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Quadratic Algebras

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Contents

Introduction

The goal of this book is to introduce the reader to some recent developments in the study of associative algebras defined by quadratic relations. More precisely, we are interested in (not necessarily commutative) algebras over a field that can be presented using a finite number of generators and (possibly nonhomogeneous) quadratic relations. This book is devoted to some aspects of the theory of such algebras, mostly evolving around the notions of Koszul algebra and Koszul duality. Its content is a mixture of known results with a few original results that we circulated since 1994 as a preprint of the same title.

One of the original motivations for the study of quadratic algebras came from the theory of quantum groups (see [**43, 77**]). Namely, quadratic algebras provide a convenient framework for "noncommutative spaces" on which quantum groups act (see [**78**]). One of the basic problems that arose in this context is how to control the growth of a quadratic algebra (e.g., measured by Hilbert series). A related question is whether there are generalizations of the Poincaré-Birkhoff-Witt theorem (for universal enveloping algebras) to more general quadratic algebras. The core of this book is our attempt to present some partial solutions. It turns out that one can shed some light on questions of this kind using the remarkable notion of Koszul algebra introduced by S. Priddy [**104**]. In fact, the study of this notion brought some dramatic changes to the area. Loosely speaking, our experience shows that general quadratic algebras behave as badly as possible, while for Koszul algebras the situation is usually much nicer. As we hope to convince the reader, the study of Hilbert series provides a good illustration of this principle.

Perhaps one of the important features of the theory of Koszul algebras is duality: for each Koszul algebra there is a dual Koszul algebra (roughly speaking, it is obtained by passing to the dual space of generators and the orthogonal space of quadratic relations). This often leads to remarkable connections between seemingly unrelated problems. For example, Koszul duality of the symmetric algebra and the exterior algebra underlies the famous description of coherent sheaves on projective spaces in terms of modules over the exterior algebra due to J. Bernstein, I. Gelfand and S. Gelfand [**27**]. More generally, in a number of situations one can prove an equivalence of derived categories of modules over Koszul dual algebras (see [**23, 11, 24, 51**]). This topic is beyond the scope of our book although we will discuss some more elementary aspects of Koszul duality.

The notion of Koszulness also proved to be a really impressive prediction tool. In many examples a few observations may suggest that some quadratic algebra is Koszul. Then this conjecture turns out to be related to some important and nontrivial features of the setting. It is also quite amazing that many important quadratic algebras naturally arising in various fields of mathematics are Koszul. Examples known to us arise in the following areas:

(i) algebraic geometry—certain homogeneous coordinate algebras are Koszul (see [**29, 37, 39, 67, 50, 72, 73, 89, 96**]);
(ii) representation theory—certain subcategories of the category $\mathcal{O}$ for a semisimple complex Lie algebra are governed by Koszul algebras (see [**19, 24**]);
(iii) noncommutative geometry—the Koszulness condition arises naturally in the theory of exceptional collections; the algebras describing certain noncommutative deformations of projective spaces are Koszul (see [**30, 31, 117**]);
(iv) topology—Steenrod algebra, cohomology algebras of formal rational $K[\pi, 1]$-spaces, holonomy algebras of supersolvable hyperplane arrangements, as well as some algebras related to configuration spaces of surfaces are Koszul; the category of perverse sheaves on a triangulated space is equivalent to modules over a Koszul algebra (see [**104, 88, 113, 28, 97, 127**]);
(v) number theory—the Milnor K-theory ring of any field (tensored with $\mathbb{Z}/l\mathbb{Z}$ for a prime l) is conjectured to be Koszul—this is a strengthening of the Bloch-Kato conjecture relating Milnor K-theory with Galois cohomology (see [**103, 102**]);
(vi) noncommutative algebra—the universal algebra generated by pseudoroots of a noncommutative polynomial is Koszul (see [**111, 93**]).

Checking the Koszul property usually requires some effort and the methods of proof vary from one case to another. Although we do not try to give a systematic exposition of these methods here, the reader will find a few sample techniques for checking Koszulness (mostly in chapter 2).

