University
LECTURE
Series

Volume 8

Gröbner Bases and Convex Polytopes

Bernd Sturmfels

American Mathematical Society

http://dx.doi.org/10.1090/ulect/008

Other Titles in This Series

Volume

8 **Bernd Sturmfels**
Gröbner bases and convex polytopes
1996

7 **Andy R. Magid**
Lectures on differential Galois theory
1994

6 **Dusa McDuff and Dietmar Salamon**
J-holomorphic curves and quantum cohomology
1994

5 **V. I. Arnold**
Topological invariants of plane curves and caustics
1994

4 **David M. Goldschmidt**
Group characters, symmetric functions, and the Hecke algebra
1993

3 **A. N. Varchenko and P. I. Etingof**
Why the boundary of a round drop becomes a curve of order four
1992

2 **Fritz John**
Nonlinear wave equations, formation of singularities
1990

1 **Michael H. Freedman and Feng Luo**
Selected applications of geometry to low-dimensional topology
1989

This page intentionally left blank

Gröbner Bases and Convex Polytopes

This page intentionally left blank

University

LECTURE

Series

Volume 8

Gröbner Bases and Convex Polytopes

Bernd Sturmfels

American Mathematical Society
Providence, Rhode Island

1991 *Mathematics Subject Classification.* Primary 13P10, 14M25;
Secondary 52B12, 90C10, 14Q99.

ABSTRACT. This book is about the interplay of computational commutative algebra and the theory of convex polytopes. A central theme is the study of toric ideals and their applications in integer programming. This book is aimed at graduate students in mathematics, computer science, and theoretical operations research.

Library of Congress Cataloging-in-Publication Data

Sturmfels, Bernd, 1962–
Gröbner bases and convex polytopes / Bernd Sturmfels.
p. cm. — (University lecture series, ISSN 1047-3998; v. 8)
Includes bibliographical references and index.
ISBN 0-8218-0487-1
1. Gröbner bases. 2. Convex polytopes. I. Title. II. Series: University lecture series (Providence, R.I.); 8.
QA251.3.S785 1995
512′.24—dc20 95-45780
CIP

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
♻ Printed on recycled paper.

10 9 8 7 6 5 4 3 2 01 00 99 98 97

Contents

This page intentionally left blank

Introduction

Gröbner bases theory provides the foundation for many algorithms in algebraic geometry and commutative algebra, with the Buchberger algorithm acting as the engine that drives the computations. Thanks to the text books by Adams-Loustaunau (1994), Becker-Weispfenning (1993), Cox-Little-O'Shea (1992) and Eisenbud (1995), Gröbner bases are now entering the standard algebra curriculum at many universities. In view of the ubiquity of scientific problems modeled by polynomial equations, this subject is of interest not only to mathematicians, but also to an increasing number of scientists and engineers.

The interdisciplinary nature of the study of Gröbner bases is reflected by the specific applications appearing in this book. These applications lie in the domains of integer programming and computational statistics. The mathematical tools to be presented are drawn from commutative algebra, combinatorics, and polyhedral geometry.

The main thread of this book centers around a special class of ideals in a polynomial ring, namely, the class of *toric ideals*. They are characterized as those prime ideals that are generated by monomial differences, or as the defining ideals of (not necessarily normal) toric varieties. Toric ideals are intimately related to recent advances in polyhedral geometry, which grew out of the theory of $\mathcal{A}$-hypergeometric functions due to Gel'fand, Kapranov and Zelevinsky (1994). A key concept is that of a *regular triangulation*. All regular triangulations of a fixed polytope are parametrized by the vertices of the *secondary polytope*.

Both the algebra and the combinatorics appearing in this book are presented as self-contained as possible. Most of the material is accessible to first-year graduate students in mathematics. The following prerequisites will be assumed throughout:

- working knowledge of the basic facts about Gröbner bases, specifically of Chapters 1–5 and 9 of (Cox-Little-O'Shea 1992), or Chapters 1–2 of (Adams-Loustaunau 1994),
- familiarity with the terminology of polyhedral geometry and linear programming, as introduced in (Schrijver 1986) or (Ziegler 1995).

