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Bojko Bakalov
Alexander Kirillov, Jr.

American Mathematical Society

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ABSTRACT. This book gives an exposition of the relations among the following three topics: monoidal tensor categories (such as a category of representations of a quantum group), 3-dimensional topological quantum field theory, and 2-dimensional modular functors, which naturally arise in 2-dimensional conformal field theory. As examples, the category of representations of a quantum group at a root of unity and the Wess-Zumino-Witten modular functor are discussed in detail.

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To our parents

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Index of Notation

$\hat{\mathfrak{g}}$ — affine Lie algebra, see (7.1.1)
$\mathfrak{g}(C - \vec{p})$ — Lie algebra of rational $\mathfrak{g}$-valued functions on the curve C, see (7.3.1)
$\mathfrak{gl}(V)$, $\widehat{\mathfrak{gl}}(V)$ — the Lie algebra of continuous endomorphisms of a Tate vector space V and its central extension, see Eq. (7.10.5), Definition 7.10.18.

$\mathrm{Hom}_{\mathcal{C}}(U, V)$ — the vector space of morphisms from U to V in an abelian category $\mathcal{C}$
H — $\bigoplus V_i \otimes V_i^*$, see (2.4.9)
$H(\Sigma) = \mathrm{H}_1(cl(\Sigma), \mathbb{R})$ — see Section 5.7
$h^\vee$ — the dual Coxeter number for $\mathfrak{g}$, see page 63

i_V — the morphism $\mathbf{1} \to V \otimes V^*$, see Definition 2.1.1
ind–$\mathcal{C}^{\boxtimes 2}$ — a completion of $\mathcal{C}^{\boxtimes 2}$, see (2.4.7)

k — a field of characteristic 0
$K(\mathcal{C})$ — the Grothendieck group (or ring) of $\mathcal{C}$, see Definition 2.1.9
$k[G]$ — the group algebra of a group G, see Section 3.2

λ_V — the isomorphism $\mathbf{1} \otimes V \xrightarrow{\sim} V$, see Definition 1.1.7
Λ_V — the set of all Lagrangian subspaces of a symplectic real vector space, see the beginning of Section 5.7
$\Lambda_\Sigma = \Lambda_{H(\Sigma)}$ — see Section 5.7
L_λ^k — irreducible integrable module over an affine Lie algebra, see Section 7.1
$\bigwedge_\alpha^{\infty/2}(V)$ — the space of semi-infinite forms, see Definition 7.10.14.

$\mathrm{Mor}\,\mathcal{C}$ — the class of morphisms in a category $\mathcal{C}$
$\mathrm{Mor}_{\mathcal{C}}(U, V)$ — the set of morphisms in $\mathcal{C}$ from U to V
M_L — a surgery of S^3 along a framed link L, see Definition 4.1.7
$\mathcal{MS}$ — the Moore–Seiberg tower, see Example 5.6.16
$\mathcal{M}_{g,n}, \overline{\mathcal{M}}_{g,n}$ — the (coarse) moduli space of pointed curves of genus g with n marked points and its compactification, see Sections 6.1, 6.2
$\mathcal{M}_g = \mathcal{M}_{g,0}$
$M_{g,n}, \overline{M}_{g,n}$ — the moduli stack of pointed curves of genus g with n marked points and its compactification, see Sections 6.1, 6.2
$M_{*,A}$ — moduli stack of pointed curves with marked points labeled by A, see (6.1.3)
$M(\Sigma)$ — see Definition 5.2.1
$\mathcal{M}(\Sigma)$ — see Theorems 5.2.3 and 5.2.10

$\mathbb{N} = \{1, 2, 3, \dots\}$ — the set of natural numbers
$[n]_i$, $[n]_i!$, $\left[{n \atop k}\right]_i$ — see (1.3.6)
N_{ij}^k — tensor product multiplicities, or fusion coefficients, see (2.4.1)

$\Omega = \sum a_i \otimes a^i$ — see (1.4.1)
$\mathrm{Ob}\,\mathcal{C}$ — the class of objects in a category $\mathcal{C}$
$\mathcal{O}_k$, $\mathcal{O}_k^{\mathrm{int}}$ — categories of level k modules and of integrable level k modules over an affine Lie algebra, see Section 7.1
$\mathcal{O}_S$ — the structure sheaf of an analytic manifold, see Section 7.2

$p^\pm$ — see (3.1.7)
P_+ — the cone of dominant integer weights, see the beginning of Section 1.3
P_+^k — dominant integer weights corresponding to integrable $\hat{\mathfrak{g}}$-modules of level k, see (7.1.3)
$\pi_1(M)$ — the (fundamental) Poincaré groupoid of a topological space M
$\mathcal{P}Teich$ — parameterized Teichmüller tower, see Example 5.6.18
$\mathcal{P}Teich_0$ — the genus zero part of $\mathcal{P}Teich$, see Remark 5.6.3

$\mathbb{Q}$ — the field of rational numbers
$\mathbb{Q}_+$ — the set of positive rational numbers
$\mathbb{Q}_-$ — the set of negative rational numbers
Q — determinant line bundle, see Definition 6.7.4

$\mathbb{R}$ — the field of real numbers
$\mathbb{R}_+$ — the set of positive real numbers
$\mathbb{R}_-$ — the set of negative real numbers
$\mathbb{R}^\times$ — the set of non-zero real numbers
R — universal R-matrix, see Example 1.2.8(iii)
R — $\bigoplus V_i \boxtimes V_i^*$, see (2.4.7)
$R^{(1)}, R^{(2)}$ — see the end of Section 2.4
ρ — the half sum of positive roots of $\mathfrak{g}$
ρ_V — the isomorphism $V \otimes \mathbf{1} \xrightarrow{\sim} V$, see Definition 1.1.7
$\mathcal{R}ep(A)$ — the category of representations of an algebra (or a group) A
$\mathcal{R}ep_f(A)$ — the category of finite-dimensional representations of A
$\mathcal{RS}(\overline{M}, M)$ — the category of flat connections with regular singularities on $\overline{M}$, see Section 6.3

σ_{VW} — commutativity isomorphism, see Definition 1.2.3
$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$
$\tilde{s}_{ij}$ — see (3.1.1)
s_{ij} — $\tilde{s}_{ij}/D$, see (3.1.16)
S_{ij} — see (3.1.32)

Lectures on Tensor Categories and Modular Functors

Bojko Bakalov and Alexander Kirillov, Jr.

This book gives an exposition of the relations among the following three topics: monoidal tensor categories (such as a category of representations of a quantum group), 3-dimensional topological quantum field theory, and 2-dimensional modular functors (which naturally arise in 2-dimensional conformal field theory). The following examples are discussed in detail: the category of representations of a quantum group at a root of unity and the Wess-Zumino-Witten modular functor.

The idea that these topics are related first appeared in the physics literature in the study of quantum field theory. Pioneering works of Witten and Moore-Seiberg triggered an avalanche of papers, both physical and mathematical, exploring various aspects of these relations. Upon preparing to lecture on the topic at MIT, however, the authors discovered that the existing literature was difficult and that there were gaps to fill.

The text is wholly expository and finely succinct. It gathers results, fills existing gaps, and simplifies some proofs. The book makes an important addition to the existing literature on the topic. It would be suitable as a course text at the advanced-graduate level.

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