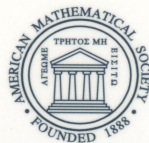


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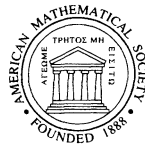
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ABSTRACT. This book is about the interplay of computational commutative algebra and the theory of convex polytopes. A central theme is the study of toric ideals and their applications in integer programming. This book is aimed at graduate students in mathematics, computer science, and theoretical operations research.

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Contents

Introduction	ix
Chapter 1. Gröbner Basics	1
Chapter 2. The State Polytope	9
Chapter 3. Variation of Term Orders	19
Chapter 4. Toric Ideals	31
Chapter 5. Enumeration, Sampling and Integer Programming	39
Chapter 6. Primitive Partition Identities	47
Chapter 7. Universal Gröbner Bases	55
Chapter 8. Regular Triangulations	63
Chapter 9. The Second Hypersimplex	75
Chapter 10. \mathcal{A} -graded Algebras	85
Chapter 11. Canonical Subalgebra Bases	99
Chapter 12. Generators, Betti Numbers and Localizations	113
Chapter 13. Toric Varieties in Algebraic Geometry	127
Chapter 14. Some Specific Gröbner Bases	141
Bibliography	155
Index	161

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Introduction

Gröbner bases theory provides the foundation for many algorithms in algebraic geometry and commutative algebra, with the Buchberger algorithm acting as the engine that drives the computations. Thanks to the text books by Adams-Loustaunau (1994), Becker-Weispfenning (1993), Cox-Little-O'Shea (1992) and Eisenbud (1995), Gröbner bases are now entering the standard algebra curriculum at many universities. In view of the ubiquity of scientific problems modeled by polynomial equations, this subject is of interest not only to mathematicians, but also to an increasing number of scientists and engineers.

The interdisciplinary nature of the study of Gröbner bases is reflected by the specific applications appearing in this book. These applications lie in the domains of integer programming and computational statistics. The mathematical tools to be presented are drawn from commutative algebra, combinatorics, and polyhedral geometry.

The main thread of this book centers around a special class of ideals in a polynomial ring, namely, the class of *toric ideals*. They are characterized as those prime ideals that are generated by monomial differences, or as the defining ideals of (not necessarily normal) toric varieties. Toric ideals are intimately related to recent advances in polyhedral geometry, which grew out of the theory of \mathcal{A} -hypergeometric functions due to Gel'fand, Kapranov and Zelevinsky (1994). A key concept is that of a *regular triangulation*. All regular triangulations of a fixed polytope are parametrized by the vertices of the *secondary polytope*.

Both the algebra and the combinatorics appearing in this book are presented as self-contained as possible. Most of the material is accessible to first-year graduate students in mathematics. The following prerequisites will be assumed throughout:

- working knowledge of the basic facts about Gröbner bases, specifically of Chapters 1–5 and 9 of (Cox-Little-O'Shea 1992), or Chapters 1–2 of (Adams-Loustaunau 1994),
- familiarity with the terminology of polyhedral geometry and linear programming, as introduced in (Schrijver 1986) or (Ziegler 1995).

The fourteen chapters are organized as follows. In the first two chapters we present some introductory Gröbner bases material, which cannot be found in the text books. Here we consider arbitrary ideals I in a polynomial ring $S = k[x_1, \dots, x_n]$, not just toric ideals. It is proved that I admits a *universal Gröbner basis*, that is, a finite subset which is a Gröbner basis for I with respect to all term orders simultaneously. This leads to the concept of the *Gröbner fan* and the *state polytope* of I . The state polytope is a convex polytope in \mathbf{R}^n whose vertices are in bijection with the distinct initial monomial ideals of I with respect to all term orders on S . In the special case where $I = \langle f \rangle$ is a principal ideal, the state

polytope coincides with the familiar Newton polytope $New(f)$. In Chapter 3 we present results and algorithms which involve the variation of term orders for a fixed ideal. In particular, it is explained how to compute the state polytope.

Chapters 4–9 deal exclusively with toric ideals. Basic algebraic features of these ideals are collected in the fourth chapter. These include degree bounds for Gröbner bases. An explicit universal Gröbner basis, the so-called *Graver basis*, is constructed.

The fifth chapter relates toric ideals to three fundamental problems associated with a (non-negative integer) linear map $\pi : \mathbf{N}^n \rightarrow \mathbf{N}^d$. Each fiber $\pi^{-1}(\mathbf{b})$, $\mathbf{b} \in \mathbf{N}^d$, consists of the lattice points in a polyhedron in \mathbf{R}^n . The three problems are: *enumeration* (list all points in $\pi^{-1}(\mathbf{b})$), *sampling* (pick a point in $\pi^{-1}(\mathbf{b})$ at random) and *integer programming* (find a point in $\pi^{-1}(\mathbf{b})$ which minimizes a given linear functional). The toric ideal $I_{\mathcal{A}}$ which is used to model these questions is the ideal of algebraic relations on the monomials with exponent vectors the columns of a matrix \mathcal{A} representing π .

Chapter 6 deals with the one-dimensional case $d = 1$. In this case the Graver basis elements of $I_{\mathcal{A}}$ correspond to *primitive partition identities*, the variety defined by $I_{\mathcal{A}}$ is a monomial curve, and the associated integer program is the *knapsack problem*. Sharp degree bounds for Gröbner bases are available in this case.