The fourteen chapters are organized as follows. In the first two chapters we present some introductory Gröbner bases material, which cannot be found in the text books. Here we consider arbitrary ideals I in a polynomial ring $S = k[x_1, \ldots, x_n]$, not just toric ideals. It is proved that I admits a *universal Gröbner basis*, that is, a finite subset which is a Gröbner basis for I with respect to all term orders simultaneously. This leads to the concept of the *Gröbner fan* and the *state polytope* of I. The state polytope is a convex polytope in $\mathbf{R}^n$ whose vertices are in bijection with the distinct initial monomial ideals of I with respect to all term orders on S. In the special case where $I = \langle f \rangle$ is a principal ideal, the state polytope coincides with the familiar Newton polytope $New(f)$. In Chapter 3 we present results and algorithms which involve the variation of term orders for a fixed ideal. In particular, it is explained how to compute the state polytope.

Chapters 4–9 deal exclusively with toric ideals. Basic algebraic features of these ideals are collected in the fourth chapter. These include degree bounds for Gröbner bases. An explicit universal Gröbner basis, the so-called *Graver basis*, is constructed.

The fifth chapter relates toric ideals to three fundamental problems associated with a (non-negative integer) linear map $\pi : \mathbf{N}^n \to \mathbf{N}^d$. Each fiber $\pi^{-1}(\mathbf{b})$, $\mathbf{b} \in \mathbf{N}^d$, consists of the lattice points in a polyhedron in $\mathbf{R}^n$. The three problems are: *enumeration* (list all points in $\pi^{-1}(\mathbf{b})$), *sampling* (pick a point in $\pi^{-1}(\mathbf{b})$ at random) and *integer programming* (find a point in $\pi^{-1}(\mathbf{b})$ which minimizes a given linear functional). The toric ideal $I_{\mathcal{A}}$ which is used to model these questions is the ideal of algebraic relations on the monomials with exponent vectors the columns of a matrix $\mathcal{A}$ representing π.

Chapter 6 deals with the one-dimensional case $d = 1$. In this case the Graver basis elements of $I_{\mathcal{A}}$ correspond to *primitive partition identities*, the variety defined by $I_{\mathcal{A}}$ is a monomial curve, and the associated integer program is the *knapsack problem*. Sharp degree bounds for Gröbner bases are available in this case.

Returning to our general discussion in Chapter 7, we present a geometric characterization of the universal Gröbner basis $\mathcal{U}_{\mathcal{A}}$, and we discuss algorithms for computing $\mathcal{U}_{\mathcal{A}}$. In Chapter 8 we establish a correspondence between the initial ideals of $I_{\mathcal{A}}$ and the regular triangulations of $\mathcal{A}$. These triangulations are parametrized by the secondary polytope $\Sigma(A)$, which is a Minkowski summand of the state polytope of $I_{\mathcal{A}}$. In Chapter 9 we apply our general theory to a specific family of toric ideals, namely those defined by the *second hypersimplex*.

In the next two chapters we venture into the realm of commutative algebra beyond toric ideals. Chapter 10 deals with Arnold's notion of *$\mathcal{A}$-graded algebras*. These are algebras with the simplest possible Hilbert function, namely, $1, 1, 1, 1, 1, 1, \ldots$. Their defining ideals are a natural generalization of initial ideals of toric ideals. In Chapter 11 we discuss *canonical subalgebra* bases (or *SAGBI bases*, as they were called by Robbiano & Sweedler (1990)). These bases need not be finite. But if they are, then they admit a nice geometric interpretation as degeneration of a parametrically presented variety into a toric variety.

In Chapter 12 we present advanced techniques for computing with toric ideals, and for applying them to integer programming. Chapter 13 aims to span a bridge to the theory of toric varieties as it exists in algebraic geometry, and, finally, in Chapter 14 the reader finds a collection of toric ideals and Gröbner bases which are dear to the author's heart.

