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**American Mathematical Society**  
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ABSTRACT. This book gives an exposition of the relations among the following three topics: monoidal tensor categories (such as a category of representations of a quantum group), 3-dimensional topological quantum field theory, and 2-dimensional modular functors, which naturally arise in 2-dimensional conformal field theory. As examples, the category of representations of a quantum group at a root of unity and the Wess-Zumino-Witten modular functor are discussed in detail.

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**To our parents**



# Contents

Introduction	1
Chapter 1. Braided Tensor Categories	9
1.1. Monoidal tensor categories	9
1.2. Braided tensor categories	15
1.3. Quantum groups	19
1.4. Drinfeld's category	21
Chapter 2. Ribbon Categories	29
2.1. Rigid monoidal categories	29
2.2. Ribbon categories	33
2.3. Graphical calculus for morphisms	35
2.4. Semisimple categories	44
Chapter 3. Modular Tensor Categories	47
3.1. Modular tensor categories	47
3.2. Example: Quantum double of a finite group	59
3.3. Quantum groups at roots of unity	62
Chapter 4. 3-Dimensional Topological Quantum Field Theory	71
4.1. Invariants of 3-manifolds	71
4.2. Topological quantum field theory	76
4.3. 1+1 dimensional TQFT	78
4.4. 3D TQFT from MTC	82
4.5. Examples	87
Chapter 5. Modular Functors	93
5.1. Modular functors	93
5.2. The Lego game	99
5.3. Ribbon categories via the Hom spaces	108
5.4. Modular functors in genus zero and tensor categories	116
5.5. Modular categories and modular functors for zero central charge	119
5.6. Towers of groupoids	124
5.7. Central extensions of modular functors	130
5.8. From 2D MF to 3D TQFT	135
Chapter 6. Moduli Spaces and Complex Modular Functors	141
6.1. Moduli spaces and complex Teichmüller tower	141
6.2. Compactification of the moduli space and gluing	147
6.3. Connections with regular singularities	152
6.4. Complex-analytic modular functors	158

6.5. Example: Drinfeld's category	161
6.6. Twisted $\mathcal{D}$ -modules	164
6.7. Complex modular functors with central charge	167
Chapter 7. Wess–Zumino–Witten Model	173
7.1. Preliminaries on affine Lie algebras	174
7.2. Reminders from algebraic geometry	175
7.3. Conformal blocks: definition	177
7.4. Flat connection in the bundle of conformal blocks	181
7.5. From local parameters to tangent vectors	188
7.6. Families of curves over a formal base	191
7.7. Coinvariants for singular curves	194
7.8. The gluing axiom for the bundle of coinvariants	195
7.9. WZW modular functor	202
7.10. Calculation of the central charge	203
Bibliography	213
Index	217
Index of Notation	219

## Bibliography

- [Ab] Abikoff, W., *The real analytic theory of Teichmüller space*, Lecture Notes in Mathematics **820**, Springer, Berlin, 1980.
- [ACK] Arbarello, E., De Concini, C., and Kac, V.G., *The infinite wedge representation and the reciprocity law for algebraic curves*, in Theta Functions — Bowdoin 1987, Part I, 171–190, Proc. Sympos. Pure Math., 49, Amer. Math. Soc., Providence, RI, 1989.
- [AP] Andersen, H.H. and Paradowski, J., *Fusion categories arising from semisimple Lie algebras*, Comm. Math. Phys. **169** (1995), 563–588.
- [APW] Andersen, H.H., Polo, P., and Wen, K., *Representations of quantum algebras*, Invent. Math. **104** (1991), 1–59.
- [Ar] Artin, M., *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.
- [At] Atiyah, M., *Topological quantum field theories*, Publ. Math. IHES **68** (1989) 175–186.
- [B1] Birman, J., *Braids, links, and mapping class groups*, Ann. Math. Stud., vol. 82, 1972.
- [B2] ———, *Mapping class groups and their relationship to braid groups*, Comm. Pure Appl. Math. **22** (1969), 213–238.
- [BD] Belavin, A.A. and Drinfeld, V.G., *Solutions of classical Yang–Baxter equation and classical Lie algebras*, Funktz. Analiz i Ego Prilozh. **16** (1982), no. 3, 1–29 (Russian); English transl. in Funct. Anal. and Appl. **16** (1982), 159–180.
- [Be] Beauville, A., *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 75–96, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996.
- [Ber] Bernstein, J. *Lectures on D-modules*, unpublished.
- [BFM] Beilinson, A., Feigin, B., and Mazur, B., *Introduction to field theory on curves*, manuscript, May 1990.
- [Bjo] Björk, J.–E. *Analytic D-modules and applications*, Mathematics and its Applications, 247, Kluwer Academic Publishers Group, Dordrecht, 1993.
- [BK] Bakalov, B. and Kirillov, A., Jr. *On the Lego–Teichmüller game*, Transf. Groups **5** (2000), no. 3, 7–44; math.GT/9809057.
- [BN] Bar-Natan, D., *Non-associative tangles*, in “Geometric topology” (Athens, GA, 1993), 139–183, AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI, 1997.
- [Bor] Borel, A. et al. *Algebraic D-modules*, Academic Press, Inc., Boston, MA, 1987.
- [BPZ] Belavin, A.A., Polyakov, A.M., and Zamolodchikov, A.B., *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Phys. **B241** (1984), 333–380.
- [BS] Beilinson, A.A. and Schechtman, V.V., *Determinant bundles and Virasoro algebras*, Comm. Math. Phys. **118** (1988), 651–701.
- [C] Crane, L., *2-d physics and 3-d topology*, Comm. Math. Phys. **135** (1991), 615–640.
- [CL] Coddington, E.A. and Levinson, N., *Theory of ordinary differential equations*, McGraw-Hill, New York, Toronto, London, 1955.
- [CP] Chari, V. and Pressley, A., *A guide to quantum groups*, Cambridge Univ. Press, Cambridge, 1995.
- [Cr] Craggs, R., *A new proof of the Reidemeister–Singer theorem on stable equivalence of Heegaard splittings*, Proc. Amer. Math. Soc. **57** (1976), 143–147.
- [Del1] Deligne, P., *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163. Springer–Verlag, Berlin–New York, 1970.
- [De2] ———, *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, pp. 111–195, Progr. Math., 87, Birkhäuser Boston, Boston, MA, 1990.
- [De3] ———, *Action du groupe des tresses sur une catégorie*, Invent. Math. **128** (1997), 159–175.