Returning to our general discussion in Chapter 7, we present a geometric characterization of the universal Gröbner basis $\mathcal{U}_{\mathcal{A}}$, and we discuss algorithms for computing $\mathcal{U}_{\mathcal{A}}$. In Chapter 8 we establish a correspondence between the initial ideals of $I_{\mathcal{A}}$ and the regular triangulations of \mathcal{A} . These triangulations are parametrized by the secondary polytope $\Sigma(\mathcal{A})$, which is a Minkowski summand of the state polytope of $I_{\mathcal{A}}$. In Chapter 9 we apply our general theory to a specific family of toric ideals, namely those defined by the *second hypersimplex*.

In the next two chapters we venture into the realm of commutative algebra beyond toric ideals. Chapter 10 deals with Arnold's notion of *\mathcal{A} -graded algebras*. These are algebras with the simplest possible Hilbert function, namely, $1, 1, 1, 1, 1, 1, \dots$. Their defining ideals are a natural generalization of initial ideals of toric ideals. In Chapter 11 we discuss *canonical subalgebra bases* (or *SAGBI bases*, as they were called by Robbiano & Sweedler (1990)). These bases need not be finite. But if they are, then they admit a nice geometric interpretation as degeneration of a parametrically presented variety into a toric variety.

In Chapter 12 we present advanced techniques for computing with toric ideals, and for applying them to integer programming. Chapter 13 aims to span a bridge to the theory of toric varieties as it exists in algebraic geometry, and, finally, in Chapter 14 the reader finds a collection of toric ideals and Gröbner bases which are dear to the author's heart.

At the end of each chapter there is a list of exercises and bibliographic notes. The exercises vary in difficulty: some are straightforward applications of the material presented, while others are more difficult and may lead to research projects. Many of the exercises assume a certain level of enthusiasm for performing computer experiments. The bibliographic notes are kept very brief. Their main purpose is to help the reader in locating a sample of the original or background literature. They are not intended to give a historical account or a complete bibliography of the respective subject areas.

This monograph grew out of a series of ten lectures given at the Holiday Symposium at New Mexico State University, Las Cruces, December 27–31, 1994. The material was updated and expanded considerably after the Holiday Symposium. In particular, Chapters 3, 12, 13 and 14 were added. I am grateful to numerous participants who supplied comments and helped me in locating errors in previous versions. Serkan Hosten did a particularly great job during the last round of proof-reading. I wish to thank all my co-authors listed in the bibliography for inspiring collaborations. Many of the techniques and results presented in this book I learned from and with them.

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Bernd Sturmfels
Berkeley, August 1995

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This page intentionally left blank

Index

- \mathcal{A} -graded algebra, 85
- Betti number, 120
- Birkhoff polytope, 145
- Buchberger algorithm, 1, 102
- canonical basis, 99
- circuit, 3, 33, 81
- Cohen-Macaulay, 135
- coherent, 85
- complex, 11
- conformal, 34
- contingency table, 40
- diagonal term order, 43
- Ehrhart polynomial, 36, 79, 133, 135
- face, 10
- fiber, 39
- Grassmann variety, 29, 103, 110, 136
- Graver basis, 35, 48, 56, 81, 150
- Graver degree, 59, 87
- Gröbner basis detection, 25
- Gröbner degree, 59, 89
- Gröbner fan, 13, 71
- Gröbner fiber, 60
- Gröbner region, 5
- Hilbert basis, 128
- Hilbert polynomial, 36, 79, 133
- hypersimplex, 75, 84
- ideal quotient, 113
- initial algebra, 99
- initial complex, 63, 102
- initial ideal, 1, 4, 85, 106
- integer programming, 39, 43, 88, 122
- knapsack problem, 52
- Lawrence lifting, 55, 149
- lexicographic (term order/
triangulation), 1, 67, 143
- lineality space, 19
- linear programming, 26, 65
- MACAULAY, 86, 89, 151
- MAPLE, 89
- marked polynomial, 26
- matroid polytope, 16
- minimal Gröbner basis, 1
- Minkowski addition/sum, 10, 61
- mono-AGA, 86, 94
- Newton polytope, 11, 19
- normal (toric variety), 127, 130, 134
- normal cone, 11
- normal fan, 11
- normalized volume, 36
- permutation matrix, 145
- primary decomposition, 122
- primitive, 33, 47, 87
- projectively normal, 132, 136
- polyhedral subdivision, 90
- polytope, 9
- random walk, 41
- reduced Gröbner basis, 1, 32, 76
- reduced lattice basis, 115
- regular (triangulation), 64
- reverse lexicographic (term order/
triangulation), 1, 113, 143, 147
- sampling, 39, 84
- secondary polytope, 71, 79
- Segre (variety), 37, 149
- standard (monomial), 1, 86, 88

state polytope, 14, 15, 19, 61, 71, 79, 102
syzygy, 107, 108, 120
tangent cone, 130
term order, 1, 5
thrackle, 79
three-dimensional matrix, 148
transportation problem, 40
triangulation, 63, 123
truncated Gröbner basis, 118
toric ideal, 31, 100
toric variety, 31, 36, 127, 129
unimodular, 69, 70, 93, 136
universal Gröbner basis, 2, 6, 15, 33, 57, 81
Veronese, 141

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