At the end of each chapter there is a list of exercises and bibliographic notes. The exercises vary in difficulty: some are straightforward applications of the material presented, while others are more difficult and may lead to research projects. Many of the exercises assume a certain level of enthusiasm for performing computer experiments. The bibliographic notes are kept very brief. Their main purpose is to help the reader in locating a sample of the original or background literature. They are not intended to give a historical account or a complete bibliography of the respective subject areas.

This monograph grew out of a series of ten lectures given at the Holiday Symposium at New Mexico State University, Las Cruces, December 27–31, 1994. The material was updated and expanded considerably after the Holiday Symposium. In particular, Chapters 3, 12, 13 and 14 were added. I am grateful to numerous participants who supplied comments and helped me in locating errors in previous versions. Serkan Hosten did a particularly great job during the last round of proof-reading. I wish to thank all my co-authors listed in the bibliography for inspiring collaborations. Many of the techniques and results presented in this book I learned from and with them.

This project was supported financially by the David and Lucile Packard Foundation and the National Science Foundation through an NYI Fellowship. Most of the writing was done during my 94/95 visit at the Courant Institute of New York University. I wish to thank my hosts at NYU and at New Mexico State University for their terrific hospitality.

Special thanks go to Hyungsook and Nina, for their support, love and cheerful energy.

Bernd Sturmfels
Berkeley, August 1995

This page intentionally left blank

Bibliography

Adams, W.W., Loustaunau, P. (1994). *An Introduction to Gröbner Bases,* American Mathematical Society, Graduate Studies in Math. Vol. III.

Arnold, V.I. (1989). A-graded algebras and continued fractions, *Communications in Pure and Applied Math,* **42**, 993–1000.

Aspinwall, P., Greene, B., Morrison, D. (1993). The monomial-divisor mirror map, *Internat. Math. Res. Notices* **12**, 319–337.

Avis, D., Fukuda, K. (1992). A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra, *Discrete Comput. Geom.* **8**, 295–313.

Batyrev, V. (1993). Quantum cohomology rings of toric manifolds, Journees de Geometrie Algebrique d'Orsay 1992, *Asterisque,* **218**, 9–34.

Bayer, D. (1982). *The Division Algorithm and the Hilbert Scheme*, Ph. D. Dissertation, Harvard University.

Bayer, D., Morrison, I. (1988). Gröbner bases and geometric invariant theory I, *J. Symbolic Computation* **6**, 209-217.

Bayer, D., Stillman, M. (1987a). A criterion for detecting m-regularity, *Inventiones Math.* **87**, 1–11.

Bayer, D., Stillman, M. (1987b). Macaulay: a computer algebra system for algebraic geometry, available by anonymous ftp from `zariski.harvard.edu`.

Becker, T., Kredel, H., Weispfenning, V. (1993). *Gröbner Bases: A Computational Approach to Commutative Algebra*, Graduate Texts in Mathematics, Springer, New York.

Billera, L.J., Filliman, P., Sturmfels, B. (1990). Constructions and complexity of secondary polytopes, *Advances in Mathematics* **83**, 155–179.

Björner, A., Las Vergnas, M., Sturmfels, B., White, N., Ziegler, G. (1993). *Oriented Matroids*, Cambridge University Press.

Bouvier, C., Gonzalez-Sprinberg, G. (1992). G-desingularisations de variétes toriques, *C.R. Acad. Sci. Paris, Ser. I* **315**, 817–820.

Bresinsky, H. (1988). Binomial generating sets for monomial curves, with applications in $\mathbf{A}^4$, *Rend. Sem. Mat. Univ. Politec. Torino* **46**, 353–370.

Bruns, W., Herzog, J. (1993). *Cohen-Macaulay Rings*, Cambridge University Press.

Buchberger, B. (1985). Gröbner bases – an algorithmic method in polynomial ideal theory, Chapter 6 in N.K. Bose (ed.) : *Multidimensional Systems Theory*, D. Reidel Publ.

