University
LECTURE
Series
Volume 30

# Combinatorial Constructions in Ergodic Theory and Dynamics

Anatole Katok

American Mathematical Society

http://dx.doi.org/10.1090/ulect/030

University
# LECTURE
Series

Volume 30

# Combinatorial Constructions in Ergodic Theory and Dynamics

Anatole Katok

**American Mathematical Society**
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 37A20; Secondary 37A25, 37C40, 37D20, 37D30.

---

For additional information and updates on this book, visit
**www.ams.org/bookpages/ulect-30**

---

**Library of Congress Cataloging-in-Publication Data**

Katok, A. B.
Combinatorial constructions in ergodic theory and dynamics / Anatole Katok.
p. cm. — (University lecture series, ISSN 1047-3998 ; v. 30)
Includes bibliographical references.
ISBN 0-8218-3496-7
1. Ergodic theory. 2. Differentiable dynamical systems. 3. Combinatorial analysis. I. Title. II. University lecture series (Providence, R. I.) ; 30.

QA611.5.K38 2003
515′.48—dc22 2003059582

---



∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 08 07 06 05 04 03

# Table of Contents

## References


[Ag] O. N. Ageev, *On ergodic transformations with homogeneous spectrum*, Jour. Dynam. Control Syst. **5** (1999), 149–152.

[Al1] S. Alpern, *New proofs that weak mixing is generic*, Invent. Math. **32** (1976), 263–279.

[Al2] S. Alpern, *Generic properties of measure preserving homeomorphisms*, in Ergodic theory, Springer LNM **729** (1979), 16–27.

[AlP] S. Alpern and V. S. Prasad, *Typical dynamics of volume preserving homeomorphisms*, Cambridge University Press, 2000.

[A1] D. V. Anosov, *Geodesic flows on Riemann manifolds with negative curvature*, (Originally published in Russian), Proceedings of the Steklov Institute of Mathematics **90** (1967), AMS.

[A2] D. V. Anosov, *Functional homology equation connected with an ergodic rotation of the circle*, Math. USSR Izvestija **7** (1973), 1257–1271.

[A3] D. V. Anosov, *Existence of smooth ergodic flows on smooth manifolds*, (Originally published in Russian), Math. USSR, Izvestija **8** (1974), 525–552.

[AK] D. V. Anosov and A. B. Katok, *New examples in smooth ergodic theory. Ergodic diffeomorphisms*, (Originally published in Russian), Transactions of the Moscow Mathematical Society **23** (1970), 1–35.

[AK2] D. V. Anosov and A. B. Katok, *New examples of ergodic diffeomorphisms of smooth manifolds. (Russian)*, Uspehi Mat. Nauk **25** (1970), no. 4, 173–174.

[BKM] D. Bernstein, A. Katok, and G. D. Mostow, *Nets and Quasiisometries*, unpublished.

[B1] R. Bowen, *Some systems with unique equilibrium states*, Mathematical Systems Theory **8** (1975), no. 3, 193–202.

[B2] R. Bowen, *Periodic orbits of hyperbolic flows*, Amer. J. Math. **94** (1972), 1–30.

[BFK] M. Brin, J. Feldman and A. Katok, *Bernoulli diffeomorphisms and group extensions of dynamical systems with non-zero characteristic exponents*, Ann. of Math. **113** (1981), 159-179.

[C1] R. V. Chacon, *Weakly mixing transformations which are not strongly mixing*, Proc. Amer. Math. Soc. **22** (1969), 559–562.

[C] R. V. Chacon, *Approximation and spectral multiplicity*, Springer LNM **160** (1970), 18–27.

[CFW] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Erg. Th. & Dynam. Syst. **1** (1981), 431–450.

[CFS] I. P. Cornfeld, S. V. Fomin, Ya G. Sinai, *Ergodic theory*, (Originally published in Russian), Springer, 1982.

[DP] D. Dolgopyat and Ya. Pesin, *Every compact manifold carries a completely hyperbolic diffeomorphism*, Erg. Th. & Dynam. Syst. **22** (2002), no. 2, 409–435.

[Fa1] B. Fayad, *Weak mixing in reparametrized linear flows on the torus*, Erg. Th. & Dynam. Syst. **22** (2002), 187–201..

[Fa2] B. Fayad, *Analytic mixing reparametrizations of the irrational flows*, Erg. Th. & Dynam. Syst. **22** (2002), 437–468.

