University LECTURE Series

Volume 10

Vertex Algebras for Beginners

Second Edition

Victor Kac

American Mathematical Society

http://dx.doi.org/10.1090/ulect/010

Selected Titles in This Series

13 **Yakov M. Eliashberg and William P. Thurston,** Confoliations, 1998
12 **I. G. Macdonald,** Symmetric functions and orthogonal polynomials, 1998
11 **Lars Gårding,** Some points of analysis and their history, 1997
10 **Victor Kac,** Vertex algebras for beginners, Second Edition, 1998
9 **Stephen Gelbart,** Lectures on the Arthur-Selberg trace formula, 1996
8 **Bernd Sturmfels,** Gröbner bases and convex polytopes, 1996
7 **Andy R. Magid,** Lectures on differential Galois theory, 1994
6 **Dusa McDuff and Dietmar Salamon,** J-holomorphic curves and quantum cohomology, 1994
5 **V. I. Arnold,** Topological invariants of plane curves and caustics, 1994
4 **David M. Goldschmidt,** Group characters, symmetric functions, and the Hecke algebra, 1993
3 **A. N. Varchenko and P. I. Etingof,** Why the boundary of a round drop becomes a curve of order four, 1992
2 **Fritz John,** Nonlinear wave equations, formation of singularities, 1990
1 **Michael H. Freedman and Feng Luo,** Selected applications of geometry to low-dimensional topology, 1989

This page intentionally left blank

University
LECTURE
Series

Volume 10

Vertex Algebras for Beginners

Second Edition

Victor Kac

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 17B69;
Secondary 17B65, 81T05, 81T40.

ABSTRACT. This book is an introduction to vertex algebras, a new mathematical structure that has appeared recently in quantum physics. It can be used by researchers and graduate students working on representation theory and mathematical physics.

Library of Congress Cataloging-in-Publication Data

Kac, Victor G., 1943–
Vertex algebras for beginners / Victor Kac. — 2nd ed.
p. cm. — (University lecture series, ISSN 1047-3998 ; v. 10)
Includes bibliographical references and index.
ISBN 0-8218-1396-X (alk. paper)
1. Vertex operator algebras. 2. Quantum field theory. 3. Mathematical physics. I. Title. II. Series: University lecture series (Providence, R.I.) ; 10.
QC174.52.O6K33 1998
512′.55—dc21 98-41276
CIP

First edition published 1996
Second edition published 1998
Second edition reprinted with corrections 2001
Printed in the United States of America.

♾ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at URL: http://www.ams.org/

10 9 8 7 6 5 4 3 2 06 05 04 03 02 01

Contents

Preface 1

Preface to the second edition 3

Chapter 1. Wightman axioms and vertex algebras 5

1.1. Wightman axioms of a QFT 5

1.2. $d = 2$ QFT and chiral algebras 8

1.3. Definition of a vertex algebra 13

1.4. Holomorphic vertex algebras 15

Chapter 2. Calculus of formal distributions 17

2.1. Formal delta-function 17

2.2. An expansion of a formal distribution $a(z, w)$ and formal Fourier transform 19

2.3. Locality of two formal distributions 24

2.4. Taylor's formula 29

2.5. Current algebras 31

2.6. Conformal weight and the Virasoro algebra 34

2.7. Formal distribution Lie superalgebras and conformal superalgebras 39

2.8. Conformal modules and modules over conformal superalgebras 50

2.9. Representation theory of finite conformal algebras 56

2.10. Associative conformal algebras and the general conformal algebra 61

2.11. Cohomology of conformal algebras 67

Chapter 3. Local fields 81

3.1. Normally ordered product 81

3.2. Dong's lemma 84

3.3. Wick's theorem and a "non-commutative" generalization 87

3.4. Bounded and field representations of formal distribution Lie superalgebras 91
3.5. Free (super)bosons 93
3.6. Free (super)fermions 98