Campillo, A., Marijuan, C. (1991). Higher relations for a numerical semigroup, *Sém. Théorie Nombres Bordeaux* **3**, 249–260.

Campillo, A., Pison, P. (1993). L'idéal d'un semi-group de type fini, *Comptes Rendues Acad. Sci. Paris*, Série I, **316**, 1303–1306.

Collart, S., Kalkbrener, M., Mall, D. (1996). Converting Bases with the Gröbner Walk, *Journal of Symbolic Computation*, to appear.

Conti, P., Traverso, C. (1991). Buchberger algorithm and integer programming, Proceedings AAECC-9 (New Orleans), Springer LNCS **539**, pp. 130-139.

Cox, D., Little, J., O'Shea, D. (1992). *Ideals, Varieties and Algorithms*, Undergraduate Texts in Mathematics, Springer, New York.

Dabrowski, R. (1996). On normality of the closure of a generic torus orbit in G/P, *Pacific J. Math.*, **172**, 321–330.

De Loera, J. (1995a). Non-regular triangulations of products of simplices, *Discrete and Computational Geometry*, to appear.

De Loera, J. (1995b). *Triangulations of Polytopes and Computational Algebra*, Ph. D. dissertation, Cornell University.

De Loera, J., Sturmfels, B., Thomas, R. (1995). Gröbner bases and triangulations of the second hypersimplex, *Combinatorica*, **15**, 409–424.

Diaconis, P., Sturmfels, B. (1993). Algebraic algorithms for sampling from conditional distributions, *Annals of Statistics*, to appear.

Diaconis, P., Graham, R., Sturmfels, B. (1995). Primitive partition identities, in: *Paul Erdös is 80. Vol II*, Janos Bolyai Society, Budapest, pp. 1–20.

Di Biase, F., Urbanke, R. (1995). An algorithm to compute the kernel of certain polynomial ring homomorphisms, *Experimental Mathematics*, **4**, 227–234.

Edelsbrunner, H. (1987). Algorithms in Combinatorial Geometry. Springer, New York.

Eisenbud, D. (1995). *Introduction to Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Mathematics, Springer, New York.

Eisenbud, D., Reeves, A., Totaro, B. (1994). Initial ideals, Veronese subrings, and rates of algebras. *Advances in Mathematics* **109**, 168–187.

Eisenbud, D., Sturmfels, B. (1996). Binomial ideals, *Duke Mathematical Journal*, **84**, 1–45.

Ewald, G., Schmeinck, A. (1993). Representation of the Hirzebruch-Kleinschmidt varieties by quadrics, *Beiträge zur Algebra und Geometrie*, **34**, 151–156.

Faugère, J.C., Gianni, P., Lazard, D., Mora, T. (1992). Efficient computation of zero-dimensional Gröbner bases by change of ordering. *J. Symbolic Computation*, **13**, 117–131.

Fulton, W. (1993). *Introduction to Toric Varieties*, Princeton University Press.

Gel'fand, I., Kapranov, M., Zelevinsky, A. (1994). *Discriminants, Resultants and Multi-Dimensional Determinants*, Birkhäuser, Boston.

Göbel, M. (1995). Computing bases for rings of permutation-invariant polynomials, *Journal of Symbolic Computation*, **19**, 285–291.

Graver, J. (1975). On the foundations of linear and integer programming I, *Mathematical Programming* **8**, 207–226.

Gritzmann, P., Sturmfels, B. (1993). Minkowski addition of polytopes: Computational complexity and applications to Gröbner bases, *SIAM J. Discrete Math.* **6**, 246–269.

Grünbaum, B. (1967). *Convex Polytopes*, Wiley Interscience, London.

Hartshorne, R. (1977). *Algebraic Geometry*, Springer Verlag, New York.

Herzog, J. (1970). Generators and relations of semigroups and semigroup rings, *Manuscripta Math.* **3**, 175–193.

Hosten, S., Sturmfels, B. (1995). GRIN: An implementation of Gröbner bases for integer programming, in "Integer Programming and Combinatorial Optimization" (E. Balas, J. Clausen, eds.), Springer Lecture Notes in Computer Science, **920**, pp. 267–276.