- [DM] Deligne, P. and Mumford, D., *The irreducibility of the space of curves of given genus*, Publ. Math. IHES **36** (1969), 75–109.
- [DPR] Dijkgraaf, R., Pasquier, V., and Roche, P., *Quasi Hopf algebras, group cohomology and orbifold models*, Nuclear Phys. B Proc. Suppl. **18B** (1990), 60–72.
- [Dr1] Drinfeld, V.G., *Quasi-Hopf algebras*, Algebra i Analiz **1**, no. 6 (1989), 114–148 (Russian), English translation in Leningrad Math. J. **1** (1990), 1419–1457.
- [Dr2] ———, *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990), no. 2, 321–342.
- [Dr3] ———, *Quantum groups*, Proc. Intern. Congr. Math., Berkeley, 1986, pp. 798–820.
- [Dr4] ———, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$* , (Russian) Algebra i Analiz **2** (1990), no. 4, 149–181; English translation in Leningrad Math. J. **2** (1991), 829–860.
- [DVVV] Dijkgraaf, R., Vafa, C., Verlinde, E., and Verlinde, H., *The operator algebra of orbifold models*, Comm. Math. Phys. **123** (1989), 485–526.
- [EFK] Etingof, P., Frenkel, I., and Kirillov, A., Jr., *Lectures on representation theory and Knizhnik–Zamolodchikov equations*, Amer. Math. Soc., Providence, RI, 1998.
- [F] Finkelberg, M., *An equivalence of fusion categories*, Geom. Funct. Anal. **6** (1996), 249–267.
- [FG] Funar, L. and Gelca, R., *On the groupoid of transformations of rigid structures on surfaces*, J. Math. Sci. Univ. Tokyo, **6** (1999), 599–646.
- [FMS] Di Francesco, P., Mathieu, P., and Sénéchal, D., *Conformal field theory*, Graduate Texts in Contemporary Physics, Springer–Verlag, New York, 1997.
- [FS] Friedan, D., and Shenker, S., *The analytic geometry of two-dimensional conformal field theory*, Nuclear Phys. **B 281** (1987), 509–545.
- [Fu] Funar, L., *2+1-D topological quantum field theory and 2-D conformal field theory*, Comm. Math. Phys. **171** (1995), 405–458.
- [G] Grothendieck, A., *Esquisse d’un programme* (1984), published in [LS], pp. 5–48. (English translation in [LS], pp. 243–283.)
- [Gab] Gabber, O., *The integrability of the characteristic variety*, Amer. J. Math., **103** (1981), 445–468.
- [Gai] Gaitsgory, D., *Notes on 2D conformal field theory and string theory*, in *Quantum fields and strings: A course for mathematicians*, vol. 2, pp. 1017–1077, Amer. Math. Soc., Providence, RI, 1999.
- [Ge1] Gervais, S., *Presentation and central extensions of mapping class groups*, Trans. Amer. Math. Soc. **348** (1996), 3097–3132.
- [Ge2] ———, *A finite presentation of the mapping class group of an oriented surface*, preprint math.GT/9811162, to appear in Topology.
- [GH] Griffiths, P. and Harris, J., *Principles of algebraic geometry*, Wiley–Interscience [John Wiley & Sons], New York, 1978.
- [GR] Gunning, R. and Rossi, H., *Analytic functions of several complex variables*, Prentice–Hall, Englewood Cliffs, NJ, 1965.
- [Ha] Hartshorne, R., *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer–Verlag, New York–Heidelberg, 1977.
- [HL] Huang, Y.-Z. and Lepowsky, J., *Intertwining operator algebras and vertex tensor categories for affine Lie algebras*, Duke Math. J. **99** (1999), 113–134.
- [HLS] Hatcher, A., Lochak, P., and Schneps, L., *On the Teichmüller tower of mapping class groups*, J. Reine Angew. Math. **521** (2000), 1–24.
- [HM] Harris, J. and Morrison, I., *Moduli of curves*, Springer–Verlag, New York, 1998.
- [HT] Hatcher, A. and Thurston, W., *A presentation for the mapping class group of a closed orientable surface*, Topology **19** (1980), 221–237.
- [Hua] Huang, Yi-Zhi, *Two-dimensional conformal geometry and vertex operator algebras*, Progress in Mathematics, **148**. Birkhäuser, Boston, MA, 1997.
- [Hum] Humphreys, J.E., *Introduction to Lie algebras and representation theory*, Springer–Verlag, New York, 1972.
- [Jan] Jantzen, J., *Lectures on quantum groups*, Graduate Studies in Mathematics, **6**, Amer. Math. Soc., Providence, RI, 1996.
- [JS] Joyal, A. and Street, R., *Braided tensor categories*, Adv. Math. **102** (1993), 20–78.
- [K1] Kac, V.G., *Infinite-dimensional Lie algebras*, Cambridge Univ. Press, 3rd ed., 1990.