[FaK] B. Fayad and A. Katok, *Constructions in elliptic dynamics*, to appear, Erg. Th. & Dynam. Syst. **23** (2003).

[FaKW] B. Fayad, A. Katok and A. Windsor, *Mixed Spectrum Reparametrizations of Linear Flows on $T^2$*, Moscow Math. Journal, **1** (2001), 521–537.

[Fe] J. Feldman, *New K-automorphisms and a problem of Kakutani*, Isr. J. Math. **24** (1976), no. 1, 16–38.

[FK] R. Feres and A. Katok, *Ergodic theory and dynamics of G-spaces*, Handbook in Dynamical Systems, vol. 1, Elsevier, 2001.

[FlFo] G. Forni and L. Flaminio, *Invariant distributions for horocycle flows*, preprint.

[Fo] G. Forni, *Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus* **146** (1997), 295–344.

[Fu] A. Furman, *Orbit equivalence rigidity*, Ann. of Math **150** (1999), 1083–1108.

[F1] H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math. **83** (1961), 573–601.

[F2] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, 1981.

[FH] A. Fathi and M. R. Herman, *Existence de difféomorphismes minimaux*, Astérisque **49** (1977), 37–59.

[G1] G. R. Goodson, *On the spectral multiplicity of a class of finite rank transformations*, Proc. Amer. Math. Soc. **93** (1985), 303–306.

[G2] G. R. Goodson, *A survey of recent results in the spectral theory of ergodic dynamical systems*, J. Dynam. Control Syst. **5** (1999), 173–226.

[GKLL] G. R. Goodson, J. Kwiatkowski, M. Lemańczyk and P. Liardet, *On the multiplicity function of ergodic group extensions of rotations*, Studia Math. **102** (1992), 157-174.

[GK] E. Gutkin and A. Katok, *Weakly mixing billiards*, Springer LNM **1345** (1988), 163–176.

[GuK] R. Gunesch and A. Katok, *Construction of weakly diffeomorphisms preserving measurable Riemannian metric and smooth measure*, Disc. and Cont. Dyn. Syst. **6** (2000), 61–83.

[H] P. R. Halmos, *Ergodic theory*, Chelsea, 1956.

[HaK] B. Hasselblatt and A. Katok, *Principal structures*, Handbook in Dynamical Systems, vol. 1A, Elsevier, 2002, pp. 1–203.

[He1] M. R. Herman, *$L^2$ regularity of measurable solutions of a finite difference equation of the circle*, preprint, University of Warwick (1976).

[He2] M. R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. IHES **49** (1979), 5–233.

[He3] M. R. Herman, *Constructions des difféomorphismes ergodiques*, unpublished manuscript.

[He4] M. R. Herman, *Construction d'un difféomorphisme minimal d'entropie topologique non nulle*, Erg. Th. & Dynam. Syst. **1** (1981), 65–76.

[He5] M. R. Herman, *Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension* 2, Comm. Math. Helv. **58** (1983), 453–502.

[Ho] B. Host, *Mixing of all orders and pairwise independent joinings of systems with singular spectrum*, Israel J. Math. **76** (1991), no. 3, 289–298.

[HK1] S. Hurder and A. Katok, *Ergodic theory and Weil measures for foliations*, Ann. Math. **126** (1987), 221–275.

[HK2] S. Hurder and A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows*, Publ. Math. IHES **72** (1990)), 5–61.

[Je] R. I. Jewett, *The prevalence of uniquely ergodic systems*, J. Math. Mech. **19** (1969-70), 717–729.

[Jo] J.-L. Journé, *On a regularity problem occurring in connection with Anosov diffeomorphisms*, Comm. Math. Phys. **106** (1986), 345–351.

[J1] A. del Junco, *A transformation with simple spectrum which is not rank one*, Canad. J. Math. **29** (1977), 655–663.

[J2] A. del Junco, *A family of counterexamples in ergodic theory*, Israel J. Math **44** (1983), 160–188.

[JR] A. Del Junco and D. Rudolph, *Kakutani equivalence of ergodic $\mathbb{Z}^n$ actions*, Erg. Th. & Dynam. Syst. **4** (1984), 89–104.