Kalkbrener, M., Sturmfels, B. (1995). Initial complexes of prime ideals, *Advances in Mathematics*, **116**, 365–376.

Kapranov, M., Sturmfels, B., Zelevinsky, A. (1992). Chow polytopes and general resultants, *Duke Mathematical Journal* **67**, 189–218.

Kapur, D., Madlener, K. (1989). A completion procedure for computing a canonical basis of a k-subalgebra. Proceedings of *Computers and Mathematics 89* (eds. Kaltofen and Watt), MIT, Cambridge, June 1989.

Kempf, G., Knudsen, F., Mumford, D., and Saint-Donat, B. (1973). *Toroidal Embeddings*, Springer Lecture Notes in Mathematics **339**.

Koelman, R. (1993a). Generators for the ideal of a projectively embedded toric surface, *Tôhoku Math. Journal* **45**, 385–392.

Koelman, R. (1993b). A criterion for the ideal of a projectively embedded toric surface to be generated by quadrics, *Beiträge zur Algebra und Geometrie*, **34**, 57–62.

Korkina, E., Post, G., Roelofs, M. (1995). Classification of generalized A-graded algebras with 3 generators, *Bulletin des Sciences Mathématiques*, **119**, 267–287.

Lee, C. (1991). Regular triangulations of convex polytopes, in *Applied Geometry and Discrete Mathematics - The Victor Klee Festschrift*, (P. Gritzmann and B. Sturmfels, eds.), American Math. Soc, DIMACS Series **4**, Providence, R.I., pp. 443–456.

Lovász, L., Plummer, M.D. (1986). *Matching Theory*, North-Holland, New York, (also in *Annals in Discrete Mathematics* **29**).

Mall, D. (1995). Combinatorics of Polynomial Ideals and Gröbner Bases, Habilitationsschrift, ETH Zürich.

Miller, J.L. (1996). Analogs of Gröbner bases in polynomial rings over a ring, *Journal of Symbolic Computation* **21**, 139–153.

Möller, H.M., Mora, T. (1984). Upper and lower bounds for the degree of Gröbner bases, EUROSAM '84, *Springer Lecture Notes in Computer Science* **174**, 172–183.

Mora, T., Robbiano, L. (1988). The Gröbner fan of an ideal, *J. Symbolic Computation* **6** 183-208.

Mount, J. (1995). *Application of Convex Sampling to Optimization and Contingency Table Generation/Counting*, Ph.D. Dissertation, Department of Computer Science, Carnegie Mellon University.

Oda, T. (1988). *Convex Bodies and Algebraic Geometry: an Introduction to the Theory of Toric Varieties.* Springer Verlag, New York.

Ollivier, F. (1991). Canonical bases: relations with standard bases, finiteness conditions, and applications to tame automorphisms, in *Effective Methods in Algebraic Geometry* (T. Mora, C. Traverso, eds). Birkhäuser, Boston, pp. 379–400.

Ostrowski, A. (1921). Über die Bedeutung der Theorie der konvexen Polyeder für die formale Algebra, *Jahresberichte Deutsche Math. Verein.* **30**, 98–99

Pottier, L. (1994). Gröbner bases of toric ideals, Rapport de recherche 2224, INRIA Sophia Antipolis, Manuscript available from
`http://www.inria.fr/safir/SAFIR/Loic.html`.

Reeves, A., Sturmfels, B. (1993). A note on polynomial reduction, *Journal of Symbolic Computation* **11**, 273–277.

Rippel, C. (1994). *Generic Initial Ideal Theory for Coordinate Rings of Flag Varieties*, Ph.D. Dissertation, UCLA.

Robbiano, L., Sweedler, M. (1990). Subalgebra bases, in *Springer Lecture Notes in Mathematics* **1430**, pp. 61-87.

Schrijver, A. (1986). *Theory of Linear and Integer Programming,* Wiley Interscience, Chichester 1986.

