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Admissible Invariant Distributions on Reductive *p*-adic Groups

Harish-Chandra

Notes by
Stephen DeBacker and Paul J. Sally, Jr.

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ABSTRACT. This book contains a faithful rendering of Harish-Chandra's lectures on admissible invariant distributions (originally delivered at the Institute for Advanced Study in the 1970s). For a p-adic field Ω, let G denote the set of Ω-rational points of a connected reductive Ω-group. Let $\mathfrak{g}$ denote the Lie algebra of G. The main purpose of these notes is to show that the character of an irreducible admissible representation of G is represented by a locally summable function on G. The proof of this result begins with a study of harmonic analysis on $\mathfrak{g}$. The key result here is that the Fourier transform of a G-invariant distribution (with compactly generated support) on $\mathfrak{g}$ is itself represented by a locally summable function on $\mathfrak{g}$. Moreover, the Fourier transform of such a distribution has an asymptotic expansion about any semisimple point of $\mathfrak{g}$. This result, and most of the work on $\mathfrak{g}$, depends heavily on Howe's Theorem (for $\mathfrak{g}$) for which a proof is provided. Finally, these results are transferred to G via Howe's "Kirillov theory." These notes are intended for advanced graduate students and mathematicians working in this area.

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Preface

Harish-Chandra first presented these notes on admissible distributions in lectures at the Institute for Advanced Study during 1973. In this preface, we provide a brief guide to the content of Harish-Chandra's notes and discuss the advances in this area of mathematics since these lectures were delivered. Of course, any such discussion will necessarily overlap Harish-Chandra's own introductory remarks (which begin on page 1).

A sketch of this material was published by Harish-Chandra in his Queen's notes [**17**]. Every statement in Harish-Chandra's Queen's notes also occurs here. Therefore, when we make a statement which occurs as an enumerated statement in the Queen's notes, we provide in parentheses the statement number appearing there (see, for example, the statement of Theorem 5.11).

A number of years ago, Harish-Chandra asked one of us (Sally) to produce a detailed version of his Queen's notes based on his own lecture notes. As was his custom, Harish-Chandra produced several versions of his lecture notes. We have made only minor changes to these, and most of these changes were with respect to the ordering. The two of us (DeBacker and Sally) carefully worked through Harish-Chandra's notes, and the version included here was typed by DeBacker. We take full responsibility for any errors.

The main results. Without further comment we adopt the terminology used by Harish-Chandra in [**20**].

Let Ω be a p-adic field of characteristic zero with ring of integers R. Let G be the group of Ω-rational points of a connected, reductive Ω-group. The group G, with its usual topology, is a locally compact, totally disconnected, unimodular group. In particular, it has a neighborhood basis of the identity consisting of compact open subgroups. Let dx denote the Haar measure on G and let G' denote the set of regular elements in G.

A complex representation (π, V) of G is *smooth* if, for each $v \in V$, there is an open subgroup K_v of G which fixes v (i.e., $\pi(k)v = v$ for all $k \in K_v$). The representation (π, V) is *admissible* if

(1) π is smooth, and
(2) for every compact open subgroup K of G, the space of K-fixed vectors has finite dimension.

Every irreducible and smooth representation is admissible [**29**]. Let (π, V) be an irreducible smooth representation of G. Denote by $C_c^\infty(G)$ the space of locally constant, compactly supported, complex-valued functions on G. For $f \in C_c^\infty(G)$, the operator

$$\pi(f) = \int_G f(x) \cdot \pi(x)\, dx$$

is an operator of finite rank. Consequently, it makes sense to define the *distribution character* of π by

$$\Theta_\pi(f) = \mathrm{tr}\ \pi(f)$$

for all $f \in C_c^\infty(G)$.