[JR2] A. del Junco and D. Rudolph, *A rank-one, rigid, simple, prime map*, Erg. Th. & Dynam. Syst. **7** (1987), no. 2, 229–247.

[Ka] S. Kakutani, *Induced measure preserving transformations*, Proc. Japan Acad. **19** (1943), 635–641.

[Kali] S. Kalikow, *Twofold mixing implies threefold mixing for rank one transformations*, Erg. Th. & Dynam. Syst. **4** (1984), no. 2, 237–259.

[KalSp] B. Kalinin and R. J. Spatzier, *Rigidity of measurable structure for actions of higher rank abelian groups*, to appear, Erg. Th. & Dynam. Syst..

[K] A. B. Katok, *Ergodic perturbations of degenerate integrable Hamiltonian systems*, (Originally published in Russian), Math. USSR, Izvestija **7** (1973), no. 3, 535-571.

[K1] A. B. Katok, *Monotone equivalence in ergodic theory*, (Originally published in Russian), Math. USSR, Izvestija **11** (1977), 99–146.

[K2] A. B. Katok, *The special representation theorem for multi-dimensional group actions*, Asterisque **49** (1977), 117-140.

[K3] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. IHES **51** (1980), 137–173.

[K4] A. Katok, *Interval exchange transformations and some special flows are not mixing*, Isr. J. Math. **35** (1980), 301-310.

[K5] A. Katok, *Bernoulli diffeomorphisms on surfaces*, Ann. of Math. **110** (1979), 529-547.

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, New York, 1995.

[KKS] A. Katok, S. Katok, and K. Schmidt, *Rigidity of measurable structure for $\mathbb{Z}^d$-actions by automorphisms of a torus*, Comm. Math. Helv. **77** (2002), 718-745.

[KK] A. Katok and A. Kononenko, *Cocycles' stability for partially hyperbolic systems*, Math. Res. Lett. **3** (1996), 191–210.

[MK] A. Katok and L. Mendoza, *Dynamical systems with nonuniformly hyperbolic behavior*, supplement to " Introduction to the modern theory of dynamical systems" (1995), 659–700.

[KR] A. Katok in collaboration with E. A. Robinson Jr., *Cocycles, cohomology and combinatorial constuctions in ergodic theory*, Smooth ergodic theory and its applications, Proc. Symp. Pure Math., vol. 69, Amer. Math. Soc., 2001, pp. 107–173.

[KSa] A. B. Katok and E. A. Sataev, *Standardness of automorphisms of transpositions of intervals and fluxes on surfaces*, (Originally published in Russian), Math. Notes Acad. Sci. USSR **20** (1977), 826–831.

[KSch] A. Katok and K. Schmidt, *The cohomology of expansive $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups*, Pacific J. Math. **170** (1995), 105-142.

[KSp1] A. Katok and R. Spatzier, *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity*, Publ. Math. IHES **79** (1994), 131-156.

[KSp2] A. Katok and R. Spatzier, *Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions*, Proc. Steklov Math. Inst. **216** (1997), 287-314.

[KS1] A. B. Katok and A. M. Stepin, *Approximations in ergodic theory*, (Originally published in Russian), Russ. Math. Surveys **22** (1967), 77–102.

[KS2] A. B. Katok and A. M. Stepin, *Metric properties of measure preserving homeomorphisms*, (Originally published in Russian), Russ. Math. Surveys **25** (1970), 191–220.

[KT1] A. Katok and J.-P. Thouvenot, *Slow entropy type invariants and smooth realization of commuting measure-preserving transformations*, Ann. Inst. H. Poincare **33** (1997), no. 3, 323–338.

[KT] A. Katok and J.-P. Thouvenot, *Spectral properties and combinatorial costructions in ergodic theory*, Handbook in Dynamical Systems, to appear, vol. 1B, Elsevier.

[Ke] M. Keane, *Interval exchange transformations*, Math. Zeitsch. **141** (1975), 25–31.

[KitSch] B. Kitchens and K. Schmidt, *Isomorphism rigidity of irreducible algebraic $\mathbb{Z}^d$-actions*, Invent. Math. **142** (2000), 559–577.

[Kc1] A. V. Kočergin, *The absence of mixing in special flows over a rotation of the circle and in flows on a two-dimensional torus*, Soviet Math. Dokl. **13** (1972), 949–952.