Schwartz, N. (1988). Stability of Gröbner bases, *J. Pure Appl. Algebra* **53**, 171–186.

Shafarevich, I.R. (1977). *Basic Algebraic Geometry*, Study Edition, Springer Verlag, Berlin.

Sinclair, A. (1993). *Algorithms for Random Generation and Counting: A Markov Chain Approach*, Birkhäuser, Boston, 1993.

Stanley, R. (1980). Decomposition of rational convex polytopes, *Annals of Discrete Math.* **6**, 333-342.

Stanley, R. (1986). *Enumerative Combinatorics.* Wadsworth & Brooks, Monterey, CA.

Stanley, R. (1987). Generalized H-vectors, intersection cohomology of toric varieties, and related results. in *Commutative Algebra and Combinatorics, Advanced Studies in Pure Math.* **11**, 187–213.

Sturmfels, B. (1991). Gröbner bases of toric varieties, *Tôhoku Math. Journal* **43**, 249-261.

Sturmfels, B. (1993a). *Algorithms in Invariant Theory*, Springer Verlag, Vienna.

Sturmfels, B. (1993b). Sparse elimination theory, in *Computational Algebraic Geometry and Commutative Algebra* [D. Eisenbud and L. Robbiano, eds.], Proceedings Cortona (June 1991), Cambridge University Press, pp. 264–298.

Sturmfels, B. (1995). On vector partition functions, *J. of Combinatorial Theory, Ser. A*, **72**, 302–309.

Sturmfels, B., Thomas, R. (1994). Variation of cost functions in integer programming, Technical Report, School of Operations Research, Cornell University.

Sturmfels, B., Weismantel, R., Ziegler, G. (1995). Gröbner bases of lattices, corner polyhedra, and integer programming, *Beiträge zur Algebra und Geometrie* , **36**, 281–298.

Tayur, S.R., and Thomas, R., and Natraj, N.R. (1995). An algebraic geometry algorithm for scheduling in the presence of setups and correlated demands, *Mathematical Programming*, **69**, 369–401.

Thomas, R. (1994). A geometric Buchberger algorithm for integer programming, *Mathematics of Operations Research*, to appear.

Thomas, R., Weismantel, R. (1995). Truncated Gröbner bases for integer programming, Preprint SC 95-09, ZIB Berlin.

Trung, N. V., Hoa, L. T. (1986). Affine semigroups and Cohen-Macaulay rings generated by monomials, *Transactions Amer. Math. Soc.*, **298**, 145–167.

Urbaniak, R., Weismantel, R., Ziegler, G. (1994). A variant of Buchberger's algorithm for integer programming. Preprint SC 94-29, ZIB Berlin.

Vlach, M. (1986). Conditions for the existence of solutions of the three-dimensional planar transportation problem, *Discrete Applied Mathematics*, **13**, 61–78.

Wagner, D.G. (1996). Singularities of toric varieties associated with finite distributive lattices, *Journal of Algebraic Combinatorics*, **5**, 149–165.

Weispfenning, V. (1987). Constructing universal Gröbner bases, in Proceedings AAEEC 5, Menorca 1987, *Springer Lecture Notes in Computer Science* **356**, pp. 408–417.

White, N. (1977). The basis monomial ring of a matroid, *Advances in Math.*, **24**, 292–279.

Ziegler, G. (1995). *Lectures on Polytopes*, Graduate Texts in Math., Springer, New York.

This page intentionally left blank

Index

This book is about the interplay of computational commutative algebra and the theory of convex polytopes. It centers around a special class of ideals in a polynomial ring: the class of toric ideals. They are characterized as those prime ideals that are generated by monomial differences or as the defining ideals of toric varieties (not necessarily normal).

The interdisciplinary nature of the study of Gröbner bases is reflected by the specific applications appearing in this book. These applications lie in the domains of integer programming and computational statistics. The mathematical tools presented in the volume are drawn from commutative algebra, combinatorics, and polyhedral geometry.

ISBN 0-8218-0487-1

9 780821 804872

ULECT/8