Motivated by the case of real reductive groups, we may ask if there exists a locally summable function F_π on G which is locally constant on G' such that

$$\Theta_\pi(f) = \int_G f(x) \cdot F_\pi(x)\, dx$$

for all $f \in C_c^\infty(G)$. It is the main purpose of these notes to provide an affirmative answer to this question. If F is an arbitrary nonarchimedean local field, then for the group $\mathrm{GL}_n(F)$ this result was established in the "tame" case by Rodier [**59**] and in the remaining cases by Lemaire [**39**]. In general, for the F-rational points of a connected reductive group defined over F, the most we can say is that the distribution character of an irreducible smooth representation is represented by a locally constant function on the set of regular elements [**22**] (see also Howe [**25**]).

One of the major results of these notes is a description of the behavior of Θ_π near a semisimple point γ of G (see Theorem 16.2). This is accomplished by deriving an asymptotic expansion for Θ_π in a neighborhood of γ. When γ is the identity element in G, we refer to this asymptotic expansion as the *local character expansion* of π. We need some definitions and notation before describing the local character expansion.

Let $\mathfrak{g}$ denote the Lie algebra of G. Let $C_c^\infty(\mathfrak{g})$ denote the space of complex-valued, locally constant, compactly supported functions on $\mathfrak{g}$. Let B be an Ω-valued, non-degenerate, symmetric, G-invariant bilinear form on $\mathfrak{g}$. Fix a non-trivial additive character χ on Ω. Let dX denote the Haar measure on the additive group of $\mathfrak{g}$ and, for $f \in C_c^\infty(\mathfrak{g})$, set

$$\hat{f}(Y) = \int_{\mathfrak{g}} f(X) \cdot \chi\big(B(X,Y)\big)\, dX$$

for $Y \in \mathfrak{g}$. The map $f \mapsto \hat{f}$ is a linear bijection of $C_c^\infty(\mathfrak{g})$ onto itself. If T is a distribution on $\mathfrak{g}$ (i.e., a linear functional on $C_c^\infty(\mathfrak{g})$), we define the Fourier transform $\widehat{T}$ of T by

$$\widehat{T}(f) = T(\hat{f})$$

for $f \in C_c^\infty(\mathfrak{g})$.

If $\mathcal{O}$ is a G-orbit in $\mathfrak{g}$ (under the adjoint action), then $\mathcal{O}$ carries a G-invariant measure which we denote by $\mu_{\mathcal{O}}$ [**55**]. It will follow from Theorem 4.4 that the Fourier transform of the distribution

$$f \mapsto \mu_{\mathcal{O}}(f)$$

for $f \in C_c^\infty(\mathfrak{g})$ is represented by a locally summable function on $\mathfrak{g}$ which is locally constant on $\mathfrak{g}'$, the set of regular elements of $\mathfrak{g}$. We denote this function by $\widehat{\mu_{\mathcal{O}}}$.

Since Ω has characteristic zero, the set of nilpotent orbits, which we denote by $\mathcal{O}(0)$, has finite cardinality. We can now state the local character expansion (see Theorem 16.2):

THEOREM. *Let π be an irreducible smooth representation of G. We can choose complex numbers $c_{\mathcal{O}}(\pi)$, indexed by $\mathcal{O} \in \mathcal{O}(0)$, such that*

$$\Theta_\pi(\exp Y) = \sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi) \cdot \widehat{\mu_{\mathcal{O}}}(Y)$$

for all $Y \in \mathfrak{g}'$ sufficiently near zero.

This remarkable theorem, which was first proved by Howe [**23**] for the general linear group, is a qualitative result that leaves many unresolved quantitative questions. For example, almost no results exist about the quantitative nature of the $c_{\mathcal{O}}$s and the $\widehat{\mu_{\mathcal{O}}}$s. Moreover, outside of some stunning work of Waldspurger [**72, 73**] and a conjecture of Hales, Moy, and Prasad [**43**] we have only limited information about the precise range in which the equality holds.