[Kc] A. V. Kočergin, *The homology of functions over dynamical systems (in Russian)*, (Originally published in Russian), Dokl. Akad. Nauk SSSR **231** (1976), 795–798.

[Ko] A. N. Kolmogorov, *On dynamical systems with integral invariant on the torus*, in Russian, Doklady Akad. Nauk SSSR **93** (1953), 763–766.

[Kr] W. Krieger, *On unique ergodicity*, Proc 6th Berkeley Symp. Prob. Stat. **II** (1972), 327–346.

[Kry] A. Krygin, *Examples of ergodic cylindrical cascades*, Math. Notes USSR Acad. Sci **16** (1974), 1180–1186.

[dl] R. de la Llave, *Analytic regularity of solutions of Livsic's cohomology equation and some applications to analytic conjugacy of hyperbolic dynamical systems*, Erg. Th. & Dynam. Syst **17** (1997), 649–662.

[dLMM] R. de la Llave, J. M. Marco, and R. Moriyón, *Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation*, Ann. of Math. **123** (1986), 537–611.

[L1] A. Livšic, *Some homology properties of U–systems*, Math. Notes USSR Acad. Sci **10** (1971), 758–763.

[L2] A. Livšic, *Cohomology of dynamical systems*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1296–1320.

[LS] A. Livšic and Ja. G. Sinaĭ, *On invariant measures compatible with smooth structure for transitive U-systems*, Soviet Math. Doklady **13** (1972), 1656–1659.

[LSa] R. U. Luz, and N. M. dos Santos,, *Cohomology-free diffeomorphisms of low-dimension tori*, Erg. Th. & Dynam. Syst. **18** (1998), 985–1006.

[Ma] R. Mañé, *An Ergodic closing lemma*, Ann. of Math. **116** (1982), 503–540.

[M] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. **115** (1982), 169–200.

[MN] J. Mathew and M. G. Nadkarni, *A measure–preserving transformation whose spectrum has a Lebesgue component of multiplicity two*, Bull. Lond. Math. Soc. **16** (1984), 402–406.

[Na] M. G. Nadkarni, *Spectral theory of dynamical systems*, Birkhäuser, 1998.

[N] J. von Neumann, *Zur Operatorenmethode in der klassischen Mechanik*, Ann. Math. **33** (1932), 587–642.

[NT] V. Niţică and A. Török, *Regularity of the transfer map for cohomologous cocycles*, Erg. Th. & Dynam. Syst. **18** (1998), no. 5, 1187–1209.

[O1] D. Ornstein, *Ergodic theory, randomness and dynamical systems*, Yale Math. monographs, vol. 5, Yale Univ. Press, 1974.

[O2] D. Ornstein, *A $K$-automorphism with no square root and Pinsker's conjecture*, Advances in Math. **10** (1973), 89–102.

[ORW] D. Ornstein, D. Rudolph and B. Weiss, *Equivalence of measure preserving transformations*, Memoirs of the AMS **37** (1982), no. 262.

[OSh] D. Ornstein and P. Shields, *An uncountable family of $K$-automorphisms*, Advances in Math. **10** (1973), 63–88.

[OS] D. Ornstein and M. Smorodinsky, *Continuous speed changes for flows*, Israel J. Math. **31** (1978), 161–168.

[OS2] D. Ornstein and M. Smorodinsky, *Ergodic flows of positive entropy can be time changed to become $K$-flows*, Israel J. Math **26** (1977), no. 1, 75–83.

[OW] D. Ornstein and B. Weiss, *Isomorphism theorem for amenable group actions*, J. d'Anal. Math. **48** (1987), 1–141.

[O] V. I. Oseledets, *On the spectrum of ergodic automorphisms*, (Originally published in Russian), Soviet Math. Dokl. **7** (1966), 776–779.

[Pa] W. Parry, *Topics in ergodic theory*, Cambridge University Press, 1981.

[Pe] K. Petersen, *Ergodic theory*, Cambridge University Press, 1983.

[Rat] M. Ratner, *Rigidity of horocycle flows*, Ann. of Math. **115** (1982), no. 3, 597–614.

[R1] E. A. Robinson, *Ergodic measure–preserving transformations with arbitrary finite spectral multiplicities*, Invent. Math. **72** (1983), 299–314.

[R2] E. A. Robinson, *Mixing and spectral multiplicity*, Ergodic Theory Dynamical Systems **5** (1985), 617–624..