- [K2] ———, *Vertex algebras for beginners*, University Lecture Series, vol. 10, Amer. Math. Soc., Providence, RI, 1997. Second ed., 1998.
- [Ka] Kassel, C., *Quantum groups*, Graduate Texts in Mathematics, 155, Springer-Verlag, New York, 1995.
- [Ke] Kerler, T., *Mapping class group actions on quantum doubles*, *Comm. Math. Phys.* **168** (1995), 353–388.
- [Ki] Kirillov, A., *On inner product in modular tensor categories*. I, *J. Amer. Math. Soc.*, **9** (1996), 1135–1170; II, *Adv. Theor. Math. Phys.* **2** (1998), no. 1, 155–180.
- [KL] Kazhdan, D. and Lusztig, G., *Tensor structures arising from affine Lie algebras*. I, *J. Amer. Math. Soc.*, **6** (1993), 905–947; II, *J. AMS*, **6** (1993), 949–1011; III, *J. AMS*, **7** (1994), 335–381; IV, *J. AMS*, **7** (1994), 383–453.
- [KM] Knudsen, F. and Mumford, D., *The projectivity of the moduli space of stable curves*. I. Preliminaries on “det” and “Div”, *Math. Scand.* **39** (1976), 19–55.
- [Kn] Knudsen, F., *The projectivity of the moduli space of stable curves*, II. *The stacks  $M_{g,n}$* , *Math. Scand.* **52** (1983), 161–199; III. *The line bundles on  $M_{g,n}$ , and a proof of the projectivity of  $\bar{M}_{g,n}$  in characteristic 0*, *Math. Scand.* **52** (1983), 200–212.
- [Ko] Kohno, T., *Topological invariants for 3-manifolds using representations of mapping class groups* I, *Topology* **31** (1992), 203–230.
- [KP] Kac, V.G. and Peterson, D.H., *Infinite-dimensional Lie algebras, theta functions and modular forms*, *Adv. in Math.* **53** (1984), no. 2, 125–264.
- [KS] Kashiwara, M. and Schapira, P., *Sheaves on manifolds*, Springer-Verlag, Berlin, 1994.
- [KT] Kac, V.G. and Todorov, I.T., *Affine orbifolds and rational conformal field theory extensions of  $W_{1+\infty}$* , *Comm. Math. Phys.* **190** (1997), 57–111.
- [KW] Kazhdan, D. and Wenzl, H., *Reconstructing monoidal categories*, I. M. Gel’fand Seminar, 111–136, *Adv. Soviet Math.* **16**, Part 2, Amer. Math. Soc., Providence, RI, 1993.
- [KZ] Knizhnik, V.G. and Zamolodchikov, A.B., *Current algebra and Wess-Zumino model in two dimensions*, *Nuclear Phys.* **B 247** (1984), 83–103.
- [L1] Lusztig, G., *Quantum deformations of certain simple modules over enveloping algebras*, *Adv. Math.* **70** (1988), 237–249.
- [L2] ———, *Introduction to quantum groups*, Birkhäuser, Boston, 1993.
- [L3] ———, *Monodromic systems on affine flag manifolds*, *Proc. R. Soc. Lond. A* **455** (1994), 231–246; *Erratum*, *Proc. R. Soc. Lond. A* **450** (1995), 731–732.
- [L4] ———, *Modular representations and quantum groups*, *Contemp. Math.*, vol 82, 1989, Amer. Math. Soc., Providence, RI, pp. 52–77.
- [L5] ———, *Unipotent representations of a finite Chevalley group of type  $E_8$* , *Quart. J. Math. Oxford Ser. (2)* **30** (1979), no. 119, 315–338.
- [L6] ———, *Leading coefficients of character values of Hecke algebras*, in “The Arcata Conference on Representations of Finite Groups” (Arcata, Calif., 1986), pp. 165–179, *Proc. Sympos. Pure Math.*, 47, Part 1, Amer. Math. Soc., Providence, RI, 1987.
- [L7] ———, *Exotic Fourier transform*, *Duke Math. J.* **73** (1994), 227–241.
- [Li] Lickorish, W.B.R., *A finite set of generators for the homeotopy group of a 2-manifold*. *Proc. Cambridge Philos. Soc.* **60** (1964), 769–778.
- [LS] Lochak, P. and Schneps, L., eds., *Geometric Galois actions. 1. Around Grothendieck’s “Esquisse d’un programme”*, London Math. Soc. Lect. Note Series, 242, Cambridge University Press, Cambridge, 1997.
- [Luo] Luo, F., *A presentation of the mapping class groups*, *Math. Res. Lett.* **4** (1997), 735–739.
- [Lyu1] Lyubashenko, V., *Modular transformations for tensor categories*, *J. Pure Appl. Algebra* **98** (1995), no. 3, 279–327.
- [Lyu2] ———, *Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, *Comm. Math. Phys.* **172** (1995), 467–516.
- [Ma] Malgrange, B., *Regular connections, after Deligne*, in “Algebraic D-modules”, A. Borel et al, Academic Press, Boston, 1987, pp. 151–172.
- [Mac] MacLane, S., *Categories for the working mathematician* (Second edition), Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.
- [MFK] Mumford, D., Fogarty, J., and Kirwan, F., *Geometric invariant theory*, 3rd ed., Springer-Verlag, Berlin, 1994.
- [Mo] Motto, M., *Maximal triads and prime decompositions of surfaces embedded in 3-manifolds*, *Trans. Amer. Math. Soc.* **331** (1992), no. 2, 851–867.