Quantitatively, this is what we know about the $c_{\mathcal{O}}$s and $\widehat{\mu_{\mathcal{O}}}$s. For the general linear group, Howe [**23**] observed that the functions $\widehat{\mu_{\mathcal{O}}}$ have a very nice form (see also [**41**]) and showed that $c_{\mathcal{O}}(\pi)$ is an integer for every irreducible supercuspidal representation π and every nilpotent orbit $\mathcal{O}$. By using results of Kazhdan [**31**], Assem [**1**] determined the functions $\widehat{\mu_{\mathcal{O}}}$ for $\mathrm{SL}_\ell(\Omega)$ with ℓ a prime. Finally, by using a result later proved in general by Huntsinger [**27**], DeBacker and Sally [**8**] and Murnaghan [**46**] evaluated an integral formula to obtain values for the $\widehat{\mu_{\mathcal{O}}}$s in the cases $\mathrm{SL}_2(\Omega)$ and $\mathrm{GSp}_4(\Omega)$.

In Theorem 22.3 Harish-Chandra derives a formula for the leading term $c_0(\pi)$ in the local character expansion of an irreducible supercuspidal representation π of G. Strengthening a conjecture of Shalika [**66**], Harish-Chandra conjectures that this formula ought to hold for all irreducible discrete series representations of G. Rogawski proved this in [**61**]. Moreover, Huntsinger [**28**] used some work of Kazhdan [**30**] to show that for an irreducible tempered representation π, $c_0(\pi)$ is zero if and only if π is not a discrete series representation.

At the other extreme, Rodier [**60**] showed (for split G) that an irreducible admissible representation π has a Whittaker model if and only if there is a regular nilpotent orbit $\mathcal{O}$ such that $c_{\mathcal{O}}(\pi)$ is not zero. Mœglin and Waldspurger [**41**] refined this result. They showed that if $\mathcal{O}$ is maximal among those nilpotent orbits for which $c_{\mathcal{O}}(\pi)$ is nonzero, then the value of $c_{\mathcal{O}}(\pi)$ is related to the dimension of a degenerate or generalized Whittaker model. There have been many applications of these results. For classical groups Mœglin [**40**] showed that if $\mathcal{O}$ is maximal among those nilpotent orbits for which $c_{\mathcal{O}}(\pi)$ is nonzero, then the orbit $\mathcal{O}$ is special. Savin [**65**] showed that, for the representations constructed by Kazhdan and Savin in [**32**], the local character expansion involves only the trivial orbit and the minimal nilpotent orbits. A representation with this property is called a minimal representation. This work was extended by Rumelhart [**63**] and Torasso [**69**]. A version of Rodier's result for covering groups of $\mathrm{GL}_n(\Omega)$ is provided and used by Flicker and Kazhdan in [**10**].

In general, the remaining $c_{\mathcal{O}}$s have been calculated explicitly in only a few cases, most notably in the work of Assem [**1**], Barbasch and Moy [**2**], and Murnaghan [**45, 46, 49, 50, 51**]. In [**13**] Hales showed that most of the basic objects of harmonic analysis—including characters and the $\widehat{\mu_{\mathcal{O}}}$s—are non-elementary. That is, at some point, their values can be calculated by counting points on hyperelliptic curves over finite fields. Perhaps this is why these objects have been so hard to quantify explicitly.

A guide to these notes. The Lie algebra $\mathfrak{g}$ is a vector space over Ω of finite dimension, and G operates on $\mathfrak{g}$ by the adjoint representation, denoted Ad. Let T be a distribution on $\mathfrak{g}$. Then, for $x \in G$, the distribution ${}^xT$ is defined by

$$ {}^xT(f) = T(f^x) $$

for $f \in C_c^\infty(\mathfrak{g})$ where

$$ f^x(X) = f(\mathrm{Ad}(x)X) $$

for $X \in \mathfrak{g}$. The distribution T is said to be G-invariant if ${}^xT = T$ for all $x \in G$. Let J denote the space of all G-invariant distributions on $\mathfrak{g}$.

For $\omega \subset \mathfrak{g}$, let $J(\omega)$ denote the space of all G-invariant distributions T such that the support of T is contained in the closure of $\mathrm{Ad}(G)\,\omega$. If L is a lattice in $\mathfrak{g}$ (i.e., a compact open R-submodule of $\mathfrak{g}$) and T is a distribution on $\mathfrak{g}$, let j_LT denote the restriction of T to $C_c(\mathfrak{g}/L)$. The following theorem, which was first conjectured by Howe in [**26**], makes nearly everything in these notes possible.

