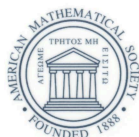


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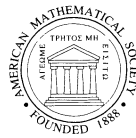
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1991 *Mathematics Subject Classification*. Primary 16W30, 16L60, 16H05;  
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ABSTRACT. This book is an introduction to Frobenius extensions of noncommutative rings, and develops five examples of recent publication: Hopf subalgebras, Frobenius algebras and solutions of YBE, invertibility and Azumaya algebras, the Jones theory of towers, idempotents and knot polynomials, and the equivalence of commutative Frobenius algebras with 2-dimensional TQFT's. The book is addressed to researchers and graduate students in algebra and related fields.

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# Contents

Preface	ix
Chapter 1. Introduction to Frobenius Extensions	1
1.1. Preliminaries	1
1.2. Ring Extensions	2
1.3. Frobenius Extensions	3
Chapter 2. The Endomorphism Ring Theorem	9
2.1. Some more preliminaries	9
2.2. Frobenius Bimodules	9
2.3. The Endomorphism Ring Theorem	11
2.4. Separability	13
2.5. A Duality	15
2.6. H-Separability	17
Chapter 3. The Jones Polynomial	21
3.1. Markov Extensions	21
3.2. The Tower of Markov Extensions	24
3.3. The Braid-like Relations	24
3.4. Representing the Braid Groups	25
3.5. The Jones Link Polynomial	27
Chapter 4. Frobenius Algebras	33
4.1. Preliminaries on Frobenius Extensions	33
4.2. Frobenius Algebras and YBE Solutions	35
4.3. Classical Frobenius Algebras and QF Rings	38
4.4. 2-Dimensional TQFT's	40
Chapter 5. Azumaya Algebras	43
5.1. The Reduced Trace	43
5.2. Separable Algebras over a Commutative Ring	44
5.3. Strongly Separable Algebras	48
5.4. Invertible Solutions of YBE	51
Chapter 6. Hopf Algebras over Commutative Rings	53
6.1. Augmented Frobenius Algebras	53
6.2. Antipodes and Integrals of Finite Projective Hopf Algebras	55
6.3. The Quantum Double	60
Chapter 7. Hopf Subalgebras	63
7.1. Frobenius Extensions of the Second Kind	63



7.2. A Freeness Theorem	67
7.3. Main Theorem	68
7.4. Two Examples	71
Appendix A. Historical Notes	75
Bibliography	79
Index	83

# Preface

These are the notes to a course of lectures I gave at Munich in the fall of 1997 and again at Göteborg the following spring. I sought to give in seven lectures a unified exposition of four recent articles in the general area of quantum groups and knot polynomials. The four articles have established an interesting connection between the notion of Frobenius algebra or the more general Frobenius extension on the one hand and Hopf subalgebras [27], solutions of the Yang-Baxter Equation [6], the Jones polynomials  $V_L(t)$  [41], and 2-dimensional topological quantum field theories [1] on the other hand. There are interesting possibilities for further interactions with the theories of Atiyah [3], Drinfel'd [21], Jones [37], Turaev [94] and Witten [99].

By a Frobenius algebra we mean a finite dimensional associative algebra  $A$  which has a nondegenerate linear functional  $\phi$ . Equivalently,  $A$  is a Frobenius algebra if  $A$  and its dual  $A^*$  are isomorphic as left  $A$ -modules via a map  $a \mapsto a\phi$ . This particular characterization generalizes to a definition of Frobenius extension of rings: a finite projective ring extension  $S \rightarrow A$  is Frobenius if  $A$  and its  $S$ -valued dual are isomorphic as  $A$ - $S$  bimodules. Our purpose then is to elaborate on these ideas, providing a relatively thorough account of Frobenius theory based on the literature by Brauer, Eilenberg, Kasch, Nakayama, Morita, and others. We also study separability in relation to Frobenius theory, which is important to two of the recent articles, [6, 41]. Then a unified exposition is given the four recent topics named above. As a bonus, an interaction of the recent topics and Frobenius theory is seen in several of the theorems and propositions.