- [MS1] Moore, G. and Seiberg, N., *Classical and quantum conformal field theory*, Comm. Math. Phys. **123** (1989), 177–254.
- [MS2] ———, *Lectures on RCFT*, Superstrings '89 (Proc. of the 1989 Trieste Spring School), M. Green et al., eds., World Sci., River Edge, NJ, 1990, pp. 1–129.
- [PS] Prasolov, V.V. and Sossinsky, A.B., *Knots, links, braids and 3-manifolds. An introduction to the new invariants in low-dimensional topology*. Translations of Mathematical Monographs **154**, Amer. Math. Soc., Providence, RI, 1997.
- [Q] Quinn, F., *Lectures on axiomatic topological quantum field theory*, in “Geometry and quantum field theory” (Park City, UT, 1991), pp. 323–453, D. Freed and K. Uhlenbeck, eds., Amer. Math. Soc., Providence, RI, IAS, Princeton, NJ, 1995.
- [RT1] Reshetikhin, N.Yu. and Turaev, V.G., *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), 1–26.
- [RT2] ———, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.
- [S] Segal, G., *Two-dimensional conformal field theories and modular functors*. IXth International Congress on Mathematical Physics (Swansea, 1988), 22–37, Hilger, Bristol, 1989.
- [Sat] Satake, I., *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363.
- [Su] Suzuki, S., *On homeomorphisms of a 3-dimensional handlebody*, Can. J. Math. **29** (1977), 111–124.
- [SV] Schechtman, V. and Varchenko, A., *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), 139–194.
- [Ta] Tate, J., *Residues of differentials on curves*, Ann. Sci. Ecole Norm. Sup. (4) **1** (1968), 149–159.
- [Tel] Teleman, C., *Lie algebra cohomology and the fusion rules*, Comm. Math. Phys. **173** (1995), 265–311.
- [TK] Tsuchiya, A. and Kanie, Y., *Vertex operators in conformal field theory on  $P^1$  and monodromy representations of braid group*, Adv. Stud. Pure Math. **16** (1988), pp. 297–372.
- [Tu] Turaev, V.G., *Quantum invariants of knots and 3-manifolds*, W. de Gruyter, Berlin, 1994.
- [TUY] Tsuchiya, A., Ueno, K., and Yamada, Y., *Conformal field theory on universal family of stable curves with gauge symmetries*, Adv. Stud. Pure Math. **19** (1992), pp. 459–566.
- [V1] Vafa, C., *Conformal theories and punctured surfaces*, Phys. Lett. **B 199** (1987), 195–202.
- [V2] ———, *Toward classification of conformal theories*, Phys. Lett. **B 206** (1988), 421–426.
- [Ve] Verlinde, E., *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. **B 300** (1988), 360–376.
- [Vi] Vistoli, A., *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670.
- [Vo] Vogel, P., *Algebraic structures on modules of diagrams*, Université Paris VII preprint, October 1996.
- [W1] Witten, E., *Topological quantum field theory*, Comm. Math. Phys. **117** (1988), 353–386.
- [W2] ———, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.
- [Waj] Wajnryb, B., *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. **45** (1983), 157–174.
- [Z] Zhu, Y., *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996), 237–302.

# Index

Abelian category, .....	9	left, .....	29
semisimple .....	9	right, .....	29
Additive category, .....	9	Extended surface, .....	95, 96
Affine Kac–Moody algebra, .....	174	Family of curves, .....	143
Associativity isomorphism, .....	12	Frobenius algebra, .....	79
Balancing axioms, .....	34	Functorial morphism, .....	11
Balancing isomorphism, .....	33	Fusion algebra, .....	52
Bialgebra, .....	12	Fusion coefficients, .....	44
cocommutative, .....	18	Fusion rule, .....	44
quasitriangular, .....	18	Gluing,	
Braid, .....	15	of manifolds, .....	71
Braid group, .....	15	for modular functor, .....	97
Braided tensor category (BTC), .....	17	for MS data, .....	109
Canonical isomorphism, .....	12	for towers, .....	124
Casimir element, .....	22	Grothendieck group, .....	32
for quantum groups, .....	34	Grothendieck ring, .....	32
Catalan number, .....	11	Groupoid, .....	94
Central charge, .....	58	Handlebody, .....	84, 136
$\mathcal{C}$ -colored ribbon tangle, .....	40	Heegaard splitting, .....	136
$\mathcal{C}$ -marked surface, .....	82, 97	Hopf algebra, .....	30
$\mathcal{C}$ -marked 3-manifold, .....	82	cocommutative, .....	18
Clifford algebra, .....	204	quasitriangular, .....	18
Commutativity isomorphism, .....	17	Kirby calculus, .....	75
Conformal blocks, .....	109, 158	Kirby–Fenn–Rourke moves, .....	75
for WZW model, .....	177	Knizhnik–Zamolodchikov (KZ) equations, .	
Conformal dimension, .....	58, 183	.....	23, 161
Connection, .....	153	Knot, .....	39
flat, .....	153	framed (ribbon), .....	40
$\log D$ , .....	153	Lantern identity, .....	58
monodromic, .....	156	Lie algebroid, .....	210
projectively flat, .....	164	Link, .....	39
with logarithmic singularities, .....	153	framed (ribbon), .....	40
with regular singularities, .....	155	special, .....	86
Coupon, .....	40	Lisse sheaf, .....	175
$\mathcal{D}$ -module, .....	155	Local system, .....	153
twisted, .....	166	Loop algebra, .....	174
Dehn twist, .....	73, 95	Mapping class group, .....	94, 95
Determinant line bundle, .....	168	pure, .....	95
Dimension, .....	39	Marking graph, .....	101, 128
Divisor with normal crossings, .....	148	Maslov index, .....	131
Double point, .....	147	Modular functor (MF), .....	93
Drinfeld associator, .....	22	$\mathcal{C}$ -extended, .....	97
Drinfeld’s category, .....	22		
Dual object, .....	29		

