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Stephen Gelbart

American Mathematical Society

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Lectures on the Arthur-Selberg Trace Formula

Stephen Gelbart

American Mathematical Society
Providence, Rhode Island

1991 *Mathematics Subject Classification.* Primary 22E55, 22E50;
Secondary 11F70, 11F72.

ABSTRACT. The Arthur-Selberg trace formula is an equality between two kinds of traces: the geometric terms given by the conjugacy classes of a group, and the spectral terms given by the induced representations. In general, these terms require a truncation in order to converge which leads to an equality of truncated kernels. The formulas are difficult in general and even the case of $GL(2)$ is nontrivial. We give a proof of Arthur's trace formula of the 1970s and 1980s with special attention given to $GL(2)$. The problem is that when the truncated terms converge, they are also shown to be polynomial in the truncation variable and expressed as "weighted" orbital and "weighted" characters. In some important cases the trace formula takes on a simple form over G; we give some examples of this, and also some examples of Jacquet's relative trace formula. This monograph is directed to the advanced graduate student or professional mathematician in this area.

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Contents

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Preface

These are Notes prepared for nine lectures given at the Mathematical Sciences Research Institute, MSRI, Berkeley during the period January–March 1995. It is a pleasant duty to record here my gratitude to MSRI, and its staff, for making possible this 1994–95 Special Year in Automorphic Forms, and for providing such a setting for work. The TEX preparation of these Notes I owe to Wendy McKay, who patiently and professionally transformed my messy scrawl into something readable; her expertise, and good cheer under the pressure of weekly deadlines, is something I shall not soon forget.

The purpose of these Notes is to describe the contents of Arthur's earlier, foundational papers on the trace formula. In keeping with the introductory nature of the lectures, we have sometimes illustrated the ideas of Arthur's general theory by applying them in detail to the case of GL(2); we have also included a few lectures on the "simple trace formula" (and its applications), and on Jacquet's relative trace formula.

I wish to thank the auditors of these lectures for their interest, and , J. Bernstein, D. Goldberg, E. Lapid, C. Rader, S. Rallis, A. Reznikov, and D. Soudry, for their helpful suggestions and explanations. Finally, I wish to thank J. Arthur, H. Jacquet, and J. Rogawski for many tutorials on these and related topics over the past year; I hope they do not mind seeing some of their comments reappear in these Notes. Some quieter thoughts owe themselves to a Minerva Grant.

Stephen Gelbart
MSRI, Spring 1995

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References

[A1] Arthur, J., *The Selberg trace formula for groups of F-rank one*, Ann. of Math. **100** (1974), 236–385.

[A2] ______, *A trace formula for reductive groups,* I: *Terms associated to classes in* $G(\mathbb{Q})$, Duke Math. J. **45** (1978), no. 4, 911–952.

[A3] ______, *A trace formula for reductive groups,* II: *Applications of a truncation operator*, Compositio Math., vol. 40, Kluwer Acad. Publ., Dordrecht, 1980, pp. 87–121.

[A4] ______, *A trace formula in invariant form*, Ann. of Math. **113** (1981), 1–74.

[A5] ______, *On a family of distributions obtained from Eisenstein series,* I: *Application of the Paley–Weiner theorem*, Amer. J. Math. **104** (1982), no. 6, 1243–1288.

[A6] ______, *On a family of distributions obtained from Eisenstein series,* II: *Explicit formulas*, Amer. J. Math. **104** (1982), no. 6, 1289–1336.

[A7] ______, *The trace formula for reductive groups*, Lectures for Journées Automorphes, Dijon, 1981.

[A8] ______, *The characters of discrete series as orbital integrals*, Invent. Math. **32** (1976), 205–261.

[A9] ______, *On the inner product of truncated Eisenstein series*, Duke. Math. J. **49** (1982), no. 1, 35–70.

[A10] ______, *A Paley–Wiener Theorem for real reductive groups*, Acta Math. **150** (1983), 1–89.

[A11] ______, *The invariant trace formula,* I: *Local theory*, J. Amer. Math. Soc. **1** (1988), no. 2, 323–383.