Chapter 4. Structure theory of vertex algebras 103
4.1. Consequences of translation covariance and vacuum axioms 103
4.2. Skewsymmetry 105
4.3. Subalgebras, ideals, and tensor products 106
4.4. Uniqueness theorem 108
4.5. Existence theorem 110
4.6. Borcherds OPE formula 111
4.7. Vertex algebras associated to formal distribution Lie superalgebras 113
4.8. Borcherds identity 116
4.9. Graded and Möbius conformal vertex algebras 119
4.10. Conformal vertex algebras 125
4.11. Field algebras 129

Chapter 5. Examples of vertex algebras and their applications 133
5.1. Charged free fermions and triple product identity 133
5.2. Boson-fermion correspondence and KP hierarchy 137
5.3. $\widehat{gl}_\infty$ and $W_{1+\infty}$ 143
5.4. Lattice vertex algebras 148
5.5. Simple lattice vertex algebras 152
5.6. Root lattice vertex algebras and affine vertex algebras 158
5.7. Conformal structure for affine vertex algebras 161
5.8. Super boson-fermion correspondence and sums of squares 168
5.9. Superconformal vertex algebras 178
5.10. On classification of conformal superalgebras 185

Bibliography 193

Index 199

Bibliography

[AM] D. Adamovich and A. Milas. Vertex operator algebras associated to modular inveriant representations for $A_1^{(1)}$. *Math. Res. Lett.*, **2** (1995), 563–575.

[A] M. Ademollo, L. Brink, A. D'Adde, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Guidiche, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Petturino. Supersymmetric strings and colour confinement. *Phys. Lett.*, **B62** (1976), 105–110.

[BBSS] F. A. Bais, P. Bouwknegt, K.Schoutens, M.Surridge. Extensions of the Virasoro algebra constructed from Kac-Moody algebras using higher order Casimir invariants. *Nucl. Phys.*, **B304** (1988), 348–370.

[BKV] B. Bakalov, V. G. Kac and A. A. Voronov. Cohomology of conformal algebras, preprint

[BD] A. A. Beilinson and V. G. Drinfeld. Chiral algebras, preprint

[Be] M. J. Bergvelt, A note on super Fock space. *J. Math. Phys.*, **30** (1983), 812–815.

[BPZ] A. Belavin, A. Polyakov, and A. Zamolodchikov. Infinite conformal symmetries in two-dimensional quantum field theory. *Nucl. Phys.*, **B241** (1984), 333–380.

[B1] R. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Natl. Acad. Sci. USA*, **83** (1986), 3068–3071.

[B2] R. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.*, **109** (1992), 405–444.

[B3] R. Borcherds. Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products. *Invent. Math.*, **120** (1995), 161–213.

[B4] R. Borcherds. Vertex algebras, preprint

[Bo] P. Bouwknegt. Extended conformal algebras from Kac-Moody algebras. In *Infinite-dimensional Lie algebras and groups*, V. G. Kac, ed., *Adv. Ser. in Math. Phys.* **7** (1989), World Scientific, 527–555.

[BS] P. Bouwknegt, K. Schoutens, eds. $\mathcal{W}$-symmetry. *Adv. Ser. in Math. Phys.*, **22**, World Scientific, 1995.

[CK1] S.-J. Cheng and V. G. Kac. A new $N = 6$ superconformal algebra. *Commun. Math. Phys.* **186** (1997), 219–231.

[CK2] S.-J. Cheng and V. G. Kac. Conformal modules, Asian J. of Math., 1 (1997),181-193. Erratum, *Asian J. Math.*, **2** (1998), 153–156.

[CKW] S.-J. Cheng, V. G. Kac and M. Wakimoto. Extensions of conformal modules. In *Topological field theory, primitive forms and related topics, Proceedings of Taniguchi symposia*, Progress in Math., Birkhäuser, 1998.