- complex, ..... 158
- genus 0, ..... 116
- non-degenerate, ..... 98
- unitary, ..... 99
- with additive central charge  $a$ , ..... 169
- with multiplicative central charge  $K$ , ..... 133
- Modular tensor category (MTC), ..... 48
- Moduli space of punctured curves, ..... 142
  - Deligne–Mumford compactification, ..... 148
- Monoidal category, ..... 12
  - strict, ..... 14
- Moore–Seiberg (MS) data, ..... 109
- Moore–Seiberg tower  $MS$ , ..... 128
  
- Natural transformation, ..... 11
- Negligible module, ..... 67
- Negligible morphism, ..... 67
  
- Parameterization, ..... 99
- Pentagon axiom, ..... 13
- Poincaré groupoid, ..... 130, 146
- Pointed curve, ..... 142
  
- Quantum dimension, ..... 44
- Quantum group, ..... 19
- Quantum Yang–Baxter equation (QYBE), ..... 18
  
- Representation of a tower, ..... 127
- Reshetikhin–Turaev (RT) invariants,
  - of links, ..... 43
  - of 3-manifolds, ..... 75
- Ribbon category, ..... 33
- Ribbon tangle, ..... 40
  - $\mathcal{C}$ -colored, ..... 40
  - generalized, ..... 40
- Riemann–Hilbert correspondence, ..... 155
- Rigidity axioms, ..... 29
- Rigid monoidal category, ..... 30
- $R$ -matrix, ..... 18
  
- Semi-infinite forms, ..... 207
- Serre relations, ..... 19
- Siegel upper half plane, ..... 170
- Simple object, ..... 9
- Stable curve, ..... 144, 148
- Stack, ..... 143
- Standard sphere, ..... 99
- Surgery, ..... 74
- Symmetric tensor category (STC), ..... 18
  
- Tangle, ..... 39
  - $\mathcal{C}$ -colored ribbon, ..... 40
  - framed (ribbon), ..... 40
- Tate central extension, ..... 205
- Tate vector space, ..... 206
- Teichmüller groupoid, ..... 95
  - complex, ..... 146
  - extended, ..... 95
- Teichmüller space, ..... 142
  
- Teichmüller tower  $\widetilde{\text{Teich}}$ , ..... 124
  - central extension  $\widetilde{\text{Teich}}$ , ..... 132
  - complex  $\text{Teich}^{\mathbb{C}}$ , ..... 146
  - in genus zero  $\text{Teich}_0$ , ..... 125
  - parameterized  $\mathcal{P}\text{Teich}$ , ..... 129
- Tensor functor, ..... 18
- Tilting module, ..... 66
  - negligible, ..... 67
- Topological quantum field theory (TQFT),
  - ..... 76
  - $\mathcal{C}$ -extended, ..... 83
  - with corners, ..... 83
- Torsor, ..... 183
- Tower functor, ..... 126
- Tower of groupoids, ..... 124, 126
- Trace, ..... 39
- Trinion (pair of pants), ..... 79
- Twist, ..... 33
  
- Unit isomorphisms, ..... 12
- Universal  $R$ -matrix, ..... 18
  
- Vassiliev invariants, ..... 43
- Verlinde algebra, ..... 52
- Verlinde formula, ..... 54
- Virasoro algebra, ..... 183
  
- Weakly ribbon category, ..... 111
- Weakly rigid monoidal category, ..... 111
- Weyl denominator, ..... 64
- Weyl formula, ..... 64
- Weyl module, ..... 62, 175
- Witt algebra, ..... 183
  
