics, Amer. Math. Soc. meeting number 882, DeKalb, IL, May 22, 1993).

For the reader who is willing to project a lot into this brief assertion of Lie's, we may say that he is asserting that the space $V = L^{-1}(0)$ of solutions of the differential equation $L = 0$ is a homogeneous space for a finite (dimensional) Lie group G. While this is obviously false if taken literally, if we take Lie's remark about general solutions to mean that V contains a Zariski dense subset which is a principal homogeneous space for G, then the Picard–Vessiot extension E for L over the field F of rational functions will be isomorphic to $F(G)$. Then Lie's comment about the striking analogy of this theory with that of Galois could mean that the correspondence theory for the differential subfields of the extension $E = F(G) \supset F$ will be a consequence of this isomorphism, which is also the main idea of the present work.

Chapter-by-chapter summary

Chapter 1: Differential Ideals

Topics include general introduction to differential rings, differential polynomial algebra, characterization of ideals differentially generated by a linear homogeneous differential operator, and the fact that the quotient of the differential polynomial algebra by such an ideal is an (ordinary) polynomial ring. Theorem 1 is proven in this chapter.

Chapter 2: The Wronskian

Topics covered are the properties of the Wronskian. The proposition following Theorem 1 and its corollary are proven in this chapter.

Chapter 3: Picard–Vessiot Extensions

Topics covered are the definition of Picard–Vessiot extensions, their construction, and their uniqueness. Theorems 2, 3, and 4 are proven in this chapter.

Chapter 4: Automorphisms of Picard–Vessiot Extensions

Topics covered are the structure of the group of automorphisms of a Picard–Vessiot extension as an algebraic group. Theorem 5 is proven in this chapter.

Chapter 5: The Structure of Picard–Vessiot Extensions

Topics covered include the structure of a Picard–Vessiot extension as the quotient field of an affine domain. Theorem 6 is proven in this chapter.

Chapter 6: The Galois Correspondence and its Consequences

Topics covered include the fundamental theorem of differential Galois theory and some applications, including equations with solvable (connected component of their) Galois group and equations solvable by quadratures, and equations with Galois group SL_n. The corollary to Theorem 6 is proven in this chapter.

Chapter 7: The Inverse Galois Problem

Topics covered include the inverse problem and derivations of the coordinate ring of an algebraic group, and the constructive solution of the inverse problem for various groups, including solvable groups.

This page intentionally left blank

This page intentionally left blank

Bibliography

[AS] Abramowitz, M. and Stegun, I., *Handbook of Mathematical Functions*, Fifth Printing with Corrections, U.S. Department of Commerce, 1966.

[B] Borel, A., *Linear Algebraic Groups*, Notes by H. Bass, W. A. Benjamin, Inc., New York, 1969.

[BB] Bialynicki-Birula, A., *On the inverse problem of Galois theory of differential fields*, Bull. Amer. Math. Soc. **16** (1963), 960-964.

[Be] Bertrand, D., *Review of Lectures on Differential Galois Theory*, Bull. Amer. Math. Soc. **33** (1996), 289-293.

[D] Deligne, P., *Catégories tannakiennes*, The Grothendieck Festschrift Volume II (P. Cartier et. al., eds.), Birkhäuser, Boston, 1990.

[K] Kaplansky, I., *An Introduction to Differential Algebra*, Second Edition, Hermann, Paris, 1976.

[Ka] Katz, N., *Exponential Sums and Differential Equations*, Annals of Mathematics Studies Number 124, Princeton University Press, Princeton, 1990.

[Ko1] Kolchin, E., *Existence theorems connected with the Picard–Vessiot theory of homogeneous linear ordinary differential equations*, Bull. Amer. Math. Soc. **54** (1948), 927–932.

[Ko2] ———, *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.

[Kov] Kovacic, J., *The inverse problem in the Galois theory of differential fields*, Ann. of Math. **89** (1969), 583-608.

[L] Levelt, A., *Differential Galois theory and tensor products*, Indag. Mathem., N.S. **1** (1990), 439–450.

[LM1] Lubotzky, A. and Magid, A., *Cohomology of unipotent and prounipotent groups*, J. of Algebra **74** (1982), 76–95.

[LM2] ———, *Free prounipotent groups*, J. of Algebra **80** (1983), 323–349.

[M] Magid, A., *Equivariant completions of rings with reductive group actions*, J. Pure and Applied Algebra **49** (1987), 173–185.

[MS] Mitschi, C. and Singer, M., *Connected linear groups as differential Galois groups*, J. of Algebra **184** (1996), 333–361.

[PR] Platonov, V. and Rapinchuk, A., *Algebraic Groups and Number Theory*, Academic Press, New York, 1994.

[P] Pommaret, J., *Differential Galois Theory*, Gordon and Breach, New York, 1983.

[R] Rosenlicht, M., *Differential extension fields of exponential type*, Pacific J. Math. **57** (1975), 289–300.

[S] Singer, M., *Algebraic relations among solutions of linear differential equations*, Trans. Amer. Math. Soc. **295** (1986), 753–763.

[T] Takeuchi, M., *A Hopf algebraic approach to the Picard–Vessiot theory*, J. of Algebra **122** (1989), 481–509.

[TT] Tretkoff, C. and Tretkoff, M., *Solution of the inverse problem of differential Galois theory in the classical case*, Amer. J. Math. **101** (1979), 1327–1332.

Lectures on Differential Galois Theory
Andy R. Magid

Differential Galois theory studies solutions of differential equations over a differential base field. In much the same way that ordinary Galois theory is the theory of field extensions generated by solutions of (one variable) polynomial equations, differential Galois theory looks at the nature of the differential field extension generated by the solutions of differential equations. An additional feature is that the corresponding differential Galois groups (of automorphisms of the extension fixing the base and commuting with the derivation) are algebraic groups. This book deals with the differential Galois theory of linear homogeneous differential equations, whose differential Galois groups are algebraic matrix groups. In addition to providing a convenient path to Galois theory, this approach also leads to the constructive solution of the inverse problem of differential Galois theory for various classes of algebraic groups. Providing a self-contained development and many explicit examples, this book provides a unique approach to differential Galois theory and is suitable as a textbook at the advanced graduate level.

ISBN 0-8218-7004-1