University
LECTURE
Series

Volume 32

The Stationary Tower

Notes on a Course by W. Hugh Woodin

Paul B. Larson

American Mathematical Society

http://dx.doi.org/10.1090/ulect/032

The Stationary Tower

Notes on a Course
by W. Hugh Woodin

This page intentionally left blank

University

LECTURE

Series

Volume 32

The Stationary Tower

Notes on a Course by W. Hugh Woodin

Paul B. Larson

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 03E40, 03E15, 03E35, 03E55, 03E60.

The author was supported in part by the Mittag-Leffler Institute, the NSERC grants of Juris Steprāns, Paul Szeptycki, and Franklin D. Tall, The Fields Institute, FAPESP (Grant #02/11551-3, University of São Paulo) and the Centre de Recerca Matemàtica of the Institut d'Estudis Catalans.

For additional information and updates on this book, visit
www.ams.org/bookpages/ulect-32

Library of Congress Cataloging-in-Publication Data

Larson, Paul B. (Paul Bradley), 1970–
The stationary tower : notes on a course by W. Hugh Woodin / Paul B. Larson.
p. cm. — (University lecture series, ISSN 1047-3998 ; v. 32)
Includes bibliographical references and index.
ISBN 0-8218-3604-8 (alk. paper)
1. Forcing (Model theory) 2. Set theory. I. Title. II. University lecture series (Providence, R.I.) ; 32.

QA9.7.L36 2004
511.3′4—dc22 2004047666

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 09 08 07 06 05 04

Contents

This page intentionally left blank

Preface

This book is an expository introduction to stationary tower forcing, a construction isolated by W. Hugh Woodin in the wake of work of Foreman, Magidor and Shelah [**12**]. The first chapter presents some relevant background material, and is also intended to serve as a warmup for readers unfamiliar with this area of set theory. In the second chapter we present the stationary tower and develop its basic properties, and in the third we present some of the major applications of this method to the study of forcing-absoluteness and sets of reals in inner models. The major results presented here include the following:

(1) If there is a proper class of Woodin cardinals, then the theory of $L(\mathbb{R})$ is invariant under all set forcing extensions;
(2) If there is a proper class of measurable Woodin cardinals, then all forcing extensions satisfying the Continuum Hypothesis agree about all Σ^2_1 sentences;
(3) If there exist infinitely many Woodin cardinals below a measurable cardinal, then the Axiom of Determinacy holds in $L(\mathbb{R})$;
(4) If there is a proper class of Woodin cardinals, then the collections of universally Baire sets of reals, ∞-homogeneously Suslin sets of reals and ∞-weakly homogeneously Suslin sets of reals coincide;
(5) If there exist a supercompact cardinal δ and a proper class of Woodin cardinals, then after any forcing making $V_{\delta+1}$ countable, the theory of the least inner model of ZF containing the ordinals and the universally Baire sets of reals is invariant under all further set forcing extensions.

Items (3), (4) and (5) here are proved using the main theorem of [**30**].

The book is mostly based on lecture notes from a graduate set theory course given by Woodin at U.C. Berkeley in the Spring of 1996. Section 2.6 is adapted from an email from Woodin to James Cummings. Section 3.2 is an updated proof of Woodin's $\underset{\sim}{\Sigma}^2_1$-absoluteness result, based on a set of his handwritten notes from 1985. Section 3.3 includes a proof from a paper by Neeman and Zapletal [**33**] and material from a preprint of Steel [**46**]. Section 3.4 is based on a set of notes written by Woodin especially for inclusion in this book. Except where noted, all of the results in Chapters 2 and 3 are due to Woodin. The history of much of the material in Chapter 1 is documented in [**30**]. The books [**19, 21**] are also good sources.

This book does not attempt to give an encyclopedic account of stationary tower forcing. In particular, a great deal of work has been done on the stationary tower which we do not discuss here; see, for instance, [**4, 5, 10, 11, 28, 31, 33, 35, 38, 53, 56, 57**].