[A12] ______, *The invariant trace formula,* II: *Global theory*, J. Amer. Math. Soc. **1** (1988), no. 3, 501–554.

[A13] ______, *A measure on the unipotent orbit*, Canad. J. Math. **37** (1985), no. 6, 1237–1274.

[A14] ______, *On a family of distributions obtained from orbits*, Canad. J. Math. **38** (1986), no. 1, 179–214.

[A15] ______, *On some problems suggested by the trace formula*, Lie Group Representations II, Lecture Notes in Mathematics, vol. 1041, Springer-Verlag, 1984, pp. 1–49.

[A16] ______, *Unipotent automorphic representations: global motivations*, Automorphic Forms, Shimura Varieties and *L*-functions, vol. 1, Academic Press, 1990, pp. 1–75.

[AC] Arthur, J. and Clozel, L., *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, Ann. of Math. Stud. No. 83, Princeton University Press, Princeton, NJ, 1975.

[BDK] Bernstein, J., Deligne, P., and Kazhdan, D, *Trace Paley–Weiner theorem for reductive p-adic groups*, J. Anal. Math. **47** (1986), 180–192.

[BDKV] Bernstein, J., Deligne, P., Kazhdan, D, and Vigneras, M.-F., *Representations des groupes sur un corps local*, Hermann, Paris, 1984.

[Clo] Clozel, L., *Invariant harmonic analysis on the Schwartz space of a reductive p-adic group*, in Harmonic Analysis on Reductive Groups, edited by W. Barker and P. Sally, Progress in Mathematics, vol. 101, Birkhäuser, Boston, 1991, pp. 101–121.

[CLL] Clozel, L., Labesse, J.-P., and Langlands, R. P., *Morning Seminar on the Trace Formula,*, Lecture Notes, Institute for Advanced Study, Princeton (1984).

[F] Y. Flicker, *Trace Formula and Base Change for* GL(3), Lecture Notes in Mathematics, vol. 927, Springer-Verlag, New York, 1982.

[Ge] Gelbart, S, *Automorphic Forms on Adele Groups*, Ann. of Math. Stud. No. 83, Princeton University Press, Princeton, NJ, 1975.

[GJ] Gelbart, S. and Jacquet, H., *Forms of* GL(2) *from the analytic point of view*, Proc. Sympos. Pure Math. (Corvallis), vol. 33 Part I, Amer. Math. Soc., Providence, RI, 1979, pp. 213–251.

[GGPS] Gelfand, I. M., Graev, M., and Piatetski-Shapiro, I., *Automorphic Functions and Representation Theory* (1969), Saunders.

[GPS] Gelfand, I. M., and Piatetski-Shapiro, I., *Automorphic functions and representation theory*, Trudy Moskov. Mat. Obš č. **12** (1963), 389–412; Trans. Moscow Math. Soc. **12** (1963), 438–464.

[Go] Godement, R., *Domaines fondamentaux des groupes arithmetiques*, Seminaire Bourbaki 257, W. A. Benjamin, Inc., New York, 1963.

[Go2] ———, *The Spectral Decomposition of Cusp Forms*, Proc. Sympos. Pure Math., vol. IX, American Mathematical Society, Providence, RI, 1966, pp. 225–234.

[Ja1] Jacquet, H., *The continuous spectrum of the relative trace formula for* GL(3) *over a quadratic extension*, Israel Journal of Math. **89**, 1–59.

[Ja2] ———, *Relative Kloosterman integrals for* GL(3), II, Canad. J. Math. **44** (1992), 1220–1240.

[JLai] Jacquet, H., and Lai, J., *A relative trace formula*, Comp. Math. **54** (1985), 243–310.

[JLR] Jacquet, H., Lai, K., and Rallis, S., *A trace formula for symmetric spaces*, Duke Math. J. **70** (1993), no. 2, 305–372.

[JL] Jacquet, H. and Langlands, R. P., *Automorphic Forms on* GL(2), Lecture Notes in Mathematics, vol. 114, Springer-Verlag, New York, 1970.

