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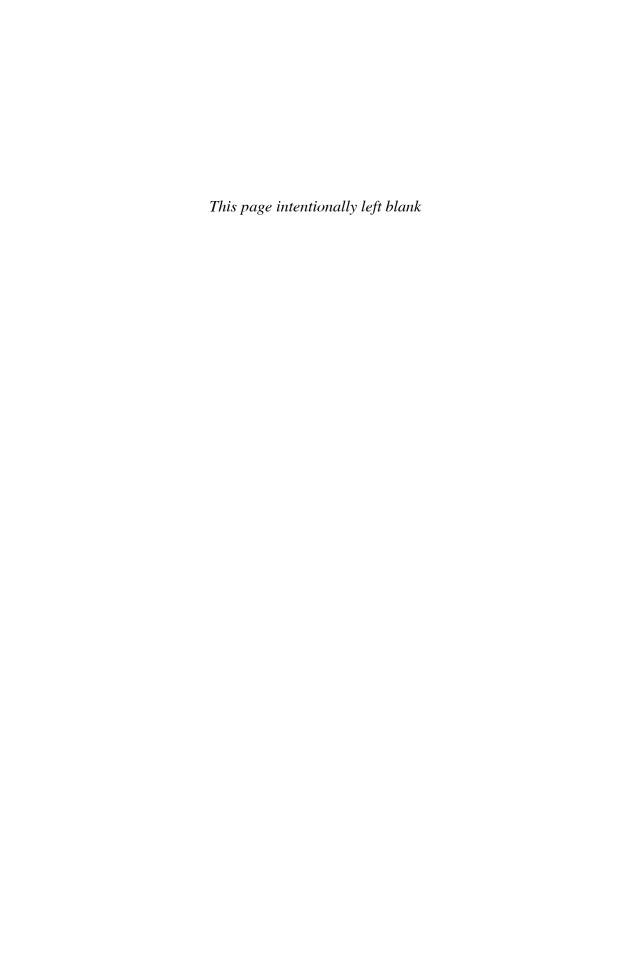
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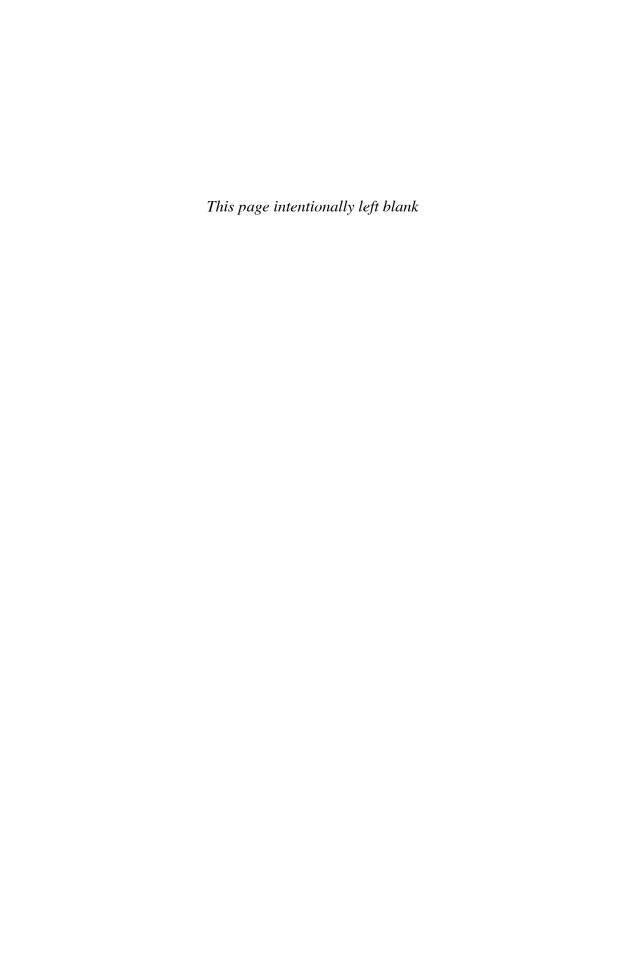
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Stephen Gelbart



