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Introduction to Mathematical Statistical Physics

R. A. Minlos
2000 Mathematics Subject Classification. Primary 82-01; Secondary 82B05.

ABSTRACT. These lectures, aimed at beginners, explain the main ideas, notions, and tools of mathematical statistical physics. The main emphasis is explaining the problems centered around the central concept of the limiting Gibbsian field. In particular, a significant part of the text is devoted to the Pirogov–Sinai theory.

The book can be used by graduate students and others who want to learn about mathematical approaches to statistical physics.
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Preliminaries

The title of the book is "Introduction to mathematical statistical physics". I would like to explain what "mathematical" means here. Mathematical physics in general, and mathematical statistical physics in particular, emerged in its present form thirty or forty years ago as an attempt of mathematicians to understand mathematical structures that form the basis of fundamental theories in physics. Since physicists and mathematicians deal with the same subject, the difference between the physical approach and the mathematical one is, from the formal point of view, not absolute. However, such a difference exists and it is mainly of a psychological nature. Roughly speaking, physicists want to get a quick explanation of an experiment, which may be nonrigorous and not correct in details. On the other hand, mathematicians are not interested in the experimental data and they want to construct a clear and noncontradictory picture of the phenomena based on physical postulates.

Of course, mathematicians studying physics convert it to mathematics with all the accepted canons: theorems, lemmas, proofs, exact definitions and so on. In addition, opportunities appear for applying many fine and abstract mathematical theories to physics.

It should be mentioned that there is an inverse influence of these studies on mathematics. The purpose and motivations of mathematical theories are more and more penetrated by the spirit of physics. But this is a separate subject, which requires a special lecture.

These lectures are aimed at beginners and emphasis is made on explaining the main ideas, notions and facts, rather than on technical tools. The inner motivation of the lectures is to introduce the reader to problems centering around the main concept of modern mathematical statistical physics — the concept of limiting Gibbsian field. In particular, the major part of these lectures is devoted to the famous Pirogov–Sinai theory, which allows us to establish the existence of several such fields for a given system (this is treated now as phase transition).

After studying these lectures the reader should turn to more detailed texts, such as the famous book by Ruelle [1], the book by Thompson [8], the monograph by Simon [9] and others (see the bibliography). Unfortunately, in these books there is no explanation of the Pirogov–Sinai theory. The only monograph where one can find such an explanation is in the book by Sinai [4]. But it is explained there in very general and concise form, which is rather difficult for beginners.

It is useful also to look in textbooks on physics.
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Introduction to Mathematical Statistical Physics

R. A. Minlos

This book presents a mathematically rigorous approach to the main ideas and phenomena of statistical physics. The introduction addresses the physical motivation, focussing on the basic concept of modern statistical physics, that is the notion of Gibbsian random fields. Properties of Gibbsian fields are analyzed in two ranges of physical parameters: “regular” (corresponding to high-temperature and low-density regimes) where no phase transition is exhibited, and “singular” (low temperature regimes) where such transitions occur.

Next, a detailed approach to the analysis of the phenomena of phase transitions of the first kind, the Pirogov-Sinai theory, is presented. The author discusses this theory in a general way and illustrates it with the example of a lattice gas with three types of particles. The conclusion gives a brief review of recent developments arising from this theory.

The volume is written for the beginner, yet advanced students will benefit from it as well. The book will serve nicely as a supplementary textbook for course study. The prerequisites are an elementary knowledge of mechanics, probability theory and functional analysis.

ISBN 0-8218-1337-4