The presentation is aimed at a graduate student who has taken introductory graduate courses in set theory and logic (as was the course the book is based on). The primary prerequisites are the basics of logic (models, elementary submodels, elementary embeddings, theories, satisfaction, quantifiers), forcing and constructibility (through Chapter VIII, §3 of [**26**], say). Infinite forcing iterations are not required, though the reader should be familiar with forcing with partial orders and with Boolean algebras, and the relationship between the two methods. Some standard forcing facts are collected in the appendix. The reader should be familiar with ultrapower constructions, which are quickly reviewed in the first section. Sharps (as presented in [**19, 21**], for instance) appear at several points in the book.

None of the results presented here is due to the author (aside from the very minor Example 2.7.10, which probably would have been obvious to any expert in determinacy who thought about it). I have, however, modified many of the original proofs from my notes and in some cases replaced them entirely (the proof of Lemma 3.1.14, for instance, uses a construction from [**27**]).

I am grateful to the many people who answered my questions on the material in this book, and who pointed out errors in earlier drafts. I would especially like to thank John Steel for letting me crib from his preprint cited above, and for enlightening me on a number of issues. Aaron Siegel and David Asperó each made a large number of helpful comments. Most of all, I thank Hugh Woodin, for all the obvious reasons, and especially for his support of this project.

0.1. Notation

Under the usual notation, which we use here, $V_0 = \emptyset$, $V_{\alpha+1}$ is the set of subsets of V_α and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for limit λ. If M is a model of ZFC, we sometimes write M_α for V_α^M. However, we sometimes use subscripts to index a family of models, such as M_α ($\alpha < \kappa$), and we hope that there will be no confusion. If $G \subset P$ is a V-generic filter, then $V_\delta[G]$ is the set of realizations by G of the P-names in V_δ, whereas $V[G]_\delta$ is $V_\delta^{V[G]}$.

If $f\colon X \to Y$ is a function and $A \subset X$, then

$$f[A] = \{y \in Y \mid \exists x \in A\ f(x) = y\}.$$

If κ is a cardinal and X is a set, $[X]^\kappa$ is the set of unordered subsets of $[X]$ of size κ, and $[X]^{<\kappa}$ and $\mathcal{P}_\kappa(X)$ both denote the set $\bigcup_{\gamma<\kappa}[X]^\gamma$. Likewise, if κ is an ordinal and and X is a set, then X^κ is the set of ordered (in ordertype κ) subsets of X of size κ (also, the set of functions from κ to X), and $X^{<\kappa}$

denotes the set $\bigcup_{\gamma<\kappa} X^\gamma$. We will frequently write expressions of the form $f[[X]^{<\omega}]$ where X is a set and f is a function with domain $[X]^{<\omega}$.

We use the usual interval notation for intervals of ordinals, so, for example, if $\alpha < \beta$ then $(\alpha, \beta) = \{\gamma < \beta \mid \gamma > \alpha\}$ and $[\alpha, \beta] = \{\gamma \leq \beta \mid \gamma \geq \alpha\}$.

As usual, $|X|$ denotes the cardinality of X, so if s is a finite sequence then $|s|$ denotes its length.

This page intentionally left blank

This page intentionally left blank

Bibliography

[1] U. Abraham, S. Shelah, *Coding with ladders a well ordering of the reals*, J. Symbolic Logic 67 (2002), no. 2, 579–597

[2] S. Banach, *Über additive Massfunktionen in abstrakten Mengen*, Fund. Math. 15 (1930), 97-101

[3] J. Barwise, *Admissible sets and structures. An approach to definability theory*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin-New York, 1975

[4] D. Burke, *Generic embeddings and the failure of box*, Proc. Amer. Math. Soc. 123 (1995), no. 9, 2867–2871

[5] D. Burke, *Precipitous towers of normal filters*, J. Symbolic Logic 62 (1997), no. 3, 741–754

[6] C.C. Chang, *Sets constructible using $L_{\kappa\kappa}$*, Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967) Amer. Math. Soc., Providence, R.I. (1971), pp. 1–8

[7] C.C. Chang, H.J. Keisler, *Model theory*, Third edition. Studies in Logic and the Foundations of Mathematics, 73. North-Holland Publishing Co., Amsterdam-New York, 1990

[8] P. Dehornoy, *Progrès récents sur l'Hypothèse du Continu [d'après Woodin]*, Séminaire Bourbaki # 915 (2002-2003)