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ABSTRACT. The Arthur-Selberg trace formula is an equality between two kinds of traces: the geometric terms given by the conjugacy classes of a group, and the spectral terms given by the induced representations. In general, these terms require a truncation in order to converge which leads to an equality of truncated kernels. The formulas are difficult in general and even the case of GL(2) is nontrivial. We give a proof of Arthur's trace formula of the 1970s and 1980s with special attention given to GL(2). The problem is that when the truncated terms converge, they are also shown to be polynomial in the truncation variable and expressed as "weighted" orbital and "weighted" characters. In some important cases the trace formula takes on a simple form over G; we give some examples of this, and also some examples of Jacquet's relative trace formula. This monograph is directed to the advanced graduate student or professional mathematician in this area.

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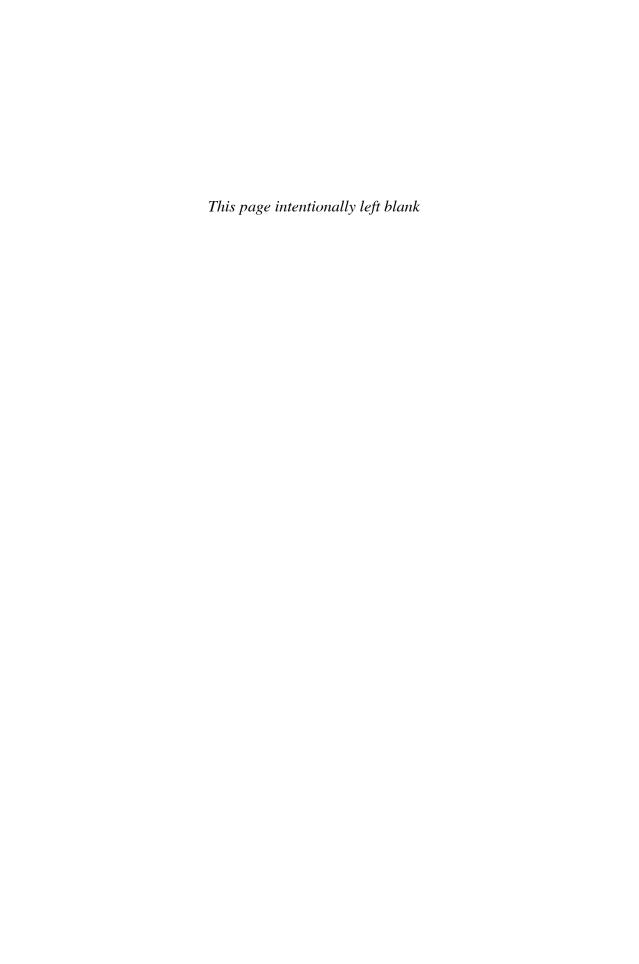
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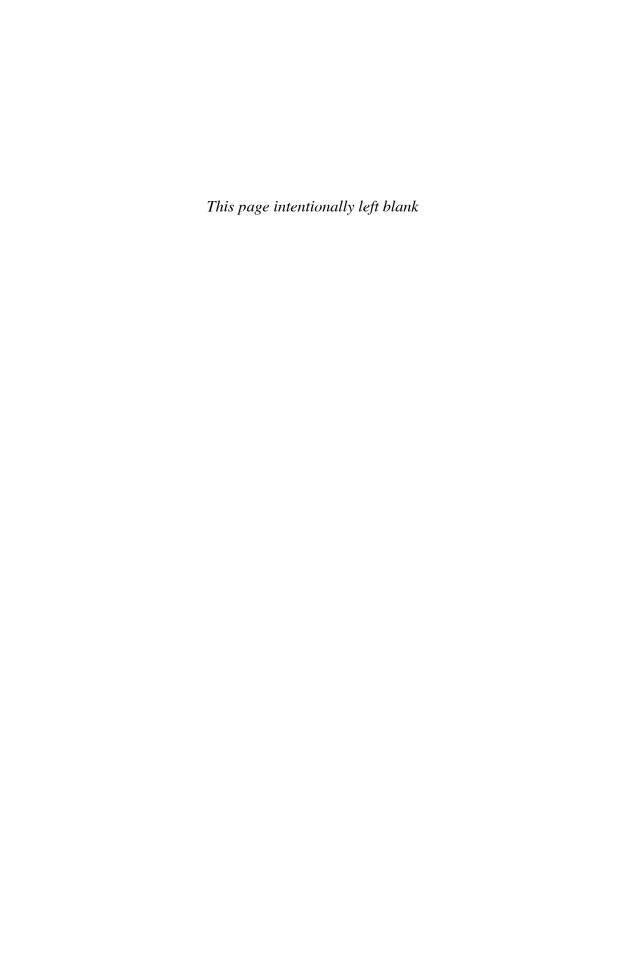
Preface