[DK] A. D'Andrea and V. G. Kac. Structure theory of finite conformal algebras, *Selecta Math.* (1998).

[DJKM] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa. Transformation groups for soliton equations. In *Nonlinear integrable systems—classical theory and quantum theory.* M. Jimbo and T. Miwa, editors, World Scientific, 1983, 39–120.

[DVVV] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde. Operator algebras of orbifold models. *Comm. Math. Phys.*, **123** (1989), 485–526.

[DL] C. Dong and J. Lepowsky. Generalized vertex algebras and relative vertex operators, *Progress in Math.* **112**, Birkhauser, Boston, 1993.

[DLM] C. Dong, H. Li and G. Mason. Compact automorphism groups of vertex operator algebras, *Int. Math. Res. Notices* **18** (1996), 913–921.

[DGM] L. Dolan, P. Goddard and P. Montague. Conformal field theory of twisted vertex operators, *Nucl. Phys.* **B338** (1990), 529–601.

[Fe] B. L. Feigin, *On the cohomology of the Lie algebra of vector fields and of the current algebra*, Selecta Math. Soviet. **7** (1988), 49–62.

[FeF] B. L. Feigin and D. B. Fuchs, *Homology of the Lie algebra of vector fields on the line.* (Russian) Funkc. Anal. i Pril. **14** (1980) no. 3, 45–60

[FF1] B. Feigin and E. Frenkel. Semi-infinite Weil complex and the Virasoro algebra. *Commun. Math. Phys.* **137** (1991), 617–639.

[FF2] B. Feigin, E. Frenkel. Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras, in Infinite Analysis, A. Tsuchiya et all, editors, *Adv. Ser. in Math. Phys.* **16** (1992), World Scientific, 197–215.

[FF3] B. Feigin, E. Frenkel. Kac-Moody groups and integrability of soliton equations. *Invent. Math.*, **120** (1995), 379–408.

[FKRW] E. Frenkel, V. Kac, A. Radul, and W. Wang. $W_{1+\infty}$ and $W(gl_N)$ with central charge N. *Comm. Math. Phys.*, **170** (1995), 337–357.

[FK] I. B. Frenkel and V. G. Kac. Basic representations of affine Lie algebras and dual resonance models. *Invent. Math.*, **62** (1980), 23–66.

[FHL] I. B. Frenkel, Y. Huang, and J. Lepowsky. On axiomatic approaches to vertex operator algebras and modules. *Mem. Amer. Math. Soc.*, **104** No 494 (1993).

[FLM] I. B. Frenkel, J. Lepowsky, and A. Meurman. *Vertex operator algebras and the Monster.* New York: Academic Press, 1988.

[FZ] I. B. Frenkel and Y. Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebra. *Duke Math. J.*, **66** No 1 (1992), 123–168.

[FMS] D. Friedan, E. Martinec, S. Shenker. Conformal invariance, supersymmetry and string theory. *Nucl. Phys.* **B271** (1986), 93–165.

[F] D. B. Fuchs. *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics. Consultants Bureau. New York, 1986.

[FST] P. Furlan, G. M. Sotkov, and I. T. Todorov. Two-dimensional conformal quantum field theory. *La Rivista del Nuovo Cimento*, **6** 1989.

[GF1] I. M. Gelfand and D. B. Fuchs. *Cohomologies of the Lie algebra of vector fields on the circle*. Funkc. Anal. i Pril. **2** (1968) no. 4, 92–93.

[GF2] I. M. Gelfand and D. B. Fuchs. *Cohomologies of the Lie algebra of formal vector fields*. Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970) 322–337.

[Ge] E. Getzler. Manin pairs and topological field theory. *Ann. Phys.* (N.Y.) **237** (1995), 161–201.

[G] P. Goddard. Meromorphic conformal field theory. In *Infinite-dimensional Lie algebras and groups*, V. G. Kac, editor, *Adv. Ser. in Math. Phys.*, **7** (1989), World Scientific, 556–587.