[9] Q. Feng, M. Magidor, W.H. Woodin, *Universally Baire Sets of Reals*, Set Theory of the Continuum (H. Judah, W. Just and W.H. Woodin, eds.), Math. Sci. Res. Inst. Publ., vol. 26, Springer-Verlag, Heidelberg, 1992, pp. 203-242

[10] M. Foreman, *Ideals and generic embeddings*, Handbook of Set Theory, to appear

[11] M. Foreman, M. Magidor, *Large cardinals and definable counterexamples to the continuum hypothesis*, Ann. Pure Appl. Logic 76 (1995), no. 1, 47–97

[12] M. Foreman, M. Magidor, S. Shelah, *Martin's Maximum. saturated ideals, and nonregular ultrafilters. Part I*, Annals of Mathematics, 127 (1988), 1-47

[13] F. Galvin, T. Jech, M. Magidor, *An ideal game*, J. Symbolic Logic 43 (1978) 285-292

[14] M. Gitik, *Nonsplitting subset of $\mathcal{P}_\kappa(\kappa^+)$*, J. Symbolic Logic 50 (1985), no. 4, 881–894

[15] N. Goldring, *Woodin cardinals and presaturated ideals*, Ann. Pure Appl. Logic 55 (1992), no. 3, 285–303

[16] S. Grigorieff, *Intermediate Submodels and Generic Extensions in Set Theory*, Annals of Mathematics 101 (1975), no. 3, 447-490

[17] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, 42. Cambridge University Press, Cambridge, 1993

[18] S. Jackson, *AD and the projective ordinals*, Cabal Seminar 81-95, 117-220, Lecture Notes in Math. 1333, Springer, Berlin 1988

[19] T. Jech, *Set Theory*, The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003

[20] T. Jech, M. Magidor, W. Mitchell, K. Prikry, *Precipitous ideals*, J. Symbolic Logic 45 (1980), no. 1, 1–8

[21] A. Kanamori, *The higher infinite. Large cardinals in set theory from their beginnings.*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1994

[22] A.S. Kechris, Y.N. Moschovakis (Eds.), *Cabal Seminar 76–77. Proceedings of the Caltech-UCLA Logic Seminar 1976–77.* Lecture Notes in Math. Vol. 689, Springer, Berlin, 1978

[23] S. Kripke, *An extension of a theorem of Gaifman-Hales-Solovay*, Fund. Math. 61 1967 29–32

[24] K. Kunen, *A model for the negation of the axiom of choice*, Cambridge Summer School in Mathematical Logic (Cambridge, 1971), pp. 489–494. Lecture Notes in Math. Vol. 337, Springer, Berlin, 1973

[25] K. Kunen, *Saturated ideals*, J. Symbolic Logic 43 (1978), no. 1, 65-76

[26] K. Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam-New York, 1983

[27] P. Larson, S. Shelah, *Bounding by canonical funtions, with CH*, J. Math. Logic 3 (2003), No. 2, 193-215

[28] A. Lewis, *Large cardinals and large dilators*, J. Symbolic Logic 63 (1998), no. 4, 1496–1510

[29] D.A. Martin, R.M. Solovay, *A basis theorem for Σ^1_3 sets of reals*, Ann. of Math. (2) 89 1969 138–159

[30] D.A. Martin, J.R. Steel, *A proof of projective determinacy*, J. Amer. Math. Soc. 2 (1989), no. 1, 71–125

[31] W.J. Mitchell, *A hollow shell: covering lemmas without a core*, Set theory (Curaçao, 1995; Barcelona, 1996), 183–198, Kluwer Acad. Publ., Dordrecht, 1998

[32] Y. Moschovakis, *Descriptive Set Theory*, Studies in Logic, North-Holland, 1980

[33] I. Neeman, J. Zapletal, *Proper forcings and absoluteness in $L(\mathbb{R})$*, Comment. Math. Univ. Carolin. 39 (1998), no. 2, 281–301

[34] D. Scott, *Measurable cardinals and constructible sets*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 9 1961 521–524

[35] D. Seabold, *Chang's conjecture and the non-stationary ideal*, Symbolic Logic 66 (2001), no. 1, 144–170