As we have already mentioned, one of the central questions studied in our book is how to generalize the Poincaré-Birkhoff-Witt-theorem (PBW-theorem) to quadratic algebras. Recall that the classical PBW-theorem for the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ can be formulated in two different ways. In the first formulation one starts with a basis of $\mathfrak{g}$ and then the theorem states that certain standard monomials in basis elements form a basis of $U\mathfrak{g}$. Another formulation simply asserts that the associated graded algebra of $U\mathfrak{g}$ with respect to the standard filtration coincides with the symmetric algebra $\mathbb{S}\mathfrak{g}$. Thus, the first way to generalize the PBW-theorem to other algebras is to modify the notion of standard monomials. Assume that we have a graded quadratic algebra (i.e., quadratic relations are homogeneous). Then using lexicographical order on the set of all monomials in generators one can define a certain set of standard monomials (depending on quadratic relations). The analogue of the PBW-theorem in this case states that if the standard monomials form a basis in the grading component of degree 3 then the same is also true for all grading components (so that we get a *PBW-basis* in our algebra). This theorem is a particular case of the so-called diamond lemma in the theory of Gröbner bases developed in works on combinatorial algebra in the late 70s (see [**26, 35, 36**]). Note that the universal enveloping algebra $U\mathfrak{g}$ can be homogenized by adding an extra central generator, so that the classical PBW-theorem would fit into this context.

Before stating the second generalization of the PBW-theorem let us say a few words about the terminology adopted in the book. We use the term "quadratic algebra" only in reference to algebras defined by homogeneous quadratic relations (because with the exception of chapter 5 we consider only such algebras). Assigning degree 1 to each generator one can view a quadratic algebra as a graded algebra $A = \bigoplus_{n\geqslant 0} A_n$ such that A_0 is the ground field and A is the quotient of the tensor algebra of A_1 by an ideal generated in degree 2. Note that sometimes (e.g., in applications

to representation theory) it is necessary to consider more general quadratic algebras such that A_0 is not necessarily equal to the ground field but rather is a semisimple algebra. We will briefly discuss algebras of this kind in section 9 of chapter 2.

Our second generalization of the PBW-theorem deals with a "nonhomogeneous quadratic algebra", i.e., an algebra with a finite number of generators and nonhomogeneous quadratic defining relations. If A is such an algebra then one can consider the natural filtration on A determined by the set of generators. Let us denote by $\mathrm{gr}A$ the associated graded algebra. On the other hand, one can truncate the relations in A leaving only their homogeneous quadratic parts. Let $A^{(0)}$ be the obtained quadratic algebra. The nonhomogeneous PBW-theorem states that the natural map $A^{(0)} \to \mathrm{gr}A$ is an isomorphism provided $A^{(0)}$ is Koszul and a certain self-consistence condition is satisfied (this result was proved independently by A. Braverman and D. Gaitsgory [**33**]). This self-consistency condition is obtained by looking at expressions of degree 3 in generators. In the case $A = U\mathfrak{g}$ it coincides with the Jacobi identity for the Lie bracket on $\mathfrak{g}$.

It is interesting that the notion of Koszulness appears also in the first generalization of the PBW-theorem: quadratic algebras having a basis of standard monomials, called *PBW-algebras*, are always Koszul (this observation goes back to S. Priddy [**104**]). However, the converse is not true: Koszul algebras are not necessarily PBW (see section 3 of chapter 4). In fact, the class of PBW-algebras is substantially smaller than that of Koszul algebras and is much easier to study. For example, the set of PBW-algebras with a given number of generators is constructible in Zariski topology while the set of Koszul algebras is often not constructible (see section 3 of chapter 4 and section 6 of chapter 6). On the other hand, there are many parallel results for both classes of algebras. Firstly, both properties can be formulated in terms of distributivity of certain lattices of vector spaces. Secondly, various natural operations with quadratic algebras, such as quadratic duality, free product, tensor product, Segre product and Veronese powers preserve both classes. The comparison between the classes of Koszul and PBW-algebras is also an important part of the present work. In our experience PBW-algebras often provide a good testing ground for guessing the general pattern that might hold for all Koszul algebras. Usually there is no problem with proving that a pattern holds for PBW-algebras; however, the case of Koszul algebras is often much harder (if at all accessible).

One of the most striking properties of Koszul algebras is the following.

Koszul Deformation Principle (V. Drinfeld [**43**]). *If a formal family of graded quadratic algebras $A(t)$ is flat in the grading components of degree $\leqslant 3$ and the algebra $A(0)$ is Koszul then the family is flat in all degrees.*

More precisely, a similar statement holds for local deformations (in Zariski topology) if we consider only a finite number of grading components (see Theorem 2.1 of chapter 6). The second version of the PBW-theorem considered above can be easily deduced from this principle. Another unexpected consequence that we derive from it is the finiteness of the number of Hilbert series of Koszul algebras with a fixed number of generators (the analogous statement for quadratic algebras is wrong). We conjecture that Hilbert series of Koszul algebras enjoy several interesting properties that can be easily checked for PBW-algebras (although we prove that these two sets of Hilbert series are different). For example, we conjecture that the Hilbert series of a Koszul algebra is always rational.