[JaYe] Jacquet, H., and Ye, Y., *Une remarque sur le changement de base quadratic*, C.R.A.S. Paris **311** (1990), 671–676.

[Kot] Kottwitz, R., *Tamagawa Numbers*, Ann. of Math. **127** (1988), 629–646.

[Kot2] Kottwitz, R., *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), 365–399; Duke Math. J. **51** (1984), 611–650.

[Lai] Lai, K. F., *Tamagawa Numbers of Reductive Algebraic Groups*, Compositio Math., vol. 41, Kluwer Acad. Publ., Dordrecht, 1980, pp. 153–188.

[Lab1] Labesse, J.-P., *La formules des traces D'Arthur-Selberg*, Seminaire Bourbaki, exposé 636, November 1984.

[Lab2] ———, *The present state of the trace formula*, Automorphic Forms, Shimura Varieties, and L-functions (L. Clozel and J. Milne, eds.), Perspectives in Math., vol. 10, Academic Press, Princeton, NJ, 1990, pp. 211–226.

[Lab3] ———, *Non-invariant base change identitites*, Preprint (Fall 1994).

[La1] Langlands, R. P., *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, vol. 544, Springer-Verlag, New York, 1976.

[La2] ———, *Base Change for* GL(2), Ann. of Math. Stud., vol. 96, Princeton University Press, Princeton, NJ, 1980.

[La3] ———, *The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups*, Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, RI, 1966, pp. 143–148.

[La4] ———, *Eisenstein series, the trace formula, and the modern theory of automorphic forms*, Number Theory, Trace Formula and Discrete Groups (Selberg Volume), Academic Press, 1989.

[La5] ———, *Les débuts d'une formule des traces stable*, Publ. Math. de L'Univ. de Paris VII, vol. 13, 1983.

[Mao] Mao, Z., *Relative Kloosterman integrals for* GL(3), III, Canad. J. Math. **45** (1993), no. 6, 1211–1230.

[MW] Moeglin, C., and Waldspurger, J.-L., *Décomposition Spectrale et Séries d'Eisenstein*, Progress in Math. Series, vol. 113, Birkhäuser Verlag, 1994.

[Ro1] Rogawski, J., *Representations of* GL(n) *and Division Algebras on a p-adic Field*, Duke. Math. J. **50** (1983), no. 1, 161–196.

[Ro2] ———, *Automorphic Representations of Unitary Groups in Three Variables*, Ann. of Math. Stud., vol. 123, Princeton University Press, Princeton, NJ.

[Ro3] ——— Rogawski, J., *The trace Paley–Wiener Theorem in the twisted case*, Trans. Amer. Math. Soc. **309** (1988), no. 1, 215–229.

[Se] Selberg, A., *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces, with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87; see also, Proc. Internat. Cong. Math. (1962), 177–189.

[Sha] Shokranian, S., *The Selberg-Arthur Trace Formula*, Lecture Notes in Mathematics, vol. 1503, Springer-Verlag, New York, 1992.

[Sha1] Shahidi, F., *Fourier transforms of intertwining operators and Plancherel measures for* GL(n), Amer. J. Math. **106** (1984), 67–111.

[Sha2] ———, *Local coefficients and normalization of intertwining operators for* GL(n), Compositio Math. **39** (1983), 271–295.

[Ye] Ye, Y., *Kloosterman integrals and base change for* GL(2), J. Reine Angew. Math. **400** (1989), 57–121.

Lectures on the Arthur-Selberg Trace Formula

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This work offers for the first time a simultaneous treatment of a general group with the case of $GL(2)$. It also treats the trace formula with the example of Jacquet's relative formula.

Features:

- Discusses why the terms of the geometric and spectral type must be truncated, and why the resulting truncations are polynomials in the truncation of value T.
- Brings into play the significant tool of (G, M) families and how the theory of Paley-Weiner is applied.
- Explains why the truncation formula reduces to a simple formula involving only the elliptic terms on the geometric sides with the representations appearing cuspidally on the spectral side (applies to Tamagawa numbers).
- Outlines Jacquet's trace formula and shows how it works for $GL(2)$.

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