In Chapters 1 and 2, and in Sections 4.1 and 7.1, we develop the theory of Frobenius extensions. In Section 4.2 we develop some more theory for Frobenius algebras over commutative rings; in Section 4.3, for the classical Frobenius algebras over fields; and in Section 6.1, for augmented Frobenius algebras over commutative rings. Sections 2.4, 2.5, 2.6, 5.3, and 7.1 study the relationship of Frobenius extensions with separability.

Chapter 3 elaborates on [41] by the author, and shows that a certain separable Frobenius extension of algebras with trace, which we introduce here as a Markov extension, has an endomorphism ring theorem and a key idempotent. We iterate this construction, generating a tower of algebras, and obtaining the characters for the braid groups that Jones obtained via  $II_1$  subfactors. We thereby derive the Jones knot and link polynomials  $V_L(t)$  [37] by a general method. The mechanism in building the tower of algebras is the endomorphism ring of the generic type of *relatively semisimple* pair given by a Markov extension of algebras over an algebraically closed field.

In Chapter 4 we study Frobenius algebras and prove a result of [6, Beidar-Fong-Stolin]. We show that the Frobenius element of the tensor-square of a Frobenius

algebra  $A$  over a commutative ring  $k$  is a solution of the Yang-Baxter Equation, though not necessarily an invertible one. The proof depends on the Nakayama automorphism of a Frobenius algebra. The question of when a solution is invertible is postponed until Chapter 5. Chapter 4 ends with a sketch of [1, Abrams] where it is proven that 2-dimensional topological quantum field theories are equivalent to commutative Frobenius algebras.

In Chapter 5 we develop the theory of Azumaya algebras in a self-contained section. We study three characterizations of Azumaya algebras which provide a simplification in the proof of two theorems. First, we prove that the solutions from Chapter 4 of the Yang-Baxter equation are invertible solutions if and only if  $A$  is an Azumaya  $k$ -algebra. Second, Azumaya algebras with invertible Hattori-Stallings rank are strongly separable, symmetric Frobenius algebras, a theorem of [16, 30, DeMeyer, Hattori].

Based on [45, author-Stolin], Chapter 6 develops a Frobenius theory for a Hopf algebra  $H$  which is finite projective over a commutative ring with trivial Picard group. Certain key properties of norms and integrals together with modular functions are already given for augmented Frobenius algebras. We then study the basic theorem of Larson-Pareigis-Sweedler [57, 75] which informs us that  $H$  is an augmented Frobenius algebra. We find an explicit Frobenius system (Frobenius homomorphism with dual base) with Nakayama automorphism for  $H$ . Then we obtain other Frobenius systems for  $H$  by applying the antipode  $S$ , its compositional inverse  $S^{-1}$  and the automorphism  $S^2$ . The Frobenius theory in Chapter 1 which compares Frobenius systems and Nakayama automorphisms of the same algebra then yields two well-known Radford formulas involving  $S^2$  and  $S^4$ . As a corollary to the first of these formulas, we show that Drinfeld's quantum double  $D(H)$  is a unimodular, symmetric algebra.

In Chapter 7 we restrict our attention to finite dimensional Hopf algebras after making a study of  $\beta$ -Frobenius extensions in Section 7.1. Based on [27], Sections 7.2 and 7.3 show explicitly that a Hopf subalgebra  $K$  of a finite dimensional Hopf algebra  $H$  is a Frobenius extension  $H/K$  twisted on one side by an automorphism  $\beta$  of  $K$ . A  $\beta$ -Frobenius system for  $H/K$  is given in terms of a relative Nakayama automorphism and the Frobenius system studied in Chapter 6. We conclude with several examples of Frobenius extensions drawn from the Taft algebra and the quantum double.

No prior knowledge of Frobenius algebras, separable algebras, or Hopf algebras is assumed of the reader, although a look at two of the early chapters of [15, Curtis-Reiner] on Frobenius and separable algebras, and the first several chapters in [90, Sweedler] on Hopf algebras will put much of this material in perspective. An index is provided at the end of this paper. A brief historical account and guide to the literature are given in Appendix A.