The study of Hilbert series of Koszul algebras led to the discovery in [**100**] of an interesting connection with the theory of discrete stochastic processes. Namely, to every Koszul algebra A one can associate a *one-dependent* stationary stochastic sequence of 0's and 1's. It is convenient to encode probabilities of various events in such a process by a linear functional $\phi : \mathbb{R}\{x_0, x_1\} \to \mathbb{R}$ on the free algebra in two variables, taking nonnegative values on all monomials and satisfying $\phi(1) = 1$. Then the condition of one-dependence is equivalent to the equation

$$\phi(f \cdot (x_0 + x_1) \cdot g) = \phi(f)\phi(g),$$

where $f, g \in \mathbb{R}\{x_0, x_1\}$. Abusing the terminology we call such a functional ϕ a *one-dependent process.* It is easy to see that ϕ is uniquely determined by the values $(\phi(x_1^n))$. Now the one-dependent process associated with a Koszul algebra A is defined by

$$\phi_A(x_1^{n-1}) = a_n/a_1^n,$$

where $a_n = \dim A_n$. Nonnegativity of values of ϕ on all monomials is equivalent to a certain system of polynomial inequalities for the numbers a_n. The fact that these inequalities are indeed satisfied for a Koszul algebra seems to be a remarkable coincidence. However, the analogy between the two theories does not end here. It turns out that under this correspondence the subclass of PBW-algebras maps to the set of so-called two-block-factor processes. The relation between all one-dependent processes and the subclass of two-block-factors was intensively studied in the 90s after it was proved in [**2**] that a one-dependent process does not have to be a two-block factor (see [**1, 118, 122**]). This topic seems to be surprisingly similar to the relation between Koszul and PBW-algebras. Motivated by this analogy we conjecture that the Hilbert series associated with every one-dependent process admits a meromorphic continuation to the entire complex plane. Rationality of Hilbert series of Koszul algebras would follow from this (by a theorem of E. Borel [**32**]). We also observe that the polynomial inequalities satisfied by the numbers $(\phi(x_1^n))$ form a subset in the well-known system of inequalities defining the notion of a *totally positive sequence* (also known as *Polya frequency sequence*). It is known that the generating series of a totally positive sequence admits a meromorphic continuation (see [**71**]). This can be considered as another hint in favor of our conjecture.

Here is the more detailed outline of the content of the book.

Chapter 1 contains some basic definitions and results concerning cohomology of graded algebras, quadratic algebras and distributivity of lattices. In particular, in section 2 we define quadratic duality for quadratic algebras and quadratic modules (we use the term "Koszul duality" when referring to this duality in the case of Koszul algebras and Koszul modules).

In chapter 2 we describe various equivalent definitions of Koszulness, including Backelin's criterion in terms of distributivity of lattices (see [**15**]). We give similar equivalent definitions for a related notion of n-Koszulness that has an advantage of being defined by a finite number of conditions. We also show that many results about quadratic and Koszul algebras have natural analogues for quadratic and Koszul *modules.* In section 5 we consider the problem of preservation of Koszulness under homomorphisms of various types between graded algebras, generalizing some results of Backelin and Fröberg [**20**]. In section 7 we give examples of projective varieties with Koszul homogeneous coordinate algebras. In section 8 we explain how to associate to a Koszul algebra A a (graded) *infinitesimal bialgebra* (or ϵ-*bialgebra*)

V_A. This construction can be viewed as a categorification of the one-dependent process ϕ_A associated with A, because the values of ϕ_A on monomials are given by dimensions of certain multigrading components of V_A. In section 9 we consider some generalizations of the notion of Koszulness including an important case of graded algebras $A = \bigoplus_{n\geqslant 0} A_n$ such that A_0 is a semisimple algebra (in the rest of the book we assume that A_0 is the ground field). We also give an interpretation of Koszul algebras in terms of monoidal functors from a certain universal (nonunital) monoidal category.

In chapter 3 we consider several natural operations on quadratic algebras and modules that preserve Koszulness and discuss the behavior of Hilbert series under these operations. Following [**20**] we consider free sums, free products, along with several types of tensor products, the Segre product $A \circ B$, the dual operation "black circle product" $A \bullet B$ and Veronese powers $A^{(n)}$. The operation $A \bullet B$ is also closely related to the internal cohomomorphism operation introduced by Manin (see [**77, 79**]). We prove that if one of the algebras is Koszul then the Hilbert series of $A \bullet B$ can be computed in terms of those of A and B and show that this is impossible if both algebras are not Koszul. An interesting application of these operations is given in section 5, where we show, following D. Piontkovskii [**92**], that Koszulness of a quadratic algebra A cannot be determined from the knowledge of the Hilbert series of A and $A^!$.