[R3] E. A. Robinson, *Transformations with highly nonhomogeneous spectrum of finite multiplicity*, Israel J. Math. **56** (1986), 75–88..

[Rud] D. Rudolph, *An example of a measure preserving map with minimal self-joinings, and applications*, J. Analyse Math. **35** (1979), 97–122.

[Ru] D. Ruelle, *Statistical mechanics on a compact set with $\mathbb{Z}^\nu$ action satisfying expansiveness and specification*, Trans. AMS **185** (1973), 237–253.

[Ry] V. V. Ryzhikov, *Transformations having homogeneous spectrum*, Jour. Dynam. Control Syst. **5** (1999), 145–148.

[Sch1] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser, 1995.

[Sch2] K. Schmidt, *The cohomology of higher-dimensional shifts of finite type*, Pacific J. Math **170** (1995), 237–269.

[Sg1] K. Sigmund, *Generic properties of invariant measures for Axiom A diffeomorphisms*, Invent. Math. **11** (1970), 99–109.

[Sg2] K. Sigmund, *On the space of invariant measures for hyperbolic flows*, Amer. J. Math. **94** (1972), no. 1, 31–37.

[Sg3] K. Sigmund, *On mixing measures for Axiom A diffeomorphisms*, Proc. AMS **36** (1972), no. 2, 497–504.

[Sh] M. Shklover, *Classical dynamical systems on the torus with continuous spectrum (Russian)*, Izv. Vyŝs. Uĉebn. Zaved. Matematika **10 (65),** (1967), 113–124.

[LSh] L. G. Shnirelman, *An example of a transformation of the plane (Russian)*, Proc. Don Polytechnic Inst (Novochekassk), Science section, Fis-math. part **14** (1930), 64–74.

[St1] A. M. Stepin, *Les spectres des systemes dynamique*, Actes, Congress Intern. Math. (1970), 941–946.

[St2] A. M. Stepin, *The relation between approximation and spectral properties of metric automorphisms*, (Originally published in Russian), Math. Notes USSR Acad. Sci. **13** (1973), 244–247.

[St3] A. M. Stepin, *Spectral properties of generic dynamical systems*, (Originally published in Russian), Math. USSR-Izv. **29** (1987), 159-282.

[T] J.-P. Thouvenot, *Entropy, isomorphism and equivalence in ergodic theory*, Handbook in Dynamical Systems, vol. 1A, Elsevier, 2002, pp. 205–238.

[V1] W. Veech, *Interval exchange transformations*, J. Analyse Math. **33** (1978), 222–272.

[V2] W. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. **115** (1982), 201–242.

[V3] W. Veech, *Periodic points and invariant pseudomeasures for toral endomorphisms*, Erg. Th. & Dynam. Syst. **6** (1986), 449–473.

[W] P. Walters, *Introduction to ergodic theory*, Springer, 1982.

[Wi] A. Windsor, *Minimal but not uniquely ergodic diffeomorphisms*, Proc. Symp. Pure Math. **69** (2001), Amer. Math. Soc., Providence, RI.

[Wt] A. Windsor, *Liouville phenomena in smooth ergodic theory*, Ph. D. thesis, Pennsylvania State University (2002).

[Z1] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, 1984.

[Z2] R. Zimmer, *Actions of semisimple groups and discrete subgroups*, Proceedings of the International Congress of Mathematicians, Amer. Math. Soc., Providence, RI, 1987, pp. 1247–1258.

Ergodic theory studies measure-preserving transformations of measure spaces. These objects are intrinsically infinite and the notion of an individual point or an orbit makes no sense. Still there is a variety of situations when a measure-preserving transformation (and its asymptotic behavior) can be well described as a limit of certain finite objects (periodic processes).

In the first part of this book this idea is developed systematically, genericity of approximation in various categories is explored, and numerous applications are presented, including spectral multiplicity and properties of the maximal spectral type. The second part of the book contains a treatment of various constructions of cohomological nature with an emphasis on obtaining interesting asymptotic behavior from approximate pictures at different time scales.

The book presents a view of ergodic theory not found in other expository sources and is suitable for graduate students familiar with measure theory and basic functional analysis.

For additional information and updates on this book, visit

**www.ams.org/bookpages/ulect-30**

ISBN 0-8218-3496-7

9 780821 834961

**ULECT/30**