THEOREM 12.1 (Theorem 2). *Let ω be a compact set in $\mathfrak{g}$ and L a lattice in $\mathfrak{g}$. Then*

$$ \dim j_L J(\omega) < \infty. $$

Although Howe [**23**] proved Theorem 12.1 only for the general linear group, Harish-Chandra [**17**] attributes this theorem to him. Consequently, Theorem 12.1 is often referred to as Howe's Theorem in the literature. Although Theorem 12.1 is used throughout Part I, the most significant applications can be found in §1.1, §4, and §5. In Part II of these notes Harish-Chandra states and proves an extension of Howe's Theorem which is used in Part III, §21. For the general linear group, Howe [**23**] was the first to prove this extension. Waldspurger [**76**] also proved this extension of Howe's Theorem in the context of weighted orbital integrals. Waldspurger's proof includes the situation under consideration in these notes.

The key to understanding elements of J is Theorem 3.1. This theorem says that the regular semisimple orbital integrals are dense in J (i.e., if $f \in C_c^\infty(\mathfrak{g})$ and $\mu_{\mathcal{O}}(f) = 0$ for all G-orbits $\mathcal{O}$ which are contained in $\mathfrak{g}'$, then $T(f) = 0$ for all $T \in J$). This result, combined with an understanding of $\widehat{\mu_{\mathcal{O}}}$ for a regular orbit $\mathcal{O}$ (§3) and Howe's Theorem, allows Harish-Chandra to prove that $\widehat{\mu_{\mathcal{O}}}$ is represented by a locally summable function on $\mathfrak{g}$ which is locally constant on $\mathfrak{g}'$ (Theorem 4.4). Waldspurger [**76**] showed that far from zero, the function $\widehat{\mu_{\mathcal{O}}}$ has a particularly nice form. Another application of Howe's Theorem and some understanding of the geometry of open and closed G-invariant neighborhoods of zero permits Harish-Chandra to write down an asymptotic expansion of $\widehat{T}$ for $T \in J(\omega)$ with ω compact. Finally, in §7 Harish-Chandra derives an explicit integral formula for the Fourier transform of a regular orbital integral. This formula lets him see that the function representing $|\eta|^{1/2} \cdot \widehat{\mu_{\mathcal{O}}}$ is locally bounded on $\mathfrak{g}$ (here η is the usual discriminant).

The techniques of §7 were extended by Huntsinger [**27**] to show that the function representing a compactly supported distribution on $\mathfrak{g}$ has an integral formula. Rader and Silberger [**54**] extended a result of Harish-Chandra [**21**] to show that the character of an irreducible discrete series representation has an integral formula which is remarkably similar to the integral formula for the Fourier transform of a regular orbital integral obtained in §7. Murnaghan [**47, 48, 50, 51**] showed that this is not an accident and that in some cases the character of a supercuspidal representation can be related to the Fourier transform of a regular orbital integral. Murnaghan's work was extended by Cunningham [**6**] and DeBacker [**7**].