[GKO] P. Goddard, A. Kent, and D. Olive. Virasoro algebras and coset space models. *Phys. Lett.*, **B152** (1985), 88–93.

[JM] M. Jimbo and T. Miwa. Solitons and infinite dimensional Lie algebras. *Publ. Res. Inst. Math. Sci.*, **19** (1983), 943–1001.

[K1] V. G. Kac. Lie superalgebras. *Adv. Math.*, **26** (1977), 8–96.

[K2] V. G. Kac. *Infinite dimensional Lie algebras*, third edition. Cambridge University Press, 1990.

[K3] V. G. Kac. Conformal superalgebras and transitive group actions on quadrics. *Commun. Math. Phys.* **186** (1997), 233–252.

[K4] V. G. Kac. Classification of infinite-dimensional simple linearly compact Lie superalgebras. *Adv. Math.* (1998).

[KL1] V. G. Kac and J. van de Leur. Super boson-fermion correspondence. *Ann. Inst. Fourier* **37** (1987), 99–137.

[KL2] V. G. Kac and J. van de Leur. On classification of superconformal algebras. In *Strings 88*, S. J. Gates et al., editors, World Scientific, 1989, 77–106.

[KL3] V. G. Kac and J. van de Leur. The n-component KP hierarchy and representation theory. *Important developments in soliton theory*, A. S. Fokas and V. E. Zakharov, editors, Springer Verlag, 1993, 302–343.

[KRad] V. G. Kac and A. D. Radul. Representation theory of the vertex algebra $W_{1+\infty}$. *Transformation groups 1* (1996), 41–70.

[KR] V. G. Kac and A. K. Raina. Bombay lectures on highest weight representations of infinite-dimensional Lie algebras. *Advanced Ser. in Math. Phys.*, **2** World Scientific, 1987.

[KT1] V. G. Kac and I. Todorov. Superconformal current algebras and their unitary representations. *Comm. Math. Phys.*, **102** (1985), 337–347.

[KT2] V. G. Kac and I. Todorov. Affine orbifolds and RCFT extensions of $W_{1+\infty}$. *Comm. Math. Phys*, **190** (1997), 57–111.

[KW1] V. G. Kac and M. Wakimoto. Modular and conformal invariance constraints in representation theory of affine algebras. *Adv. Math.*, **70** (1988), 156–234.

[KW2] V. G. Kac and M. Wakimoto. Integrable highest weight modules over affine superalgebras and number theory. In *Progress in Math.*, **123** (1994), 415–456.

[KWang] V. G. Kac and W. Wang. Vertex operator superalgebras and their representations. *Contemp. Math.*, **175** (1994), 161–191.

[KS] Y. Kazama and H. Suzuki. New $N = 2$ superconformal field theories and superstring compactification. *Nucl. Phys.*, **B321** (1989), 232–268.

[KV] T. Kimura and A. A. Voronov. The cohomology of algebras over moduli spaces. In *Progress in Math.* **129**, Birkhauser, Boston, 1995, 305–334.

[Li] H.-S. Li. Local systems of vertex operators, vertex superalgebras and modules. *J. Pure Appl. Algebra*, **109** (1996), 143–195.

[LZ] B. H. Lian and G. J. Zuckerman. Commutative quantum operator algebras. *J. Pure Appl. Algebra*, **100** (1995), 117–140.

[Man] S. Mandelstam. Dual resonance models. *Phys. Reports*, **13** (1974), 259–353.

[M] O. Mathieu. Classification des algèbras de Lie graduées simple de croissance ≤ 1. *Invent. Math.*, **86** (1986), 371–426.

[MN] A. Matsuo and K. Nagatomo. On axioms for a vertex algebra and the locality of quantum fields, preprint.

[MP] A. Meurman and M. Primc. Annihilating fields of standard modules of $gl(2, \mathbb{C})^{\sim}$ and combinatiorial identities, preprint.