Chapter 4 is devoted to PBW-algebras. We start by giving a proof of the PBW-theorem for quadratic algebras that gives a criterion for the existence of a PBW-basis (as we have mentioned before, this is really a particular case of the diamond lemma). Then we prove that PBW-algebras are Koszul and give a criterion of the PBW-property in terms of distributivity of lattices in the spirit of Backelin's criterion of Koszulness. We also check that the class of PBW-algebras is stable under quadratic duality and under all operations considered in chapter 3. Then we discuss Hilbert series of PBW-algebras. We show that the Hilbert series of a PBW-algebra is a generating function for the number of paths in a finite oriented graph and hence is rational. In section 7 we prove a generalization of the PBW-theorem involving filtrations with values in an ordered semigroup. In section 8 we consider commutative PBW-algebras. We prove that they are Koszul and compute their Hilbert series. We also present some examples showing that the sets of Hilbert series of PBW-algebras and Koszul algebras are different. In section 9 we discuss a generalization of the classes of Koszul and PBW-algebras from graded algebras to $\mathbb{Z}$-algebras. In section 11 we consider 3-dimensional elliptic Sklyanin algebras. We prove that they are Koszul but do not admit a PBW-basis even viewed as $\mathbb{Z}$-algebras.

In chapter 5 we consider nonhomogeneous quadratic algebras. For these algebras we prove in section 2 the PBW-theorem involving an analogue of the Jacobi identity and Koszulness of the corresponding homogeneous quadratic algebra. We also prove in section 3 a version of this theorem for nonhomogeneous quadratic modules. In section 4 we consider an analogue of quadratic duality for the nonhomogeneous case. It turns out that the dual object to a nonhomogeneous quadratic algebra is a so-called CDG-algebra (curved DG-algebra). In section 5 we give some examples of nonhomogeneous quadratic algebras and modules. In particular, we list all solutions of the analogue of the Jacobi identity in the case of the quadratic relations corresponding to a free commutatative superalgebra, and consider an

example related to the PBW-theorem for quantum universal enveloping algebras (Example 6). The remainder of this chapter is devoted to various cohomological calculations related to nonhomogeneous quadratic duality.

Chapter 6 is devoted to the Koszul Deformation Principle for quadratic algebras and some of its consequences, such as finiteness of the number of Hilbert series of Koszul algebras with a fixed number of generators. Furthermore, in section 3 we give an explicit bound on this number and in section 7 we prove that the number of such Hilbert series is finite even if the ground field is allowed to vary. In section 4 we discuss some results on generic algebras among quadratic algebras with a given number of generators and relations. In section 5 we consider examples of possible Hilbert series for algebras with a small number of generators and relations. Section 6 contains counterexamples from [**56**] showing that the set of Koszul algebras is not constructible and that the set of Hilbert series of quadratic algebras with a given number of generators is infinite.

In chapter 7 we explain the connection between Koszul algebras and one-dependent processes. We start by formulating several conjectures on Hilbert series of Koszul algebras, such as the rationality conjecture. Then we derive a system of polynomial inequalities satisfied by the numbers $a_n = \dim A_n$ for a Koszul algebra A. The polynomials of a_n appearing in these inequalities express the dimensions of multigrading components of the ϵ-bialgebra V_A. Then we show that these inequalities allow one to associate a one-dependent process to the sequence (a_n). We show that Koszul duality corresponds to the natural duality on one-dependent processes and also introduce analogues of some other operations on Koszul algebras for one-dependent processes. In section 5 we show that the one-dependent process associated with a PBW-algebra is a two-block-factor and that every two-block-factor can be approximated by those obtained from PBW-algebras. In section 7 we review the notion of a Hilbert space representation of a one-dependent process due to V. de Valk [**121**]. In section 8 we discuss the conjecture that the Hilbert series of a one-dependent process can be extended meromorphically to the entire complex plane. We show that this series always admits a meromorphic continuation to the disk $|z| < 2$ (it converges for $|z| < 1$) and prove the conjecture for two-block-factor processes. In section 9 we give a construction due to B. Tsirelson of a one-dependent process associated with an arbitrary quadratic algebra and a Hermitian form on the space of generators. In section 10 we consider an analogue for Koszul *modules* of the construction of a one-dependent process from a Koszul algebra.

In the Appendix we recall some definitions concerning DG-algebras, DG-modules and Massey products.

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