I thank the Carlsberg Foundation in Copenhagen, Heidelberg University, the Graduate College in Mathematical Physics at Ludwig-Maximilians University in Munich, and NorFA in Oslo for their support of this project at various stages. I also thank Professors D. Kastler, J. Cuntz, B. Pareigis, and J. Brzezinski for inviting me to Marseilles, Heidelberg, Munich, and Göteborg, respectively. I thank too the participants of my two courses of lectures for their attention and valuable comments, especially J. Brzezinski, F. Kasch, R. Larson, B. Pareigis, P. Schauenburg, H.-J. Schneider, Y. Sommerhäuser and A. A. Stolin. A special thanks to Marit in whose home in Nesbyen much of my work on these notes took place.

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# Index

- Azumaya algebra, 45
  - strongly separable, 51
- bimodule
  - central generator, 18
  - centrally projective, 17
  - faithfully balanced, 2, 10
  - Frobenius, 9
  - left dual  ${}^*M$ , 2
  - right dual  $M^*$ , 1
- braid group  $B_n$ , 25
- Casimir element, 44
- center  $Z(R)$  of ring  $R$ , 2
- central separable algebra, 45
- centralizer of a ring extension  $C_A(S)$ , 3
- co-ring, 34
- coideal subalgebra, 67
- conditional expectation, 14
- derivative  $\frac{dE}{dF}$ , 59
- double centralizer result, 7, 20
- E-index, 7
- E-multiplication, 5
- faithfully flat, 3
- Frobenius algebra, 35
  - augmented, 53
- Frobenius element  $\sum_i x_i \otimes y_i$ , 5
- Frobenius extension, 3
  - symmetric, 6
- Frobenius homomorphism  $E$ , 4
- Frobenius isomorphism, 4
- Frobenius system  $(E, x_i, y_i)$ , 4
- group-like element, 55
- H-equivalent modules, 17
- H-separable extension, 18
- Hattori-Stallings (HS) rank  $r(P_R)$ , 1
- Hecke relation, 30
- Hopf algebra, 55
  - almost cocommutative, 60
  - quasi-triangular, 61
- Hopf module, 55
- Hopf subalgebra  $K \subseteq H$ , 63
  - normal, 68
- Hopf-Galois extension, 68
- integral, 53
- invertible solution of YBE, 37
- Jones idempotent  $e_n$ , 24
- Jones polynomial  $V_L(t)$ , 29
- Larson-Pareigis-Sweedler Theorem, 55
- Markov extension, 21
- Markov system, 21
- Markov trace, 21
- mirror image link, 30
- modular function  $m$ , 54, 57
- Morita's Lemma, 2
- Nakayama automorphism, 6, 36
- Nichols-Zoeller Theorem, 67
- nondegenerate linear functional, 38
- norm, 53
- open problem, 16, 34, 58, 72, 77
- progenerator, 2
- projective base  $(f_i, x_i)$ , 1
- QF extension, 34
- QF ring, 38
- quantum double, 60, 72
- Quantum Yang-Baxter Equation, 61
- reduced trace, 43
- reflexive module, 2
- relative injective module, 34
- relative projective module, 33
- relative semisimplicity, 33
- ring extension  $A/S$ , 2
- ring with perfect duality, 16
- semisimple extension, 33
- separability idempotent, 45
- separable algebra, 44
- separable base, 49
- skein relation, 30
- split extension, 14
- strongly separable algebra, 49
- subfactors, 7
- symmetric algebra, 36

symmetric separability idempotent, 49

Taft algebra, 71

topological quantum field theory, 40

tower of Markov extensions, 24

universal  $R$ -matrix, 60

Yang-Baxter Equation (YBE), 37

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## New Examples of Frobenius Extensions

Lars Kadison

This volume is based on the author's lecture courses to algebraists at Munich and at Göteborg. He presents, for the first time in book form, a unified approach from the point of view of Frobenius algebras/extensions to diverse topics, such as Jones' subfactor theory, Hopf algebras and Hopf subalgebras, the Yang-Baxter Equation and 2-dimensional topological quantum field theories.

### Other Features:

- Initial steps toward a theory of noncommutative ring extensions.
- Self-contained sections on Azumaya algebras and strongly separable algebras.
- Applications and generalizations of Morita theory and Azumaya algebra due to Hirata and Sugano.

Understanding the text requires no prior background in Frobenius algebras or Hopf algebras. An index and a thorough list of further references are included. There is an appendix giving a brief historical guide to the literature.

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