- Yang–Baxter equation, ..... 18



# Index of Notation

- $\alpha_{UVW}$  — associativity isomorphism, see Definition 1.1.7  
 $\mathcal{A}_{\mathcal{L}}$  — the sheaf of first-order differential operators in a line bundle  $L$ , see Example 6.6.3  
 $B_n$  — the braid group in  $n$  strands, see Definition 1.2.1  
 $\langle\langle \cdot, \cdot \rangle\rangle$  — see the beginning of Section 1.3  
 $\langle \cdot, \cdot \rangle$  — see the beginning of Section 1.4  
 $(\cdot, \cdot)$  — the pairing between a vector space and its dual  
 $\langle W_1, \dots, W_n \rangle = \text{Hom}_{\mathbb{C}}(\mathbf{1}, W_1, \dots, W_n)$  — see (5.3.2)  
 $\overline{N}$  — the manifold  $N$  with reversed orientation  
 $\mathbb{C}$  — the field of complex numbers  
 $\mathbb{C}^\times$  — the set of non-zero complex numbers  
 $\mathbb{C}_q = \mathbb{C}(q^{1/P}/Q)$  — see the beginning of Section 1.3  
 $\mathcal{C}$  — a category  
 $\mathcal{C}^{\text{op}}$  — the opposite (dual) category to  $\mathcal{C}$   
 $\mathcal{C}_1 \boxtimes \mathcal{C}_2$  — tensor product of additive categories, see Definition 1.1.15  
 $\mathcal{C}^{\boxtimes n}, \mathcal{C}^{\boxtimes A}$  — tensor product of  $\mathcal{C}$  with itself, see Definition 1.1.15  
 $\mathcal{C}(\mathfrak{g})$  — the category of finite-dimensional representations of  $U_q(\mathfrak{g})$  over  $\mathbb{C}_q$  that have a weight decomposition (1.3.13)  
 $\mathcal{C}(\mathfrak{g}, \varkappa)$  — the category of finite-dimensional representations of  $U_q(\mathfrak{g})|_{q=e^{\pi i/(m\varkappa)}}$  over  $\mathbb{C}$  possessing a weight decomposition, see Theorem 1.3.2  
 $\mathcal{C}^{\text{int}} \equiv \mathcal{C}^{\text{int}}(\mathfrak{g}, \varkappa)$  — the category of tilting modules over  $U_q(\mathfrak{g})$  modulo negligible morphisms, see Definition 3.3.19  
 $c_{ij} = \delta_{ij}^*$  — charge conjugation matrix, see Theorem 3.1.7  
 $C_{ij}$  — see (3.1.34)  
 $\overline{C}$  — the Weyl chamber, see (3.3.6)  
 $C^\vee$  — normalization of a singular curve  $C$ , see Section 7.7  
 $\mathcal{C}(V)$  — the Clifford algebra of  $V \oplus V^*$  for a Tate vector space  $V$ , see Definition 7.10.3  
 $\Delta$  — comultiplication, see Example 1.1.8(iii)  
 $\Delta_\lambda$  — conformal dimensions, see (7.4.7)  
 $\mathcal{D}_S$  — sheaf of differential operators on  $S$   
 $\mathcal{D}_{\mathcal{L}^c}$  — twisted sheaf of differential operators, see Definition 6.6.5  
 $\mathcal{D}_S^0$  — see (6.3.5)  
 $\mathcal{D}(\mathfrak{g}, \varkappa)$  — Drinfeld's category, see Theorem 1.4.5  
 $\det L = \bigwedge^{\dim L} L$  — the top exterior power of a vector space  $L$ , see (6.7.1)  
 $\dim$  — dimension, see (2.3.9)  
 $\dim_q$  — quantum dimension, see (2.3.10)  
 $\delta, \delta_V$  — isomorphism  $V \xrightarrow{\sim} V^{**}$ , see Definition 2.2.1  
 $d_i$  —  $\dim V_i$ , see (2.4.4)  
 $D$  — Casimir element in  $U(\mathfrak{g})$ , see (1.4.4)  
 $D = \sqrt{p^+ p^-}$  in an MTC, see (3.1.15)  
 $D(G)$  — the Drinfeld double of  $k[G]$ , see Section 3.2  
 $DV$  — dual of  $V$  in the category of  $\hat{\mathfrak{g}}$ -modules of level  $k$ , see Section 7.1  
 $D^n$  —  $n$ -disk  
 $D^{(n)}$  —  $n$ -th infinitesimal neighborhood of a divisor  $D \subset S$ , see Section 7.6  
 $\partial M$  — the boundary of a manifold  $M$   
 $\sqcup$  — disjoint union  
 $\text{End}_{\mathcal{C}}(U) = \text{Mor}_{\mathcal{C}}(U, U)$  — the set of endomorphisms of  $U$  in  $\mathcal{C}$   
 $\varepsilon$  — counit, see Example 1.1.8(iii)  
 $e_V$  — evaluation morphism  $V^* \otimes V \rightarrow \mathbf{1}$ , see Definition 2.1.1  
 $f^*$  — the dual morphism to  $f$ , see (2.1.15)  
 $F^{-1}(T)$  — Reshetikhin–Turaev invariant, see Corollary 2.3.11  
 $\mathcal{F}un(\mathcal{C})$  — see Example 5.6.11  
 $\gamma$  — the antipode of a Hopf algebra, see Example 2.1.4  
 $\Gamma(\Sigma), \Gamma_{g,n}$  — mapping class group, see Definition 5.1.7(ii)  
 $\Gamma_g = \Gamma_{g,0}$   
 $\Gamma'(\Sigma), \Gamma'_{g,n}$  — pure mapping class group, see Definition 5.1.7(ii)  
 $\mathfrak{g}$  — simple finite-dimensional Lie algebra over  $\mathbb{C}$ , see Section 1.3