[NS] A. Neveu and J. H. Schwarz. Factorizable dual model of pions. *Nucl. Phys.*, **B31** (1971), 86–112.

[OS] N. Ohta and H. Suzuki. $N = 2$ superconformal models and their free field realizations. *Nucl. Phys.*, **B332** (1990), 146–168.

[R1] J.F. Ritt. Differential groups and formal Lie theory for an infinite number of parameters. *Ann. of Math.* (2), **52** (1950), 708–726.

[R2] J.F. Ritt. Differential groups of order two. *Ann. of Math.*, **53** (1951), 491–519.

[RS] P. Ramond and J. H. Schwarz. Classification of dual model gauge algebras. *Phys. Lett.*, **64B** (1976), 75–77.

[Se] J.-P. Serre. Lie algebras and Lie groups. Benjamin, 1965.

[Sk] T. H. R. Skyrme. Kinks and the Dirac equation. *J. Math. Phys.*, **12** (1971), 1735–1743.

[S] H. Sugawara. A field theory of currents. *Phys. Rev.* **176** (1968), 2019–2025.

[T] K. Thielemans. A Mathematica package for computing operator product expansions. *Internat. Mod. Phys. C*, **2** No 3 (1991), 787–798.

[Wa] W. Wang. Rationality of Virasoro vertex operator algebras. *Duke Math. J.*, **71** No 1 (1993), 197–211.

[W] A. S. Wightman. Quantum field theory in terms of vacuum expectation values. *Phys. Rev.*, **101** (1956), 860–866.

[Wi] K. Wilson. OPE and anomalous dimensions in the Thirring model. *Phys. Rev. D*, **2** (1970), 1473–1477.

[Z] Y. Zhu. Modular invariance of characters of vertex operator algebras. *Journal AMS*, **9** (1996), 237–302.

This page intentionally left blank

Index

λ-action, 53
λ-bracket, 40
λ-product, 27
2-cocycle, 46, 144, 152

affine central charge, 95
affine Kac-Moody algebra, 32
affine vertex algebra, 122
affinization of a conformal algebra, 41
affinization of a loop algebra, 32
affinization of a vertex algebra, 108
annihilation algebra, 56
associative conformal algebra, 61
associative superalgebra, 24
automorphism of a vertex algebra, 107

basic cohomology, 70
Borcherds commutator formula, 112
Borcherds identity, 117
Borcherds OPE formula, 111
boson-fermion correspondence, 139
bosonization, 134
bounded representation, 92
bracket, 24

Casimir operator, 163
causal order, 5
central charge, 37, 125
character, 136, 170
charge, 135
charge decomposition, 135
charge operator, 135
charged free fermions, 34, 133, 155
chiral algebra, 11
Clifford affinization, 33
conformal group, 7
conformal linear map, 65
conformal module, 50, 52, 56
conformal quantum field theory, 7
conformal superalgebra, 39
conformal vector, 125
conformal vertex algebra, 125
conformal weight, 34, 127, 186
coset model, 113
current algebra, 31
current conformal superalgebra, 47
currents, 32

Dedekind η-function, 137

denominator identity, 136, 176
derivation of a vertex algebra, 107
Dong's lemma, 84
dual bases, 94
dual Coxeter number, 164

eigendistribution, 34
energy, 135
energy operator, 135
energy-momentum field, 125
Existence theorem, 110
extended annihilation algebra, 56

field, 14, 81, 82
field algebra, 129
field representation, 91
finite conformal algebra, 40
finite formal distribution algebra, 45
finite module, 54, 59
formal delta-function, 18
formal distribution, 17
formal distribution Lie superalgebra, 37, 91
formal Fourier transform, 21
forward cone, 5
free boson, 31, 94
free bosonic vertex algebra, 116
free fermionic vertex algebra, 116
free fermions, 99
free field theory, 88
free neutral fermion, 34
Frenkel-Kac construction, 160

general conformal algebra, 66
general linear field algebra, 86
generating set of fields, 111
Goddard uniqueness theorem, 108
Goddard-Kent-Olive construction, 167
graded conformal superalgebra, 186
graded vertex algebra, 119
group GL_∞, 142