[36] S. Shelah, *Proper and improper forcing*, second edition. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998

[37] S. Shelah, W.H. Woodin, *Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable*, Israel J. Math. 70 (1990), no. 3, 381–394

[38] S. Shelah, J. Zapletal, *Canonical models for $\aleph_1$-combinatorics*, Ann. Pure Appl. Logic 98 (1999), no. 1-3, 217–259

[39] J.R. Shoenfield, *The problem of predicativity*, Essays on the foundations of mathematics, Magnes Press, Hebrew University, Jerusalem (1961) 132-139

[40] E.C. Smith, Jr., A. Tarski, *Higher degrees of distributivity and completeness in Boolean algebras*, Trans. Amer. Math. Soc. 84 (1957), 230–257

[41] R.M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. of Math. (2) 92 1970 1–56

[42] R.M. Solovay *The independence of DC from AD*, in [**22**], pp. 171–183

[43] J.R. Steel, *Inner models with many Woodin cardinals*, Ann. Pure Appl. Logic 65 (1993), no. 2, 185–209

[44] J.R. Steel, *The core model iterability problem*, Lecture Notes in Logic, 8. Springer-Verlag, Berlin, 1996

[45] J.R. Steel, *An outline of inner model theory*, Handbook of Set Theory, to appear

[46] J.R. Steel, *The derived model theorem*, preprint, January 2003

[47] J.R. Steel, R. Van Wesep, *Two consequences of determinacy consistent with choice*, Trans. Amer. Math. Soc. 272 (1982), no. 1, 67–85

[48] S. Todorcevic, *Universally meager sets and principles of generic continuity and selection in Banach spaces*, preprint, October, 2003

[49] S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. 16 (1930), 140–150

[50] R. Van Wesep, *Wadge degrees and descriptive set theory*, in [**22**], pp. 151–170

[51] J. Vickers, P. Welch, *On elementary embeddings from an inner model to the universe*, J. Symbolic Logic 66 (2001), no. 3, 1090–1116

[52] W.H. Woodin, *Supercompact cardinals, sets of reals, and weakly homogeneous trees*, Proc. Nat. Acad. Sci. U.S.A. 85 (1988), no. 18, 6587–6591

[53] W.H. Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, DeGruyter Series in Logic and Its Applications, vol. 1, 1999

[54] W.H. Woodin, *The Continuum Hypothesis*, Proceedings of the 2000 Logic Colloquium, to appear

[55] W.H. Woodin, *Set Theory after Russell; The journey back to Eden*, Proceedings of the 2003 Russell Conference, to appear

[56] Y. Yoshinobu, *On strength of precipitousness of some ideals and towers*, J. Math. Soc. Japan 51 (1999), no. 3, 535–541

[57] J. Zapletal, *Terminal notions in set theory*, Ann. Pure Appl. Logic 109 (2001), no. 1-2, 89–116

This page intentionally left blank

Index

The stationary tower is an important method in modern set theory, invented by Hugh Woodin in the 1980s. It is a means of constructing generic elementary embeddings and can be applied to produce a variety of useful forcing effects.

Hugh Woodin is a leading figure in modern set theory, having made many deep and lasting contributions to the field, in particular to descriptive set theory and large cardinals. This book is the first detailed treatment of his method of the stationary tower that is generally accessible to graduate students in mathematical logic. By giving complete proofs of all the main theorems and discussing them in context, it is intended that the book will become the standard reference on the stationary tower and its applications to descriptive set theory.

The first two chapters are taken from a graduate course Woodin taught at Berkeley. The concluding theorem in the course was that large cardinals imply that all sets of reals in the smallest model of set theory (without choice) containing the reals are Lebesgue measurable. Additional sections include a proof (using the stationary tower) of Woodin's theorem that, with large cardinals, the Continuum Hypothesis settles all questions of the same complexity as well as some of Woodin's applications of the stationary tower to the studies of absoluteness and determinacy.

The book is suitable for a graduate course that assumes some familiarity with forcing, constructibility, and ultrapowers. It is also recommended for researchers interested in logic, set theory, and forcing.

For additional information
and updates on this book, visit
www.ams.org/bookpages/ulect-32

ISBN 0-8218-3604-8

9 780821 836040

ULECT/32