- $\hat{\mathfrak{g}}$  — affine Lie algebra, see (7.1.1)  
 $\mathfrak{g}(C - \bar{p})$  — Lie algebra of rational  $\mathfrak{g}$ -valued functions on the curve  $C$ , see (7.3.1)  
 $\mathfrak{gl}(V)$ ,  $\widehat{\mathfrak{gl}}(V)$  — the Lie algebra of continuous endomorphisms of a Tate vector space  $V$  and its central extension, see Eq. (7.10.5), Definition 7.10.18.
- $\text{Hom}_{\mathcal{C}}(U, V)$  — the vector space of morphisms from  $U$  to  $V$  in an abelian category  $\mathcal{C}$   
 $H$  —  $\bigoplus V_i \otimes V_i^*$ , see (2.4.9)  
 $H(\Sigma) = H_1(\text{cl}(\Sigma), \mathbb{R})$  — see Section 5.7  
 $h^V$  — the dual Coxeter number for  $\mathfrak{g}$ , see page 63
- $i_V$  — the morphism  $\mathbf{1} \rightarrow V \otimes V^*$ , see Definition 2.1.1  
 $\text{ind-}\mathcal{C}^{\boxtimes 2}$  — a completion of  $\mathcal{C}^{\boxtimes 2}$ , see (2.4.7)
- $k$  — a field of characteristic 0  
 $K(\mathcal{C})$  — the Grothendieck group (or ring) of  $\mathcal{C}$ , see Definition 2.1.9  
 $k[G]$  — the group algebra of a group  $G$ , see Section 3.2
- $\lambda_V$  — the isomorphism  $\mathbf{1} \otimes V \xrightarrow{\sim} V$ , see Definition 1.1.7  
 $\Lambda_V$  — the set of all Lagrangian subspaces of a symplectic real vector space, see the beginning of Section 5.7  
 $\Lambda_{\Sigma} = \Lambda_{H(\Sigma)}$  — see Section 5.7  
 $L_{\lambda}^k$  — irreducible integrable module over an affine Lie algebra, see Section 7.1  
 $\Lambda_{\alpha}^{\infty/2}(V)$  — the space of semi-infinite forms, see Definition 7.10.14.
- $\text{Mor } \mathcal{C}$  — the class of morphisms in a category  $\mathcal{C}$   
 $\text{Mor}_{\mathcal{C}}(U, V)$  — the set of morphisms in  $\mathcal{C}$  from  $U$  to  $V$   
 $M_L$  — a surgery of  $S^3$  along a framed link  $L$ , see Definition 4.1.7  
 $\mathcal{MS}$  — the Moore–Seiberg tower, see Example 5.6.16  
 $\mathcal{M}_{g,n}, \overline{\mathcal{M}}_{g,n}$  — the (coarse) moduli space of pointed curves of genus  $g$  with  $n$  marked points and its compactification, see Sections 6.1, 6.2  
 $\mathcal{M}_g = \mathcal{M}_{g,0}$   
 $M_{g,n}, \overline{M}_{g,n}$  — the moduli stack of pointed curves of genus  $g$  with  $n$  marked points and its compactification, see Sections 6.1, 6.2  
 $M_{*,A}$  — moduli stack of pointed curves with marked points labeled by  $A$ , see (6.1.3)  
 $M(\Sigma)$  — see Definition 5.2.1  
 $\mathcal{M}(\Sigma)$  — see Theorems 5.2.3 and 5.2.10
- $\mathbb{N} = \{1, 2, 3, \dots\}$  — the set of natural numbers  
 $[n]_i, [n]_i!, \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_i$  — see (1.3.6)  
 $N_{ij}^k$  — tensor product multiplicities, or fusion coefficients, see (2.4.1)
- $\Omega = \sum a_i \otimes a^i$  — see (1.4.1)  
 $\text{Ob } \mathcal{C}$  — the class of objects in a category  $\mathcal{C}$   
 $\mathcal{O}_k, \mathcal{O}_k^{\text{int}}$  — categories of level  $k$  modules and of integrable level  $k$  modules over an affine Lie algebra, see Section 7.1  
 $\mathcal{O}_S$  — the structure sheaf of an analytic manifold, see Section 7.2
- $p^{\pm}$  — see (3.1.7)  
 $P_+$  — the cone of dominant integer weights, see the beginning of Section 1.3  
 $P_+^k$  — dominant integer weights corresponding to integrable  $\hat{\mathfrak{g}}$ -modules of level  $k$ , see (7.1.3)  
 $\pi_1(M)$  — the (fundamental) Poincaré groupoid of a topological space  $M$   
 $\mathcal{PTeich}$  — parameterized Teichmüller tower, see Example 5.6.18  
 $\mathcal{PTeich}_0$  — the genus zero part of  $\mathcal{PTeich}$ , see Remark 5.6.3
- $\mathbb{Q}$  — the field of rational numbers  
 $\mathbb{Q}_+$  — the set of positive rational numbers  
 $\mathbb{Q}_-$  — the set of negative rational numbers  
 $Q$  — determinant line bundle, see Definition 6.7.4
- $\mathbb{R}$  — the field of real numbers  
 $\mathbb{R}_+$  — the set of positive real numbers  
 $\mathbb{R}_-$  — the set of negative real numbers  
 $\mathbb{R}^{\times}$  — the set of non-zero real numbers  
 $R$  — universal  $R$ -matrix, see Example 1.2.8(iii)  
 $R = \bigoplus V_i \boxtimes V_i^*$ , see (2.4.7)  
 $R^{(1)}, R^{(2)}$  — see the end of Section 2.4  
 $\rho$  — the half sum of positive roots of  $\mathfrak{g}$   
 $\rho_V$  — the isomorphism  $V \otimes \mathbf{1} \xrightarrow{\sim} V$ , see Definition 1.1.7  
 $\text{Rep}(A)$  — the category of representations of an algebra (or a group)  $A$   
 $\text{Rep}_f(A)$  — the category of finite-dimensional representations of  $A$   
 $\mathcal{RS}(\overline{M}, M)$  — the category of flat connections with regular singularities on  $\overline{M}$ , see Section 6.3
- $\sigma_{VW}$  — commutativity isomorphism, see Definition 1.2.3  
 $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$   
 $\tilde{s}_{ij}$  — see (3.1.1)  
 $s_{ij} = \tilde{s}_{ij}/D$ , see (3.1.16)  
 $S_{ij}$  — see (3.1.32)