These are Notes prepared for nine lectures given at the Mathematical Sciences Research Institute, MSRI, Berkeley during the period January–March 1995. It is a pleasant duty to record here my gratitude to MSRI, and its staff, for making possible this 1994–95 Special Year in Automorphic Forms, and for providing such a setting for work. The TEX preparation of these Notes I owe to Wendy McKay, who patiently and professionally transformed my messy scrawl into something readable; her expertise, and good cheer under the pressure of weekly deadlines, is something I shall not soon forget.

The purpose of these Notes is to describe the contents of Arthur's earlier, foundational papers on the trace formula. In keeping with the introductory nature of the lectures, we have sometimes illustrated the ideas of Arthur's general theory by applying them in detail to the case of GL(2); we have also included a few lectures on the "simple trace formula" (and its applications), and on Jacquet's relative trace formula.

I wish to thank the auditors of these lectures for their interest, and , J. Bernstein, D. Goldberg, E. Lapid, C. Rader, S. Rallis, A. Reznikov, and D. Soudry, for their helpful suggestions and explanations. Finally, I wish to thank J. Arthur, H. Jacquet, and J. Rogawski for many tutorials on these and related topics over the past year; I hope they do not mind seeing some of their comments reappear in these Notes. Some quieter thoughts owe themselves to a Minerva Grant.

Stephen Gelbart MSRI, Spring 1995



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Lectures on the Arthur-Selberg Trace Formula

Stephen Gelbart

The Arthur-Selberg trace formula is an equality between two kinds of traces: the geometric terms given by the conjugacy classes of a group and the spectral terms given by the induced representations. In general, these terms require a truncation in order to converge, which leads to an equality of truncated kernels. The formulas are difficult in general and even the case of GL(2) is nontrivial. The book gives proof of Arthur's trace formula of the 1970s and 1980s, with special attention given to GL(2). The problem is that when the truncated terms converge, they are also shown to be polynomial in the truncation variable and expressed as "weighted" orbital and "weighted" characters. In some important cases the trace formula takes on a simple form over G. The author gives some examples of this, and also some examples of Jacquet's relative trace formula.

This work offers for the first time a simultaneous treatment of a general group with the case of GL(2). It also treats the trace formula with the example of Jacquet's relative formula.

Features:

- Discusses why the terms of the geometric and spectral type must be truncated, and why the resulting truncations are polynomials in the truncation of value *T*.
- Brings into play the significant tool of (G, M) families and how the theory of Paley-Weiner is applied.
- Explains why the truncation formula reduces to a simple formula involving only the elliptic terms on the geometric sides with the representations appearing cuspidally on the spectral side (applies to Tamagawa numbers).
- Outlines Jacquet's trace formula and shows how it works for GL(2).