Following Shalika [**66**], in §8 Harish-Chandra develops the theory of what have become known as Shalika germs. In [**56, 57, 58**] Repka explicitly computed the Shalika germs corresponding to the regular and subregular unipotent orbits of $\mathrm{GL}_n(\Omega)$ and $\mathrm{SL}_n(\Omega)$ on the set of regular elliptic elements. (Kim [**33, 34, 35**] and Kim and So [**36**] partially computed the regular and subregular Shalika germs for $\mathrm{Sp}_4(\Omega)$.) For regular unipotent orbits, these results were extended to all groups by Shelstad [**67**], and for subregular unipotent orbits, they were extended to all groups by Hales [**15**]. For $\mathrm{GL}_n(\Omega)$, Rogawski [**62**] stated and proved (in some cases) a conjecture about the values of the Shalika germs evaluated at certain elliptic elements. Rogawski's conjecture for the Shalika germs of $\mathrm{GL}_n(\Omega)$ was confirmed by Waldspurger in [**74**]. This work was used by Murnaghan and Repka [**52**] to investigate which Shalika germs contribute to expansions about singular elliptic elements. For $\mathrm{GL}_n(\Omega)$, Waldspurger [**75**] provided an algorithm for computing Shalika germs which significantly extended the results of his earlier paper [**74**]. Courtès [**5**] extended this work of Waldspurger's to the group $\mathrm{SL}_n(\Omega)$. A few groups have had their Shalkia germs nearly completely worked out: Sally and Shalika [**64, 66**] computed them for $SL_2(\Omega)$ on the elliptic set, Langlands and Shelstad [**38**] calculated most of them for $SU(3, \Omega)$, and Hales [**14, 16**] worked them out for $GSp_4(\Omega)$ and $Sp_4(\Omega)$. There are many interesting questions surrounding Shalika germs and the theory of endoscopy which are beyond the scope of this discussion (see [**37**] and [**71**]).

Finally, in Part III Harish-Chandra studies admissible distributions on G. The goal of this section is to provide a way to transfer the results of Part I and Part II, which were concerned with G-invariant distributions on $\mathfrak{g}$, to admissible distributions on G. Suppose that we are only concerned with the behavior of our distribution on G near the identity. The definition of an admissible distribution, which Harish-Chandra attributes to the work of Howe (see [**20**, §16], and [**24, 25, 26**]), combined with some results derived from Howe's "Kirillov theory" ([**24, 25**]), allows him to derive from an admissible distribution on G a distribution on $\mathfrak{g}$ which

(1) satisfies the hypothesis of the extension of Howe's Theorem and
(2) is related to the original distribution on G via the exponential map.

Harish-Chandra is then able to conclude that near the identity, Θ_π is represented by a locally summable function on G.

There have been some generalizations of these notes to different settings. In [**4**], Clozel showed that the main results of these notes hold for non-connected groups. Clozel's paper also includes almost all of Part III of these notes. In [**11**] and [**12**], Hakim extended the content of these notes to certain symmetric spaces. However, in general, not everything in these notes can be extended to symmetric spaces. Rader and Rallis [**53**] generalized some of what can be carried over and provided counterexamples for those results which cannot be extended to the symmetric space setting. For further analysis of the symmetric space situation, see the work of Bosman [**3**] and Flicker [**9**]. Finally, Vignéras [**70**] explored the situation for modular representations.

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List of Symbols

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Admissible Invariant Distributions on Reductive p-adic Groups

Harish-Chandra †,
Notes by Stephen DeBacker, and Paul J. Sally, Jr.

Harish-Chandra presented these lectures on admissible invariant distributions for p-adic groups at the Institute for Advanced Study in the early 1970s. He published a short sketch of this material as his famous "Queen's Notes". This book, which was prepared and edited by DeBacker and Sally, presents a faithful rendering of Harish-Chandra's original lecture notes.

The main purpose of Harish-Chandra's lectures was to show that the character of an irreducible admissible representation of a connected reductive p-adic group G is represented by a locally summable function on G. A key ingredient in this proof is the study of the Fourier transforms of distributions on $\mathfrak{g}$, the Lie algebra of G. In particular, Harish-Chandra shows that if the support of a G-invariant distribution on $\mathfrak{g}$ is compactly generated, then its Fourier transform has an asymptotic expansion about any semisimple point of $\mathfrak{g}$.

Harish-Chandra's remarkable theorem on the local summability of characters for p-adic groups was a major result in representation theory that spawned many other significant results. This book presents, for the first time in print, a complete account of Harish-Chandra's original lectures on this subject, including his extension and proof of Howe's Theorem.

In addition to the original Harish-Chandra notes, DeBacker and Sally provide a nice summary of developments in this area of mathematics since the lectures were originally delivered. In particular, they discuss quantitative results related to the local character expansion.

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