- Sets* — the groupoid with objects finite sets, and morphisms bijections, see Section 5.6
- $S_{\alpha\beta}$  — clutching map, see (6.2.7)
- $S_n$  — the symmetric group on  $n$  objects
- $S^n$  —  $n$ -sphere
- $S_{0,n}$  — standard sphere, see (5.2.1)
- $Sp_D$  — specialization functor for connections with regular singularities, see Lemma 6.3.15
- $\text{tr}$  — trace, see Definition 2.3.2
- $\text{tr}_q$  — quantum trace, see (2.3.10)
- $\theta, \theta_V$  — balancing isomorphism, or twist, see (2.2.7)
- $\theta_i$  —  $\theta_{V_i} = \theta_i \text{id}_{V_i}$ , see (2.4.4)
- $\theta, \theta^V$  — the highest root of  $\mathfrak{g}$  and the corresponding coroot
- $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$
- $\Theta_S$  — the sheaf of vector fields on  $S$ , see Section 7.4
- Teich* — Teichmüller groupoid, see Definition 5.1.7(i)
- Teich* — Teichmüller tower, see the beginning of Section 5.6
- Teich*<sub>0</sub> — Teichmüller tower in genus zero, see Remark 5.6.3
- Teich*<sup>stab</sup> — the subgroupoid of *Teich* of stable surfaces, see Theorem 6.1.13
- Teich*<sup>C</sup> — complex Teichmüller groupoid, see Definition 6.1.11
- $\widetilde{\text{Teich}}$  — central extension of the Teichmüller tower, see Definition 5.7.6
- $T_V = \pi_1(\Lambda_V)$  — the Poincaré groupoid of  $\Lambda_V$ , see the beginning of Section 5.7
- $T_\Sigma = T_{\Lambda_\Sigma}$  — see Section 5.7
- $T(\Sigma), T_{g,n}$  — Teichmüller space, see Theorem 6.1.6
- $T_g = T_{g,0}$  — see Theorem 6.1.2
- $t_{ij}$  —  $\delta_{ij}\theta_i$ , see Theorem 3.1.7
- $T_{ij}$  — see (3.1.33)
- $\tau(M), \tau(M, \Omega)$  — Reshetikhin–Turaev invariants, see Theorems 4.1.12 and 4.1.16
- $\tau(M), \tau(\partial M)$  — TQFT, see Definition 4.2.1
- $\tau(N), \tau(\Sigma)$  — modular functor, see Definitions 5.1.1.1, 5.1.13
- $\tau(\Sigma; \{W_\alpha\})$  — see Definition 5.1.13
- $\tau(C, \vec{p}, V_1, \dots, V_n)$  — the vector space of coinvariants for WZW model, see Definition 7.3.1
- $\tau(C_S, \vec{p}, V_1, \dots, V_n)$  — the sheaf of coinvariants corresponding to a family  $C_S$  of curves over  $S$ , see (7.4.2)
- $\mathbf{1}$  — unit object, see Definition 1.1.7
- $U(\mathfrak{g})$  — the universal enveloping algebra of a Lie algebra  $\mathfrak{g}$
- $U_q(\mathfrak{g})$  — quantum group, see Definition 1.3.1
- $U_q(\mathfrak{g})_{\mathbb{Z}}$  — a  $\mathbb{Z}$ -form of  $U_q(\mathfrak{g})$ , see (1.3.16)
- $U(\hat{\mathfrak{g}})_k = U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(K - k)$  — see Section 7.1
- $\text{Vec}(k)$  — the category of  $k$ -vector spaces
- $\text{Vec}_f(k)$  — the category of finite-dimensional  $k$ -vector spaces
- $V_i$  ( $i \in I$ ) — representatives of the equivalence classes of simple objects in a semi-simple abelian category
- $V_\lambda^k$  — Weyl module over affine Lie algebra, see (7.1.2)
- Vir* — the Virasoro algebra, see (7.4.8)
- $V_\lambda$  — simple finite-dimensional module over  $\mathfrak{g}$  or  $U_q(\mathfrak{g})$  ( $q$  is a formal variable) with highest weight  $\lambda$
- $V_\lambda$  — Weyl module over  $U_q(\mathfrak{g})$  ( $q$  is a root of unity), see Definition 3.3.2
- $V^*$  — right dual to  $V$ , see Definition 2.1.1
- $*V$  — left dual to  $V$ , see (2.1.7), (2.1.8)
- $V_\alpha$  — a lattice in a Tate vector space  $V$ , see Definition 7.10.10
- $W$  — the Weyl group of  $\mathfrak{g}$
- $W^a$  — the affine Weyl group, see Theorem 3.3.6
- $\mathbb{Z}$  — the ring of integers
- $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$  — the set of nonnegative integers
- $\mathbb{Z}_- = \{0, -1, -2, \dots\}$  — the set of nonpositive integers
- $Z(G)$  — the center of a group  $G$
- $\zeta = (p^+ / p^-)^{1/6}$  in an MTC, see (3.1.15)

## Lectures on Tensor Categories and Modular Functors

Bojko Bakalov and Alexander Kirillov, Jr.

This book gives an exposition of the relations among the following three topics: monoidal tensor categories (such as a category of representations of a quantum group), 3-dimensional topological quantum field theory, and 2-dimensional modular functors (which naturally arise in 2-dimensional conformal field theory). The following examples are discussed in detail: the category of representations of a quantum group at a root of unity and the Wess-Zumino-Witten modular functor.

The idea that these topics are related first appeared in the physics literature in the study of quantum field theory. Pioneering works of Witten and Moore-Seiberg triggered an avalanche of papers, both physical and mathematical, exploring various aspects of these relations. Upon preparing to lecture on the topic at MIT, however, the authors discovered that the existing literature was difficult and that there were gaps to fill.

The text is wholly expository and finely succinct. It gathers results, fills existing gaps, and simplifies some proofs. The book makes an important addition to the existing literature on the topic. It would be suitable as a course text at the advanced-graduate level.

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