Hamiltonian, 34, 119
holomorphic vertex algebra, 15
homomorphism of vertex algebras, 106

ideal of a vertex algebra, 107
induced module, 91
infinitesimal translation operator, 14, 103
inner automorphism of a vertex algebra, 121
integral lattice, 148
integration by parts, 17
invariant bilinear form, 31
irregular ideal, 44

Jacobi triple product identity, 136

Kac-Todorov model, 180
KP hierarchy, 141, 142

lattice vertex algebra, 148, 154
Lie algebra $\mathcal{D}$ of regular differential operators on $\mathbb{C}\setminus\{0\}$, 145
Lie algebra $\widehat{gl}_\infty$, 144
Lie algebra $\widetilde{gl}_\infty$, 143
Lie algebra gl_∞, 140
Lie superalgebra, 24
light cone coordinates, 8
linear field algebra, 86
local formal distribution, 21
local linear field algebra, 86
locality, 24
locality axiom, 14, 103
loop algebra, 32

Minkowski space-time, 5
Möbius-conformal vertex algebra, 123
module over a conformal algebra, 52
mutually local formal distributions, 24

n-th product of fields, 82, 84
n-th product of formal distributions, 26
$N = 1$ superconformal vector, 179

$N = 1$ superconformal vertex algebra, 180
$N = 1$ vertex superalgebra, 183
$N = 2$ conformal superalgebra, 186
$N = 2$ superconformal Lie algebra, 182
$N = 2$ superconformal vertex algebra, 182
$N = 4$ conformal superalgebra, 187
Neveu-Schwarz algebra, 179
Neveu-Schwarz conformal superalgebra, 185
"non-commutative" Wick formula, 90, 116
normally ordered product, 81, 87

OPE coefficients, 21
operator product expansion (OPE), 21, 26
orbifold model, 113
oscillator algebra, 31

parity, 13
Pauli matrices, 187
Poincaré group, 5
positive and negative parts of a formal distribution, 25
primary field, 127

quantum field theory, 5
quasiassociativity, 119, 130
quasiprimary field, 123

reduced cohomology, 70
regular formal distributions Lie superalgebra, 114
residue, 17
root lattice, 159
root system, 159

singular vector, 93
skew-supersymmetric bilinear form, 33
skewsymmetry of a vertex algebra, 105
space of states, 14
space-like separated subsets, 5
state-field correspondence, 14
strongly generating set of fields, 111
subalgebra of a vertex algebra, 106
Sugawara construction, 165
super boson-fermion correspondence, 172
superaffine vertex algebra, 122
superaffinization, 33
superconformal Lie algebra, 181
supercurrent, 33
supercurrent algebra, 115
superdimension, 14
superfields, 183
superspace, 13
supersymmetric bilinear form, 32

Taylor's formula, 29, 30, 82
tensor product of vertex algebras, 107
translation covariance axiom, 14, 103
twisted group algebra, 152

universal affine vertex algebra, 115
universal vertex algebra associated to a formal distribution Lie superalgebra, 115

vacuum axiom, 14, 103
vacuum subalgebra, 110
vacuum vector, 14
Veneziano field, 139
Verma module, 93
vertex algebra, 14, 103
vertex algebra $W_{1+\infty,c}$, 147
vertex operator, 12, 139, 155
Virasoro algebra, 36
Virasoro conformal algebra, 48
Virasoro field, 125
Virasoro formal distribution, 37
Virasoro vertex algebra, 128

W-algebra, 113
Weyl affinization, 94
Wick theorem, 87

ISBN 0-8218-1396-X
9 780821 813966
ULECT/10.R